23. DEFINITE INTEGRATION

1. INTRODUCTION

Let f(x) be a continuous function defined on a closed interval [a, b] and $\int f(x)\,dx = F(x) + c$ then $\int_a^b f(x)\,dx = [F(x)]_a^b$ or $\int_a^b f(x)\,dx = F(b) - F(a)$ is called the definite integral of f(x) within limits a and b. The interval [a, b] is called the range of integration. Every definite integral has a unique solution.

Note: $\int_a^b f(x)\,dx = F(b) - F(a)$ also represents the net area of the curve f(x) with x-axis. $\int_0^{\pi/2} \sin^2 x\,dx$

Sol: $\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4}$

Illustration 1: If $\int_0^1 (3x^2 + 2x + k)\,dx = 0,$ find the value of k. **(JEE MAIN)**

Sol: Here the answer of the definite integral $\int_0^1 \left[3x^2 + 2x + k\right]dx$ is already given i.e. 0 hence by using simple integral formulas we can solve it and by comparing it to 0, we will obtain the value of k.

Here, we have, $\int_0^1 (3x^2 + 2x + k)\,dx = 0$

$\left[3\frac{x^3}{3} + 2\frac{x^2}{2} + kx\right]_0^1 = 0$; $\left[x^3 + x^2 + kx\right]_0^1 = 0$

(1 + 1 + k) – (0 + 0 + 0) = 0 ; 2 + k = 0 ⇒ k = –2

Illustration 2: Evaluate: $\int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2)\,dx.$ **(JEE MAIN)**

Sol: As we know $\int_a^b \{f(x) \pm g(x)\}dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$. Hence by using this method we can solve the given definite integral.

We have, $\int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2)\,dx = 2\int_0^{\frac{\pi}{4}} \sec^2 x\,dx + \int_0^{\frac{\pi}{4}} x^3\,dx + 2\int_0^{\frac{\pi}{4}} dx$

$= 2\left[\tan x\right]_0^{\pi/4} + \left[\frac{x^4}{4}\right]_0^{\pi/4} + 2[x]_0^{\pi/4} = 2\left(\tan\frac{\pi}{4} - \tan 0\right) + \left[\frac{(\pi/4)^4}{4} - 0\right] + 2\left[\frac{\pi}{4} - 0\right]$

$$= 2(1-0)+\left(\frac{\pi^4}{4^5}-0\right)+\frac{\pi}{2} = 2+\frac{\pi^4}{1024}+\frac{\pi}{2}$$

2. PROPERTIES OF DEFINITE INTEGRALS

Property 1

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

Here x is a dummy variable; it can be replaced by any other variable t, u,........

$$\int_0^{\pi/2} \sin(x)\,dx = \int_0^{\pi/2} \sin t\,dt = \int_0^{\pi/2} \sin u\,du =$$

This is similar to the summation property $\sum_{T=1}^{10} r^2 = \sum_{T=1}^{10} t^2 = \sum_{U=1}^{10} u^2 = ...$

Property 2

$$\int_a^b f(x)\,dx = -\int_b^a f(x)dx$$

i.e. the interchange of limits of a definite integral changes only its sign.

Property 3

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad (a < c < b)$$

Generally, this property is used when the integrand has two or more rules in the integration interval

$\Rightarrow \quad \int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + ... + \int_{c_n}^b f(x)\,dx$ where $a < c_1 < c_2 < .... c_n < b$.

Illustration 3: Evaluate: $\int_1^4 f(x)dx$, where $f(x) = \begin{cases} 2x+8, & 1 \le x \le 2 \\ 6x, & 2 \le x \le 4 \end{cases}$ **(JEE MAIN)**

Sol: Here as we know, $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $(a < c < b)$. Hence by using this property and solving by using the integral formula we can solve it.

We have, $I = \int_1^4 f(x)\,dx$

$$= \int_1^2 f(x)\,dx + \int_2^4 f(x)dx = \int_1^2 (2x+8)dx + \int_2^4 6x\,dx$$

$$= \left[x^2+8x\right]_1^2 + \left[3x^2\right]_2^4 = \left[(2)^2+8(2)-(1)^2-8(1)\right] + \left[3(4)^2-3(2)^2\right]$$

= 11 + 36 = 47.

Illustration 4: Evaluate : $\int_0^2 |1-x|\,dx$ **(JEE MAIN)**

Sol: Here $|1-x| = \begin{cases} 1-x, \text{ when } 0 \le x \le 1 \\ x-1, \text{ when } 1 \le x \le 2 \end{cases}$ therefore, similar to the problem above, we can solve it.

$$|1-x| = \begin{cases} 1-x, \text{ when } 0 \le x \le 1 \\ x-1, \text{ when } 1 \le x \le 2 \end{cases}$$

$$\therefore\ I = \int_0^1 (1-x)\,dx + \int_1^2 (x-1)\,dx = \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2 = (1/2 - 0) + (0 + 1/2) = 1$$

Property 4

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Property 5

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Application: $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx = \frac{b-a}{2}$

PLANCESS CONCEPTS

With the help of the above property, the following integrals can be obtained.

$$\int_0^{\pi/2} f(\sin x)dx = \int_0^{\pi/2} f(\cos x)dx \ ; \ \int_0^{\pi/2} f(\tan x)dx = \int_0^{\pi/2} f(\cot x)dx$$

$$\int_0^{\pi/2} f(\sin 2x)\sin x\ dx = \int_0^{\pi/2} f(\sin 2x)\cos x\,dx \ ; \ \int_0^1 f(\log x)dx = \int_0^1 f[\log(1-x)]dx$$

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}dx = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}dx = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \frac{\tan^n x}{1+\tan^n x}dx = \int_0^{\pi/2} \frac{\cot^n x}{1+\cot^n x}dx = \frac{\pi}{4} \ ; \ \int_0^{\pi/2} \frac{1}{1+\tan^n x}dx = \int_0^{\pi/2} \frac{1}{1+\cot^n x}dx = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \operatorname{cosec}^n x}dx = \int_0^{\pi/2} \frac{\operatorname{cosec}^n x}{\operatorname{cosec}^n x + \sec^n x}dx = \frac{\pi}{4} \ ; \ \int_0^{\pi/4} \log(1+\tan x)dx = \frac{\pi}{8}\log 2$$

$$\int_0^{\pi/2} \log \cot x\ dx = \int_0^{\pi/2} \log \tan x\ dx = 0$$

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Illustration 5: Prove that $\int_0^1 \cot^{-1}(1-x+x^2)\,dx = 2\int_0^1 \tan^{-1} x.dx$ **(JEE MAIN)**

Sol: As we know $\cot^{-1}\left(\frac{a}{b}\right) = \tan^{-1}\left(\frac{b}{a}\right)$ and $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ by using these two formulae we can solve the given problem.

$$\int_0^1 \cot^{-1}(1-x+x^2)\,dx = \int_0^1 \tan^{-1}\left[\frac{1}{1-x+x^2}\right] = \int_0^1 \tan^{-1}\left[\frac{1+x-x}{1-x(1-x)}\right] = \int_0^1 \tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right)dx$$

$$= \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1}(1-x)dx = 2\int_0^1 \tan^{-1} x.dx$$

$$\left(\because \tan^{-1}\left(\frac{a+b}{1-ab}\right) = \tan^{-1} a + \tan^{-1} b\right)$$

Illustration 6: Find the value of $\int_0^1 \log\left(\frac{1}{x}-1\right) dx$ **(JEE MAIN)**

Sol: Here $\log\left(\frac{1-x}{x}\right) = \log(1-x) - \log(x)$ and $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ by using these two formulae we can solve it.

$$\int_0^1 \log\left(\frac{1-x}{x}\right) dx = \int_0^1 \log\,(1-x)dx - \int_0^1 \log\,(x)dx = \int_0^1 \log\left[1-(1-x)\right]dx - \int_0^1 \log x\,dx = \int_0^1 \log x\,dx - \int_0^1 \log x\,dx$$

$$= \int_0^1 \log\,(x)dx - \int_0^1 \log\,(x)dx = 0$$

Illustration 7: Evaluate: $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x}dx$ **(JEE MAIN)**

Sol: As $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ therefore we can write $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x}dx$ in the form of

$\int_0^{\pi/2} \frac{a\sin(\pi/2-x) + b\cos(\pi/2-x)}{\sin(\pi/2-x) + \cos(\pi/2-x)}dx$ and then adding these two equations we can solve the given problem.

$$I = \int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x}dx \quad \text{... (i)}$$

$$I = \int_0^{\pi/2} \frac{a\sin(\pi/2-x) + b\cos(\pi/2-x)}{\sin(\pi/2-x) + \cos(\pi/2-x)}dx = \int_0^{\pi/2} \frac{a\cos x + b\sin x}{\sin x + \cos x}dx \quad \text{... (ii)}$$

Adding (i) and (ii),

$$\therefore 2I = \int_0^{\pi/2} \frac{(a+b)(\sin x + \cos x)}{\sin x + \cos x}dx = \int_0^{\pi/2} (a+b)dx = (a+b)\pi/2 \Rightarrow I = (a+b)\,\pi/4$$

Illustration 8: Show that $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$ **(JEE ADVANCED)**

Sol: This problem is similar to the problem above.

Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx$ ... (i)

By property 4, we have

$$I = \int_0^{\pi/2} \frac{\sin^2((\pi/2)-x)}{\sin((\pi/2)-x) + \cos((\pi/2)-x)}dx = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x}dx \quad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx \Rightarrow I = \frac{1}{2}\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{1}{(1/\sqrt{2})\sin x + (1/\sqrt{2})\cos x} dx$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{1}{\cos(\pi/4)\sin x + \sin(\pi/4)\cos x} dx = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{1}{\sin\left(x + (\pi/4)\right)} dx$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \operatorname{cosec}\left[x + \frac{\pi}{4}\right] dx = \frac{1}{2\sqrt{2}}\left\{\log\tan\left[\frac{x}{2} + \frac{\pi}{8}\right]\right\}_0^{\pi/2}$$

$$= \frac{1}{2\sqrt{2}}\left\{\log\tan\left[\frac{\pi}{4} + \frac{\pi}{8}\right] - \log\tan\frac{\pi}{8}\right\} = \frac{1}{2\sqrt{2}}\log\left(\frac{\tan(3\pi/8)}{\tan(\pi/8)}\right) = \frac{1}{2\sqrt{2}}\log\left(\frac{\cot(\pi/8)}{\tan(\pi/8)}\right)$$

$$= \frac{2}{2\sqrt{2}}\log\cot\frac{\pi}{8} = \frac{1}{\sqrt{2}}\log(\sqrt{2} + 1)$$

Illustration 9: Evaluate : $\int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\tan x}}{1 + \sqrt{\tan x}} dx$ **(JEE ADVANCED)**

Sol: By putting $\tan x = \frac{\sin x}{\cos x}$ and using the property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, we can solve the given problem.

Let $I = \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\tan x}}{1 + \sqrt{\tan x}} dx \quad \Rightarrow \quad I = \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$ ... (i)

On applying $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ we get

$$I = \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\sin\left((3\pi/4) - (\pi/4) - x\right)}}{\sqrt{\cos\left((3\pi/4) - (\pi/4) - x\right)} + \sqrt{\sin\left((3\pi/4) - (\pi/4) - x\right)}} dx$$

$$= \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\sin\left((\pi/2) - x\right)}}{\sqrt{\cos\left((\pi/2) - x\right)} + \sqrt{\sin\left((\pi/2) - x\right)}} dx$$

$$= \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx + \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_{-\pi/4}^{3\pi/4} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$= \int_{-\pi/4}^{3\pi/4} dx = [x]_{-\pi/4}^{3\pi/4} = \left[\frac{3\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \left[\frac{3\pi}{4} + \frac{\pi}{4}\right] = \pi \Rightarrow I = \frac{\pi}{2}$$

Illustration 10: The value of $\int_0^{\pi/2} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) dx$ is **(JEE ADVANCED)**

Sol: Similar to the problems above, we can write $\int_0^{\pi/2} \log\left(\frac{4 + 3\sin x}{4 + 3\cos x}\right) dx$ as

$\int_0^{\pi/2} \log\left(\frac{4 + 3\sin\left((\pi/2) - x\right)}{4 + 3\cos\left((\pi/2) - x\right)}\right) dx$ and then by adding these two equations we can solve the given problem.

Let $I = \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$

On applying property 5, we get

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\sin((\pi/2)-x)}{4+3\cos((\pi/2)-x)}\right)dx$$

$$= \int_0^{\pi/2} \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx = -\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx = -I \Rightarrow I = 0$$

Thus, $\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx = 0$

Illustration 11: $I = \int_0^{\pi/2} \frac{dx}{4+5\sin x}$ **(JEE ADVANCED)**

Sol: Let $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and then by putting $\tan\frac{x}{2} = t$, we can solve the given problem.

$$I = \int_0^{\pi/2} \frac{dx}{4+5\ (2\tan(x/2)/1+\tan^2(x/2))} = \int_0^{\pi/2} \frac{\sec^2(\pi/2)dx}{4+4\tan^2(\pi/2)+10\tan(\pi/2)}$$

Let $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2} = dt$

$$\Rightarrow \int_0^1 \frac{2dt}{4+4t^2+10t} = \frac{1}{2}\int_0^1 \frac{dt}{(t+(1/2))(t+2)} = \frac{1}{3}\int_0^1 \frac{1}{(t+(1/2))} - \frac{1}{(t+2)}dt = \frac{1}{3}\left[\ln\frac{t+(1/2)}{t+2}\right]_0^1 = \frac{1}{3}\log 2$$

Illustration 12: Evaluate : $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ **(JEE ADVANCED)**

Sol: Let $\tan x = \frac{\sin x}{\cos x}$ and then using property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, we can solve the given problem.

$$\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad \text{... (i)}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}dx \quad [\because \text{ here } a+b=\pi/2]$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \quad \text{... (ii)}$$

$$\therefore\ 2I = \int_{\pi/6}^{\pi/3} 1\,dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6} \Rightarrow I = \frac{\pi}{12}$$

Property 6

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & \text{if } f(-x) = f(x) \text{ (even function)} \\ 0 & \text{if } f(-x) = -f(x) \text{ (odd function)} \end{cases}$$

Note: This property is to be used if the integrand is either an even or odd function of x

Illustration 13: $\int_{-\pi/2}^{\pi/2} \cos^2 x\,dx$ is equal to **(JEE MAIN)**

Sol: As $\int_{-\pi/2}^{\pi/2} \cos^2 x\,dx = 2\int_{0}^{\pi/2} \cos^2 x\,dx$, therefore using property 7 we can solve it.

Here $I = 2\int_{0}^{\pi/2} \cos^2 x\,dx \; \{\because f(-x) = f(x)\}$; $\int_{0}^{\pi/2} (1+\cos 2x)dx = \left\{x + \frac{\sin 2x}{2}\right\}_0^{\pi/2} = \frac{\pi}{2}$

Illustration 14: $\int_{-1}^{1} \frac{x^3 \sin(1+x^2)}{1+x^2} dx$ is equal to **(JEE ADVANCED)**

Sol: Here by using the property $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & \text{if } f(-x) = f(x) \text{ (even function)} \\ 0 & \text{if } f(-x) = -f(x) \text{ (odd function)} \end{cases}$

Here $f(x) = \frac{x^3 \sin(1+x^2)}{1+x^2}$ & $f(-x) = -\frac{x^3 \sin(1-x^2)}{1+x^2}$

$\because f(x) = -f(x)$

$\therefore I = 0$

Property 7: $\int_{0}^{2a} f(x)dx = \begin{cases} 2\int_{0}^{a} f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

Note: The above property is used to halve the limits

Illustration 15: Evaluate : $\int_{0}^{2\pi} \frac{\sin 2\theta}{a - b\cos\theta} d\theta$ **(JEE MAIN)**

Sol: Let $\int_{0}^{2a} f(x)dx = \begin{cases} 2\int_{0}^{a} f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$. Hence by using this property we can solve the given problem.

Let $I = \int_{0}^{2\pi} \frac{\sin 2\theta}{a - b\cos\theta} d\theta \rightarrow$ Let $f(\theta) = \frac{\sin 2\theta}{a - b\cos\theta}$

$$f(2\pi - \theta) = \frac{\sin 2(2\pi - \theta)}{a - b\cos(2\pi - \theta)} = \frac{-\sin 2\theta}{a - b\cos\theta} = -f(\theta)$$

By property 7, we have

$$\therefore \int_{0}^{2\pi} \frac{\sin 2\theta}{a - b\cos\theta} d\theta = 0$$

Illustration 16: Evaluate $\int_0^{2\pi} x\sin^4 x \cos^6 x\, dx$ **(JEE ADVANCED)**

Sol: Similar to the problem above.

$$I = \int_0^{2\pi} x\sin^4 x \cos^6 x\, dx = \int_0^{2\pi} (2\pi - x)\sin^4 x \cos^6 x\, dx$$

$$2I = 2\pi\int_0^{2\pi} \sin^4 x\ \cos^6 x\ dx\ ;\quad I = 2\pi\int_0^{\pi} \sin^4 x\ \cos^6 x\ dx\ ;$$

$$I = 4\pi\int_0^{\pi/2} \sin^4 x\ \cos^6 x\ dx\ ;\quad I = 4\pi\int_0^{\pi/2} \cos^4 x\sin^6 x\ ;$$

$$\Rightarrow I = \frac{2\pi}{16}\int_0^{\pi/2} (\sin 2x)^4 dx \Rightarrow 2x = t \Rightarrow dx = \frac{dt}{2}$$

$$\Rightarrow I = \frac{\pi}{16}\int_0^{\pi} \sin^4 t\ dt = \frac{\pi}{8}\int_0^{\pi/2} \sin^4 t\, dt \qquad \Rightarrow I = \frac{\pi}{8}\left[\frac{1}{2}\int_0^{\pi/2}\left(\sin^4 t + \sin^4 t\right)dt\right] = \frac{\pi}{8}.\frac{1}{2}.\frac{3\pi}{8} = \frac{3\pi^2}{128}$$

Property 8: If f(x) = f(x + a) (i.e. f(x) is a function with period a), then $\int_0^{na} f(x)dx = n\int_0^{a} f(x)dx$

Illustration 17: Evaluate: $\int_0^{4\pi} \sin^8 x\ dx$ **(JEE MAIN)**

Sol: Here $\sin^8(\pi - x) = \sin^8 x$, therefore by using this property, we can solve the given problem.

$$I = 4\int_0^{\pi} \sin^8 x\ dx\ = 8\int_0^{\pi/2} \sin^8 x\ dx = 8\frac{7.5.3.1}{8.6.4.2}.\frac{\pi}{2} = \frac{35\pi}{32}$$

Illustration 18: Evaluate: $\int_0^{2\pi} \cos^5 x\ dx$ **(JEE ADVANCED)**

Sol: Let $I = \int_0^{2\pi} \cos^5 x\ dx$

Let $f(x) = \cos^5 x$

$f(2\pi - x) = \cos^5(2\pi - x) = \cos^5 x = f(x)$

Then $\int_0^{2\pi} \cos^5 x\ x\ dx = 2\int_0^{\pi} \cos^5 x\ dx$

Now, $f(\pi - x) = \cos^5(\pi - x) = (-\cos x)^3 = -\cos^5 x$

$= -f(x)\ ;\ \int_0^{\pi} \cos^5 x\ dx = 0$

Hence $\int_0^{2\pi} \cos^5 x\ dx = 0$

Property 9

$$\int_a^{a+nT} f(x)dx = n\int_0^{T} f(x)dx$$ (if f(x + T) = f(x), and nÎN i.e. f(x) is a function with period T)

$$\int_{a+mT}^{b+nT} f(x)dx = (n-m)\int_0^{T} f(x)dx + \int_a^{b} f(x)dx \qquad m,n \in I$$

Illustration 19: $I = \int_0^{200\pi} \sqrt{1+\cos x}\ dx$ **(JEE MAIN)**

Sol: $I = \sqrt{2}\int_0^{200\pi}\left|\cos\frac{x}{2}\right|dx \qquad \frac{x}{2} = t$

$\Rightarrow I = 2\sqrt{2}\int_0^{100\pi}|\cos t|\,dt = 200\sqrt{2}\int_0^{\pi}|\cos t|\,dt = 400\sqrt{2}$

Property 10: $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = h'(x)\,f(h(x)) - g'(x)\,f(g(x))$

Corollary (1): $\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = h'(x)\,f(h(x))$ [a is any constant independent of x]

Corollary (2): $\frac{d}{dx}\int_a^{x} f(t)\,dt = f(x)$

Property 11: $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

Property 12: If $f(x) \ge 0$ on [a, b], then $\int_a^b f(x)dx \ge 0$

This property is also called the domination law.

There are a few more properties which might be helpful in solving problems

1. Shift property: $\int_a^b f(x)dx = \int_{a\pm c}^{b\pm c} f(x)\,dx$

2. Reflection property: $\int_a^b f(x)dx = -\int_{-a}^{-b} f(-x)\,dx$

3. Expansion/Contraction property: $\int_a^b f(x)dx = k\int_{a/k}^{b/k} f(x)\,dx \quad \forall\, k > 0$

PLANCESS CONCEPTS

$\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}} = \pi$ if $(\beta > \alpha)$

$\int_\alpha^\beta \sqrt{(x-\alpha)(x-\beta)}\,dx = \frac{\pi}{8}(\beta-\alpha)^2$

$\int_a^b \sqrt{\frac{x-a}{b-x}}\,dx = \frac{\pi}{2}(b-a)$

If f(t) is an odd function, then $\phi(x) = \int_a^x f(t)\,dt$ is an even function.

If f(x) is an even function, then $\phi(x) = \int_a^x f(t)\,dt$ is an odd function.

Every continuous function defined on [a, b] is integrable over [a, b]

Every monotonic function defined on [a, b] is integrable over [a, b]

PLANCESS CONCEPTS

Change of variables: If the function f(x) is continuous on [a, b] and the function $x = \phi(t)$ is continuously differentiable on the interval $[t_1, t_2]$ and $a = \phi(t_1)$, $b = \phi(t_2)$, then

$$\int_a^b f(x)\,dx = \int_{t_1}^{t_2} f(\phi(t))\phi'(t)\,dt.$$

Nitish Jhawar (JEE 2009 AIR 7)

3. SOME SPECIAL INTEGRALS

3.1 Walli's Formula

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)\,(n-3)....2}{n(n-2)....1} \quad \text{(if n is odd positive integer)}$$

$$= \frac{(n-1)\,(n-3).....1}{n(n-2).....2}\left(\frac{\pi}{2}\right) \quad \text{(if n is even positive integer)}$$

Illustration 20: Evaluate $\int_0^{\pi/2} \cos^7 x\,dx$ **(JEE MAIN)**

Sol: By using Walli's formula we can solve the given problem.

$$I = \frac{6.4.2}{7.5.3} = \frac{16}{35}$$

3.2 Gamma Function

$$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{\Gamma\left((m+1)/2\right)\Gamma\left((n+1)/2\right)}{2\Gamma\left((m+n+2)/2\right)}$$

where $\Gamma(n)$ is called the gamma function

OR

$$\int_0^{\pi/2} \sin^m \cos^n x\,dx = \frac{((m-1)\,(m-3).....(2\text{ or }1))\,(n-1)\,((n-3)...(2\text{ or }1))}{(m+n)\,(m+n-2).......(2\text{ or }1)}$$

(if m and n both are not simultaneously even positive integers)

$$\frac{((m-1)\,(m-3)....(1))((n-1)(n-3)....(1))}{(m+n)(m+n-2)....(2)}\left(\frac{\pi}{2}\right) \quad \text{(if m and n are both even positive integers)}$$

Illustration 21: Evaluate $I = \int_0^{\pi/2} \sin^4 x \cos^5 x\,dx$. **(JEE MAIN)**

Sol: Using the gamma function formula i.e.

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\Gamma\left((m+1)/2\right)\Gamma\left((n+1)/2\right)}{2\Gamma\left((m+n+2)/2\right)}$$

We can solve it.

$$I = \frac{r\left((4+1)/2\right)\Gamma\left((5+1)/2\right)}{2\Gamma\left((4+5+2)/2\right)} = \frac{\Gamma\left(5/2\right)\Gamma(3)}{2\Gamma\left(11/2\right)} = \frac{\left((3/2).(1/2)\right)\left(2.1\right)}{2\left((9/2).(7/2).(5/2).(3/2).(1/2)\right)} = \frac{8}{315}$$

4. NEWTON LEIBNITZ FORMULA

In calculus, **Leibnitz's rule** for differentiation under the integral sign named after Gottfried Leibnitz tells us that if we have an integral $\int_{y_0}^{y_1} f(x,y)dy$ then for x in (x_0, x_1) the derivative of this integral is thus expressible as

$$\frac{d}{dx}\left(\int_{y_0}^{y_1} f(x,y)dy\right) = \int_{y_0}^{y_1} f_x(x,y)dy$$

provided that f and its partial derivative f_x are both continuous over a region in the form $[x_0. x_1] \times [y_0, y_1]$.

5. SUMMATION OF SERIES BY INTEGRATION (LIMIT AS A SUM)

To find the sum of an infinite series with the help of definite integration, the following formula is used

$$\lim_{n\to\infty} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)\frac{1}{n} = \int_0^1 f(x)\,dx$$

The following method is used to solve the questions on summation of series.

(i) After writing (r – 1)th or rth term of the series, express it in the form $\frac{1}{n} f\left(\frac{r}{n}\right)$.

Therefore the given series will take the form as $\lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right)$

(ii) Now write $\int$ in place of $\lim_{n\to\infty} \Sigma$ and x in place of $\frac{r}{n}$ and dx in place of n. We get summation in the form of integral $\int_0^1 f(x)\,dx$.

Also we can write $\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} [f(a) + f(a+h) + ... + f(a+\left(n-1\right)h)]$ $\left[\text{where } h = \frac{b-a}{n}\right]$

Illustration 22: Evaluate $\lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}\right]$ **(JEE MAIN)**

Sol: By using the summation of series by integration formula i.e $\lim_{n\to\infty} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)\frac{1}{n} = \int_0^1 f(x)\,dx$ we can solve it.

Limit = $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n+r} = \lim_{n\to\infty} \sum\left(\frac{1}{1+(r/n)} . \frac{1}{n}\right) = \int_0^1 \frac{1}{1+x} dx = [\log(1+x)]_0^1 = \log 2$

Illustration 23: $\lim_{n\to\infty} \frac{1^{100} + 2^{100} + 3^{100} .... n^{100}}{n^{101}}$ **(JEE MAIN)**

Sol: By observing the given problem, we can say that it's a sum of an infinite series so by using the summation of series by integration formula we can solve it.

$T_r = \frac{r^{100}}{n^{101}} = \frac{1}{n}\times\left(\frac{r}{n}\right)^{100}$; $S = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{100}$; $= \int_0^1 x^{100}dx = \frac{1}{101}$

Illustration 24: Find the value of $\lim_{n\to\infty}\left[\frac{n}{(n+1)^2}+\frac{n}{(n+2)^2}+.....+\frac{1}{4n}\right]$ **(JEE ADVANCED)**

Sol: Here $t_r = \frac{n}{(n+r)^2} = \frac{1}{n}\frac{1}{[1+(r/n)]^2}$, therefore similar to the problem above, we can solve it.

Therefore the given series $= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{[1+(r/n)]^2}\cdot\frac{1}{n} = \int_0^1\frac{1}{(1+x)^2}dx$

Given series $= \int_0^1\frac{1}{(1+x)^2}dx = \left[-\frac{1}{1+x}\right]_0^1 = \frac{-1}{2}+1 = \frac{1}{2}$

Evaluate the following definite integrals as the limit of sums.

Illustration 25: $\int_a^b \cos x\,dx$ **(JEE ADVANCED)**

Sol: Here $\int_a^b f(x)dx = \lim_{n\to\infty}\frac{b-a}{n}[f(a)+f(a+h)+....+f(a+(n-1)h)]$ where f(x) = cos x and $h = \frac{b-a}{n}$

$$\therefore \int_a^b \cos x\,dx = \lim_{n\to\infty}\frac{b-a}{n}[\cos a+\cos(a+h)+....+\cos(a+(n-1)h)]$$

$$= \lim_{n\to\infty}\frac{b-a}{n}\cdot\left[\frac{\cos(a+((n-1)/2).h)\,.\,\sin(nh/2)}{\sin(h/2)}\right]$$

$$= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\cdot\frac{\cos\left(a+\frac{n-1}{2}\cdot\frac{(b-a)}{n}\right).\sin\left(\frac{n.(b-a)}{2n}\right)}{\sin\left(\frac{b-a}{2n}\right)}$$

$$= \lim_{n\to\infty} 2\cdot\frac{b-a}{2n}\cdot\frac{\cos(a+(1-(1/n))/2)\,(b-a))\,.\,\sin((b-a)/2)}{\sin((b-a)/2n)}$$

$$= \lim_{n\to\infty} 2\cdot\frac{\cos\left(a+(1-(1/n))\,((b-a)/2)\right).\sin((b-a)/2)}{\sin((b-a)/2n)/((b-a)/2n)}$$

$$= 2\cos\left(\frac{b+a}{2}\right)\sin\left(\frac{b-a}{2}\right) = \sin b - \sin a$$

Illustration 26: $\int_1^2 (x^2+x)dx$ **(JEE ADVANCED)**

Sol: Similar to the problem above.

$$h = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$\int_a^b f(x)dx = \lim_{n\to\infty}\frac{b-a}{n}[f(a)+f(a+h)+....+f(a+(n-1)h)]$$

$$\int_1^2 (x^2+x)\,dx = \lim_{n\to\infty}\frac{1}{n}[f(1)+f(1+h)+....+f(1+(n-1)h)]$$

$$= \lim_{n\to\infty}\frac{1}{n}[(1^2+1)+\{(1+h)^2+(1+h)\}+....+\{(1+(n-1)h)^2+(1+(n-1)h)\}]$$

$$= \lim_{n\to\infty}\frac{1}{n}[1^2.n+h(1+2+...+(n-1))+1\,.\,n+2h(1+2+...+(n-1))+h^2(1^2+2^2+...(n-1)^2)]$$

Here $h = \frac{1}{n}$

$$= \lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}\frac{(n-1)(n)}{2}+n+\frac{2}{n}\,.\,\frac{n(n-1)}{2}+\frac{1}{n^2}\frac{(n-1)n(2n-1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[1+\frac{\left(1-(1/n)\right)(1)}{2}+1+\frac{2\left(1-(1/n)\right)}{2}+\frac{\left(1-(1/n)\right)(1)\left(2-(1/n)\right)}{6}\right]$$

$$= 1+\frac{1}{2}+1+1+\frac{1}{3} = \frac{23}{6}$$

6. INTEGRAL WITH INFINITE LIMITS

If a function f(x) is continuous for $a \le x < \infty$, then by definition,

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \qquad \text{... (i)}$$

If there exists a finite limit on the right-hand side of (i), then the improper integral is said to be convergent; otherwise it is divergent.

Geometrically, the improper integral (i) for f(x) > 0, is the area of the figure bounded by the graph of the function y = f(x), the straight line x = a, and the x-axis. Similarly, we can define

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx \text{ and } \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx$$

7. IMPORTANT RESULTS

If $f(x) \ge 0$ and $a < b$, then $\int_a^b f(x)dx \ge 0$, e.g. $\int_0^{\pi/2} \sin x\ dx = 1$

If $f(x) \ge 0$ and $a < b$, then $\int_b^a f(x)dx \le 0$, e.g. $\int_{\pi/2}^{0} \cos x\ dx = -1$

If $f(x) \le 0$ and $a < b$, then $\int_b^a f(x)dx \ge 0$, e.g. $\int_{\pi/2}^{0} \sin x\ dx = 1$

$\int_0^x [x]dx = \int_0^1 (0)dx + \int_1^2 (1)dx + \int_2^3 2dx + ... + \int_{[x]}^{x} [x]dx$, where [ ] denotes the greatest integer of x.

$$\int_0^{\pi/2} \log(\sin x)dx = \int_0^{\pi/2} \log(\cos x)dx = -\frac{\pi}{2}\log 2$$

$$\int_0^{\pi/2} \log(\tan x)dx = \int_0^{\pi/2} (\cot x)dx = 0$$

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx = \int_0^a f(x)dx + \int_0^a f(a+x)dx$$

$\int_a^b [x]dx = (b-a)\int_0^1 x\,dx$, where [ ] denotes the fractional part of x.

e.g., $\int_0^5 [x]\,dx = 5\int_0^1 x\,dx = \frac{5}{2}$

Integral of an inverse function is given by $\int_{f(a)}^{f(b)} f^{-1}(y)dy = bf(b) - af(a) - \int_a^b f(x)\,dx$

Derivation of the given formula is given in the solved examples

8. GEOMETRICAL APPLICATION

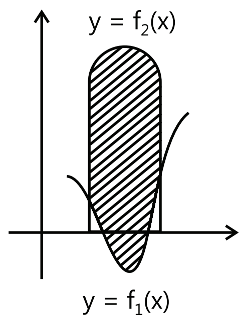


Figure 23.1

The area of the figure bounded by the graphs of two continuous functions $y = f_1(x)$ and $y = f_2(x)$, $f_1(x) \le f_2(x)$, and two straight lines x= a and x = b is determined by the formula $S = \int_a^b (f_2(x) - f_1(x))dx$. It is sometimes convenient to use formulae analogous to x.with respect to y, i.e., regarding x as a function of y. In particular, the area bounded by the curve x =f(y), the y-axis and the two abscissae y = c and y = d is given by $\int_c^d f\ (y)dy$. The area of the figure bounded by the graphs of two continuous functions $x = f_1(y)$ and $f_2(y)$ (with $f_1(y) \le f_2(y)$), and the two straight lines y = c, y = d is given by $\int_c^d (f_2(y) - f_1(y))dy$

From the view of geometry we get an important inequality as if $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

FORMULAE SHEET

Important results

1. $\int_a^b \{f(x) \pm g(x) \pm h(x)\}dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx + \int_a^b h(x)dx$	2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ (a < c < b)	4. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
5. $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(-x) = f(x) \text{ (even function)} \\ 0 & \text{if } (-x) = -f(x) \text{ (odd function)} \end{cases}$	6. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
7. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$	8. $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\ dt = h'(x)\,f(h(x)) - g'(x)\,f(g(x))$
9. $\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx$ (if f(x + T) = f(x), and $n \in N$ i.e. f(x) is a function with period T)	10. If f(x) = f(x + a) then $\int_0^{na} f(x)dx = n\int_0^a f(x)dx$

11. $\left\|\int_a^b f(x)\,dx\right\| \le \int_a^b \|f(x)\|\,dx$	12. $\int_a^b f(x)dx = k\int_{a/k}^{b/k} f(x)dx \quad \forall\, k > 0$
13. $\frac{d}{dx}\left(\int_{y_0}^{y_1} f(x,y)dy\right) = \int_{y_0}^{y_1} f_x(x,y)dy$ (Leibnitz formula)	

Definite integral of rational functions

1. $\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$	2. $\int_0^\infty \frac{x^{p-1}dx}{1+x} = \frac{\pi}{\sin(p\pi)},\; 0<p<1$
3. $\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2}\cos^2 x\,dx = \frac{\pi}{4}$	4. $\int_0^\infty \frac{\sin(px)}{x}dx = \begin{cases}\pi/2 & p>0\\ 0 & p=0\\ -\pi/2 & p<0\end{cases}$
5. $\int_0^\infty \frac{\sin^2 px}{x^2} = \frac{\pi p}{2}$	6. $\int_0^{2x} \frac{dx}{a+b\sin x} = \frac{2\pi}{\sqrt{a^2-b^2}}$
7. $\int_0^\infty \sin ax^2 dx = \int_0^\infty \cos(ax^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}$	8. $\int_0^\infty \frac{\sin x}{\sqrt{x}}dx = \int_0^\infty \frac{\cos x}{\sqrt{x}}dx = \sqrt{\frac{\pi}{2}}$
9. $\int_0^\infty \frac{\tan x}{x}dx = \frac{\pi}{2}$	

Advanced formulas

1. $\int_0^\pi \sin(mx)\,.\,\sin(nx)dx = \begin{cases}0 & m,n \text{ integers and } m\ne n\\ \pi/2 & m,n \text{ integers and } m=n\end{cases}$
2. $\int_0^\pi \cos(mx)\,.\,\cos(nx)dx = \begin{cases}0 & m,n \text{ integers and } m\ne n\\ \pi/2 & m,n \text{ integers and } m=n\end{cases}$
3. $\int_0^\pi \sin(mx).\cos(nx)dx = \begin{cases}0 & m,n \text{ integers and } m+n \text{ odd}\\ 2m/(m^2-n^2) & m,n \text{ integers and } m+n \text{ even}\end{cases}$
4. $\int_0^{\pi/2} \sin^{2m} x\,dx = \int_0^{\pi/2}\cos^{2m} x\,dx = \frac{1.3.5....2m-1}{2.4.6....2m}\frac{\pi}{2}$

Definite integrals of exponential functions

1. $\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$	2. $\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$
3. $\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$	4. $\int_0^\infty x^n e^{-ax}dx = \frac{\Gamma(n+1)}{a^{n+1}}$

5. $\int_0^\infty x^m e^{-ax^2} dx = \dfrac{\Gamma\left(\frac{m+1}{2}\right)}{2a^{(m+1)/2}}$	6. $\int_0^\infty \dfrac{x\,dx}{e^x - 1} = \dfrac{\pi^2}{6}$
7. $\int_0^\infty \dfrac{x^{n-1}}{e^x - 1} dx = \Gamma(n)\left(\dfrac{1}{1^n} + \dfrac{1}{2^n} + \dfrac{1}{3^n} + ...\right)$	8. $\int_0^\infty \dfrac{x\,dx}{e^x + 1} = \dfrac{\pi^2}{12}$
9. $\int_0^\infty \dfrac{x^{n-1}}{e^x + 1} dx = \Gamma(n)\left(\dfrac{1}{1^n} - \dfrac{1}{2^n} + \dfrac{1}{3^n} - ....\right)$	10. $\int_0^\infty \dfrac{e^{-ax} - e^{-bx}}{x \sec(px)} dx = \dfrac{1}{2}\ln\left(\dfrac{b^2 + p^2}{a^2 + p^2}\right)$
11. $\int_0^\infty \dfrac{e^{-ax} - e^{-bx}}{x\csc(px)} dx = \arctan\dfrac{b}{p} - \arctan\dfrac{a}{p}$	12. $\int_0^\infty \dfrac{e^{-ax}(1-\cos x)}{x^2} dx = \text{arc}\cot a - \dfrac{a}{2}\ln(a^2+1)$

Solved Examples

JEE Main/Boards

Example 1: Evaluate:

(i) $\displaystyle\int_0^a \frac{dx}{\sqrt{(a^2/4) - (x-(a/2))^2}}$ (ii) $\displaystyle\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}dx$

Sol: (i) As we know $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$, therefore by using this formula we can solve the given problem.

(ii) Put x = a cos θ : θ ∈ [0, p] and solve it using the appropriate formula.

(i) $\displaystyle\int_0^a \frac{dx}{\sqrt{(a^2/4) - (x-(a/2))^2}}$

$= \left(\sin^{-1}\dfrac{x-(a/2)}{(a/2)}\right)_0^a ; = \left(\sin^{-1}\dfrac{2x-a}{a}\right)_0^a$

$= [\sin^{-1} 1 - \sin^{-1}(-1)] = 2\sin^{-1}(1) = 2 \times \dfrac{\pi}{2} = \pi$. (ii)

Then dx = –a sin θ dθ. Hence,

$\displaystyle\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}dx = \int_{\pi}^{0}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-a\sin\theta)d\theta$

$\displaystyle= a\int_0^{\pi}\sqrt{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}}\,.\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$

$\displaystyle= a\int_0^{\pi} 2\sin^2\frac{\theta}{2}d\theta = a\int_0^{\pi}(1-\cos\theta)d\theta$

$= a(\theta - \sin\theta)_0^{\pi} = a(\pi) = a\pi.$

Example 2: Evaluate $\displaystyle\int_0^{\pi/2}\frac{\sin x}{\sin x + \cos x}dx$

Sol: Let $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

By using this we can write $\displaystyle\int_0^{\pi/2}\frac{\sin x}{\sin x + \cos x}dx$

as $\displaystyle\int_0^{\pi/2}\frac{\sin[(\pi/2)-x]}{\sin[(\pi/2)-x]+\cos[(\pi/2)-x]}dx$ and by adding we can get the result.

$\displaystyle I = \int_0^{\pi/2}\frac{\sin[(\pi/2)-x]}{\sin[(\pi/2)-x]+\cos[(\pi/2)-x]}dx$

$\displaystyle= \int_0^{\pi/2}\frac{\cos x}{\cos x + \sin x}dx$

$\displaystyle\therefore\ 2I = \int_0^{\pi/2}\frac{\sin x + \cos x}{\sin x + \cos x}dx = \int_0^{\pi/2}dx = \frac{\pi}{2}$

$\therefore\quad I = \dfrac{\pi}{4}$

Example 3: Evaluate $\int_0^1 \log\left(\frac{1}{x}-1\right)dx$

Sol: Here $\log\left(\frac{1-x}{x}\right)=\log(1-x)-\log(x)$ and $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$ by using these two formulae we can solve it.

$$I=\int_0^1 \log\left(\frac{1}{x}-1\right)dx$$

(Put x = $\cos^2 t$: cos t > 0; then dx = –2 cos t sin t dt)

$$=-\int_{\pi/2}^{0} \log(\sec^2 t-1)\,.\,2\cos t\,\sin t\,dt$$

$$=\int_0^{\pi/2} \log(\tan^2 t)\,.\,\sin 2t\,dt \;=2\int_0^{\pi/2} \sin 2t\,.\,\log(\tan t)dt$$

$$=2\int_0^{\pi/2} \log(\cot t)\,.\,\sin 2t\,dt$$

$$\therefore\; 2I=2\int_0^{\pi/2} \log(\tan t\,.\,\cot t)\times \sin 2t\,dt=0$$

Example 4: Evaluate:

(i) $I=\int_0^{\pi} |\cos x|\,dx$

(ii) $I=\int_{-2}^{1} |2x+1|\,dx$

(iii) $I=\int_1^4 f(x)\,dx$, where $f(x)=\begin{cases}4x+3, & 1\le x\le 2\\ 3x+5, & 2<x\le 4\end{cases}$

Sol: (i) Here |cos(π – x)| = |cos x| hence $|\cos x|=\cos x$ therefore using the formula $\int \cos x=\sin x$ we can solve it.

(ii) By putting $2x+1=z$ we can solve it.

(iii) As $\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$ (a < c < b)

By using this formula we can obtain the result.

(i) $I=2\int_0^{\pi/2} |\cos x|\,dx$

$$=\;2\int_0^{\pi/2} \cos x\,dx=2(\sin x)_0^{\pi/2}=2(1)=2$$

(ii) $I=\int_{-2}^{1} |2x+1|\,dx$ (put $2x+1=z$)

$$=\frac{1}{2}\int_{-3}^{3} |z|\,dz=\int_0^3 |z|\,dz=\frac{9}{2}.$$

(iii) $I=\int_1^2 f(x)dx+\int_2^4 f(x)dx$

$$=\;\int_1^2 (4x+3)dx+\int_2^4 (3x+5)dx$$

$$=\;(2x^2+3x)_1^2+\left(\frac{3x^2}{2}+5x\right)_2^4$$

= 9 + 28 = 37.

Example 5: Evaluate I = $\int_0^{1.7} [x^2]\,dx$, where [x] is the greatest integer function

Sol: $[x^2]$ takes constant values 0, 1, 2 in intervals (0, 1), $(1, \sqrt{2})$, $(\sqrt{2}, \sqrt{3})$ respectively. By substituting these values we will get the required result.

$$I=\;\int_0^1 [x^2]dx+\int_1^{\sqrt{2}} [x^2]dx+\int_{\sqrt{2}}^{1.7} [x^2]dx$$

$$=\;\int_0^1 0\,dx+\int_1^{\sqrt{2}} 1\,dx+\int_{\sqrt{2}}^{1.7} 2\,dx$$

$$=\;0+(\sqrt{2}-1)+2(1.7-\sqrt{2})=2.4-\sqrt{2}$$

Example 6: Let f(x) be an odd function in the interval $\left[-\frac{T}{2},\frac{T}{2}\right]$ with period T, prove that F(x) = $\int_a^x f(t)dt$ is a periodic function with period T.

Sol: As f(x) is an odd function.

$$F(x+T)=\int_a^{x+T} f(t)dt=\int_a^x f(t)dt+\int_x^{x+T} f(t)dt=F(x)+I(x)$$

where I(x) = $\int_x^{x+T} f(t)dt=\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt=0$ (since f is an odd function). Hence F(x) is a periodic function with period T.

Example 7: Evaluate $\int_0^{\pi} \theta\,\sin^2\theta\cos^2\theta\,d\theta$

Sol: As we know, $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$, hence by using this formula we can evaluate it.

Let I = $\int_0^{\pi} \theta\,\sin^2\theta\cos^2\theta\,d\theta$

$$= \int_0^{\pi} (\pi - \theta)\sin^2(\pi - \theta)\cos^2(\pi - \theta)d\theta$$

$$= \int_0^{\pi} (\pi - \theta)\sin^2\theta\cos^2\theta\, d\theta$$

$$= \pi\int_0^{\pi} \sin^2\theta\cos^2\theta\, d\theta - \int_0^{\pi} \theta\sin^2\theta\cos^2\theta\, d\theta$$

$$= \pi\int_0^{\pi} \left(\frac{\sin 2\theta}{2}\right)^2 d\theta - I$$

$$\Rightarrow\ 2I = \frac{\pi}{4}\int_0^{\pi} \sin^2 2\theta d\theta = \frac{\pi}{4}\int_0^{\pi}\left(\frac{1-\cos 4\theta}{2}\right)d\theta$$

$$= \frac{\pi}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi} = \frac{\pi^2}{8}$$

$$\therefore\ I = \frac{\pi^2}{16}$$

Example 8: Evaluate $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n-1}\frac{1}{n}\sqrt{\left(\frac{n+r}{n-r}\right)}$

Sol: Here by using the limit as a sum method we can solve the given problem.

$$\lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{n}\sqrt{\left(\frac{n+r}{n-r}\right)}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{n}\sqrt{\frac{1+r/n}{1-r/n}} = \int_0^1\sqrt{\frac{1+x}{1-x}}dx$$

$$= \int_0^1 \frac{1+x}{\sqrt{1-x^2}}dx = \int_0^1\frac{dx}{\sqrt{1-x^2}} + \int_0^1\frac{x\,dx}{\sqrt{1-x^2}}$$

$$= [\sin^{-1}x - \sqrt{1-x^2}]_0^1$$

$$= [\sin^{-1}1 - 0] - [\sin^{-1}0 - 1] = \frac{\pi}{2}+1$$

Example 9: Integrate : $I = \int_{\pi/4}^{3\pi/4}\frac{\theta}{1+\sin\theta}d\theta$

Sol: As $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ hence we can write $\int_{\pi/4}^{3\pi/4}\frac{\theta}{1+\sin\theta}d\theta$ as $\int_{\pi/4}^{3\pi/4}\frac{\pi-\theta}{1+\sin\theta}d\theta$ and then by putting $\theta = \frac{\pi}{2}+y$ we can solve the given problem.

$$I = \int_{\pi/4}^{3\pi/4}\frac{\theta}{1+\sin\theta}d\theta = \int_{\pi/4}^{3\pi/4}\frac{\pi-\theta}{1+\sin\theta}d\theta$$

$$2I = \pi\int_{\pi/4}^{3\pi/4}\frac{d\theta}{1+\sin\theta}\ ;\ \text{Put } \theta = \frac{\pi}{2}+y$$

$$= \pi\int_{-\pi/4}^{\pi/4}\frac{dy}{1+\cos y} = 2\pi\int_0^{\pi/4}\frac{dy}{1+\cos y}$$

$$I = \frac{\pi}{2}\int_0^{\pi/4}\sec^2\frac{y}{2}dy = \pi\left[\tan\frac{y}{2}\right]_0^{\pi/4} = \pi\tan\frac{\pi}{8}$$

JEE Advanced/Boards

Example 1: Show that $1 < \int_0^1 e^{x^2}dx < e$.

Sol: e^{x^2} is an increasing function in [0, 1]. Further, $e^0 \le e^{x^2} \le e^1\ \forall\ x \in [0, 1]$

$$\therefore\ \int_0^1 1\,dx < \int_0^1 e^{x^2}dx < \int_0^1 e\,dx$$

or $1 < \int_0^1 e^{x^2}dx < e$.

Example 2: If $F(x) = \int_0^{x^2}\frac{t^2-5t+4}{4+e^{2t}}dt$, find the critical points of F(x).

Sol: By using Leibnitz rule we can write

$$F(x) = \int_0^{x^2}\frac{t^2-5t+4}{4+e^{2t}}dt,$$

as $F'(x) = \frac{(x^2)^2-5x^2+4}{4+e^{2x^2}}\cdot(2x) = 0.$

By Leibnitz Rule,

$$F'(x) = \frac{(x^2)^2-5x^2+4}{4+e^{2x^2}}\cdot(2x)$$

$F'(x) = 0$

$\Rightarrow\quad (x^4 - 5x^2 + 4)\,x = 0$

$\Rightarrow\quad (x^2 - 4)(x^2 - 1)\,x = 0$

$\Rightarrow\quad x = 0, \pm1, \pm2$

These are the critical points of F(x).

Example 3: Evaluate: $\int_0^{\pi/2} \log \sin x\, dx$

Sol: We can write $\int_0^{\pi/2} \log \sin x\, dx$

As $\int_0^{\pi/2} \log \sin\left(\frac{\pi}{2}-x\right)dx$ and then by adding these two integration we can obtain the result.

$$I = \int_0^{\pi/2} \log \sin x\, dx = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2}-x\right)dx$$

$$= \int_0^{\pi/2} \log \cos x\, dx$$

$$\therefore 2I = \int_0^{\pi/2} (\log \sin x + \log \cos x)dx = \int_0^{\pi/2} \log(\sin x \cos x)dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx = \int_0^{\pi/2} \log \sin 2x\, dx - \int_0^{\pi/2} \log 2\, dx$$

$$= -\frac{\pi}{2}\log 2 + \int_0^{\pi/2} \log \sin 2x\, dx \text{ (Put } 2x = t)$$

$$= -\frac{\pi}{2}\log 2 + \frac{1}{2}\int_0^{\pi} \log \sin t\, dt$$

$$= -\frac{\pi}{2}\log 2 + \frac{1}{2}(2)\int_0^{\pi/2} \log \sin t\, dt.$$

$$\therefore 2I = -\frac{\pi}{2}\log 2 + I \Rightarrow I = -\frac{\pi}{2}\log 2$$

Example 4: Evaluate: (i) $I = \int_1^3 (x^2 + x)dx$

(ii) $I = \int_a^b \sin x\, dx$ as limit of a sum.

Sol: By using the limit as a sum method we can solve the problems above.

(i) $f(x) = x^2 + x$, $a = 1$, $b = 3$, $nh = 3 - 1 = 2$

$$I = \lim_{n\to\infty} h\sum_{r=1}^{n} f(a+rh)$$

$$= \lim_{n\to\infty} h\sum_{r=1}^{n} ((a+rh)^2 + (a+rh))$$

$$= \lim_{n\to\infty} h\left(\sum_{r=1}^{n} r^2h^2 + rh(2a+1) + (a^2+a)\right)$$

$$= \lim_{n\to\infty} h\left(\frac{n(2+h)\,(4+h)}{6} + (2a+1)\,\frac{n(2+h)}{2} + n(a^2+a)\right)$$

$$= \lim_{n\to\infty}\left(\frac{2(2+h)\,(4+h)}{6} + (2a+1)\,\frac{2(2+h)}{2} + 2(a^2+a)\right)$$

$$= \frac{8}{3} + 6 + 4 = \frac{38}{3}$$

(ii) $I = \int_a^b \sin x\, dx$

$nh = b - a$; $I = \lim_{h\to 0} h\left(\sum_{r=1}^{n} \sin(a+rh)\right)$

$$= \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{h}{2}}\left(\sum_{r=1}^{n} 2\sin\frac{h}{2}\sin(a+rh)\right)$$

$$= \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{h}{2}}\left(\sum_{r=1}^{n} \cos\left(a+hr-\frac{h}{2}\right) - \cos\left(a+hr+\frac{h}{2}\right)\right)$$

$$= \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{h}{2}}\left(\cos\left(a+\frac{h}{2}\right) - \cos\left(a+nh+\frac{h}{2}\right)\right)$$

$= \cos a - \cos b$

Example 5: Evaluate $I = \int_{-1}^{3} (|x-2| + 2[x])dx$, where [x] is the greatest integer function.

Sol: By putting $x - 2 = y$ and it is negative in interval -3 to -1 and positive in interval 0 to 1.

$$I_1 = \int_{-1}^{3} |x-2|dx \text{ ; Put } x - 2 = y$$

$$\int_{-3}^{1} |y|dy = \int_{-3}^{-1} -y\, dy + 2\int_0^1 y\, dy$$

$$= -\frac{1}{2}[y^2]_{-3}^{-1} + [y^2]_0^1 = 4 + 1 = 5$$

$$I_2 = \int_{-1}^{3} [x]\, dx$$

$$= \int_{-1}^{0} -dx + \int_0^1 0\, dx + \int_1^2 dx + \int_2^3 2\, dx = -1 + 0 + 1 + 2 = 2$$

$\therefore I = I_1 + 2I_2 = 9$

Example 6: Show that $I = \int_0^{\infty} \frac{x \log x}{(1+x^2)^2}dx = 0$

Sol: By splitting the given integration into two intervals i.e. from 0 to 1 and then 1 to ∞ we can solve the given problem.

$$\int_0^\infty \frac{x\log x}{(1+x^2)^2}dx = \int_0^1 \frac{x\log x}{(1+x^2)^2}dx + \int_1^\infty \frac{x\log x}{(1+x^2)^2}dx$$

Put x = 1/y in the second integral

$$\therefore \int_1^\infty \frac{x\log x}{(1+x^2)^2}dx = \int_1^0 \frac{y^4\log y}{y^3(1+y^2)^2}dy = -\int_0^1 \frac{y\log y}{(1+y^2)^2}dy$$

Thus $I = \int_0^1 \frac{x\log x}{(1+x^2)^2}dx - \int_0^1 \frac{y\log y}{(1+y^2)^2}dy = 0$

Example 7: If $I = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1-x^2}\right)dx,$ then find its value.

Sol: We can write $\cos^{-1}\left(\frac{2x}{1-x^2}\right)$ as $\cos^{-1}\left(\frac{-2x}{1-x^2}\right)$

$= \left(\pi - \cos^{-1}\frac{2x}{1+x^2}\right)$ and then by solving we will get the result.

$$I = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1-x^2}\right)dx = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{-2x}{1-x^2}\right)dx$$

$$= \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4}\left(\pi - \cos^{-1}\frac{2x}{1+x^2}\right)dx$$

$$2I = \pi \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4}dx = 2\pi \int_0^{\frac{1}{\sqrt{3}}} \frac{x^4}{1-x^4}dx$$

$$I = \pi(-1)\int_0^{\frac{1}{\sqrt{3}}} \left(1 - \frac{1}{1-x^4}\right)dx$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2}\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1-x^2} + \frac{1}{1+x^2}dx$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2}\cdot\frac{\pi}{6} + \frac{\pi}{4}\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1-x} + \frac{1}{1+x}dx$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi^2}{12} + \frac{\pi}{4}\left(\log\frac{|1+x|}{|1-x|}\right)_0^{\frac{1}{\sqrt{3}}}$$

$$= \frac{\pi^2}{12} - \frac{\pi}{\sqrt{3}} + \frac{\pi}{4}\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$

Example 8: Evaluate $\int_a^b (px+q)dx$ as a limit of a sum

Sol: Here as f(x) = px + q, therefore using the limit as sum method we can solve the given problem.

$$I = \int_a^b (px+q)dx$$

$= \lim_{h\to 0} h[f(a) + f(a + h) + \ldots.. + f(a + (n - 1)h)]$

$= \lim_{h\to 0} h[(pa + q) + \{p(a + h) + q\} + \ldots\ldots +$

$\{p(a + (n - 1)h) + q\}]$

$= \lim_{h\to 0} h[p(a + a + \ldots\ldots. + a) + ph(1 + 2 + \ldots\ldots + (n - 1))$

$+ q(1 + 1 + \ldots\ldots + 1)]$

$= \lim_{h\to 0} h\left[pna + \frac{1}{2}pnh(n-1) + qn\right]$

$= \lim_{h\to 0}\left[hpna + \frac{1}{2}pnh(hn-h) + qnh\right]$ .....(i)

Since, h = (b – a)/n, or nh = b – a, we obtain from (i)

$I = \lim_{h\to 0}\left[(pa+q)(b-a) + \frac{p}{2}(b-a)(b-a-h)\right]$

$= (pa + q)(b - a) + \frac{p}{2}(b - a)^2$

$= \frac{p}{2}(b - a)(2a + b - a) + q(b - a)$

$= \frac{p}{2}(b^2 - a^2) + q(b - a).$

Example 9: If $U_n = \int_0^\pi \frac{1-\cos n\pi}{1-\cos x}dx$ where n is a positive integer or zero, then show that $U_{n+2} + U_n = 2U_{n+1}$.

Hence show that $\int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta}d\theta = \frac{n\pi}{2}$

Sol: Here $U_n = U_{n+2} - U_{n+1}$ therefore by substituting n+2 and n+1 in place of n and solving we will get the required result.

$$\because\ U_n = \int_0^\pi \frac{1-\cos nx}{1-\cos x}dx$$

$\therefore U_{n+2} - U_{n+1}$

$$= \int_0^\pi \frac{\{(1-\cos(n+2)x)\} - \{1-\cos(n+1)x\}}{(1-\cos x)}dx$$

$$= \int_0^\pi \frac{\cos(n+1)x - \cos(n+2)x}{(1-\cos x)}dx$$

$$= \int_0^{\pi} \frac{2\sin\left(n+(3/2)\right)x\sin(x/2)}{2\sin^2(x/2)}$$

$$\Rightarrow U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{\sin\left(n+(3/2)\right)x}{\sin(x/2)}dx \quad .....(i)$$

Similarly

$$\Rightarrow U_{n+1} - U_n = \int_0^{\pi} \frac{\sin\left(n+(1/2)\right)x}{\sin(x/2)}dx \quad ....(ii)$$

from (1) and (2), we get

$(U_{n+2} - U_{n+1}) - (U_{n+1} - U_n)$

$$= \int_0^{\pi} \frac{\sin\left(n+(3/2)\right)x - \sin\left(n+(1/2)\right)x}{\sin(x/2)}$$

$$= \int_0^{\pi} \frac{2\cos(n+1)x\ \sin(x/2)}{\sin(x/2)}dx = 2\left\{\frac{\sin(n+1)x}{(n+1)}\right\}_0^{\pi} = 0$$

$\therefore U_{n+2} + U_n = 2U_{n+1}$

Hence proved

Now $U_{n+2} - U_{n+1} = U_{n+1} - U_n$.

Similarly implies

$U_{n+2} - U_{n+1} = U_{n+1} - U_n = U_n - U_{n-1} = ... = U_1 - U_0$

$\therefore U_n - U_{n-1} = U_1 - U_0 = \pi - 0$

$\Rightarrow U_n = \pi + U_{n-1}$

$= \pi + \pi + U_{n-2}$

$= 2\pi + U_{n-2}$

$U_n = n\pi + U_0$ ......(3) [$\because U_0 = 0$]

$U_n = np$

Hence $\therefore \int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta}d\theta = \int_0^{\pi/2} \frac{1-\cos 2n\theta}{1-\cos 2\theta}d\theta$

Put $2\theta = x \therefore d\theta = \frac{dx}{2}$

Hence $\int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta}d\theta = \frac{1}{2}\int_0^{\pi} \frac{1-\cos nx}{1-\cos x}dx$

$= \frac{1}{2}U_n = \frac{1}{2}n\pi$ {from (1)}

Example 10: Solve $\int_{-\pi/4}^{\pi/4} \frac{x+(\pi/4)}{2-\cos 2x}dx$.

Sol: By splitting $\int_{-\pi/4}^{\pi/4} \frac{x+(\pi/4)}{2-\cos 2x}dx$

$= \int_{-\pi/4}^{\pi/4} \frac{x}{2-\cos 2x}dx + \frac{\pi}{4}\int_{-\pi/4}^{\pi/4} \frac{1}{2-\cos 2x}dx$ and as we know $\left(\frac{x}{2-\cos 2x}\right)$ is an odd function

therefore $\int_{-\pi/4}^{\pi/4} \frac{x}{2-\cos 2x}dx = 0.$

Therefore $0 + \frac{\pi}{4}2\int_0^{\pi/4} \frac{1}{2-\cos 2x}dx$

This is because $\left(\frac{x}{2-\cos 2x}\right)$ is an odd function,

whereas $\left(\frac{1}{2-\cos 2x}\right)$ is an even function

$$= \frac{\pi}{2}\int_0^{\pi/4} \frac{dx}{2-((1-\tan^2 x)/(1+\tan^2 x))}$$

$$= \frac{\pi}{2}\int_0^{\pi/4} \frac{(1+\tan^2 x)dx}{2(1+\tan^2 x)-(1-\tan^2 x)} = \frac{\pi}{2}\int_0^{\pi/4} \frac{\sec^2 x\,dx}{1+3\tan^2 x}$$

Now let $\tan x = t \quad \therefore \sec^2 x\,dx = dt$

$$\Rightarrow \frac{\pi}{2}\int_0^1 \frac{dt}{1+3t^2} = \frac{\pi}{2\sqrt{3}}\left(\tan^{-1}\sqrt{3}t\right)_0^1 = \frac{\pi^2}{6\sqrt{3}}$$

Example11: Show that

$$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}$$

Sol: Since $0 < x < 1$

so $\frac{1}{\sqrt{4-x^2}} < \frac{1}{\sqrt{4-x^2-x^3}} < \frac{1}{\sqrt{4-2x^2}}$

Hence by using the property:

If $f(x) \le g(x)$ on [a, b], then $\int_a^b f(x)dx \le \int_a^b g(x)dx$ we can solve the given problem.

Integrate the above relation

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^2}} < \int_0^1 \frac{dx}{\sqrt{4-2x^2}}$$

$$\left(\sin^{-1}\frac{x}{2}\right)_0^1 < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{1}{\sqrt{2}}\left(\sin^{-1}\frac{x}{\sqrt{2}}\right)_0^1$$

$$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}.$$

Hence proved.

JEE Main/Boards

Exercise 1

Q.1 $\int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}}$

Q.2 $\int_0^{\pi/2} \frac{dx}{(4\sin^2 x + 5\cos^2 x)}$

Q.3 $\int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x\cos x}dx$

Q.4 $\int_0^1 |5x-3|\,dx$

Q.5 $\int_1^3 f(x)dx$, where $f(x) = \begin{cases} 2x+1, & 1 \le x \le 2 \\ x^2+1, & 2 \le x \le 3 \end{cases}$

Q.6 $\int_{-\pi/4}^{\pi/4} |\sin x|dx$

Q.7 $\int_0^{\pi} \frac{x}{(1+\sin^2 x)}dx$

Q.8 Evaluate using limit of a sum: $\int_0^2 (x^2+1)dx$

Q.9 Evaluate: $\int_0^{\pi/2} |\sin x - \cos x|\,dx$

Q.10 If f and g are continuous function on [0, a] satisfying f(x) = f(a – x) and g(x) + g(a– x) = 2 then, show that

$\int_0^a f(x)g(x)dx = \int f(x)dx.$

Q.11 Evaluate: $\int_0^{100\pi} \sqrt{1-\cos 2x}\,dx$

Q.12 (i) Show that if f(t) is an odd function then $\int_0^a f(t)\,dx$ is an even function w.r.t. x.

(ii) Can $\int_a^x f(t)\,dt$ be an odd function if f(t)dt is an even function?

Q.13 If f(x) = $\int_0^{x^2} \sqrt{1+t^2}\,dt$, then find the value of f'(x).

Q.14 Evaluate $\int_{-\pi/2}^{\pi/2} \frac{\pi+4x^2}{-\cos\left(|x|+(\pi/3)\right)}dx.$

Q.15 If f(x) = $\int_1^x \frac{\log t}{l+t}dt$ then prove that

$f(x) + f\left(\frac{1}{x}\right) = \frac{1}{2}(\log x)^2.$

Q.16 $\int_{\frac{1}{x}}^{2t} |\log x|\,dt$

Q.17 $\int_0^x \frac{\sin\left(n+(1/2)\right)x}{2\sin(x/2)}dx,\ n \in N.$

Q.18 If F(x) = $\int_{\frac{5x}{4}}^{x} (3\sin t + 4\cos t)dx$. Find the least value of F(x) on the interval $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$.

Q.19 If I_A = $\int_0^{\frac{\pi}{4}} \tan^n \theta\, d\theta$, n ∈ N, then find $n(I_{n-1} + I_{n+1})$ and I_B.

Q.20 If "a" is a positive integer, solve for "a"

$\int_0^a \left(a^2\left(\frac{\cos 3x}{4}+\frac{3}{4}\cos x\right)+a\sin x - 20\cos x\right)dx \le \frac{-a^3}{3}.$

Q.21 If f(x) = sin x, then find its mean value on (–2, 0).

Q.22 Evaluate $I = \int_0^{\pi} \frac{1}{x+\sqrt{a^2-x^2}}dx.$

Q.23 Show that $I = \int_0^{a^2} \left[\sqrt{x}\right]dx = \frac{n(n-1)(4\pi+1)}{6}$, where [x] is the greatest integer function.

Q.24 Show that I = $\int_0^{nx+\lambda} |\sin x|\,dx = 2n+1-\cos\lambda,\ n \in N,$ $0 \le \lambda < \pi$.

Q.25 Show that I = $\int_0^{\pi} \frac{\pi \sin 2x, \sin\left((\pi/2)\cos x\right)}{2x-\pi} dx = \frac{8}{x^2}$.

Q.26 Let f and g be function satisfying the following conditions:

(i) f(0) = 1 (ii) f(x) = g(x), g′(x) = f(x)

(iii) g(0) = 0 (iv) $g(x) \geq 0 \; \forall \; x \in R$

Find f(1).

Q.27 Show that

(i) $\int_0^{\pi} \log(1+\cos x)dx = \pi \log(1/2)$;

(ii) $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}} = \frac{\pi}{12}$

Q.28 Prove that

$\int_0^{\pi} \frac{dx}{1-2a\cos x + x^2} - \frac{\pi}{1-a^2}$ or $\frac{\pi}{a^2-1}$; a > 0,

According as a < 1 or a > 1.

Q.29 (i) Evaluate $\lim_{n\to 0} \frac{\int_0^a x\,dx}{\alpha \sin \alpha}$

(ii) If y = $x\int_x^a \log dx$, Find $\frac{dy}{dx}$ at x = e.

Q.30 Find the intervals of increase of f(x) defined by f(x) = $\int_0^{\alpha} (t^2+2t)\,(t^2-1)dt$.

Exercise 2

Single Correct Choice Type

Q.1 $\int_{-1}^{1} f(x)\,dx$ is equal to where f(x) = $\begin{cases} 1-2x, & x<0 \\ 1+2x, & x \geq 0 \end{cases}$

(A) 4 (B) –4 (C) 2 (D) –2

Q.2 $\int_{-1}^{1} e^{|x|}\,dx$ equals

(A) 2e (B) 2e – 1 (C) 2e – 2 (D) e – 2

Q.3 $\int_0^1 [x]dx$ equals ; where [·] is G.I.F.

(A) 0 (B) 2 (C) 3 (D) 1

Q.4 $\int_0^x |\cos x|dx$ equals

(A) 1 (B) 2 (C) 3 (D) 4

Q.5 $\int_{-2}^{2} |2x+3|dx$ equals

(A) $\frac{25}{2}$ (B) 0 (C) $\frac{25}{4}$ (D) $\frac{25}{3}$

Q.6 $\int_{-2}^{2} |1-x^2|\,dx =$

(A) 2 (B) 4 (C) 6 (D) 8

Q.7 The point of extremum of $\int_0^{x^2} \frac{t^2-5t+4}{2+e^t} dt$ are

(A) x = –2 (B) x = 1

(C) x = 0 (D) All of the above

Q.8 The point of intersection

$F_1(x) = \int_2^x (2t-5)\,dt$ and $F_2(x) = \int_0^x 2t\,dt$, are -

(A) $\left(\frac{6}{5}, \frac{36}{25}\right)$ (B) $\left(\frac{2}{3}, \frac{4}{9}\right)$ (C) $\left(\frac{1}{3}, \frac{1}{9}\right)$ (D) $\left(\frac{1}{5}, \frac{1}{25}\right)$

Q.9 If f and g are continuous function on [0, a) satisfying f(x) = f(a – x) and g(x)+g(a–x)=2,then I = $\int_0^a f(x)g(x)dx =$

(A) $\int_0^a f(x)dx$ (B) $\int_a^0 f(x)dx$

(C) $2\int_0^a f(x)dx$ (D) None of these

Q.10 The value of integral $\int_0^{\log 5} \frac{e^x\sqrt{e^x-1}}{e^x+3} dx =$

(A) 3 + 2p (B) 4 – p

(C) 2 + p (D) None of these

Q.11 The value of the integral $\int_{-\alpha}^{\pi} \sin mx \sin nx\,dx$ for m ≠ n (m, n ∈ I), is -

(A) 0 (B) p (C) π/2 (D) 2p

Q.12 $\int_{1/e}^{e} |\log x| dx =$

(A) $1-\frac{1}{e}$ (B) $2\left(1-\frac{1}{e}\right)$

(C) $e^{-1}-1$ (D) None of these

Q.13 $\int_{0}^{\pi} \frac{dx}{1-2a\cos x + a^2} =$

(A) $\frac{\pi}{2(1-a^2)}$ (B) $\pi(1-a^2)$

(C) $\frac{\pi}{1-a}$ (D) None of these

Q.14 $\int_{0}^{1} (1-x)^9 dx =$

(A) x (B) $\frac{1}{10}$ (C) $\frac{11}{10}$ (D) 2

Q.15 $\int_{0}^{\pi} \frac{dx}{\left(x+\sqrt{x^2+1}\right)^3} =$

(A) $\frac{3}{8}$ (B) $\frac{1}{8}$ (C) $-\frac{3}{8}$ (D) None of these

Q.16 If [x] denotes the greatest integer less than or equal to x, then the value $\int_{1}^{5} [|x-3|] dx$ is -

(A) 1 (B) 2 (C) 4 (D) 8

Q.17 $\int_{-\pi/2}^{\pi/2} \frac{\sin x}{1+\cos^2 x} e^{-\cos^2 x} dx$ is equal to -

(A) $2e^{-1}$ (B) 1 (C) 0 (D) None of these

Q.18 The value of

$\int_{-1/2}^{1/2} \left[\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2 - 2\right]^{1/2} dx$ equal

(A) log (4/3) (B) 2 log (4/3)

(C) 4 log (4/3) (D) –4 log (4/3)

Q.19 Let f(x) = x – [x], for every real number x, where [x] is integral pat of x. Then $\int_{-1}^{1} f(x) dx$ is

(A) 1 (B) 2 (C) 0 (D) ½

Q.20 If [x] stands for the greatest integer function, the value of $\int_{4}^{10} \frac{[x^2]}{[x^2-28x+196]+[x^2]} dx$ is

(A) 0 (B) 1 (C) 3 (D) None of these

Q.21 The value of $\int_{-1}^{3} (|x-2|+[x]) dx$ is ([x] stands for greatest integer less than or equal to x)

(A) 7 (B) 5 (C) 4 (D) 3

Q.22 $\int_{0}^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ is equal to

(A) $\frac{\pi}{2}$ (B) $\sqrt{2}\log(\sqrt{2}+1)$

(C) $\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$ (D) None of these

Q.23 If $u_{10} - \int_{0}^{\pi/2} x^{10} \sin x \, dx$ then the value of $u_{10} + 90\, u_8$ is

(A) $9\left(\frac{\pi}{2}\right)^8$ (B) $\left(\frac{\pi}{2}\right)^9$ (C) $10\left(\frac{\pi}{2}\right)^9$ (D) $9\left(\frac{\pi}{2}\right)^9$

Q.24 For any integer n, the integral $\int_{0}^{\pi} e^{\sin^2 x} \cos^3 (2n+1)x \, dx$ has the value

(A) π (B) 1 (C) 0 (D) None of these

Q.25 The value of $\int_{-\pi/2}^{\pi/2} \sin(\log(x+\sqrt{x^2+1})) dx$ is

(A) 1 (B) –1 (C) 0 (D) None of these

Q.26 The value of α ∈ (-π, 0) satisfying $\sin\alpha + \int_{\alpha}^{2\alpha} \cos 2x \, dx = 0$ is

(A) –π/2 (B) –p (C) –π/3 (D) 0

Q.27 If f(x) = $\int_{x^2}^{x^4} \sin\sqrt{t} \, dt$, then $f^{-1}(x)$ equals

(A) $\sin x^2 - \sin x$ (B) $4x^3 \sin x^2 - 2x \sin x$

(C) $x^4 \sin x^2 - x \sin x$ (D) None of these

Q.28 $\int_0^{\pi} x\sin x\cos^4 x\,dx =$

(A) $\frac{\pi}{10}$ (B) $\frac{\pi}{5}$ (C) $-\frac{\pi}{5}$ (D) None of these

Q.29 If f(x) = $ae^{2x} + be^{x} + cx$, satisfies the conditions f(0) = –1, f′(log 2) = 31, $\int_0^{\log 4}(f(x)-cx)dx = \frac{39}{2}$, then

(A) a = 5, b = 6, c = 3 (B) a = 5, b = –6, c = 3

(C) a = –5, b = 6, c = 3 (D) None of these

Q.30 $\int_{-\pi/4}^{\pi/4} \frac{e^x \sec^2 x\,dx}{e^{2x}-1}$ is equal to

(A) 0 (B) 2 (C) e (D) None of these

Q.31 $\int_{-1}^{a} \log_a\left(x+\sqrt{1+x^2}\right)dx$ is equal to

(A) 2 $\log_a a$ (B) 0

(C) $\log_a 2$ + log a (D) None of these

Q.32 The value of $\int_{-2} \frac{\sin^2 x}{\left[(x/\pi)\right]+(1/2)}dx$, where [x] = the greatest integer less than or equal to x, is

(A) 1 (B) 0 (C) 4 – sin 4 (D) None of these

Q.33 If f(x) = $\int_0^x \log(1+t^2)dt$ then the value of f′(1) is equal to

(A) 2 (B) 0 (C) 1 (D) None of these

Q.34 $\int_0^x \frac{dx}{1+3^{\cos x}}$ is equal to

(A) π (B) 0 (C) $\frac{\pi}{2}$ (D) None of these

Previous Years' Questions

Q.1 The value of the integral $\int_0^{\pi/2}\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx$ is ***(1983)***

(A) π/4 (B) π/2 (C) p (D) None of these

Q.2 For any integer n, the integral $\int_0^{x} e^{\cos^2 x}\cos^3(2n+1)x\,dx$ has the value ***(1985)***

(A) π (B) 1 (C) 0 (D) None of these

Q.3 Let f: R → R be a differentiable function and f(1) = 4. Then, the value of $\lim_{x\to 1}\int_4^{f(x)}\frac{2t}{x-1}dt$ is ***(1990)***

(A) 8f′ (1) (B) 4f′ (1) (C) 2f′ (1) (D) f′ (1)

Q.4 The value of $\int_0^{\pi/2}\frac{dx}{1+\tan^3 x}$ is ***(1993)***

(A) 0 (B) 1 (C) π/2 (D) π/4

Q.5 The value of $\int_0^{2\pi}[2\sin x]dx$ where [·] represents the greatest integral function, is ***(1995)***

(A) $-\frac{5\pi}{3}$ (B) – π (C) $\frac{5\pi}{3}$ (D) –2p

Q.6 $\int_0^x f(t)dt + x + \int_x^1 t f(t)dt$, then the value of f(1) is ***(1998)***

(A) $\frac{1}{2}$ (B) 0 (C) 1 (D) $-\frac{1}{2}$

Q.7 $\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x}$ is equal to ***(1999)***

(A) 2 (B) –2 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

Q.8 If for a real number y, [y] is the greatest integer less than or equal to y, then the value of the integral $\int_{\pi/2}^{3\pi/2}[2\sin x]dx$ is ***(1999)***

(A) –π (B) 0 (C) $-\frac{\pi}{2}$ (D) $\frac{\pi}{2}$

Q.9 The value of $\int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}dx, a>0$, is ***(2001)***

(A) π (B) aπ (C) $\frac{\pi}{2}$ (D) 2π

Q.10 Let f : (0, ∞) → R and F(x) = $\int_0^{\pi} f(t)dt$, If $F(x^2) = x^2(1+x)$, then f(4) equals ***(2001)***

(A) $\frac{5}{4}$ (B) 7 (C) 4 (D) 2

Q.11 Let f(x) = $\int_1^x \sqrt{2-t^2}dt$. Then, the real value of x if it satisfies $x^2 - f'(x) = 0$ are ***(2002)***

(A) ±1 (B) $\pm\frac{1}{\sqrt{2}}$ (C) $\pm\frac{1}{2}$ (D) 0 and 1

Q.12 Let T > 0 be a fixed real number. Suppose, f is a continuous function such that for all $x \in R$, f(x + T) = f(x). If I = $\int_0^T f(x)dx$, then the value of $\int_3^{3+3T} f(2x)dx$, ***(2002)***

(A) $\frac{3}{2}I$ (B) I (C) 3I (D) 6I

Q.13 If f(x) = $\int_{x^2}^{x^2+1} e^{-t^2}dt$, then f(x) increases in ***(2003)***

(A) (2, 2) (B) No value of x (C) $(0, \infty)$ (D) $(-\infty, 0)$

Q.14 The value of the integral $\int_0^1 \sqrt{\frac{1-x}{1+x}}dx$ is ***(2004)***

(A) $\frac{\pi}{2}+1$ (B) $\frac{\pi}{2}-1$ (C) –1 (D) 1

Q.15 Match the conditions expressions in column I with statement in column II ***(2007)***

Column I	**Column II**
(A) $\int_{-1}^{1}\frac{dx}{1+x^2}$	(p) $\frac{1}{2}\log\left(\frac{2}{3}\right)$
(B) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$	(q) $2\log\left(\frac{2}{3}\right)$
(C) $\int_2^3 \frac{dx}{1-x^2}$	(r) $\frac{\pi}{3}$
(D) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$	(s) $\frac{\pi}{2}$

Q.16 The value of $\int_{-2}^{2}|1-x^2|\,dx$ is..... ***(1989)***

Q.17 The value of $\int_{\pi/4}^{3\pi/4}\frac{x}{1+\sin x}dx$..... ***(1993)***

Q.18 The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ is..... ***(1994)***

Q.19 Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}}dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}}dx$.

Then which one of the following is true? ***(2008)***

(A) $I > \frac{2}{3}$ and $J > 2$ (B) $I < \frac{2}{3}$ and $J < 2$

(C) $I < \frac{2}{3}$ and $J > 2$ (D) $I > \frac{2}{3}$ and $J < 2$

Q.20 $\int_0^x [\cot x]dx$, $[\cdot]$ denotes the greatest integer function, is equal to ***(2009)***

(A) $\frac{\pi}{2}$ (B) 1 (C) -1 (D) $-\frac{\pi}{2}$

Q.21 Let $p(x)$ be a function defined on R such that $p(x) = p(1-x)$ for all $p(0) = 1$ $p(1) = 41$.Then $\int_0^1 p(x)dx$ equals. ***(2010)***

(A) 21 (B) 41 (C) 42 (D) $\sqrt{41}$

Q.22 The value of $\int_0^1 \frac{8\log(1+x)}{1+x^2}dx$. is ***(2011)***

(A) $\frac{\pi}{8}\log 2$ (B) $\frac{\pi}{2}\log 2$ (C) $\log 2$ (D) $\pi\log 2$

Q.23 If $g(x) = \int_0^x \cos 4t\,dt$, then $g(x+\pi)$ equals ***(2012)***

(A) $\frac{g(x)}{g(\pi)}$ (B) $g(x)+g(\pi)$

(C) $g(x)-g(\pi)$ (D) $g(x).g(\pi)$

Q.24 Statement-I : The value of the integral $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$ is equal to $\pi/6$.

Statement-II: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ ***(2013)***

(A) Statement-I is true; statement-II is true; statement-II is a correct explanation for statement-I.

(B) Statement-I is true; statement-II is true; statement-II is not a correct explanation for statement-I.

(C) Statement-I is true; statement-II is false.

(D) Statement-I is false; statement-II is true.

Q.25 The integral $\int\left(1+x-\frac{1}{x}\right)e^{e+\frac{1}{x}}dx$ is equal to ***(2014)***

(A) $(x+1)e^{x+\frac{1}{x}}+c$ (B) $-xe^{x+\frac{1}{x}}+c$

(C) $(x-1)^{x+\frac{1}{x}}+c$ (D) $xe^{x+\frac{1}{x}}+c$

Q.26 The integral $\int_2^4 \frac{\log x^2}{\log x^2+\log(36-12x+x^2)}dx$ is equal to ***(2015)***

(A) 1 (B) 4 (C) 1 (D) 6

JEE Advanced/Boards

Exercise 1

Q.1 $\int_0^1 e^{\tan^{-1}x}\sin^{-1}(\cos x)dx.$

Q.2 Prove that :

(i) $\int_0^1 \sqrt{(x-\alpha)(\beta-x)}dx - \frac{(\beta-\alpha)^2 x}{8}$

(ii) $\int_0^a \sqrt{\frac{x-\alpha}{\beta-x}}dx = (\beta-\alpha)\frac{\pi}{2}$

(iii) $\int_0^a \frac{dx}{x\sqrt{(x-\alpha)(\beta-x)}} = \frac{\pi}{\sqrt{\alpha\beta}}$ where α, β > 0

(iv) $\int_0^b \frac{x\,dx}{\sqrt{(x-\alpha)(\beta-x)}} = (\alpha+\beta)\frac{\pi}{2}$ where α < β

Q.3 (i) Let $\beta(\Pi) = \int_0^{n\pi}\sqrt{1-\sin t}\,dt.$

Find the value of β(2) – β(1).

(ii) Determine a positive integer n ≤ 5, such that $\int_0^1 e^x(x-1)^n\,dx = 16-6e.$

Q.4 (i) $\int_0^{\pi/2} e^x\left[\cos(\sin x)\cos^2\frac{x}{2}+\sin(\sin x)\sin^2\frac{x}{2}\right]dx$

(ii) $\int_0^{\pi}\left\{(1+x)e^x+(1-x)e^{-x}\right\}\ln x\,dx.$

Q.5 If P = $\int_0^{\infty}\frac{x^2}{1+x^4}dx; Q = \int_0^{\infty}\frac{x\,dx}{1+x^4}$ and R = $\int_0^{\infty}\frac{dx}{1+x^4}$

then prove that

(i) $Q = \frac{\pi}{4}$,

(ii) P = R

(iii) $P-\sqrt{2}\,Q+R = \frac{\pi}{2\sqrt{2}}$

Q.6 $\int_1^2 \frac{(x^2-1)dx}{x^2\sqrt{2x^4-2x^2+1}} = \frac{u}{v}$ where u and v are in their lowest form. Find the value of $\frac{(1000)u}{V}$

Q.7 Let h(x) = (fog) (x) + K where K is any constant. If $\frac{d}{dx}(h(x)) = -\frac{\sin x}{\cos^{-2}(\cos x)}$ then compute the value of j(0) where j(x)

$\int_{g(x)}^{f(x)}\frac{f(t)}{g(t)}dt$, where f and g are trigonometric functions.

Q.8 $\int_0^{\pi/2}\sqrt{\frac{1-\sin 2x}{1+\sin 2x}}dx$

Q.9 If the value of the definite integral $I = \int_0^2 (3x^2-3x+1)$ $\cos(x^3-3x^3+4x-2)$ dx can be expressed in the form as p(sin q where p, q ∈ N, then find (p + q).

Q.10 $\int_{-\sqrt{2}}^{\sqrt{3}}\frac{2x^7+3x^6-10x^5-7x^3-12x^2}{x^2+2}dx$.

Q.11 For a ≥ 2, if the value of the definite integral $\int_0^a \frac{dx}{a^2+(x-(1x))^2}$ equals $\frac{x}{5050}$. Find the value of a.

Q.12 $\int_{-2}^{2}\frac{x^2-x}{\sqrt{x^2+4}}$.

Q.13 Let u = $\int_0^{\pi/4}\left(\frac{\cos x}{\sin x-\cos x}\right)^2 dx$ and

v = $\int_0^{\pi/4}\left(\frac{\sin x+\cos x}{\cos x}\right)^2 dx$. Find the value of $\frac{v}{u}$

Q.14 $\int_0^{\pi/4}\frac{x\,dx}{\cos x(\cos x+\sin x)}$.

Q.15 $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{x^2-x+1}dx$

Q.16 $\int_1^{\frac{1+\sqrt{5}}{2}}\frac{x^2+1}{x^4-x^2+1}\ln\left(1+x-\frac{1}{x}\right)dx$

Q.17 $\lim_{x\to 0} n^2 \int_{-1\pi}^{1\pi} (2010\sin x + 2012\cos x)\,|x|\,dx$.

Q.18 Find the value of the definite integral $\int_0^{\pi} |\sqrt{2}\sin x + 2\cos x|\,dx$.

Q.19 If $\int_0^{\pi} \sqrt{(\cos x + \cos 2x + \cos 3x)^2 + (\sin x + \sin 2x + \sin 3x)^2}\,dx$ has the value equal to $\left(\frac{\pi}{k} + \sqrt{w}\right)$. w are positive integer. Find the value of $(k^2 + w^2)$.

Q.20 $\int_0^1 \frac{1-x}{1+x} \frac{dx}{\sqrt{x+x^2+x^3}}$

Q.21 $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin\left(\frac{\pi}{4}+x\right)}\,dx$.

Q.22 A continuous real function f satisfies f(2x) = 3 f(x) $\forall x \in R$.

If $\int_0^{\pi} f(x)dx = 1$, then compute the value of definite integral $\int_1^2 f(x)dx$.

Q.23 The value of $\int_{-1}^{3} \{|x-2| + [x]\}\,dx$, where [x] denotes the greatest integer less than or equal to x is.

Q.24 $\int_1^0 \sin^{-1}\frac{2x}{1+x^2}\,dx$.

Q.25 $\int_0^1 \frac{(ax+b)\sec x\tan x}{4+\tan^2 x}\,dx\ (a,b>0)$

Q.26 $\int_0^{\pi} \frac{(2x+3)\sin x}{(1+\cos^2 x)}\,dx$.

Q.27 Evaluate $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}$

Q.28 If $\int_0^{n\pi} \frac{x|\sin x|}{I+|\cos x|}\,dx\ (n \in N)$ is equal to $100\pi\log 2$, then the value of n.

Q.29 Evaluate $\int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}\,dx$.

Q.30 $\int_0^{\pi} \frac{\ln(I+ax)}{1+x^2}\,dx,\ a \in N$.

Q.31 $\int_0^{\frac{\ln 3}{2}} \frac{e^x+1}{e^{2x}+1}\,dx$.

Q.32 If $\int_0^a \sqrt{x}\,dx = 2a\int_0^{\pi/2} \sin^3 x\,dx$, find the value of $\int_a^{a+1} x\,dx$.

Q.33 Let α, β be the distinct positive roots of the equation tan x = 2x then evaluate $\int_0^1 (\sin_{\alpha} x \,.\, \sin_{\beta} x)dx$, independent of α and β.

Q.34 Show that $\int_0^{p+q} |\cos x|\,dx = 2q + \sin p$ where $q \in N$ & $-\frac{\pi}{2} < p < \frac{\pi}{2}$.

Q.35 Show that the sum of the two integrals

$\int_{-1}^{-\pi} e^{(x+1)^2}\,dx + 3\int_{1/3}^{2/3} e^{(x-2x)^2}\,dx$ is zero.

Q.36 Let F(x) = max (sin px, cos px). Find the value of $\frac{\pi}{4\sqrt{2}} \int_{-10}^{10} F(x)dx$.

Q.37 $\int_0^{\pi/2} \tan^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]dx$.

Q.38 Comment upon the nature of roots of the quadratic equation $x^2 + 2x = k + \int_0^1 |t+k|\,dt$ dependent on the value of $k \in R$.

Q.39 $\int_{-1}^{1} \frac{(2x^{232} + x^{998} + 4x^{1668}\sin x^{691})}{1+x^{666}}$

Q.40 $\pi\int_0^{\pi} \frac{x^2\sin 2x \,.\, \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}\,dx$

Q.41 Evaluate $\int_{1/3}^{1} \frac{\left(x - x^3\right)^{1/3}}{x^4} dx$

Q.42 $\lim_{x\to\infty} \frac{1}{n^2} \sum_{k=0}^{n=1} \left[k \int_{k}^{k-1} \sqrt{(x-k)(k+1-x)} dx \right]$

Q.43 Let I = $\int_0^{\pi/2} \frac{\cos x + 4}{3\sin x + 4\cos x + 25} dx$ and

I = $\int_0^{\pi/2} \frac{\sin x + 3}{3\sin x + 4\cos x + 25} dx$.

If 25 I = aπ + b ln $\frac{c}{d}$ where a, b, c and d ∈ N and $\frac{c}{d}$ is not a perfect square of a rational then find the value of (a + b + c + d).

Q.44 Let y = f(x) be a quadratic function with f(2) = 1. Find the value of the integral

$\int_{2-\pi}^{2+\pi} f(x).\sin\left(\frac{x-2}{2}\right) dx$.

Exercise 2

Single Correct Choice Type

Q.1 $\int_0^2 |x^2 + 2x - 3| dx$ equals

(A) 5/3 (B) 7/3 (C) 4 (D) 0

Q.2 The correct evaluation of $\int_0^{\pi/2} \left|\sin\left(x - \frac{\pi}{4}\right)\right| dx$ is -

(A) $2+\sqrt{2}$ (B) $2-\sqrt{2}$ (C) $-2+\sqrt{2}$ (D) 0

Q.3 The correct evaluation of $\int_0^{\pi} |\sin^4 x| dx$ is -

(A) $\frac{8\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{4\pi}{3}$ (D) $\frac{3\pi}{8}$

Q.4 $\int_0^{1.5} [x^2] dx$, where [·] denotes the greatest integer function, equals -

(A) $2+\sqrt{2}$ (B) $2-\sqrt{2}$

(C) $-2+\sqrt{2}$ (D) $-2-\sqrt{2}$

Q.5 Solve $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2 \sin^2 x} dx$

(A) $\frac{\pi^2}{2ab}$ (B) $\frac{\pi^2}{4ab}$

(C) $\frac{\pi^2}{3ab}$ (D) $\frac{\pi}{5ab}$

Q.6 $\int_0^{\pi/4} \frac{\sec x}{1 + 2\sin^2 x}$ is equal to -

(A) $\frac{1}{3}\left|\log(\sqrt{2}+1) + \frac{\pi}{2\sqrt{2}}\right|$ (B) $\frac{1}{3}\left|\log(\sqrt{2}+1) - \frac{\pi}{2\sqrt{2}}\right|$

(C) $3\left|\log(\sqrt{2}+1) - \frac{\pi}{2\sqrt{2}}\right|$ (D) $3\left|\log(\sqrt{2}+1) + \frac{\pi}{2\sqrt{2}}\right|$

Q.7 If $\int_0^1 e^{x^2}(x-\alpha)dx = 0$, then

(A) 1 < α < 2 (B) α < 0

(C) 0 < α < 1 (D) None of these

Q.8 $\int_b^{\pi/2} \{x - [\sin x]\} dx$ is equal to -

(A) $\frac{\pi^2}{8}$ (B) $\frac{\pi^2}{8} - 1$ (C) $\frac{\pi^2}{8} - 2$ (D) None of these

Q.9 The value of the integral $\int_b^{100} \sin\{x - [x]\}\pi dx$ is -

(A) $\frac{100}{\pi}$ (B) $\frac{200}{\pi}$ (C) 100 π (D) 200 π

Q.10 The value of the integral $\int_0^{\infty} \frac{x\log x}{(1+x^2)^2} dx$ is -

(A) 1 (B) 0 (C) 2 (D) None of these

Q.11 $\lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)......\left(1+\frac{n}{n}\right)\right]^{1/n}$ is equal to -

(A) e/4 (B) 4/e (C) 2/e (D) None of these

Q.12 The solution of the equation

$\int_{\log 2}^{x} \frac{1}{\sqrt{e^x - 1}} dx = \frac{\pi}{6}$

(A) x = log 4 (B) x = log 2

(C) x = log $\left(\frac{1}{4}\right)$ (D) None of these

Q.13 The value of

$$\lim_{n\to\infty}\left(\frac{1}{1+n^3}+\frac{4}{8+n^3}+\frac{9}{27+n^3}+\ldots+n \text{ terms}\right) \text{ is -}$$

(A) $\frac{1}{3}\log 2$ (B) 0 (C) $\frac{1}{3}\log 3$ (D) None of these

Q.14 $\lim_{n\to\infty}$

$$\left\{\frac{n^2}{(n^2+1^2)^{3/2}}+\frac{n^2}{(n^2+2^2)^{3/2}}+\ldots+\frac{n^2}{[n^2+(n-1)^2]^{3/2}}\right\} \text{ is}$$

equal -

(A) $-\frac{1}{\sqrt{2}}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{2}$ (D) None of these

Q.15 $\lim_{n\to\infty}\left\{\frac{\sqrt{n}}{\sqrt{n^2}}+\frac{\sqrt{n}}{\sqrt{(n+4)^3}}+\frac{\sqrt{n}}{\sqrt{(n+8)^3}}+\ldots.+\frac{\sqrt{n}}{\sqrt{[n+4(n-1)]^3}}\right\}$

is equal -

(A) $\frac{1}{10}\left[5-\sqrt{5}\right]$ (B) $\left[5-\sqrt{5}\right]$ (C) $\frac{1}{5}\left[5-\sqrt{5}\right]$ (D) 0

Q.16 $\lim_{n\to\infty}\left\{\tan\left(\frac{\pi}{2n}\right)\tan\left(\frac{2\pi}{2n}\right)\tan\left(\frac{3\pi}{2n}\right)\ldots\tan\left(\frac{n\pi}{2n}\right)\right\}^{\frac{1}{6}}$

is equal -

(A) 0 (B) 1 (C) –1 (D) 2

Previous Years' Questions

Q.1 The integral $\int_{1/2}^{1/2}\left([x]+\log\left(\frac{1+x}{1-x}\right)\right)dx$ equals ***(2002)***

(A) $-\frac{1}{2}$ (B) 0 (C) 1 (D) $\log\left(\frac{1}{2}\right)$

Q.2 If $I(m,n) = \int_0^1 t^m(1+t)^n dt$, then the expression for I(m,n) in terms of I(m + 1, n – 1) is ***(2003)***

(A) $\frac{2^n}{m+1}-\frac{n}{m+1}I(m+1,n-1)$

(B) $\frac{n}{m+1}I(m+1,n-1)$

(C) $\frac{2^n}{m+1}+\frac{n}{m+1}I(m+1,n-1)$

(D) $\frac{m}{m+1}I(m+1,n-1)$

Q.3 Let f be a non-negative function defined on the interval [0, 1] . If $\int_0^x\sqrt{1-(f'(t))^2}\,dt$ = $\int_0^x f(t)dt,\ 0\le x\le 1$ and f(0) = 0, then ***(2009)***

(A) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

(B) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

(C) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

(D) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

Q.4 The value of

$$\int_{\sqrt{\log 2}}^{\sqrt{\log 3}}\frac{x\sin x^2}{\sin x^2+\sin(\log 6-x^2)}dx \text{ is -}$$ ***(2011)***

(A) $\frac{1}{4}\log\frac{3}{2}$ (B) $\frac{1}{2}\log\frac{3}{2}$ (C) $\log\frac{3}{2}$ (D) $\frac{1}{6}\log\frac{3}{2}$

Q.5 Let $S_n = \sum_{k=0}^{n}\frac{n}{n^2+kn+k^2}$ and

$T_n = \sum_{k=0}^{n-1}\frac{n}{n^2+kn+k^2}$, for n = 1, 2, 3 ….. , then ***(2008)***

(A) $S_n<\frac{\pi}{3\sqrt{3}}$ (B) $S_n>\frac{\pi}{3\sqrt{3}}$

(C) $T_n<\frac{\pi}{3\sqrt{3}}$ (D) $T_n>\frac{\pi}{3\sqrt{3}}$

Q.6 If $I_n = \int_{-\pi}^{\pi}\frac{\sin nx}{(1+\pi^x)\sin x}dx$, n = 0, 1, 2,…, then ***(2009)***

(A) $I_n = I_{n+2}$ (B) $\sum_{m=1}^{10}I_{2m+1}=10\pi$

(C) $\sum_{m=1}^{10}I_{2m}=0$ (D) $I_n = I_{n+1}$

Q.7 The value(s) of $\int_0^1\frac{x^4(1-x)^4}{1+x^2}dx$ is (are) ***(2010)***

(A) $\frac{22}{7}-\pi$ (B) $\frac{2}{105}$ (C) 0 (D) $\frac{71}{15}-\frac{3\pi}{2}$

Paragraph for Q.8

Read the following passage and answer the questions. For every function f(x) which is twice differentiable, these will be good approximation of

$\int_a^b f(x)dx = \left(\frac{b-a}{2}\right)\{f(a) + f(b)\}$,

for more accurate results for $c \in (a, b)$,

$F(c) = \frac{c-a}{2}[f(a)-f(c)] + \frac{b-c}{2}[f(b)-f(c)]$

When $c = \frac{a+b}{2}$

$\int_a^b f(x)dx = \frac{b-a}{4}\{f(a)+f(b)+2f(c)\}dx$ ***(2006)***

Q.8 Good approximation of $\int_0^{\pi/2} \sin x\, dx$, is ***(2003)***

(A) $\pi/4$ (B) $\pi(\sqrt{2}+1)/4$

(C) $\pi(\sqrt{2}+1)/8$ (D) $\pi/8$

Q.9 If f″(x) < 0, "x∈ (a, b), and (c,f(c)) is point of maxima where c∈(a, b), then f′(c) is - ***(2009)***

(A) $\frac{f(b)-f(a)}{b-a}$ (B) $3\left(\frac{f(b)-f(a)}{b-a}\right)$

(C) $2\left(\frac{f(b)-f(a)}{b-a}\right)$ (D) 0

Q.10 If $\lim_{t\to a}\frac{\int_a^1 f(x)dx - ((t-a)/2)\{f(t)+f(a)\}}{(t-a)^3} = 0$, then degree of polynomial function f(x) at most is - ***(2002)***

(A) 0 (B) 1 (C) 3 (D) 2

Q.11 For any real number x, let [x] denotes the largest integer less than or equal to x. Let f be a real valued function defined on the interval

[–10, 10] by f(x) = $\begin{cases} x-[x] & \text{if } [x] \text{ is odd} \\ 1+[x]-x & \text{if } [x] \text{ is even} \end{cases}$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos p x\, dx$ is .... ***(2010)***

Q.12 For x > 0, let f(x) = $\int_1^x \frac{\log t}{1+t}dt$. Find the function f(x) + f(1/x) and show that f(e) +f(1/e) = 1/2. Here, $\ln t = \log_e t$ ***(2000)***

Q.13 If f is an even function, then prove that

$\int_0^{\pi/2} f(\cos 2x)\cos x\, dx = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\, dx$ ***(2003)***

Q.14 Evaluate $\int_{-\pi/3}^{\pi/3}\frac{\pi+4x^3}{2-\cos(|x|+(\pi/3))}dx$. ***(2004)***

Q.15 Evaluate

$\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\, dx$ ***(2005)***

Q.16 The value of $\frac{(5050)\int_0^1 (1-x^{50})^{100}dx}{\int_0^1 (1-x^{50})^{101}dx}$ ***(2006)***

Q.17 Let $g(x) = \int_0^x \frac{f'(t)}{1+t^2}dt$, then which of the following is true? ***(2008)***

(A) $g(x)$ is positive on $(-\infty,0)$ and negative on $(\infty,0)$

(B) $g(x)$ is negative on $(-\infty,0)$ and positive on $(0,\infty)$

(C) $g(x)$ changes sign on both $(-\infty,0)$ and ∴

(D) $g(x)$ does not change sign on (∞,∞)

Q.18 $\int_{-1}^{1} g'(x)dx =$ ***(2008)***

(A) $2g(-1)$ (B) 0 (C) $-2g(1)$ (D) $2g(1)$

Q.19 The total number of distinct $x \in [0,1]$ for which $\int_0^x \frac{t^2}{1+t^4}dt = 2x-1$ is ***(2016)***

Q.20 Let $f R \to R$ be a continuous function which satisfies $f(x) = \int_0^x f(t)dt$. Then the value of f(In5) is ***(2009)***

Q.21 Let f be a non-negative function defined on the interval $[0,1]\int_0^x \sqrt{1-(f'(t))^2}dt = \int_0^x f(t)dt$ $0 \le x \le 1$ and $f(0) = 0$ then ***(2009)***

(A) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(B) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(C) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

(D) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

Q.22 Match the statements/expressions in column I with the open intervals in column II. ***(2009)***

Column I	Column II
(A) Interval contained in the domain of definition of non-zero	(p) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
(B) Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)dx$	(q) $\left(0,\frac{\pi}{2}\right)$
(C) Interval in which at least one of the points of local maximum of $\cos^2 x+\sin$ lies	(r) $\left(\frac{\pi}{8},\frac{\pi}{2}\right)$
(D) Interval in which $\tan^{-1}(\sin x+\text{cox})$ is increasing	(s) $\left(0,\frac{\pi}{2}\right)$
	(t) $(-\pi,\pi)$

Q.23 Match the statements in column I with those in column II. ***(2010)***

Column I	Column II
(A) A line from the origin meets the lines $\frac{x-2}{1}=\frac{y-1}{-2}=\frac{z+1}{1}$ and $\frac{x-\frac{8}{3}}{2}=\frac{y+3}{-1}=\frac{z-1}{1}$ at P and Q respectively. If length $PQ=d$ Then d^2 is	(p) -4
(B) The values of x satisfying $\tan^{-1}(x+3)-\tan^{-1}(x-3)=\sin^{-1}\left(\frac{3}{5}\right)$	(q) 0
(C) Non-zero vectors $\vec{a},\vec{b}$ and $\vec{c}$ satisfy $\vec{a}.\vec{c}=0(\vec{b}-\vec{a}).(\vec{b}-\vec{c})=0$ and possible values of are	(r) 4
(D) Let f be the function on $[-\pi,\pi]$ given by $f(0)=9$ and $f(x)=\sin\left(\frac{9x}{2}\right)/\sin\left(\frac{x}{2}\right)x\neq 0$ The value of $\frac{2}{\pi}\int_{-\pi}^{\pi} f(x)dx$ is	(r) 5
	(s) 6

Q.24 The value of $\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\log(1+n)}{t^4+4}dt$ is ***(2010)***

(A) 0 (B) $\frac{1}{12}$ (C) $\frac{1}{24}$ (D) $\frac{1}{64}$

Q.25 The value (s) of $\int_0^1 \frac{x^{4(1-x)^4}}{1+x^2}dx$ is (are) ***(2010)***

(A) $\frac{22}{7}-\pi$ (B) $\frac{2}{105}$

(C) 0 (D) $\frac{71}{15}-\frac{3\pi}{2}$

Q.26 The value of $\int_{\sqrt{\log 2}}^{\sqrt{\log 3}} \frac{x\sin x^2}{\sin x^2+\sin(\log 6-x^2)}dx$ is ***(2011)***

(A) $\frac{1}{4}\log\frac{3}{2}$ (B) $\frac{1}{2}\log\frac{3}{2}$ (C) $\log\frac{3}{2}$ (D) $\frac{1}{6}\log\frac{3}{2}$

Q.27 The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2+\log\frac{\pi+x}{\pi-x}\right)\cos x dx$ is ***(2012)***

(A) 0 (B) $\frac{\pi^2}{2}-4$ (C) $\frac{\pi^2}{2}+4$ (D) $\frac{\pi^2}{2}$

Q.28 The following integral $\int_{\pi/4}^{\pi/2} (2\operatorname{cosec} x)^{17} dx$ is equal to ***(2014)***

(A) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$

(B) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$

(C) $\int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{16} du$

(D) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16} du$

Q.29 Match the following: ***(2014)***

List I	List II
(i) The number of polynomials $f(x)$ with non negative integer coefficients of degree ≤ 2 satisfying $f(0)=0$ and $\int_0^1 f(x)dx = 1$, is	(p) 8
(ii) The number of points in the interval $\left[-\sqrt{13}, \sqrt{13}\right]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	(q) 2
(iii) $\int_{-2}^{2} \frac{3x^2}{(1+e^x)} dx$ equals	(r) 4
(iv) $\frac{\left(\int_{-1/2}^{1/2} \cos 2x.\log\left(\frac{1+x}{1-x}\right)dx\right)}{\int_0^{1/2} \cos 2x.\log\left(\frac{1+x}{1-x}dx\right)}$ equals	(s) 0

Codes:

	i	ii	iii	iv
(A)	r	q	s	p
(B)	q	r	s	p
(C)	r	q	p	s
(D)	q	r	p	s

Q.30 The value of $\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}(1-x^2)\right\} dx$ is ________ ***(2014)***

Q.31 If $\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$

Where $\tan^{-1}$ takes only principal values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is ***(2015)***

Q.32 The option(s) with the values of *a* and *L* that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t\left(\sin^6 at + \cos^4 at\right)}{\int_0^{\pi} e^t\left(\sin^6 at + \cos^4 at\right)} = L?$$ ***(2015)***

(A) $a = 2, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$ (B) $a = 2, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

(C) $a = 4, L = \frac{e^{4\pi}-1}{e^{\pi}-1}$ (D) $a = 4, L = \frac{e^{4\pi}+1}{e^{\pi}+1}$

Q.33 The correct statement(s) is(are) ***(2015)***

(A) $f'(1) < 0$

(B) $f'(2) < 0$

(C) $f'(x) \ne 0$ for any $x \in (1,3)$

(D) $f(x) = 0$ for some $x \in (1,3)$

Q.34 Let $f: R \to R$ be a function defined by

$f(x) = \begin{Bmatrix} [x] & x \le 2 \\ 0 & x > 2 \end{Bmatrix}$ where $[x]$ is the greatest integer less than or equal to *x*.

If $I = \int_{-1}^{2} \frac{xf(x^2)}{2+f(x+1)} dx$ *dx*, then the value of (4*I* - 1) is

(2015)

Q.35 The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^x} dx$ is equal to ***(2016)***

(A) $\frac{\pi^2}{4} - 2$ (B) $\frac{\pi^2}{4} + 2$

(C) $\pi^2 - e^{\frac{\pi}{2}}$ (D) $\pi^2 + e^{\frac{\pi}{2}}$

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q.3 Q.8 Q.12
Q.17 Q.21 Q.23
Q.26 Q.28

Exercise 2

Q.9 Q.12 Q.17
Q.20 Q.23 Q.29
Q.32 Q.34

Previous Years' Questions

Q.4 Q.8 Q.11

JEE Advanced/Boards

Exercise 1

Q.2 Q.7 Q.10
Q.15 Q.22 Q.27
Q.32 Q.34 Q.44

Exercise 2

Q.2 Q.7 Q.10
Q.12 Q.15

Previous Years' Questions

Q.1 Q.4 Q.6
Q.7 Q.10 Q.15

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $\frac{\pi}{6}$

Q.2 $\frac{\pi}{4\sqrt{5}}$

Q.3 $\frac{\pi}{3\sqrt{3}}$

Q.4 $\frac{13}{10}$

Q.5 $\frac{34}{3}$

Q.6 $2-\sqrt{2}$

Q.7 $\frac{\pi^2}{2\sqrt{2}}$

Q.8 $\frac{14}{3}$

Q.9 $2(\sqrt{2}-1)$

Q.11 $200\sqrt{2}$

Q.12 (ii) Not necessary

Q.13 $2x\sqrt{1+x^4}$

Q.14 $\frac{4\pi}{\sqrt{3}}\tan^{-1}\left(\frac{1}{2}\right)$

Q.16 $2-\frac{2}{e}+2e\log 2$

Q.17 $\frac{\pi}{2}$

Q.18 $\frac{3}{2}-2\sqrt{3}+\frac{1}{\sqrt{2}}$

Q.19 $1, \frac{5}{12}-\frac{1}{2}\log 2$

Q.20 a = 1, 2, 3 or 4

Q.21 –1	**Q.22** $\frac{\pi}{4}$	**Q.26** $\frac{e^2+1}{2e}$
Q.29 (i) $\frac{1}{2}$; (ii) 1+e	**Q.30** $(-\infty, -2) \cup (-1, 0) \cup (1, \infty)$	

Exercise 2

Single Correct Choice Type

Q.1 A	**Q.2** C	**Q.3** C	**Q.4** B	**Q.5** A	**Q.6** B
Q.7 D	**Q.8** A	**Q.9** A	**Q.10** B	**Q.11** A	**Q.12** B
Q.13 C	**Q.14** B	**Q.15** A	**Q.16** B	**Q.17** C	**Q.18** C
Q.19 A	**Q.20** C	**Q.21** A	**Q.22** C	**Q.23** C	**Q.24** C
Q.25 C	**Q.26** C	**Q.27** B	**Q.28** B	**Q.29** B	**Q.30** A
Q.31 B	**Q.32** C	**Q.33** C	**Q.34** C		

Previous Years' Questions

Q.1 A	**Q.2** C	**Q.3** A	**Q.4** D	**Q.5** A	**Q.6** A
Q.7 A	**Q.8** C	**Q.9** C	**Q.10** C	**Q.11** A	**Q.12** C
Q.13 D	**Q.14** B	**Q.15** A → s ; B → s ; C → p ; D → r		**Q.16** 4	**Q.17** $\pi(\sqrt{2}-1)$
Q.18 $\frac{1}{2}$	**Q.19** B	**Q.20** D	**Q.21** A	**Q.22** D	**Q.23** B C
Q.24 D	**Q.25** D	**Q.26** C			

JEE Advanced/Boards

Exercise 1

Q.1 $\frac{\pi^2}{8}-\frac{\pi}{4}(1+\log 2)+\frac{1}{2}$	**Q.3** (a) 4 (b) n = 3	
Q.4 (i) $\frac{1}{2}[e^{\pi/2}(\cos 1+\sin 1)-1]$ (ii) $e^{1+e}+e^{1-e}+e^{-e}-e^{e}+e-e^{-1}$		**Q.6** 125
Q.7 1 – sec(1)	**Q.8** ln 2	**Q.9** 4
Q.10 $\frac{\pi}{2\sqrt{2}}-\frac{16\sqrt{2}}{5}$	**Q.11** 2525	**Q.12** $4\sqrt{2}-4\ln\left|1+\sqrt{2}\right|$
Q.13 4	**Q.14** $\frac{\pi}{8}\log 2$	**Q.15** $\frac{\pi^2}{6\sqrt{3}}$
Q.16 $\frac{\pi}{8}\log 2$	**Q.17** 2012	**Q.18** $2\sqrt{6}$
Q.19 153	**Q.20** $\frac{\pi}{3}$	**Q.21** $\frac{\pi(a+b)}{2\sqrt{2}}$

Q.22 5

Q.23 90

Q.24 $\frac{\pi}{\sqrt{3}}$

Q.25 $\frac{(a\pi+2b)\pi}{3\sqrt{3}}$

Q.26 $\frac{\pi(\pi+3)}{2}$

Q.27 $=\frac{1}{2}\left[x\right]_0^{\pi/2}=\frac{1}{2}\left(\frac{\pi}{2}-0\right)=\frac{\pi}{4}$

Q.28 10

Q.29 $\frac{1}{2}\left[\frac{\pi}{2}-\log 2\right]$

Q.30 $\tan^{-1}(a)\,.\,\log\sqrt{1+a^2}$

Q.31 $\frac{1}{2}\left[\frac{\pi}{6}+\log 3-\log 2\right]$

Q.32 $\frac{9}{2}$

Q.33 0

Q.36 5

Q.37 $\frac{3\pi^2}{16}$

Q.38 Real and distinct $\forall\ k \in R$

Q.39 $\frac{\pi+4}{666}$

Q.40 8

Q.41 6

Q.42 $\frac{\pi}{16}$

Q.43 62

Q.44 I = 8 as $\int_0^{\pi/2} y\sin y\,dy = 1$

Exercise 2

Single Correct Choice Type

Q.1 C	**Q.2** B	**Q.3** D	**Q.4** B	**Q.5** A	**Q.6** A
Q.7 C	**Q.8** A	**Q.9** B	**Q.10** B	**Q.11** B	**Q.12** A
Q.13 A	**Q.14** B	**Q.15** A	**Q.16** B		

Previous Years' Questions

Q.1 A	**Q.2** A	**Q.3** C	**Q.4** A	**Q.5** A, D	**Q.6** A, B, C
Q.7 A	**Q.8** C	**Q.9** A	**Q.10** B	**Q.11** 4	

Q.12 $f(e)+f\left(\frac{1}{e}\right)=\frac{1}{2}(\log e)^2=\frac{1}{2}$

Q.13 $I=\sqrt{2}\int_0^{\pi/4} f(\sin 2t)\cos t\,dt$

Q.14 $\frac{4\pi}{\sqrt{3}}\tan^{-1}\left(\frac{1}{2}\right)$

Q.15 $\frac{24}{5}\left(e\cos\left(\frac{1}{2}\right)+\frac{e}{2}\sin\left(\frac{1}{2}\right)-1\right)$

Q.16 5051

Q.17 B

Q.18 D

Q.19 A

Q.20 0

Q.21 C

Q.22 $A\rightarrow$ p, q, s; $B\rightarrow$ p, t; $C\rightarrow$ p, q, r, t; $D\rightarrow$ s

Q.23 A

Q.24 B	**Q.25** A		**Q.26** A	**Q.27** B	**Q.28** A
Q.29 D	**Q.30** 2	**Q.31** 9	**Q.32** A, C	**Q.33** A, B, C	**Q.34** –2
Q.35 A					

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $\int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}} = \int_{1/4}^{1/2} \frac{dx}{\sqrt{\frac{1}{4}-\left(x - \frac{1}{2}\right)^2}}$

$= \sin^{-1}\frac{\left(x-\frac{1}{2}\right)}{1/2}\Bigg|_{1/4}^{1/2} = \sin^{-1}0 - \sin^{-1}\left(\frac{-1/4}{1/2}\right)$

$= \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Sol 2: $\int_0^{\pi/2} \frac{dx}{4\sin^2 x + 5\cos^2 x} = \int_0^{\pi/2} \frac{dx}{4+\cos^2 x}$

$= \int_0^{\pi/2} \frac{dx}{\frac{9}{2}+\frac{\cos 2\theta}{2}} = \int_0^{\pi/2} \frac{2dx}{9+\frac{1-\tan^2\theta}{1+\tan^2\theta}}$

$= 2\int_0^{\pi/2} \frac{\sec^2\theta d\theta}{10+8\tan^2\theta}$

$= \int_0^{\pi/4} \frac{2\sec^2\theta d\theta}{10+8\tan^2\theta} + \int_{\pi/4}^{\pi/2} \frac{2\operatorname{cosec}^2\theta}{10\cot^2\theta+8}d\theta$

$= \int_0^{\pi/4} \frac{\sec^2\theta d\theta}{5+4\tan^2\theta} + \int_{\pi/4}^{\pi/2} \frac{\operatorname{cosec}^2\theta}{5\cot^2\theta+4}d\theta$

$= \int_0^1 \frac{dt}{5+4t^2} + \int_1^0 -\frac{dt}{5t^2+4} = \int_0^1 \frac{dt}{5+4t^2} + \int_0^1 \frac{dt}{5t^2+4}$

$= \frac{1}{4}\times\frac{1}{\sqrt{5}/2}\tan^{-1}\frac{t}{\sqrt{5}/2}\Bigg|_0^1 + \frac{1}{5}\times\frac{1}{2/\sqrt{5}}\tan^{-1}\frac{t}{2/\sqrt{5}}\Bigg|_0^1$

$= \frac{1}{2\sqrt{5}}\tan^{-1}\frac{2}{\sqrt{5}} + \frac{1}{2\sqrt{5}}\tan^{-1}\frac{\sqrt{5}}{2}$

$= \frac{1}{2\sqrt{5}}\left[\tan^{-1}\frac{2}{\sqrt{5}} + \cot^{-1}\frac{2}{\sqrt{5}}\right]$

$= \frac{1}{2\sqrt{5}}\times\frac{\pi}{2} = \frac{\pi}{4\sqrt{5}}$

Sol 3: $\int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x\cos x}dx$

$I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{1+\sin x\cos x}dx$ or $I = \int_0^{\pi/2} \frac{\cos^2 x}{1+\sin x\cos x}dx$

$\therefore 2I = \int_0^{\pi/2} \frac{1}{1+\sin x\cos x}dx = \int_0^{\pi/2} \frac{\sec^2 x dx}{1+\tan^2 x+\tan x}$

$= \lim_{x\to\infty}\int_0^x \frac{dt}{1+t^2+t} = \lim_{x\to\infty}\int_0^x \frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2t+1}{\sqrt{3}}\Bigg|_0^\infty$

$= \frac{2}{\sqrt{3}}\left[\frac{\pi}{2}-\frac{\pi}{6}\right] = \frac{2\pi}{3\sqrt{3}}$

$\therefore I = \frac{\pi}{3\sqrt{3}}$

Sol 4: $\int_0^{3/5}(3-5x)dx + \int_{3/5}^1 (5x-3)dx$

$= 3x - \frac{5}{2}x^2\Bigg|_0^{3/5} + \frac{5x^2}{2} - 3x\Bigg|_{3/5}^1$

$= \frac{9}{5}-\frac{5}{2}\times\frac{9}{25}+\left(\frac{5}{2}-3\right)-\left(\frac{5}{2}\times\frac{9}{25}-\frac{9}{5}\right)$

$= \frac{9}{5}-\frac{9}{10}-\frac{1}{2}-\frac{9}{10}+\frac{9}{5}$

$= \frac{18}{5}-\frac{9}{5}-\frac{1}{2} = \frac{9}{5}-\frac{1}{2} = \frac{13}{10}$

Sol 5: $\int_1^2 (2x+1)dx + \int_2^3 (x^2+1)dx$

$= x^2 + x\Big|_1^2 + \frac{x^3}{3}+x\Big|_2^3$

$= (4 + 2 - 2) + (9 + 3) - \left(\frac{8}{3}+2\right)$

$= 4 + 12 - 2 - \frac{8}{3} = 14 - \frac{8}{3} = \frac{34}{3}$

Sol 6: $\int_{-\pi/4}^{\pi/4} \sin x dx = 2\int_0^{\pi/4} \sin x dx = 2\left[-\cos x\Big|_0^{\pi/4}\right]$

$= 2\left[-\left(\frac{1}{\sqrt{2}}-1\right)\right] = 2\left[1-\frac{1}{\sqrt{2}}\right] = 2-\sqrt{2}$

Sol 7: $\int_0^{\pi} \frac{x}{(1+\sin^2 x)}dx = \int_0^{\pi} \frac{\pi - x}{(1+\sin^2 x)}dx = I$

$\therefore 2I = \pi\int_0^{\pi} \frac{1}{1+\sin^2 x}dx = 2\pi\int_0^{\pi/2} \frac{1}{1+\sin^2 x}dx$

$\therefore I = \pi\int_0^{\pi/2} \frac{1}{1+\frac{1-\cos 2x}{2}}dx = \pi\int_0^{\pi/2} \frac{2}{3-\cos 2x}dx$

$= 2\pi\int_0^{\pi/2} \frac{dx}{3-\frac{(1-\tan^2 x)}{1+\tan^2 x}} = 2\pi\int_0^{\pi/2} \frac{\sec^2 x dx}{2+4\tan^2 x}$

$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\sec^2 x dx}{\frac{1}{2}+\tan^2 x} = \frac{\pi}{2}\left[\int_0^{1} \frac{dt}{\frac{1}{2}+t^2} - \int_{\pi/4}^{\pi/2} \frac{\operatorname{cosec}^2 x dx}{\frac{1}{2}\cot^2 x+1}\right]$

$= \frac{\pi}{2}\left[\left[\frac{1}{\frac{1}{\sqrt{2}}}\tan^{-1}\frac{t}{\frac{1}{\sqrt{2}}}\right]_0^1 - \int_1^0 \frac{dt}{\frac{1}{2}t^2+1}\right]$

$= \frac{\pi}{2}\left[\sqrt{2}\tan^{-1}\sqrt{2} + 2\int_0^1 \frac{dt}{t^2+2}\right]$

$= \frac{\pi}{2}\left[\sqrt{2}\tan^{-1}\sqrt{2} + \frac{2}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}\right]_0^1$

$= \frac{\pi}{2}\left[\sqrt{2}\tan^{-1}\sqrt{2} + \sqrt{2}\tan^{-1}\frac{1}{\sqrt{2}}\right]$

$= \frac{\pi}{2}\times\sqrt{2}\times\frac{\pi}{2} = \frac{\pi^2}{2\sqrt{2}}$

Sol 8: $\int_0^2 (x^2+1)dx$

$h = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

$\therefore I = \lim_{n\to 0}\sum_{r=1}^{n} hf(a+rh) = \lim_{n\to 0}\sum_{r=1}^{n} h((rh)^2+1)$

$= \lim_{n\to 0}\sum_{r=1}^{n}(r^2h^3+h)$

$= \lim_{n\to\infty}\left[h^3\times\frac{n(n+1)(2n+1)}{6}+hn\right]$

$= \lim_{n\to\infty}\left[\frac{8}{n^3}\frac{(n)(n+1)(2n+1)}{6}+\frac{2}{n}\times n\right]$

$= \frac{8\times 1\times 2}{6}+2 = \frac{14}{3}$

Sol 9: $\int_0^{\pi/2} \left|\sin x - \cos x\right| dx$

$= \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)dx$

$= \sin x + \cos x\Big|_0^{\pi/4} + (-\cos x - \sin x\Big|_{\pi/4}^{\pi/2}$

$= \left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1\right) - \left(1-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\right)$

$= \sqrt{2} - 1 - 1 + \sqrt{2} = 2\sqrt{2} - 2$

Sol 10: f(x) = f(a – x)

g(x) + g(a – x) = 2

$\int_0^a f(x)g(x)dx = \int_0^a f(a-x)g(a-x)dx = I$

$\therefore 2I = \int_0^a \left[f(x)g(x)+f(a-x)g(a-x)\right]dx = \int_0^a \left[f(x)\times 2\right]dx$

$I = \int_0^a f(x)dx$

Sol 11: $\int_0^{100\pi} \sqrt{1-\cos 2x}dx$

$\int_0^{100\pi} \sqrt{2\sin^2 x}dx = \sqrt{2}\int_0^{100\pi} \sqrt{\sin^2 x}dx$

Q$\sin^2(\pi - x) = \sin^2 x$

$\because I = 100\sqrt{2}\int_0^{\pi} \sqrt{\sin^2 x}dx = 100\sqrt{2}\int_0^{\pi} |\sin x|dx$

Also $\left|\sin(\pi - x)\right| = |\sin x|$

$\therefore I = 200\sqrt{2}\int_0^{\pi/2} |\sin x|dx$

$= 200\sqrt{2}\left(-|\cos x|\right)_0^{\pi/2} = 200\sqrt{2}$

Sol 12: (i) f(–t) = –f(t)

$g(x) = \int_a^x f(t)dt$

$g(-x) = \int_a^{-x} f(t)dt = \int_a^{-a} f(t)dt + \int_{-a}^{-x} f(t)dt$

Qf(t) = odd function

So $\int_a^{-a} f(t)dt = 0$

$\therefore \int_a^{-x} f(t)dt = \int_{-a}^{-x} f(t)dt$

Put t = –p

$= -\int_a^x f(-p)dp \quad \because f(-p) = -f(p)$

$= \int_a^x f(p)dp$

$\therefore g(-x) = g(x)$

(ii) f(t) = f(–t)

$g(x) = \int_a^x f(t)dt$; $g(-x) = \int_a^{-x} f(t)dt$

Put t = –p

$= -\int_{-a}^x f(-p)dp = -\int_{-a}^x f(p)dp = -\int_{-a}^x f(t)dt$

$\therefore g(-x) = \int_x^{-a} f(t)dx$

$\therefore g(x) + g(-x) = \int_a^x f(t)dt + \int_x^{-a} f(t)dt = \int_a^{-a} f(t)dt$

$\therefore$ It is not necessary that if f(t) is even then $\int_a^x f(t)dt$ is odd

Sol 13: $f(x) = \int_a^{x^2} \sqrt{1+t^2}\,dt$

$f'(x) = \sqrt{1+x^4}\,dx^2 = 2x\sqrt{1+x^4}$

Sol 14: $I = \int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3 dx}{2-\cos\left(|x| + \frac{\pi}{3}\right)}$

$I = \int_{-\pi/3}^{\pi/3} \frac{\pi - 4x^3}{2-\cos\left(|x| + \frac{\pi}{3}\right)}dx$

$\therefore 2I = 2\pi \int_{-\pi/3}^{\pi/3} \frac{dx}{2-\cos\left(|x| + \frac{\pi}{3}\right)}$

$\Rightarrow I = 2\pi \int_0^{\pi/3} \frac{dx}{2 - \frac{\left(1-\tan^2\left(\frac{x}{2}+\frac{\pi}{6}\right)\right)}{\left(1+\tan^2\left(\frac{x}{2}+\frac{\pi}{6}\right)\right)}}$

$= \frac{2\pi}{3}\int_0^{\pi/3} \frac{\sec^2\left(\frac{x}{2}+\frac{\pi}{6}\right)dx}{\frac{1}{3}+\tan^2\left(\frac{x}{2}+\frac{\pi}{6}\right)}$

Put $\tan\left(\frac{x}{2}+\frac{\pi}{6}\right) = t$

$\frac{1}{2}\sec^2\left(\frac{x}{2}+\frac{\pi}{6}\right)dx = dt$

$\therefore I = \frac{4\pi}{3}\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dt}{\frac{1}{3}+t^2} = \frac{4\pi}{3} \times \sqrt{3}\tan^{-1}\sqrt{3}t\Big|_{1/\sqrt{3}}^{\sqrt{3}}$

$= \frac{4\pi}{\sqrt{3}}\left[\tan^{-1}3 - \tan^{-1}1\right] = \frac{4\pi}{\sqrt{3}}\left[\tan^{-1}\left(\frac{3-1}{1+3\times 1}\right)\right]$

$= \frac{4\pi}{3}\tan^{-1}\frac{1}{2}$

Sol 15: $f(x) = \int_1^x \frac{\log t}{1+t}dt$

To Prove. $f(x) + f\left(\frac{1}{x}\right) = \frac{1}{2}(\log x)^2$

Put $t = \frac{1}{p} \Rightarrow dt = -\frac{1}{p^2}dp$

$f\left(\frac{1}{x}\right) = \int_1^x \frac{\log\frac{1}{p}}{1\left(1+\frac{1}{p}\right)}\left(-\frac{1}{p^2}\right)dp$

$f\left(\frac{1}{x}\right) = \int_1^x \frac{\log p}{p(1+p)}dp = \int_1^x \frac{\log t}{t(t+1)}dt$

$\therefore f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\log t}{t} dt$

$= (\log)^2\Big|_1^x - \int \frac{\log t}{t} dt$

$\therefore 2I = (\log x)^2$

$\therefore f(x) + f\left(\frac{1}{x}\right) = \frac{1}{2}(\log x)^2$

Sol 16: $-\int_{1/e}^{1} \log x dx + \int_1^{2e} \log x dx$

$= -(x\log x - x)_{1/e}^{1} + (x\log x - x)\Big|_1^{2e}$

$= -\left[0 - 1 - \left[\frac{1}{e}\log\frac{1}{e} - \frac{1}{e}\right]\right]$

$+ [(2e\log 2e - 2e) - (0 - 1)]$

$= 1 - \frac{1}{e} - \frac{1}{e} + 2e\log 2 + 2e - 2e + 1$

$= 2 - \frac{2}{e} + 2e\log 2$

Sol 17: $\int_0^{\pi} \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\frac{x}{2}} dx \quad n \in N$

$2\sin\left(n + \frac{1}{2}\right)x \cos\frac{x}{2} = \sin(nx + 2) + \sin(nx)$

$= \frac{1}{2}\int_0^{\pi} \frac{\sin nx + \sin(nx + x)}{\sin x} dx$

$= \frac{1}{2}\int_0^{\pi} \frac{\sin(n+1)x + \sin nx}{\sin x} dx$

If n is odd

$I = \frac{1}{2}\int_0^{\pi} \frac{\sin nx - \sin(n+1)x}{\sin x} dx$

$\therefore 2I = \int_0^{\pi} \frac{\sin nx}{\sin x} dx = \pi \qquad \Rightarrow I = \frac{\pi}{2}$

If n is even

$2I = \int_0^{\pi} \frac{\sin(n+1)x}{\sin x} dx = \pi; \quad I = \frac{\pi}{2}$

Sol 18: $F(x) = \int_{5\pi/4}^{x} (3\sin t + 4\cos t) dt$

$= 3(-\cos t)\Big|_{5\pi/4}^{x} + 4\sin t\Big|_{5\pi/4}^{x}$

$= 3\left[-\left(\cos x - \frac{1}{\sqrt{2}}\right)\right] + 4\left[\sin x - \frac{1}{\sqrt{2}}\right]$

$= -3\cos x + \frac{3}{\sqrt{2}} + 4\sin x - \frac{4}{\sqrt{2}}$

$= \left(\frac{4\sin x - 3\cos x}{5}\right)5 - \frac{1}{\sqrt{2}}$

From interval $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$ $\sin x < \cos x$

$\therefore$ We get min value of $x = \frac{4\pi}{3}$

$\therefore$ Min value $= -4 \times \frac{\sqrt{3}}{2} + \frac{3}{2} - \frac{1}{\sqrt{2}} = \frac{3}{2} - 2\sqrt{3} - \frac{1}{\sqrt{2}}$

Sol 19: $I_n = \int_0^{\pi/4} \tan^n \theta d\theta$

$I_{n-1} + I_{n+1} = \int_0^{\pi/4} \left(\tan^{n-1}\theta + \tan^{n+1}\theta\right) d\theta$

$= \int_0^{\pi/4} (\tan^{n-1}\theta)\sec^2\theta d\theta$

$\tan\theta = t \Rightarrow \sec^2 q d\theta = dt$

$= \int_0^1 t^{n-1} dt = \frac{t^n}{n}\Big|_0^1 = \frac{1}{n}$

$\therefore \quad n(I_{n-1} + I_{n+1}) = n \times \frac{1}{n} = 1$

$I_7 = \int_0^{\pi/4} \tan^7 \theta d\theta$

$= \int_0^{\pi/4} \tan^5\theta \sec^2 d\theta - \int_0^{\pi/4} \tan^3\theta \sec^2\theta d\theta$

$+ \int_0^{\pi/4} \tan\theta \sec^2\theta d\theta - \int_0^{\pi/4} \tan\theta d\theta$

$= \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - \log\sqrt{2} = \frac{5}{12} - \frac{1}{2}\log 2$

Sol 20: $\int_0^{\pi/2}\left\{a^2\left(\frac{\cos 3x}{4}+\frac{3}{4}\cos x\right) + a\sin x - 20\cos x\right\}dx \le -\frac{a^2}{3}$

$\int_0^{\pi/2}\left\{a^2\cos^3 x + a\sin x - 20\cos x\right\}dx \le -\frac{a^2}{3}$

$= a^2\left[+\frac{1}{12}\sin^3 x\Big|_0^{\pi/2} + \frac{3}{4}(+\sin x)\Big|_0^{\pi/2}\right]$

$+ a(-\cos x)\Big|_0^{\pi/2} - 20\sin x\Big|_0^{\pi/2} \le -\frac{a^2}{3}$

$= a^2\left[-\frac{1}{12}+\frac{3}{4}\right] + a - 20 \le -\frac{a^2}{3}$

$\Rightarrow a^2 + a - 20 \le 0$

$(a+5)(a-4) \le 0 \because a \in [-5, 4]$

$\therefore$ a is +ve interger

So a = 1, 2, 3 or 4

Sol 21: f(x) = sin x

Mean value of sin x from [–2, 0)

$\therefore \int_{-2}^{0}\frac{\sin x}{2}dx = \frac{-1[0+2]}{2} = -1$

Sol 22: $I = \int_0^a \frac{1}{x+\sqrt{a^2-x^2}}dx$

$x = a\cos\theta$

$dx = -a\sin q\, d\theta$

$\int_{\pi/2}^{a}\frac{-a\sin\theta d\theta}{a\cos\theta + a\sin\theta} = \int_0^{\pi/2}\left(\frac{\sin\theta}{\cos\theta+\sin\theta}\right)d\theta$

$= \int_0^{\pi/2}\left(\frac{\cos\theta}{\cos\theta+\sin\theta}\right)d\theta$

$\Rightarrow I = \frac{1}{2}\left[\int_0^{\pi/2}d\theta\right] = \frac{1}{2}\left[\frac{\pi}{2}\right] = \frac{\pi}{4}$

Sol 23: $I = \int_0^1 0dx + \int_1^4 1dx + \int_4^9 2dx .... + \int_{(n-1)^2}^{n^2}(n-1)dx$

$\therefore \sum_{n=0}^{n}\int_{(n-1)^2}^{n^2}(n-1)dx$

$= \sum(n-1)x\Big|_{(n-1)^2}^{n^2} = \sum(n-1)\left(n^2-(n-1)^2\right)$

$= \sum(n-1)(2n-1) = \sum(2n^2-3n+1)$

$= \frac{2n(n+1)(2n+1)}{6} - \frac{3n(n+1)}{2} + n$

$= \frac{n(n+1)}{2}\left[\frac{4n+2}{3}-3\right] + n = \frac{n(n+1)(4n-7)}{6} + n$

$= n\left(\frac{4n^2-3n-7+6}{6}\right) = \frac{n(4n^2-3n-1)}{6}$

$= \frac{n(n-1)(4n+1)}{6}$

Sol 24: $\int_0^{n\pi+\lambda}|\sin x|dx = 2n + 1 - \cos\lambda$

$n \in N, 0 \le \lambda < p$

LHS $= \int_0^{\lambda}|\sin x|dx + \int_{\lambda}^{n\pi+\lambda}|\sin x|dx$

$= -\cos x\Big|_0^{\lambda} + n\int_0^{\pi}|\sin x|dx$

$= -(\cos\lambda - 1) + 2n\int_0^{\pi/2}\sin x dx$

$= 2n + 1 - \cos\lambda$

Sol 25: $I = \int_0^{\pi}\frac{x\sin 2x.\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$ ..... (i)

$I = \int_0^{\pi}\frac{(\pi-x)(-\sin 2x)\sin\left(\frac{\pi}{2}(-\cos x)\right)dx}{2(\pi-x)-\pi}$ ..... (ii)

On adding (i) and (ii)

$2I = \int_0^{\pi}\frac{(2x-\pi)\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}dx$

$= \int_0^{\pi}\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)dx$

or $I = \int_0^{\pi/2}\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)dx$

Let $\frac{\pi}{2}\cos x = t$

$-\frac{\pi}{2}\sin x dx = dt$ or $\sin x dx = -\frac{2}{\pi}dt$

$$I = -\frac{2}{\pi}\int_{\pi/2}^{0} 2\times\frac{2t}{\pi}\sin t\,dt$$

$$= \frac{8}{\pi^2}\int_0^{\pi/2} t\sin t\,dt = \frac{8}{\pi^2}\left[t(-\cos t)\Big|_0^{\pi/2} + \int_0^{\pi/2}\cos t\,dt\right]$$

$$= \frac{8}{\pi^2}\left[0 + \sin t\Big|_0^{\pi/2}\right] = \frac{8}{\pi^2}$$

Sol 26: Let f(x) = $K_1e^x + K_2e^{-x}$

$g(x) + f'(x) = K_1e^x - K_2e^{-x}$

$\therefore g'(x) = K_1e^x + K_2e^{-x} = f(x)$

$\therefore f(0) = 1 \Rightarrow K_1 + K_2 = 1$

Also $g(0) = 0 \Rightarrow K_1 - K_2 = 0$

$K_1 = K_2 = \frac{1}{2}$

$$\therefore f(x) = \frac{e^x + e^x}{2}$$

$$\therefore f(1) = \frac{e + \frac{1}{e}}{2} = \frac{e^2 + 1}{2e}$$

Sol 27: (i) $\int_0^{\pi}\log(1+\cos x)dx$

$$= \int_0^{\pi}\log(1-\cos x)dx = I$$

$$\therefore 2I = \int_0^{\pi}\log(1-\cos^2 x)dx$$

$$= 2\int_0^{\pi}\log\sin x\,dx = 4\int_0^{\pi/2}\log\sin x\,dx$$

$$= 4\times\left(\frac{-\pi}{2}\right)\log 2$$

$2I = -2\pi\log 2$

$\therefore I = -\pi\log 2 = \pi\log\frac{1}{2}$

(ii) $$\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}} = \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot\left(\frac{\pi}{2}-x\right)}}$$

$$= \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$$

$$\therefore 2I = \int_{\pi/6}^{\pi/3}\left(\frac{\sqrt{\tan x}+1}{1+\sqrt{\tan x}}\right)dx = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6}$$

$$\therefore I = \frac{\pi}{12}$$

Sol 28: $\int_0^{\pi}\frac{dx}{1-2a\cos x + a^2}$

$$= \int_0^{\pi}\frac{dx}{1+a^2-\frac{2a\left(1-\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}}}$$

$$= \int_0^{\pi}\frac{\sec^2\frac{x}{2}dx}{(1+a^2)\left(1+\tan^2\frac{x}{2}\right)-2a\left(1-\tan^2\frac{x}{2}\right)}$$

$$= \int_0^{\pi}\frac{\sec^2\frac{x}{2}dx}{(1+a^2-2a)+\tan^2\frac{x}{2}(1+a^2+2a)}$$

Putting $\frac{x}{2} = t$

$\sec^2\frac{x}{2}dx = 2dt$

$$\int_0^{\infty}\frac{dt}{(1+a)^2t^2+(1-a)^2} = \frac{1}{1+a^2}\int_0^{\infty}\frac{dt}{t^2+\left(\frac{1-a}{1+a}\right)^2}$$

$$= \frac{1}{(1+a^2)}\times\frac{1+a}{|1-a|}\tan^{-1}\frac{t}{\left|\frac{1-a}{1+a}\right|}\Bigg|_0^{\infty}$$

$$= \frac{1}{1-a^2}\frac{\pi}{2} \text{ If } a < 1$$

$$= \frac{1}{a^2-1}\frac{\pi}{2} \text{ if } a > 1$$

Sol 29: (i) $$\lim_{\alpha\to 0}\frac{\int_0^{\alpha}x\,dx}{\alpha\sin x} = \lim_{\alpha\to 0}\frac{\alpha^2}{2\alpha\sin\alpha}$$

$$= \frac{1}{2}\lim_{\alpha\to 0}\frac{1}{\left(\frac{\sin\alpha}{\alpha}\right)} = \frac{1}{2}$$

(ii) $y = x^{\int_1^x \ell n t\,dt}$

$$\log y = \left(\int_1^x \log t\,dt\right)\log x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\int_1^x \log t\,dt + (\log x)(\log x)$$

$$\frac{dy}{dx} = x^{\int_1^x \log t dt}\left[\log^2 x + \frac{1}{x}\int_1^x \log t dt\right]$$

$$\left.\frac{dy}{dx}\right|_{x=e} = e^{\int_1^e \log t dt}\left[\log^2 e + \frac{1}{e}\int_1^e \log t dt\right]$$

$$= e(e\log e - e - (-1))\left[\log^2 e + \frac{1}{e}(e\log e - e\ (-1))\right]$$

$$= e\left[\frac{1}{e}+1\right] = e+1$$

Sol 30: $f(x) = \int_1^x (t^2+2t)(t^2-1)dt$

$f'(x) = (x^2 + 2x)(x^2 - 1) > 0$

$x(x + 2)(x - 1)(x + 1) > 0$

$\therefore x \in (-\infty, -2) \cup (-1, 0) \cup (1, \infty)$

Exercise 2

Single Correct Choice Type

Sol 1: (A) $\int_{-1}^{0} (1-2x)dx + \int_0^1 (1+2x)dx$

$= \left. x-x^2\right|_{-1}^{0} + \left. x+x^2\right|_0^1 = 0 - [-1 - 1] + [1 + 1] = 4$

Sol 2: (C) $\int_{-1}^{0} e^{-x}dx + \int_1^1 e^x dx$

$= \left. -e^{-x}\right|_{-1}^{0} + \left. e^x\right|_0^1 - [1- e^{+1}] + [e^1 - 1]$

$= e^{+1} + e^1 - 2 = 2e - 2$

Sol 3: (C) $\int_0^3 [x]dx = \int_0^1 0dx + \int_1^2 1dx + \int_2^3 2dx$

$= 0 + 1 + 2 = 3$

Sol 4: (B) $\int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} -\cos x dx$

$= \left. \sin x\right|_0^{\pi/2} - \left. \sin x\right|_{\pi/2}^{\pi} = 1 - [0 - 1] = 2$

Sol 5: (A) $\int_{-2}^{-3/2} -(2x+3)dx + \int_{-3/2}^{2} (2x+3)dx$

$= -\left[x^2+3x\right]_{-2}^{-3/2} + \left. x^2 + 3x\right|_{-3/2}^{2}$

$= -\left[\frac{9}{4}-\frac{9}{2}-(4-6)\right] + \left[4+6-\left(\frac{9}{4}-\frac{9}{2}\right)\right]$

$= \frac{9}{4} - 2 + 10 + \frac{9}{4} = \frac{9}{2} + 8 = \frac{25}{2}$

Sol 6: (B) $\int_{-2}^{2} |1-x^2|\,dx = 2\int_0^2 \left|1-x^2\right|dx$

$= 2\left[\int_0^1 (1-x^2)dx + \int_1^2 (x^2-1)dx\right]$

$= 2\left[\left. x-\frac{x^3}{3}\right|_0^1 + \left. \frac{x^3}{3}-x\right|_1^2\right]$

$= 2\left[1-\frac{1}{3}+\left(\frac{8}{3}-2\right)-\left(\frac{1}{3}-1\right)\right]$

$= 2\left[\frac{2}{3}+\frac{2}{3}+\frac{2}{3}\right] = 4$

Sol 7: (D) $f'(x) = \left(\frac{x^4-5x^2+4)}{2+e^{x^2}}\right)\times 2x = 0$

$x = 0$ or $(x^2 - 4)(x^2 - 1) = 0$

$\therefore x = 0, x = \pm 2, x = \pm 1$

Sol 8: (A) $F_1(x) = \int_2^x (2t-5)dt = \left. t^2-5t\right|_2^x$

$= x^2 - 5x - (4 - 10) = x^2 - 5x + 6$

$F_2(x) = \int_0^x 2tdt = x^2$

$\therefore x^2 = x^2 - 5x + 6 \Rightarrow x = \frac{6}{5}, y = \frac{36}{25}$

Sol 9: (A) $f(x) = f(a-x)$

$g(x) = 2 - g(a-x)$

$I = \int_0^a f(x)\,g(x)\,dx = \int_0^a f(a-x).(2-g(a-x))\,dx$

$\Rightarrow 2\int_0^a f(a-x)dx - \int_0^a f(a-x).g(a-x).dx$

Put $a - x = t$

$-dx = dt$

$\Rightarrow -2\int_a^0 f(t)\,dt - \int_a^0 -f(t).g(t)dt$

$$\Rightarrow -2\int_a^0 f(t).dt + \int_a^0 f(t).g(t).dt$$

$$\Rightarrow I = 2\int_0^a f(t).dt - \int_0^a f(t).g(t)\,dt$$

$$I = 2\int_0^a f(t).dt - I \Rightarrow 2I = 2\int_0^a f(t)\,.dt\ I = \int_0^a f(x)dx$$

Sol 10: (B) $\int_0^{\log 5} \frac{e^x\sqrt{e^x - 1}}{e^x + 3}dx$

$e^x + 3 = t$

$e^x dx = dt$

$$\int_4^8 \frac{\sqrt{t-4}}{t}dt$$

$t = 4\sec^2\theta$; $dt = 8\sec^2\theta\tan\theta dq$

$$\int_0^{\pi/4} \frac{2\tan\theta \times 8\sec^2\theta\tan\theta}{4\sec^2\theta}d\theta = \int_0^{\pi/4} 4(\sec^2\theta - 1)d\theta$$

$$= 4\tan\theta\Big|_0^{\pi/4} - 4\theta\Big|_0^{\pi/4} = 4 - p$$

Sol 11: (A)

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = 2\int_0^{\pi} \sin mx \sin nx dx$$

$$= \int_0^{\pi} [\cos(m-n)x - \cos(m+n)x]dx = 0$$

Sol 12: (B) $-\int_{1/e}^{1} \log x dx + \int_1^e \log x dx$

$$= -\left[x\log x - x\Big|_{1/e}^{1}\right] + x\log x - x\Big|_1^e$$

$$= -\left[(-1) - \left(\frac{1}{e}\log\frac{1}{e} - \frac{1}{e}\right)\right] + \left[(e\log e - e) - (-1)\right]$$

$$= 1 + \left(-\frac{1}{e}\right) - \frac{1}{e} + e - e + 1$$

$$= 2 - \frac{2}{e} = 2\left(1 - \frac{1}{e}\right)$$

Sol 13: (C) $\int_0^{\pi} \frac{dx}{1 - 2a\cos x + a^2}$

$$\Rightarrow \text{Put } \cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$$

$$\Rightarrow \int_0^{\pi} \frac{\sec^2(x/2).dx}{(1+a^2)(1+\tan^2(x/2)) - 2a(1 - \tan^2 x/2)}$$

Put $\tan(x/2) = t$

$$\frac{1}{2}.\sec^2(x/2)\,dx = dt \Rightarrow 2\int_0^{\infty} \frac{dt}{(1-a^2) + (1+a^2)t^2}$$

$$\Rightarrow \frac{2}{1-a}.\tan^{-1}.\frac{(1+a)t}{(1-a)}\Bigg]_0^{\infty} \Rightarrow \frac{2}{1-a}\left(\tan^{-1}\infty - \tan^{-1}0\right)$$

$$\Rightarrow \frac{2}{1-a}.\frac{\pi}{2} \Rightarrow \frac{\pi}{1-a}$$

Sol 14: (B) $\int_0^{\pi/4} \frac{\sec x}{1 + 2\sin^2 x}dx = \int_0^1 (1 - (1-x)^9 dx$

$$= \int_0^1 x^9 dx = \frac{1}{10}$$

Sol 15: (A) $\int_0^{\infty} \frac{dx}{\left(x + \sqrt{x^2+1}\right)^3}$

$x = \tan\theta$

$$\int_0^{\pi/2} \frac{\sec^2\theta d\theta}{(\tan\theta + \sec\theta)^3} = \int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)^3}d\theta$$

$1 + \sin\theta = t$

$\cos\theta d\theta = dt$

$$\int_1^2 \frac{dt}{t^3} = -\frac{1}{2t^2}\Bigg|_1^2 = -\frac{1}{2}\left[\frac{1}{4} - 1\right] = \frac{3}{8}$$

Sol 16: (B) $\int_1^5 \left[|x-3|\right]dx$

$$\int_1^2 1dx + \int_2^3 0dx + \int_3^4 0dx + \int_4^5 1dx$$

$1 + 1 = 2$

Sol 17: (C) $I = \int_{-\pi/2}^{\pi/2} \frac{\sin x}{1 + \cos^2 x}e^{-\cos^2 x}dx$

$$I = \int_{-\pi/2}^{\pi/2} \frac{-\sin x}{1 + \cos^2 x}e^{-\cos^2 x}dx$$

$\Rightarrow 2I = 0 \Rightarrow I = 0$

Sol 18: (C) $I = \int_{-1/2}^{1/2} \left[\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2 - 2\right]^{1/2} dx$

$$I = 2\int_0^{1/2}\left|\frac{x+1}{x-1}-\frac{x-1}{x+1}\right|^{2/2}dx$$

$$= 2\int_0^{1/2}\left|\left(\frac{x+1}{x-1}-\frac{x-1}{x+1}\right)\right|dx = 2\int_0^{1/2}\left|\frac{4x}{x^2-1}\right|dx$$

$$= -4\log\left|(x^2-1)\right|_0^{1/2} = -4\log\frac{3}{4}$$

Sol 19: (A) $\int_{-1}^{1}\{x-[x]\}dx$

$$= \left.\frac{x^2}{2}\right|_{-1}^{1} - \left[\int_{-1}^{0}(-1)dx+\int_0^1 0dx\right]$$

$$= 0 - [-(0 + 1) + 0] = 1$$

Sol 20: (C) $I = \int_4^{10}\frac{[x^2]}{[(x-14)^2]+[x^2]}dx$

$$I = \int_4^{10}\frac{[(14-x)^2]}{[x^2]+[(x-14)^2]}dx$$

$$\therefore 2I = \int_4^{10}dx = 10-4 = 6 \Rightarrow I = 3$$

Sol 21: (A) $\int_{-1}^{3}\left(|x-2|+[x]\right)dx$

$$= \int_{-1}^{2}(2-x)dx+\int_2^3(x-2)dx + \int_{-1}^{0}(-1)dx$$

$$+ \int_0^1 0dx+\int_1^2 1dx+\int_2^3 2dx$$

$$= \left.2x-\frac{x^2}{2}\right|_{-1}^{2} + \left.\frac{x^2}{2}-2x\right|_2^3 - 1(0+1) + 1 + 2$$

$$= [4-2]-\left(-2-\frac{1}{2}\right)+\left(\frac{9}{2}-6\right)-(2-4)+2$$

$$= 2+\frac{5}{2}-\frac{3}{2}+4 = 6+1 = 7$$

Sol 22: (C) $\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx = \int_0^{\pi/2}\frac{\cos^2 x}{\sin x+\cos x}dx$

$$\therefore 2I = \int_0^{\pi/2}\frac{1}{\sin x+\cos x}dx = \frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{dx}{\cos\left(x-\frac{\pi}{4}\right)}$$

$$= \frac{2}{\sqrt{2}}\int_0^{\pi/4}\sec\left(x-\frac{\pi}{4}\right)dx$$

$$= \frac{1}{\sqrt{2}}\log\left[\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right]_0^{\pi/4}$$

$$2I = \frac{2}{\sqrt{2}}\left[\log\left(\sqrt{2}+1\right)\right] \Rightarrow I = \frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$$

Sol 23: (C) $\mu_{10} = \int_0^{\pi/2}x^{10}\sin x dx$

$$\mu_8 = \int_0^{\pi/2}x^8\sin x dx$$

$$\mu_{10} = x^{10}(-\cos x)\Big|_0^{\pi/2} - \int_0^{\pi/2}(-\cos x)10x^9 dx$$

$$\mu_{10} = \left(\frac{\pi}{2}\right)^{10}(0)+\int_0^{\pi/2}\cos x 10.x^9 dx$$

$$= 10\left[\int_0^{\pi/2}\cos x.x^9 dx\right]$$

$$= 10\left[x^9\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2}9x^8\sin x dx\right]$$

$$= 10\times\left(\frac{\pi}{2}\right)^9 - 90\mu_8$$

$$\therefore \mu_{10}+90\mu_8 = 10\left(\frac{\pi}{2}\right)^9$$

Sol 24: (C) $\int_0^{\pi}e^{\sin^2 x}\cos^3(2n+1)x dx$

$$I = \int_0^{\pi}e^{\sin^2 x}\cos^3\left((2n+1)\pi-(2n+1)x\right)dx$$

$$= -\int_0^{\pi}\sin^2 x\cos^3\left((2n+1)x\right)dx$$

$$\therefore 2I = 0 \quad \Rightarrow I = 0$$

Sol 25: (C) $I = \int_{-\pi/2}^{\pi/2}\sin\log\left(x+\sqrt{x^2+1}\right)dx$

$$= \int_{-\pi/2}^{\pi/2}\sin\log\left(\sqrt{x^2+1}-x\right)dx$$

$$= \int_{-\pi/2}^{\pi/2}\sin\log\frac{1}{\sqrt{x^2+1}+x}dx$$

$= -\int_{-\pi/2}^{\pi/2} \sin\log\left(\sqrt{x^2+1}+x\right)dx = -I$

$\therefore 2I = 0 \Rightarrow I = 0$

Sol 26: (C) $\sin\alpha + \int_{\alpha}^{2\alpha} \cos 2x dx = 0$

$\Rightarrow \sin\alpha + \frac{1}{2}\sin 2x\Big|_{\alpha}^{2\alpha} = 0$

$\Rightarrow \sin\alpha + \frac{1}{2}[\sin 4\alpha - \sin 2\alpha] = 0$

$\Rightarrow \sin\alpha + \cos 3\alpha \sin\alpha = 0$

$\Rightarrow \sin\alpha = 0$ or $\cos 3\alpha = -1$

$\Rightarrow \alpha = n\pi$, or $3\alpha = (2n+1)\pi$

$\therefore \alpha = -\frac{\pi}{3}$

Sol 27: (B) $f'(x) = \sin\sqrt{x^4}\,dx^4 - \sin\sqrt{x^2}\,dx^2$

$= 4x^3\sin x^2 - 2x\sin x$

Sol 28: (B) $\int_0^{\pi} x\sin x\cos^4 x dx = \int_0^{\pi} (\pi - x)\sin x\cos^4 x dx$

$\therefore 2I = \pi\int_0^{\pi} \sin x\cos^4 x dx$

Let $\cos x = t$

$-\sin x dx = dt$

$2I = -\pi\int_1^{-1} t^4 dt$

$2I = \pi\int_{-1}^{1} t^4 dt = \frac{\pi}{5}[1+1]$

$\therefore I = \frac{\pi}{5}$

Sol 29: (B) $f(0) = a + b = -1$

$f'(x) = 2ae^{2x} + be^x + c$ ..... (i)

$f'(\log 2) = 8a + 2b + c = 31$ ..... (ii)

$\int_0^{\log 4} \left(f(x)-(x)\right)dx = \int_0^{\log 4}\left(ae^{2x}+be^x\right)dx$

$= \frac{a}{2}e^{2x} + be^x\Big|_0^{\log 4} = 8a + 4b - \left(\frac{a}{2}+b\right) = \frac{39}{2}$

$= 15a + 6b = 39$ ..... (iii)

$\Rightarrow 9a = 45$

$a = 5; b = -6; c = 3$

Sol 30: (A) $I = \int_{-\pi/4}^{\pi/4} \frac{e^{-x}\sec^2 x dx}{e^{-2x}-1} = \int_{-\pi/4}^{\pi/4} \frac{e^x \sec^2 x}{1-e^{2x}}dx$

$\therefore 2I = \int_{-\pi/4}^{\pi/4}\left(\frac{e^x\sec^2 x}{e^{2x}-1} - \frac{e^x\sec^2 x}{e^{2x}-1}\right)dx = 0$

$I = 0$

Sol 31: (B) $I = \int_{-1}^{a} \log\left(x+\sqrt{1+x^2}\right)$

$= \log\left(\frac{-1}{x-\sqrt{1+x^2}}\right) = \log\left(\frac{1}{\sqrt{1+x^2}-x}\right)$

$= \int_{-1}^{a} -\log\left(\sqrt{1+x^2}-x\right)$

$\therefore 2I = \int_{-1}^{a} \log(x^2+1-x^2) = \int_{-1}^{a} \log 1 = 0$

Sol 32: (C) $\int_{-2}^{0} \frac{\sin^2 x}{-\frac{1}{2}}dx + \int_0^2 \frac{\sin^2 x}{\frac{1}{2}}dx$

$= -\frac{2}{2}\int_{-2}^{0}\frac{1-\cos 2x}{-\frac{1}{2}}dx + \frac{2}{2}\int_0^2 (1-\cos 2x)dx$

$= -1[2] + \frac{1}{2}\sin 2x\Big|_{-2}^{0} + 1[2] - \frac{1}{2}\sin 2x\Big|_0^2$

$= -2 + \frac{1}{2}[0 + \sin 4] + 2 - \frac{1}{2}[\sin 4] = 4 - \sin 4$

Sol 33: (C) $f(x) = \int_0^x \log(1+t^2)dt$

$f'(x) = \log(1+x^2)$

$f''(x) = \frac{1}{1+x^2}\times 2x$

$\therefore f''(1) = \frac{2}{2} = 1$

Sol 34: (C) $\int_0^{\pi}\frac{dx}{1+3^{\cos x}}$

$= \int_0^{\pi}\frac{dx}{1+3\cos^{(\pi-x)}} = \int_0^{\pi}\frac{dx}{1+3^{-\cos x}}$

$$= \int_0^{\pi} \frac{3^{\cos x}}{1+3^{\cos x}} dx$$

$$\therefore 2I = \int_0^{\pi} dx = \pi$$

$$\therefore I = \frac{\pi}{2}$$

Previous Years' Questions

Sol 1: (A) Let $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}} dx$ ....(i)

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\cot x}+\sqrt{\tan x}} dx \quad ....(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} I\,dx$$

$$\therefore I = \frac{\pi}{4}$$

Sol 2: (C) Let $I = \int_0^{\pi} e^{\cos^2 x}.\cos^3\{(2n+1)x\}dx$

Using $\int_0^a f(x)dx$

$$= \begin{cases} 0, & f(a-x) = -f(x) \\ 2\int_0^{a/2} f(x)dx, & f(a-x) = f(x) \end{cases}$$

Again, let f(x) = $e^{\cos^2 x}.\cos^3\{(2n+1)x\}$

$\therefore$ f(π – x) = $(e^{\cos^2 x})$ {– cos³ (2n + 1)x} = – f(x)

$\therefore$ I = 0

Sol 3: (A) $\lim_{x\to 1} \int_4^{f(x)} \frac{2t}{x-1}dt = \lim_{x\to 1} \frac{\int_4^{f(x)} 2t\,dt}{x-1}$

(using L' Hospital's rule)

$$= \lim_{x\to 1} \frac{2f(x).f'(x)}{1} = 2f(1).f'(1)$$

= 8f′(1) [∵ f(1) = 4]

Sol 4: (D) Let $I = \int_0^{\pi/2} \frac{1}{1+\tan^3 x} dx$

$$= \int_0^{\pi/2} \frac{1}{1+\frac{\sin^3 x}{\cos^3 x}} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x+\sin^3 x} dx \quad .....(i)$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^3\left(\frac{\pi}{2}-x\right)}{\cos^3\left(\frac{\pi}{2}-x\right)+\sin^3\left(\frac{\pi}{2}-x\right)} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x+\cos^3 x} dx \quad .....(ii)$$

On adding Eqs (i) and (ii), we get $2I = \int_0^{\pi/2} 1\,dx$

$$2I = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

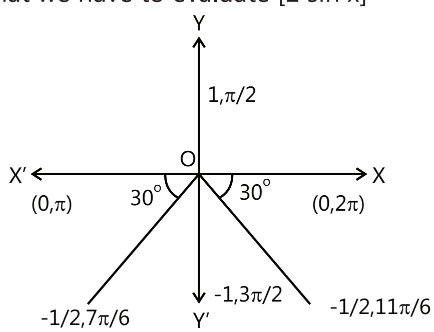
Now, $\int_0^1 f(x)dx = \frac{2A}{\pi}$

$$\Rightarrow \int_0^1 \left\{A\sin\left(\frac{\pi x}{2}\right)+B\right\}dx = \frac{2A}{\pi}$$

$$\Rightarrow \left[-\frac{2A}{\pi}\cos\frac{\pi x}{2}+Bx\right]_0^1 = \frac{2A}{\pi}$$

$$\Rightarrow B+\frac{2A}{\pi} = \frac{2A}{\pi} \Rightarrow B = 0$$

Sol 5: (A) It is a questions of greatest integer function. We have subdivide the interval π to 2π as under keeping in view that we have to evaluate [2 sin x]



We known that, $\sin\frac{\pi}{6} = \frac{1}{2}$,

$$\sin\left(\pi+\frac{\pi}{6}\right) = \sin\frac{7\pi}{6} = -\frac{1}{2}$$

$$\sin\frac{11\pi}{6} = \sin\left(2\pi-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$$

$$\sin\frac{9\pi}{6} = \sin\frac{3\pi}{6} = -1$$

Hence, we divide the interval π to 2π as

$\left(\pi,\frac{7\pi}{6}\right),\left(\frac{7\pi}{6},\frac{11\pi}{6}\right),\left(\frac{11\pi}{6},2\pi\right)$

sin x = $\left(0,-\frac{1}{2}\right),\left(-1,-\frac{1}{2}\right),\left(-\frac{1}{2},0\right)$

2sin x = (0, –1), (–2, –1), (–1, 0)

[2sinx] x = –1

$= \int_{\pi}^{7\pi/6}[2\sin x]dx + \int_{7\pi/6}^{11\pi/6}[2\sin x]dx$

$+ \int_{11\pi/6}^{2\pi}[2\sin x]dx$

$= \int_{\pi}^{7\pi/6} -1\,dx + \int_{7\pi/6}^{11\pi/6} -2\,dx + \int_{11\pi/6}^{2\pi} -1\,dx$

$= -\frac{\pi}{6} - 2\left(\frac{4\pi}{6}\right) - \frac{\pi}{6} = -\frac{10\pi}{6} = -\frac{5\pi}{3}$

Sol 6: (A) Given, $\int_0^x f(t)dt = x + \int_x^1 t\, f(t)dt$

On differentiating both sides w.r.t. x, we get

f(x) 1 = 1 – xf(x) . 1

$\Rightarrow$ (1 + x) f(x) = 1

$\Rightarrow$ f(x) = $\frac{1}{1+x}$

$\Rightarrow$ f(1) = $\frac{1}{1+1} = \frac{1}{2}$

Sol 7: (A) Let $I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ .....(i)

$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos(\pi - x)}$

$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1-\cos x}$ ....(ii)

On adding Eqs. (i) and (ii), we get

$2I = \int_{\pi/4}^{3\pi/4}\left(\frac{1}{1+\cos x} + \frac{1}{1-\cos x}\right)dx$

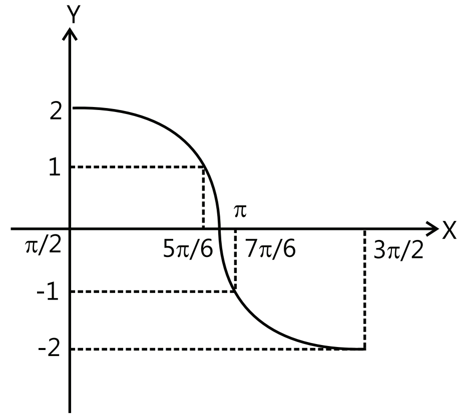
$\Rightarrow 2I = \int_{\pi/4}^{3\pi/4}\left(\frac{2}{1-\cos^2 x}\right)dx$

$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \operatorname{cosec}^2 x\, dx = [-\cot x]_{\pi/4}^{3\pi/4}$

$= \left[-\cot\frac{3\pi}{4} + \cot\frac{\pi}{4}\right]$ = – (–1) + 1 = 2

Sol 8: (C) The graph of y = 2sin x for $\pi/2 \le x \le 3\pi/2$ is given in figure. From the graphs, it is clear that

$$[2\sin x] = \begin{cases} 2, & \text{if } x = \pi/2 \\ 1, & \text{if } \pi/2 < x \le 5\pi/6 \\ 0, & \text{if } 5\pi/6 < x \le \pi \\ -1, & \text{if } \pi < x \le 7\pi/6 \\ -2, & \text{if } 7\pi/6 < x \le 3\pi/2 \end{cases}$$



Therefore, $\int_{\pi/2}^{3\pi/2}[2\sin x]dx$

$= \int_{\pi/2}^{5\pi/2} dx + \int_{5\pi/6}^{\pi} 0\,dx + \int_{\pi}^{7\pi/6}(-1)dx$

$+ \int_{7\pi/6}^{3\pi/2}(-2)dx$

$= \left[x\right]_{\pi/2}^{5\pi/6} + \left[-x\right]_{\pi}^{7\pi/6} + \left[-2x\right]_{7\pi/6}^{3\pi/2}$

$= \left(\frac{5\pi}{6} - \frac{\pi}{2}\right) + \left(-\frac{7\pi}{6} + \pi\right)$

$+ \left(\frac{-2.3\pi}{2} + \frac{2.7\pi}{6}\right)$

$= \pi\left(\frac{5}{6} - \frac{1}{2}\right) + \pi\left(1 - \frac{7}{6}\right) + \pi\left(\frac{7}{3} - 3\right)$

$= \pi\left(\frac{5-3}{6}\right) + \pi\left(-\frac{1}{6}\right) + \pi\left(\frac{7-9}{3}\right) = -\frac{\pi}{2}$

Sol 9: (C) Let $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}dx$ ....(i)

$= \int_{\pi}^{-\pi} \frac{\cos^2(-x)}{1+a^{-x}}d(-x)$

$\Rightarrow I = \int_{-\pi}^{\pi} a^x \frac{\cos^2 x}{1+a^x}dx$ ....(ii)

On adding Eqs. (i) and (ii), we get

$2I = \int_{-\pi}^{\pi}\left(\frac{1+a^x}{1+a^x}\right)\cos^2 x\,dx$

$= \int_{-\pi}^{\pi}\cos^2 x\,dx = 2\int_0^{\pi}\frac{1+\cos 2x}{2}dx$

$= \int_0^{\pi}(1+\cos 2x)dx$

$= \int_0^{\pi}1\,dx = \int_0^{\pi}\cos 2x\,dx$

$= \left[x\right]_0^x + 2\int_0^{\pi/2}\cos 2x\,dx$

$= \pi + 0$

$\Rightarrow 2I = \pi \Rightarrow I = \pi/2$

Sol 10: (C) Given, $F(x) = \int_0^{\pi} f(t)dt$

By Leibniz rule,

$F'(x) = f(x)$ ...(i)

But $F(x^2) = x^2(1+x) = x^2 + x^3$ (given)

$\Rightarrow F(x) = x + x^{3/2}$

$\Rightarrow F'(x) = 1 + \frac{3}{2}x^{1/2}$

$\Rightarrow f(x) = F'(x) = 1+\frac{3}{2}x^{1/2}$ [from Eq. (i)]

$\Rightarrow f(4) = 1 + \frac{3}{4}(4)^{1/2}$

$\Rightarrow f(4) = 1 + \frac{3}{2} \times 2 = 4$

Sol 11: (A) Given, $f(x) = \int_1^x \sqrt{2-t^2}\,dt$

$\Rightarrow f'(x) = \sqrt{2-x^2}$

Also $x^2 - f'(x) = 0$

$\therefore x^2 = \sqrt{2-x^2}$

$\Rightarrow x^4 = 2 - x^2 \Rightarrow x^4 + x^2 - 2 = 0$

$\Rightarrow x = \pm 1$

Sol 12: (C) $\int_3^{3+3T} f(2x)dx$ put $2x = y \Rightarrow dx = \frac{1}{2}dy$

$\therefore \frac{1}{2}\int_6^{6+6T} f(y)dy = \frac{6I}{2} = 3I$

Sol 13: (D) Given, $f(x) = \int_{x^2}^{x^2+1} e^{-t^2}dt$

On differentiating both sides using Newton's Leibnitz formula , we get

$f'(x) = e^{-(x^2+1)^2}\left\{\frac{d}{dx}(x^2+1)\right\} - e^{-(x^2)^2}\left\{\frac{d}{dx}(x^2)\right\}$

$= e^{-(x^2+1)^2}.\,2x - e^{-(x^2)^2}.\,2x$

$= 2xe^{-(x^4+2x^2+1)}(1-e^{2x^2+1})$

[where, $e^{2x^2+1} > 1,\ \forall$ x and $e^{-(x^4+2x^2+1)} > 0\forall x$]

$\therefore f'(x) > 0$

which shows $2x < 0$ or $x < 0$

$\Rightarrow x \in (-\infty, 0)$

Sol 14: (B) $I = \int_0^1\sqrt{\frac{1-x}{1+x}}dx = \int_0^1\frac{1-x}{\sqrt{1-x^2}}dx$

$\int_0^1\frac{1}{\sqrt{1-x^2}}dx - \int_0^1\frac{x}{\sqrt{1-x^2}}dx$

$I = \left[\sin^{-1}x\right]_0^1 + \int_1^0\frac{t}{t}dt$

(where, $t^2 = 1 - x^2 \Rightarrow t\,dt = -x\,dx$)

$I = (\sin^{-1}1 - \sin^{-1}0) + \left[t\right]_1^0 = \frac{\pi}{2} - 1$

Sol 15: (A) Let $I = \int_{-1}^{1}\frac{dx}{1+x^2}$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\,dq$

$\therefore I = 2\int_0^{\pi/4}d\theta = \frac{\pi}{2}$

(B) Let $I = \int_0^1\frac{dx}{\sqrt{1-x^2}}$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\,dq$

$\therefore I = \int_0^{\pi/2}1\,d\theta = \frac{\pi}{2}$

(C) $\int_2^3\frac{dx}{1-x^2} = \frac{1}{2}\left[\log\left(\frac{1+x}{1-x}\right)\right]_2^3$

$= \frac{1}{2}\left[\log\left(\frac{4}{-2}\right) - \log\left(\frac{3}{-1}\right)\right] = \frac{1}{2}\left[\log\left(\frac{2}{3}\right)\right]$

(D) $\int_1^2\frac{dx}{x\sqrt{x^2-1}} = \left[\sec^{-1}x\right]_1^2 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

Sol 16: $\int_{-2}^{2}|1-x^2|\,dx$

$= \int_{-2}^{-1}(x^2-1)dx + \int_{-1}^{1}(1-x^2)dx$

$+\int_{1}^{2}(x^2-1)dx$

$= \left[\frac{x^3}{3}-x\right]_{-2}^{-1} + \left[x-\frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{x^3}{3}-x\right]_{1}^{2}$

$= \left(-\frac{1}{3}+1+\frac{8}{3}-2\right)+\left(1-\frac{1}{3}+1-\frac{1}{3}\right)$

$+\left(\frac{8}{3}-2-\frac{1}{3}+1\right)$

$= 4$

Sol 17: Let $I = \int_{\pi/4}^{3\pi/4}\frac{x}{1+\sin x}dx$ ... (i)

$\Rightarrow I = \int_{\pi/4}^{3\pi/4}\frac{\left(\frac{\pi}{4}+\frac{3\pi}{4}-x\right)}{1+\sin\left(\frac{\pi}{4}+\frac{3\pi}{4}-x\right)}dx$

$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

$\Rightarrow I = \int_{\pi/4}^{3\pi/4}\frac{\pi-x}{1+\sin(\pi-x)}dx$

$\Rightarrow I = \int_{\pi/4}^{3\pi/4}\frac{\pi}{1+\sin x}dx \; - \int_{\pi/4}^{3\pi/4}\frac{x}{1+\sin x}dx$

$\Rightarrow 1 = \pi\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\sin x}-1$ [from Eq. (i)]

$\Rightarrow 1 = \frac{\pi}{2}\int_{\pi/4}^{3\pi/4}\frac{dx}{(1+\sin x)}$

$\Rightarrow 1 = \frac{\pi}{2}\int_{\pi/4}^{3\pi/4}\frac{(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$

$= \frac{\pi}{2}\int_{\pi/4}^{3\pi/4}\frac{(1-\sin x)}{1-\sin^2 x}dx$

$= \frac{\pi}{2}\int_{\pi/4}^{3\pi/4}\left(\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}\right)dx$

$= \frac{\pi}{2}\int_{\pi/4}^{3\pi/4}(\sec^2 x-\sec x.\tan x)dx$

$= \frac{\pi}{2}\left[\tan x-\sec x\right]_{\pi/4}^{3\pi/4}$

$= \frac{\pi}{2}[-1-1-(-\sqrt{2}-\sqrt{2})]$

$= \frac{\pi}{2}(-2+2\sqrt{2})$

$= \pi(\sqrt{2}-1)$

Sol 18: Let $I = \int_2^3\frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ ... (i)

$\Rightarrow I = \int_2^3\frac{\sqrt{2+3-x}}{\sqrt{(2+3)-(5-x)}+\sqrt{2+3-x}}dx$

$\Rightarrow I = \int_2^3\frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx$ ... (ii)

On adding Eqs. (i) and (ii), we get

$\Rightarrow\ 2I = \int_2^3\frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx$

$\Rightarrow\ 2I = \int_2^3 1\,dx = 1\ \Rightarrow\ I = \frac{1}{2}$

Sol 19: (B)

$I = \int_0^1\frac{\sin x}{\sqrt{x}}dx < \int_0^1\frac{x}{\sqrt{x}}dx = \int_0^1\sqrt{x}dx = \frac{2}{3}x^{3/2}\Big|_2^2 = \frac{2}{3}$

$\Rightarrow I < \frac{2}{3}$

$J = \int_0^1\frac{\cos x}{\sqrt{x}}dx < \int_0^1\frac{1}{\sqrt{x}}dx\,|_0^1 = 2$

$\therefore J \le 2.$

Sol 20: (D) Let $\int_0^x\left[\cot x\right]dx$ ... (i)

$= \int_0^x\left[\cot\left(\pi-x\right)\right]dx, \int_0^x\left[-\cot x\right]dx$ ... (ii)

Adding (1) and (2)

$2I = \int_0^x\left[\cot x\right]dx + \int_0^x\left[-\cot x\right]dx = \int_0^x(-1)dx$

$\left[\begin{array}{l}\because [x]+[-x] = -1 \text{ if } x \notin Z \\ \qquad\qquad\quad = 0 \text{ if } x \in Z\end{array}\right]$

$= \left[-x\right]_0^x = -\pi$

$\therefore = -\frac{\pi}{2}$

Sol 21: (A) $p'(x) = p'(1-x)$

$\Rightarrow p(x) = -p(1-x) + c$

at $x = 0$

Now $p(0) = -p(1-x) + 42$

$\Rightarrow p(x) + p(1-x) = 42$

$I = \int_0^1 p(x)dx \int_0^1 p(1-x)dx$

$2I = \int_0^1 (42)dx \quad \Rightarrow I = 21.$

Sol 22: (D) $I = 8\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

$= 8\int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta d\theta (\text{let } x = \tan\theta)$

$= \int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta = \int_0^{\frac{\pi}{4}} \log 2 d\theta - \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)d\theta$

$= 8\log 2\frac{\pi}{4} - 1$

$2I = 2\pi\log 2$

$I = \pi\log 2$

Sol 23: (B, C) $g(x) = \int_0^x \cos 4t\, dt$

$\Rightarrow g'(x) = \cos 4x \quad \Rightarrow g(x) = \frac{\sin 4x}{4} + k$

$\Rightarrow g(x) = \frac{\sin 4x}{4}[\because g(0) = 0]$

$= g(x) + g(\pi) = g(x) - g(\pi) \ (\because g(\pi) = 0)$

Sol 24: (D) $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}$

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\, dx}{1+\sqrt{\tan x}}$

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\, dx}{1+\sqrt{\tan x}}$

$2I = \int_{\pi/6}^{\pi/3} dx$

$\Rightarrow \ I = \frac{1}{2}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{\pi}{12},$ statement-1is false

$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ it is property

Sol 25: (D) $\int \left\{ e^{\left(x-\frac{1}{x}\right)} + \left(1-\frac{1}{x^2}\right)e^{e+\frac{1}{x}} \right\} dx$

$= x.e^{x+\frac{1}{x}} + c$

As $\int (xf'(x) + f(x))dx(x) + c$

Sol 26: (C) $I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(36-12x+x^2)}dx$

$I = \frac{2}{2}\int_2^4 \frac{\log x}{\log x + \log(6-x)}dx$ ... (i)

$I = \int_2^4 \frac{\log(6-x)}{\log x(6-x) + \log}dx \quad \left\{\int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right\}$ ...(ii)

Equation (i) & (ii) gives

$= \int_2^4 \frac{\log x + \log(6-x)}{\log x + \log(6-x)}dx = \int_2^4 dx = 2$

Hence I = 1

JEE Advanced/Boards

Exercise 1

Sol 1: $\int_0^1 e^{\tan^{-1}x}\sin^{-1}(\cos x)dx.$

$\int_0^1 (\tan^{-1}x)\sin^{-1}\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx$

$= \int_0^1 \left(\frac{\pi}{2}\tan^{-1}x - x\tan^{-1}x\right)dx$

$$= \frac{\pi}{2}\left[x\tan^{-1}x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}dx\right] - \int_0^1 x\tan^{-1}x dx$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right] - \left[(\tan^{-1}x)\frac{x^2}{2}\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2+1-1}{1+x^2}dx\right]$$

$$= \frac{\pi^2}{8} - \frac{\pi}{4}\log - \frac{1}{2}\times\frac{\pi}{4} + \frac{1}{2}\left[(x)_0^1 - \tan^{-1}\Big|_0^1\right]$$

$$= \frac{\pi^2}{8} - \frac{\pi}{4}\log - \frac{\pi}{8} + \frac{1}{2} - \frac{1}{2}\times\frac{\pi}{4}$$

$$= \frac{\pi^2}{8} - \frac{\pi}{4}((\log 2) + 1) + \frac{1}{2}$$

Sol 2: (i) Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$dx = 2(\beta - \alpha)\sin\theta\cos\theta\, d\theta$

$$I = \int_\alpha^b \sqrt{(x-\alpha)(\beta-x)}dx$$

$$= \int_0^{\pi/2} \sqrt{(\beta-\alpha)\cos^2\theta(\beta-a)\sin^2\theta}$$

$\times (\beta - \alpha)\sin 2\theta d\theta$

$$= \frac{(\beta-\alpha)^2}{2}\int_0^{\pi/2} \sin^2 2\theta d\theta$$

$$= \frac{(\beta-\alpha)^2}{4}\int_0^{\pi/2} (1-\cos 4\theta)d\theta$$

$$= \frac{(\beta-\alpha)^2}{4}\times\frac{\pi}{2} = \frac{(\beta-\alpha)^2\pi}{8}$$

(ii) $I = \int_\alpha^\beta \sqrt{\frac{(x-\alpha)}{(\beta-\alpha)}}dx$

$$= \int_0^{\pi/2} \sqrt{\frac{(\beta-\alpha)\cos^2\theta}{(\beta-\alpha)\sin^2\theta}}\times(\beta-\alpha)\sin 2\theta d\theta$$

$$= 2(\beta-\alpha)\int_0^{\pi/2} \frac{\cos\theta}{\sin\theta}\times\sin\theta\cos\theta d\theta$$

$$= 2(\beta-\alpha)\int_0^{\pi/2}\left(\frac{1+\text{co}2\theta}{2}\right)d\theta$$

$$= (\beta-\alpha)\frac{\pi}{2}$$

(iii) $I = \int_0^{\pi/2} \frac{2(\beta-\alpha)\sin\theta\cos\theta d\theta}{(\alpha\cos^2\theta+\beta\sin^2\theta)\times(\beta-\alpha)\cos\theta\sin\theta}$

$$= \int_0^{\pi/2} \frac{2d\theta}{\alpha\cos^2\theta+\beta\sin^2\theta} = \frac{1}{\beta}\int_0^{\pi/2}\frac{2\sec^2\theta d\theta}{\frac{\alpha}{\beta}+\tan^2\theta}$$

Put $\tan\theta = t$

$$\frac{2}{\beta}\int_0^\infty \frac{dt}{\frac{\alpha}{\beta}+t^2} = \frac{2}{\beta}\times\frac{1}{\sqrt{\frac{\alpha}{\beta}}}\tan^{-1}\frac{t}{\sqrt{\frac{\alpha}{\beta}}}\Bigg|_0^\infty$$

$$= \frac{2}{\sqrt{\alpha\beta}}\times\frac{\pi}{2} = \frac{\pi}{\sqrt{\alpha\beta}}$$

(iv) $I = \int_0^{\pi/2} \frac{(\alpha\cos^2\theta+\beta\sin^2\theta)\times(\beta-\alpha)\sin 2\theta d\theta}{(\beta-\alpha)\sin\theta\cos\theta}$

$$= 2\int_0^{\pi/2}(\alpha\cos^2\theta+\beta\sin^2\theta)d\theta$$

$$= 2\int_0^{\pi/2}\left((\beta-\alpha)\sin^2\theta+\alpha\right)d\theta$$

$$= 2\alpha\times\frac{\pi}{2} + 2\int_0^{\pi/2}\frac{(\beta-\alpha)}{2}(1-\cos 2\theta)d\theta$$

$$= \alpha\pi + 2\left(\frac{\beta-\alpha}{2}\right)\int_0^{\pi/2}(1-\cos 2\theta)d\theta$$

$$= \alpha\pi + \frac{\beta-\alpha}{2}\times\frac{\pi}{2}\times 2 = \left(\frac{\beta}{2}+\frac{\alpha}{2}\right)\pi = (\alpha+\beta)\frac{\pi}{2}$$

Sol 3: (i) $\int_0^{2\pi}\sqrt{1-\sin t}dt - \int_0^\pi \sqrt{1-\sin t}dt$

$$= \frac{1}{2}\int_0^{2\pi}\left(\sqrt{1-\sin t}+\sqrt{1+\sin t}\right)dt - \int_0^\pi\sqrt{1-\sin t}dt$$

$$= \int_0^\pi \sqrt{1-\sin t}+\sqrt{1+\sin t}dt - \int_0^\pi\sqrt{1-\sin t}dt$$

$$= \int_0^\pi \sqrt{1-\sin t}dt$$

$$= \int_0^\pi \left|\sin\frac{t}{2}+\cos\frac{t}{2}\right|dt$$

$$= -2\cos\frac{t}{2}+2\sin\frac{t}{2}\Big|_0^\pi$$

$= -2[0 - 1] + 2[1 - 0] = 4$

(ii) $\int_0^1 e^x (x-1)^n$

$= \left[(x-1)^n e^x \Big|_0^1 - n\int_0^1 (x-1)^{n-1} e^x dx \right]$

$= - (-1)^n - n\left[(x-1)^{n-1} e^x \Big|_0^1 - (n-1)\int_0^1 (x-1)^{n-2} e^x dx \right]$

$= -(-1)^n + n(-1)^{n-1}$

$+ n(n-1)\left[(x-1)^{n-2} e^x \Big|_0^1 - (n-2)\int_0^1 (x-1)^{n-3} e^x dx \right]$

$= -(-1)^n + n(-1)^{n-1} - n(n-1)(-1)^{n-2}$

$- n(n-1)(n-2)\int_0^1 (x-1)^{n-3} e^x dx$

Taking n = 3

$= -(-1)^3 + 3(-1)^2 - 3(3-1)(-1)^1 - 3(2)(1)\int_0^1 e^x dx$

$= +1 + 3 + 6 - 6(e^1 - 1)$

$= 16 - 6e$

Sol 4: (i) $\int_0^{\pi/2} e^x \left\{ \cos(\sin x)\cos^2\frac{x}{2} + \sin(\sin x)\sin^{-2}\frac{x}{2} \right\} dx$

$\frac{1}{2}\int_0^{\pi/2} e^x \left\{ \cos(\sin x)[\cos x + 1] + \sin(\sin x)[1 - \cos x] \right\} dx$

$\frac{1}{2}\int_0^{\pi/2} e^x \left\{ [\cos(\sin x) + \sin(\sin x)] + \cos x[\cos(\sin x) - \sin(\sin x)] \right\}$

Put cos(sinx) + sin(sinx) = t

(–sin(sinx)cosx + cos(sinx)cosx)dx = dt

$\frac{1}{2}\int_0^{\pi/2} e^x \{f(x) + f'(x)\} dx$

$= \frac{1}{2} e^x f(x)\Big|_0^{\pi/2} = \frac{1}{2} e^x \{\cos(\sin x) + \sin(\sin x)\}\Big|_0^{\pi/2}$

$= \frac{1}{2}\left[e^{\pi/2}(\cos 1 + \sin 1) - e^{\circ}(\cos 0) \right]$

$= \frac{1}{2}\left[e^{\pi/2}(\cos 1 + \sin 1) - 1 \right]$

(ii) $\int_1^e \left\{ (1+x)e^x + (1-x)e^{-x} \right\} \log x dx$

logx = t

$x = e^t \Rightarrow dx = e^t dt$

$\int_0^1 \left\{ \{1 + e^t\} \right\} e^{e^t} + \{1 - e^t\} e^{-e^t} dt$

$= \int_0^1 \left\{ \left(e^{e^t} + e^{-e^t} \right) t + \left(e^t e^{e^t} - e^t e^{-e^t} \right) t \right\} e^t dt$

Sol 5: $R = \int_0^\infty \frac{dx}{1+x^4}$

Put $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2} dt$

$= \int_\infty^0 \frac{-t^2}{1+t^2} dt = \int_0^\infty \frac{x^2}{1+x^2} = P$

$\therefore 2I = 2P = \int_0^\infty \frac{1+x^2}{1+x^4} dx$

$= \int_0^\infty \frac{1+x^{-2}}{\left(1-\frac{1}{x}\right)^2 + 2} dx$

$= \int_{-\infty}^\infty \frac{dt}{t^2+2}$ (Put $x - \frac{1}{x} = t$)

$\therefore 2I = \frac{1}{\sqrt{2}} \tan^{-1} \frac{1}{\sqrt{2}}\Big|_{-\infty}^{\infty}$

$= \frac{1}{\sqrt{2}}\left[\frac{\pi}{2} + \frac{\pi}{2} \right] = \frac{\pi}{\sqrt{2}}$

$\therefore I = \frac{\pi}{2\sqrt{2}}$

$\int_0^\infty \frac{x dx}{1+x^4}$

Put $x^2 = t \Rightarrow 2xdx = dt$

$= \frac{1}{2}\int_0^\infty \frac{dt}{1+t^2} = \frac{1}{2}\tan^{-1} t\Big|_0^\infty = \frac{\pi}{2}$

$\therefore P + R - \sqrt{2}Q = \frac{\pi}{2\sqrt{2}} + \frac{\pi}{2\sqrt{2}} - \frac{\sqrt{2}\pi}{4}$

$= \frac{\pi}{\sqrt{2}} - \frac{\pi}{2\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$

Sol 6: $\int_1^2 \frac{(x^2-1)}{x^5\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}dx$

Put $2-\frac{2}{x^2}+\frac{1}{x^4}=t \Rightarrow \left(\frac{4}{x^3}-\frac{4}{x^5}\right)dx = dt$

$\frac{1}{4}\int_1^{\frac{25}{16}}\frac{dt}{\sqrt{t}} = \frac{1}{4}\times 2\sqrt{t}\Big|_1^{\frac{25}{16}} = \frac{1}{2}\left[\frac{5}{4}-1\right]=\frac{1}{8}$

$\therefore \frac{1000}{8} = 125$

Sol 7: h(x) = fog(x) + k

$\frac{dh(x)}{dx} = f'\{g(x)\}g'(x) = \frac{-\sin x}{\cos^2(\cos x)}$

$j(x) = \int_{g(x)}^{f(x)}\frac{f(t)}{g(t)}dt$

$h(x) = -\int\frac{\sin x}{\cos^2(\cos x)}dx = \int\frac{dt}{\cos^2 t} = \tan t$

= tan(cosx) + c

$\therefore$ f(x) = tanx, g(x) = cosx

$J(x) = \int_{\cos x}^{\tan x}\frac{\tan t}{\cos t}dt$

$j(0) = \int_1^0 \frac{\sin t}{\cos^2 t}dt$

$\cos t = u \Rightarrow -\sin t\,dt = du$

$= \int_0^1\frac{du}{u^2} = \frac{-1}{u}\Big|_1^{\cos 1} = -\left[\frac{1}{\cos 1}-1\right] = 1-\sec 1$

Sol 8: $\int_0^{\pi/2}\sqrt{\frac{1-\sin 2x}{1+\sin 2x}}dx = \int_0^{\pi/2}\sqrt{\left(\frac{1-\tan x}{1+\tan x}\right)^2}dx$

$= \int_0^{\pi/2}\left|\tan\left(x-\frac{\pi}{4}\right)\right|dx = 2\log\sec\left(x-\frac{\pi}{4}\right)\Big|_0^{\pi/4}$

$= 2\ell n\sqrt{2} = \log 2$

Sol 9: $2I = \int_0^2 (3x^2-3x+1)\cos(x^3-3x^2+4x-2)dx$

$+ \int_0^2 (3x^2-9x+7)\cos(x^3-3x^2+4x-2)dx$

$2I = 2\int_0^2 (3x^2-6x+4)\cos(x^3-3x^2+4x-2)dx$

Put $x^3-3x^2+4x-2=t$

$I = \int_{-2}^{2}\cos t\,dt = \sin t\Big|_{-2}^{2} = \sin 2+\sin 2 = 2\sin 2$

$\therefore p = q = 2 \Rightarrow p+q = 4$

Sol 10: $I = \int_{-\sqrt{2}}^{\sqrt{2}}\frac{3x^6-12x^2+1}{x^2+2}dx$

$= \int_{-\sqrt{2}}^{\sqrt{2}}\frac{3x^6+6x^4-6x^4-12x^2+1}{x^2+2}dx$

$= \int_{-\sqrt{2}}^{\sqrt{2}}\frac{3x^4(x^2+2)-6x^2(x^2+2)+1}{x^2+2}dx$

$= \int_{-\sqrt{2}}^{\sqrt{2}}3x^4-6x^2+\frac{1}{x^2+2}dx$

$= 2\times 3\left[\frac{x^5}{5}-\frac{2}{3}x^3\Big|_0^{\sqrt{2}}\right] + 2\times\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}\Big|_0^{\sqrt{2}}$

$= 6\left[\frac{4\sqrt{2}}{5}-\frac{4\sqrt{2}}{3}\right]+\frac{\pi}{2\sqrt{2}} = \frac{-16\sqrt{2}}{5}+\frac{\pi}{2\sqrt{2}}$

Sol 11: $\int_0^\infty\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\Big|_0^\infty$

$\frac{1}{a}\frac{\pi}{2} = \frac{\pi}{5050} \Rightarrow a = 2525$

Sol 12: $\int_{-2}^{2}\frac{x^2-x}{\sqrt{x^2+4}}dx = \int_{-2}^{2}\frac{x^2+x}{\sqrt{x^2+4}}dx$

$\therefore I = \int_{-2}^{2}\frac{x^2}{\sqrt{x^2+4}}dx = 2\int_0^2\frac{x^2}{\sqrt{x^2+4}}dx$

$= 2\int_0^2\left(\sqrt{x^2+4}-\frac{4}{\sqrt{x^2+4}}\right)dx$

$= 2\left[\frac{x}{2}\sqrt{x^2+4}+2\log\left|x+\sqrt{x^2+4}\right| - 8\log\left|x+\sqrt{x^2+4}\right|\right]_0^2$

$= 2\sqrt{8}-4\log\left|2+2\sqrt{2}\right|+4\log 2$

$= 4\sqrt{2}-4\log\left|1+\sqrt{2}\right|$

Sol 13: $u = \frac{1}{2}\int_0^{\pi/4}\left(\frac{\cos x}{\sin\left(x+\frac{\pi}{4}\right)}\right)^2 dx$

$= \frac{1}{2}\int_0^{\pi/4}\left(\frac{\cos\left(\frac{\pi}{4}-x\right)}{\cos x}\right)^2 dx$

$v = 2\int_0^{\pi/4}\left(\frac{\cos\left(\frac{\pi}{4}-x\right)}{\cos x}\right)^2$

$\therefore \frac{v}{u} = \frac{2}{1/2} = 4$

Sol 14: $\frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{xdx}{\cos x\cos\left(\frac{\pi}{4}-x\right)}$

$= \frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{\left(\frac{\pi}{4}-x\right)dx}{\cos\left(\frac{\pi}{4}-x\right)\cos x}$

$\therefore 2I = \frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{\frac{\pi}{4}dx}{\cos x\cos\left(\frac{\pi}{4}-x\right)}$

$I = \frac{\pi}{8\sqrt{2}}\int_0^{\pi/4}\frac{dx}{\cos x\cos\left(\frac{\pi}{4}-x\right)}$

$= \frac{\pi}{8}\int_0^{\pi/4}\frac{dx}{\cos^2 x+\cos x\sin x}$

$= \frac{\pi}{8}\int_0^{\pi/4}\frac{\sec^2 xdx}{1+\tan x}$

$= \frac{\pi}{8}\int_0^1\frac{dt}{1+t} = \frac{\pi}{8}\log(1+t)\Big|_0^1 = \frac{\pi}{8}\log 2$

Sol 15: $\int_0^1\frac{\sin^{-1}\sqrt{x}}{x^2-x+1}dx$

Let $\sin^{-1}\sqrt{x} = t$

$\frac{1}{\sqrt{1-x}}\times\frac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow dx = \left(\sqrt{1-\sin^2 t}\right)2\sin t dt = \sin 2t dt$

$\int_0^{\pi/2}\frac{t\sin 2t dt}{\sin^4 t-\sin^2 t+1}$

$= \int_0^{\pi/2}\frac{t\sin 2t dt}{1-\sin^2 t\cos^2 t} = \int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-t\right)\sin 2t dt}{1-\sin^2 t\cos^2 t}$

$\therefore 2I = \frac{\pi}{2}\int_0^{\pi/2}\frac{\sin 2t dt}{1-\sin^2 t\cos^2 t}$

$2I = \frac{\pi}{2}\int_0^{\pi/2}\frac{\sin 2t dt}{1-\frac{\sin^2 2t}{4}} = 4.\frac{\pi}{2}\int_0^{\pi/2}\frac{\sin 2t dt}{3+\cos^2 2t}$

$2I = \left[2\pi\int_1^{-1}\left(\frac{dt}{3+t^2}\right)\right]-\frac{1}{2}$

$\therefore I = \frac{\pi}{2}\int_{-1}^1\frac{dt}{3+t^2} = \frac{\pi}{2}\times\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\Big|_{-1}^1$

$= \frac{1}{2}\frac{\pi}{\sqrt{3}}\left[\frac{\pi}{6}+\frac{\pi}{6}\right] = \frac{\pi^2}{6\sqrt{3}}$

Sol 16: $\int_{\frac{1+\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}}\frac{x^2+1}{x^2\left(x^2+\frac{1}{x^2}-1\right)}\log\left(1+x-\frac{1}{x}\right)dx$

$= \int_1^{\frac{1+\sqrt{5}}{2}}\frac{1+x^2}{\left(\left(x-\frac{1}{x}\right)^2+1\right)}\log\left(1+\left(x-\frac{1}{x}\right)\right)dx$

$x-\frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$

$= \int_0^1\frac{\log(1+t)}{(t^2+1)}dt$

$t = \tan\theta$

$= \int_0^{\pi/4}\log(1+\tan\theta)d\theta$

$= \int_0^{\pi/4}\log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$

$= \int_0^{\pi/4}\left[\log(2)-\log(1+\tan\theta)\right]d\theta$

$$\therefore 2I = \int_0^{\pi/4} \log 2 d\theta = \frac{\pi}{4}\log 2 \Rightarrow I = \frac{\pi}{8}\log 2$$

Sol 17: $\lim_{n\to\infty} n^2 \int_{-1/n}^{1/n} (2010\sin x + 2012\cos x)\,|x|\,dx$

$$= \lim_{n\to\infty} n^2 \int_{-1/n}^{1/n} (2012\cos x)\,|x|\,dx$$

$$= 2012 \lim_{n\to\infty} 2n^2 \int_0^{1/n} x\cos x dx$$

$$= 2012\times 2 \lim_{n\to\infty} n^2 \left[x\sin x\Big|_0^{1/n} - \int_0^{1/n} \sin x dx \right]$$

$$= 2012\times 2 \lim_{n\to\infty} n^2 \left[\frac{1}{n}\sin\frac{1}{n} + \cos\frac{1}{n} - 1 \right]$$

$$= 2012 \times 2 \lim_{n\to\infty} \left[\frac{\sin\frac{1}{n}}{\frac{1}{n}} + \frac{\cos\frac{1}{n} - 1}{\frac{1}{n^2}} \right]$$

$$= 2012 \times 2\left[1 - \frac{1}{2}\right] = 2012$$

Sol 18: $\int_0^{\pi} \left|\sqrt{2}\sin x + 2\cos x\right| dx$

$\sqrt{2}\sin x + 2\cos x > 0$

$\Rightarrow \tan x > -\sqrt{2}$

$\therefore x < \pi - \tan^{-1}\sqrt{2}$

$$= \int_0^{\pi-\tan^{-1}\sqrt{2}} \left(\sqrt{2}\sin x + 2\cos x\right) dx$$

$$+ \int_{\pi}^{\pi-\tan^{-1}\sqrt{2}} \left(\sqrt{2}\sin x + 2\cos x\right) dx$$

$$= -\sqrt{2}\cos x\Big|_0^{\pi-\tan^{-1}\sqrt{2}} + 2\sin x\Big|_0^{\pi-\tan^{-1}\sqrt{2}}$$

$$+ 2\sin x\Big|_{\pi}^{\pi-\tan^{-1}\sqrt{2}} - \sqrt{2}\cos x\Big|_{\pi}^{\pi-\tan^{-1}\sqrt{2}}$$

$$= -\sqrt{2}[-\cos\tan^{-1}\sqrt{2} - 1] + 2\sin\tan^{-1}\sqrt{2}$$

$$+ 2[\sin\tan^{-1}\sqrt{2}] - \sqrt{2}\left[-\cos\tan^{-1}\sqrt{2} + 1\right]$$

$$= 2\sqrt{2}\times\frac{1}{\sqrt{3}} + 4\times\frac{\sqrt{2}}{\sqrt{3}} = \frac{6\sqrt{2}}{\sqrt{3}} = 2\sqrt{6}$$

Sol 19: cosx + cos3x = 2cos2xcosx

sinx + sin3x = 2sin2xcosx

$$\therefore I = \int_0^{\pi} \sqrt{(2\cos x + 1)^2\left[\cos^2 2x + \sin^2 2x\right]}dx$$

$$= \int_0^{\pi} |2\cos x + 1|\,dx$$

$$= \int_0^{2\pi/3} (2\cos x + 1)dx + \int_{2\pi/3}^{\pi} (-2\cos x - 1)dx$$

$$= 2\sin x\Big|_0^{2\pi/3} + \frac{2\pi}{3} - 2\sin x\Big|_{2\pi/3}^{\pi} - \left(n - \frac{2\pi}{3}\right)$$

$$= 2\left[\frac{\sqrt{3}}{2}\right] + \left(\frac{4\pi}{3} - \pi\right) - 2\left(0 - \frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{3} + \sqrt{3} + \frac{\pi}{3} = 2\sqrt{3} + \frac{\pi}{3} = \sqrt{12} + \frac{\pi}{3}$$

$\therefore$ w = 12, k = 3

$\Rightarrow k^2 + w^2 = 9 + 144 = 153$

Sol 20: $\int_0^1 \frac{(1-x)(1+x)}{x(1+x)(1+x)} \frac{dx}{\sqrt{\frac{1}{x} + 1 + x}}$

$$= \int_0^1 \frac{1-x^2}{x^2} \left(\frac{1}{\left(\frac{1}{x} + 1\right)(1+x)} \right) \frac{dx}{\sqrt{\frac{1}{x} + x + 1}}$$

$$= \int_0^1 \frac{x^{-2} - 1}{\left(\frac{1}{x} + x + 1\right)} \frac{dx}{\sqrt{\frac{1}{x} + x + 1}}$$

Put $\frac{1}{x} + x + 1 = t \Rightarrow -\int_{\infty}^{3} \frac{dt}{(t+1)\sqrt{t}}$

Put $t = \tan^2\theta \Rightarrow dt = 2\tan\theta\sec^2\theta d\theta$

$$= -\int_{\pi/2}^{\pi/3} \frac{2\tan\theta\sec^2\theta}{\sec^2\theta\tan} d\theta = 2\int_{\pi/3}^{\pi/2} d\theta$$

$$= 2\times\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{\pi}{3}$$

Sol 21: $\int_0^{\pi/2} \left(\frac{a\sin x + b\cos x}{\sin x + \cos x}\right)\sqrt{2}dx$

$$= \int_0^{\pi/2} \frac{(a\cos x + b\sin x)\sqrt{2}}{\sin x + \cos x} dx$$

$\therefore\ 2I=\sqrt{2}(a+b)\frac{\pi}{2}\Rightarrow I=\frac{(a+b)\pi}{2\sqrt{2}}$

Sol 22: $\int_0^1 f(x)dx=1$

$\Rightarrow\int_0^1\frac{f(2x)}{3}dx\ =1\Rightarrow\int_0^1 f(2x)dx=3$

$\therefore 2I\ \int_0^{\pi/2}\frac{(a+b)(\sin x+\cos x)}{(\sin x+\cos x)}\sqrt{2}dx$

Put 2x = t ⇒ dx = $\frac{dt}{2}$

$\int_1^2 f(t)\frac{dt}{2}\ =3\Rightarrow\int_1^2 f(t)dt\ =6$

$\therefore\int_1^2 f(t)dt\ =6-1=5$

Sol 23: $\int_{-1}^{3}\ \{|x-2|+[x]\}dx=\int_{-1}^{0}\ \{|x-2|+[x]\}dx+$

$\int_0^{-1}\ \{|x-2|+[x]\}dx=\int_1^2\ \{|x-2|+[x]\}dx$

$+\int_2^3\ \{|x-2|+[x]\}dx$

$\int_{-1}^{0}\ (2-x-1)dx+\int_0^1\ (2-x+0)dx+$

$\int_1^2\ (2-x+1)dx+\int_2^3\ (x-2+2)dx+$

$=x-\frac{x^2}{2}\Big|_{-1}^{0}+2x-\frac{x^2}{2}\Big|_0^1+3x-\frac{x^2}{2}\Big|_1^2+\frac{x^2}{2}\Big|_2^3$

$=-\left(-1-\frac{1}{2}\right)+\left(2-\frac{1}{2}\right)+(6-2)-\left(3\ -\frac{1}{2}\right)+\frac{9}{2}-2$

$=-\left(-1-\frac{1}{2}\right)+\left(2-\frac{1}{2}\right)+(6-2)-\left(3\ -\frac{1}{2}\right)+\frac{9}{2}-2$

$=7$

Sol 24: x = tanθ

dx = secθdθ

$\int_0^{\pi/3}\left(\sin^{-1}\frac{2\tan\theta}{1+\tan\theta}\right)\sec^2\theta d\theta$

$=\int_0^{\pi/4}2\theta\sec^2\theta d\theta\ +\int_{\pi/4}^{\pi/3}(\pi-2\theta)\sec^2\theta d\theta$

$=-2\left[\theta\tan\theta\Big|_{\pi/4}^{\pi/3}-\int_{\pi/4}^{\pi/3}\tan\theta d\theta\right]+\pi\tan\theta\Big|_{\pi/4}^{\pi/3}$

$+2\left[\theta\tan\theta\Big|_0^{\pi/4}-\int_0^{\pi/4}\tan\theta d\theta\right]$

$=-2\left[\left(\frac{\pi}{3}\times\sqrt{3}-\frac{\pi}{4}\right)\right]\ +\ \left[\log 2-\frac{1}{2}\log 2\right]^2$

$+\ \pi(\sqrt{3}-1)\ +2\left[\frac{\pi}{4}\right]-\left[\frac{1}{2}\log 2\right]2$

$=-\frac{2\pi}{\sqrt{3}}+\frac{\pi}{2}+\log 2+\ \sqrt{3}\,\pi-\pi+\frac{\pi}{2}\ -\log 2$

$=\frac{\pi}{\sqrt{3}}$

Sol 25: $\int_0^{\pi}\frac{(ax+b)\sec x\tan x}{4+\tan^2 x}dx$

$\frac{2I}{a\pi+2b}=\left[\int_0^{\pi}\frac{\sec x\tan x}{4+\tan^2 x}dx\right]$

$=\int_1^{-1}\frac{dt}{3+t^2}\ =\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\Big|_1^{-1}$

$=\frac{1}{\sqrt{3}}\left[\pi-\frac{\pi}{6}-\frac{\pi}{6}\right]\ =\frac{1}{\sqrt{3}}\times\frac{2\pi}{3}\quad\therefore I=\frac{(a\pi+2b)\pi}{3\sqrt{3}}$

Sol 26: $\int_0^{\pi}\frac{(2x-3)\sin x}{(1+\cos^2 x)}dx$

$2I=\int_0^{\pi}\frac{(2\pi+6)\sin x}{1+\cos^2 x}dx$

$\frac{I}{\pi+3}\ =\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx\ =\int_1^{-1}\frac{-dt}{1+t^2}\ =\int_1^{1}\frac{dt}{1+t^2}$

$=\ \tan^{-1}t\Big|_{-1}^{1}\ =\frac{\pi}{4}\ +\frac{\pi}{4}\Rightarrow I=(\pi+3)\ \frac{\pi}{2}$

Sol 27: Let $f(x)=\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}$ ....(i)

Then, $f\left(\frac{\pi}{2}-x\right)=\frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}$

$$= \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} \quad \text{...(ii)}$$

Now, $f(x)+f\left(\frac{\pi}{2}-x\right)=\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}=1$

$$\therefore I=\frac{1}{2}\int_0^{\pi 2}\left[f(x)+f\left(\frac{\pi}{2}-x\right)\right]dx=\frac{1}{2}\int_0^{\pi/2} x\,dx$$

$$=\frac{1}{2}[x]_0^{\pi/2}=\frac{1}{2}\left(\frac{\pi}{2}-0\right)=\frac{\pi}{4}$$

Sol 28: $2I = \int_0^{n\pi}\frac{n\pi\,|\sin x|}{1+(\cos x)}dx$

$$2I = n^2\pi\int_0^{\pi}\frac{\sin x}{1+\cos x}dx = 2n^2\pi\int_0^{\pi/2}\frac{\sin x}{1+\cos x}dx$$

$$= 2n^2\pi\int_1^0\frac{-dt}{1+t} = 2n^2\pi\log(t+1)\Big|_0^1 = 2n^2\pi\log 2$$

$\therefore I = n^2\pi\log 2 = 100\pi\log 2$

$\therefore n = 10$

Sol 29: $\int_0^{\pi/2}\frac{\cos x}{1+\cos x+\sin x}dx$

$$=\int_0^{\pi/2}\frac{\cos x}{(1+\cos x)+\sin x}dx$$

$$=\int_0^{\pi/2}\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}dx$$

$$=\int_0^{\pi/2}\frac{1-\tan^2\frac{x}{2}}{2+2\tan\frac{x}{2}}dx\ s$$

[Dividing numerator and denominator By $\cos^2\frac{x}{2}$]

$$=\frac{1}{2}\int_0^{\pi/2}\frac{\left(1-\tan\frac{x}{2}\right)\left(1+\tan\frac{x}{2}\right)}{1+\tan\frac{x}{2}}dx$$

$$=\frac{1}{2}\int_0^{\pi/2}\left(1-\tan\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\left[x+2\log\cos\frac{x}{2}\right]_0^{\pi/2}$$

$$=\frac{1}{2}\left[\left(\frac{\pi}{2}+2\log\cos\frac{\pi}{4}\right)-(0+2\log 1)\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{2}+2\log\frac{1}{\sqrt{2}}\right]=\frac{1}{2}\left[\frac{\pi}{2}+\log\frac{1}{2}\right]=\frac{1}{2}\left[\frac{\pi}{2}-\log 2\right]$$

Sol 30: $\int_0^a\frac{\log(1+ax)}{1+x^2}dx$

$ax = \tan\theta \Rightarrow abx = \sec^2\theta\,d\theta$

$$\int_0^{\tan^{-1}a^2}\frac{a\log(1+\tan\theta)}{(a^2+\tan^2\theta)}\times\sec^2\theta = d\theta$$

$$a\left[\log(1+\tan\theta)\int_0^{\tan^{-1}a^2}\frac{\sec^2\theta}{a^2+\tan^2\theta}d\theta\Big|_0^{\tan^{-1}a^2} - \int_0^{\tan^{-1}a^2}\frac{\sec^2\theta}{(1+\tan\theta)}\int_0^{\tan^{-1}\theta}\frac{\sec^2\theta}{a^2+\tan^2\theta}d\theta\right]$$

$$a\left[\log(1+\tan\theta)\times\frac{1}{a}\tan^{-1}\frac{\tan\theta}{\alpha}\Big|_0^{\tan^{-1}a^2}\right.$$

$$-\int_0^{\tan^{-1}a^2}\left(\frac{\sec^2\theta}{1+\tan\theta}\times\frac{1}{a}\tan^{-1}\frac{\tan\theta}{a}\right)d\theta$$

$2I = \log(1+a^2)\tan^{-1}a$

$I = \tan^{-1}a\log\sqrt{1+a^2}$

Sol 31: $\int_0^{\frac{\ell n3}{2}}\frac{e^x+1}{e^{2x}+1}dx = \int_0^{\frac{\ell n3}{2}}\frac{e^x dx}{e^{2x}+1} + \int_0^{\frac{\ell n3}{2}}\frac{1}{e^{2x}+1}dx$

$e^{2x} = t$

$2e^{2x}\,dx = dt$

$dx = \frac{1}{2t}dt$

$$\tan^{-1}e^x\Big|_0^{\frac{\log 3}{2}} + \frac{1}{2}\int_1^3\frac{1}{(t+1)t}dt$$

$$= \tan^{-1}\sqrt{3} - \frac{\pi}{4} + \frac{1}{2}\left[\int_1^3\left(\frac{1}{t}-\frac{1}{t+1}\right)dt\right.$$

$= \frac{1}{2}\left[\frac{\pi}{6} + \log 3 - \log 2\right]$

Sol 32: Given, $\int_a^a \sqrt{x}\, dx = 2a \int_0^{\pi/2} \sin^3 x\, dx$

$\Rightarrow \left[\frac{x^{3/2}}{3/2}\right]_0^a = 2a \int_0^{\pi/2} \frac{3\sin x - \sin 3x}{4} dx$

$[\because \sin 3x = 3\sin x - 4\sin^3 x]$

$\Rightarrow \frac{2}{3}\left[a^{3/2} - 0\right] = \frac{a}{2}\left[3(-\cos x) - \left(-\frac{\cos 3x}{3}\right)\right]_0^{\pi/2}$

$\Rightarrow \frac{2}{3}a^{3/2} = \frac{a}{2}\left[-3\left(\cos\frac{\pi}{2} - \cos 0\right) + \frac{1}{3}\left(\cos\frac{3\pi}{2} - \cos 0\right)\right]$

$\Rightarrow \frac{2}{3}a^{3/2} = \frac{a}{2}\left(-3(0-1) + \frac{1}{3}(0-1)\right)$

$\Rightarrow \frac{2}{3}a^{3/2} = \frac{4a}{3} \Rightarrow a\sqrt{a} - 2a = 0$

$\Rightarrow a\left(\sqrt{a} - 2\right) = 0 \Rightarrow a = 0$ or $\sqrt{a} = 2 \Rightarrow a = 0$ or $a = 4$

When a = 0:

$\int_a^{a+1} x\, dx = \int_0^1 x\, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}(1-0) = \frac{1}{2}$

When a = 4:

$\int_0^{a+1} x\, dx = \int_4^5 x\, dx = \left[\frac{x^2}{2}\right]_4^5 = \frac{1}{2}(25-16) = \frac{9}{2}$

Sol 33: tan x = 2x

$\frac{1}{2}\int_0^1 [\cos(\alpha-\beta)x - \cos(\alpha+\beta)x]dx$

$\frac{1}{2}\left[\left.\frac{\sin(\alpha-\beta)x}{(\alpha-\beta)}\right|_0^1 - \left.\frac{\sin(\alpha+\beta)x}{(\alpha+\beta)}\right|_0^1\right]$

$\frac{1}{2}\left[\frac{\sin(\alpha-\beta)}{\alpha-\beta} - \frac{\sin(\alpha-\beta)}{(\alpha+\beta)}\right]$

$\sin\alpha = 2\alpha\cos a$

$\sin\beta = 2\beta\cos b$

$= \frac{1}{2}\left[\left(\frac{\sin\alpha\frac{\sin\beta}{2\beta} - \frac{\sin\alpha}{2\alpha}\sin\beta}{\alpha-\beta}\right) - \left(\frac{\frac{\sin\alpha \operatorname{isn}\beta}{2\beta} + \frac{\cos\alpha\cos\beta}{2\alpha}}{(\alpha+\beta)}\right)\right]$

$= \frac{1}{2}\sin\alpha\sin\beta\left[\frac{1}{2\alpha\beta} - \frac{1}{2\alpha\beta}\right] = 0$

Sol 34: $\int_0^{p+q\pi} |\cos x|\, dx$

$\int_0^p \cos x + \int_p^{p+q\pi} |\cos x|\, dx$

$= \sin x\Big|_0^p + \int_0^{q\pi} (\cos x)dx$

$= \sin p + q \times 2\int_0^{q\pi} (\cos x)dx$

$= 2q + \sin p$

Sol 35: $\int_{-4}^{-5} e^{(x+5)^2} dx + 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2} dx$

Let x + 5 = t and $3\left(x - \frac{2}{3}\right) = t$

$= \int_1^0 e^{t^2} dt + (-1)\int_1^0 e^{t^2} dt = 0$

Put t = –2

$\int_0^1 e^{t^2} dt + (-1)\int_1^0 e^{z^2} dz = 0$

Sol 36: $\sin \pi x > \cos \pi x$

$2n\pi + \frac{\pi}{4} < \pi x < 2n\pi + \frac{\pi}{4}$

$2n + \frac{1}{4} < x < 2n + \frac{1}{4}$

$\therefore \frac{\pi}{4\sqrt{2}} \int_{-10}^{10} F(x)dx$

$= \frac{\pi}{4\sqrt{2}} \times 2 \times 10\int_0^1 f(x)dx$

$= \frac{5\pi}{\sqrt{2}}\left[\int_0^{1/4} \cos\pi x dx + \int_{1/4}^1 \sin\pi x dx\right]$

$= \frac{5\pi}{\sqrt{2}}\left[\frac{1}{\pi}\left[\sin\frac{\pi}{4} - 0\right] - \frac{1}{\pi}\left[\cos\pi - \cos\frac{\pi}{4}\right]\right]$

$$= \frac{5\pi}{\sqrt{2}}\left[\frac{1}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}}\right] = 5$$

Sol 37: $\int_0^{\pi/2} \tan^{-1}\left[\frac{1+\sin x-(1-\sin x)}{1+\sin x+(1-\sin x)-2\sqrt{1-\sin^2 x}}\right]dx$

$$= \int_0^{\pi/2} \tan^{-1}\frac{2\sin x}{2-2\cos x}dx = \int_0^{\pi/2} \tan^{-1}\frac{2\sin x}{2-2\cos x}dx$$

$$= \int_0^{\pi/2} \tan^{-1}\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)dx = \int_0^{\pi/2}\left(\frac{\pi}{2}-\frac{x}{2}\right)dx$$

$$= \frac{\pi^2}{4}-\frac{1}{2}\times\frac{\pi^2}{4}\times\frac{1}{2} = \frac{\pi^2}{4}-\frac{\pi^2}{16} = \frac{3\pi^2}{16}$$

Sol 38: $x^2 + 2x = k + \int_0^1 |t+k|dt$

$t = k = 0 \Rightarrow dt = dU$

$$kt\int_k^{k+1} udu = \frac{1}{2}\left[(k+1)^2-k^2\right]$$

$$= \frac{2k+1}{2}$$

$x^2 + 2x = \frac{4k+1}{2} \Rightarrow x^2 + 2x - \left(\frac{4k+1}{2}\right) = 0$

$\Rightarrow x = -2 \pm \frac{\sqrt{4+2(4k+1)}}{2a}$

$\Rightarrow$ x = real and distinct

Sol 39: $I = \int_{-1}^{1} \frac{2x^{332}+x^{998}+4x^{1668}\sin x^{691}}{1+x^{666}}dx$

$$I = \int_{-1}^{1}\frac{2x^{332}+x^{998}}{1+x^{666}}dx = 2\int_0^1\frac{2x^{332}+x^{998}}{1+x^{666}}dx$$

$$= 2\left[\int_0^1\left(\frac{x^{332}}{1+x^{666}}+x^{332}\right)dx\right]$$

$$= 2\frac{1}{333}+2\int_0^1\frac{x^{332}}{1+(x^{333})^2}dx$$

$$= \frac{2}{333}+2\left[\int_0^1\left(\frac{dt}{1+t^2}\right)\right]\frac{1}{333}$$

$$= \frac{2}{333}+\frac{2}{333}\tan^{-1}t\Big|_0^1 = \frac{2}{333}\left[1+\frac{\pi}{4}\right] = \frac{\pi+4}{666}$$

Sol 40: $2I = \pi\int_0^{\pi}\frac{[x^2-(x-\pi)^2]\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)dx}{2x-\pi}$

$$= \pi^2\int_0^{\pi}\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)dx$$

Let $\frac{\pi}{2}\cos x = t \Rightarrow -\frac{\pi}{2}\sin xdx = dt$

$$= -\pi^2\times\frac{2}{\pi}\int_{\pi/2}^{-\pi/2} 2.\frac{2}{\pi}t\sin tdt$$

$$2I = 8\int_{-\pi/2}^{\pi/2} t\sin tdt$$

$$I = 4\left[-t\cos t\Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2}\cos tdt\right] = 4[+2] = 8$$

Sol 41: $\int_{1/3}^{1}\frac{(x-x^3)^{1/3}}{x^4}dx = \int_{1/3}^{1}\frac{\left[x^3\left(\frac{1}{x^2}-1\right)\right]^{1/3}}{x^4}$

$$dx = \int_{1/3}^{1}\frac{\left(\frac{1}{x^2}-1\right)^{1/3}}{x^3}dx$$

Put $\frac{1}{x^2}-1 = t$, then $-\frac{2}{x^3}dx = dt$ or $\frac{1}{x^3}dx = -\frac{1}{2}dt$

When $x = 1, t = \frac{1}{1^2}-1 = 0$ and when $x = \frac{1}{3}, t = 9-1 = 8$

Now, $\int_{1/3}^{1}\frac{(x-x^3)^{1/3}}{x^4}dx = -\frac{1}{2}\int_8^0 t^{1/3}\,dt = -\frac{1}{2}\left[\frac{t^{4/3}}{\frac{4}{3}}\right]_8^0$

$$= -\frac{3}{8}\left[0-8^{4/3}\right] = -\frac{3}{8}\left[-2^4\right] = -\frac{3}{8}(-16) = 6$$

Sol 42: $\lim_{n\to\infty}\frac{1}{n^2}\left[\sum_{k=0}^{n-1}k\int_k^{k+1}\sqrt{(x-k)(k+1-x)}dx\right]$

$x - k = t$

$$\int_0^1\sqrt{t(1-t)}dt = \int_0^1\sqrt{\left(\frac{1}{2}\right)^2-(t-1)^2}dx$$

$$= \frac{1}{2}\left(t-\frac{1}{2}\right)\sqrt{\frac{1}{4}-\left(t-\frac{1}{2}\right)^2} + \frac{1}{8}\sin^{-1}\frac{\left(t-\frac{1}{2}\right)}{\frac{1}{2}}\Bigg|_0^1$$

$= \frac{\pi}{2}\times\frac{1}{8}+\frac{\pi}{2}\times\frac{1}{8}=\frac{\pi}{8}$

$\therefore \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=0}^{n-1}k\times\frac{\pi}{8} = \int_0^1\frac{\pi}{8}x\,dx=\frac{\pi}{8}\times\frac{1}{2}=\frac{\pi}{16}$

Sol 43: $I = \int_0^{\pi/2}\frac{\sin x+3}{5\sin(x+\alpha)+25}dx \quad \cos\alpha = \frac{3}{5}$

$4I + 3J = \int_0^{\pi/2}\frac{4\cos x+3\sin x+25}{4\cos x+3\sin x+25}dx=\frac{\pi}{2}$

$3I - 4J = \int_0^{\pi/2}\frac{3\cos x-4\sin x}{4\cos x+3\sin x+25}dx$

$= \log\,(4\cos x+3\sin x+25)\Big|_0^{\pi/2}$

$= \log(28) - \log(29) = \log\frac{28}{29}$

$16I + 9I = 2\pi + 3\log\frac{28}{29} = 2\pi + 3\log\frac{28}{29}$

$a + b + c + d = 2 + 3 + 28 + 29 = 62$

Sol 44: $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

$f'(2) = 4a + b = 1$

$f'(2) = 4a + b = 1$

$\int_{2-\pi}^{2+\pi} f(x)\sin\left(\frac{x-2}{2}\right)dx = -\int_{2-\pi}^{2+\pi} f(4-x)\text{in}\frac{(x-2)}{2}dx$

$2I = -\int_{2-\pi}^{2+\pi}\left[f(x)-f(4-x)\right]\sin\frac{(x-2)}{2}dx$

$= -\int_{2-\pi}^{2+\pi}\left\{\begin{matrix}ax^2+bx+c\\ -\left[a(4-x)^2+b(4-x)+c\right]\end{matrix}\right\}\sin\left(\frac{x-2}{2}\right)dx$

$= \int_{2-\pi}^{2+\pi}\left[\begin{matrix}a(x-4+x)(x+4-x)\\ +b(x-4+x)\end{matrix}\right]\sin\left(\frac{x-2}{2}\right)dx$

$= \int_{2-\pi}^{2+\pi}\left(a(2x-4)4+2bx-4b\right)\sin\left(\frac{x-2}{2}\right)dx$

$= \int_{2-\pi}^{2+\pi}(8ax+2bx-4)\sin\frac{(x-2)}{2}dx$

$= 4\int_{2-\pi}^{2+\pi}\frac{(x+2)}{2}\sin\left(\frac{x-2}{2}\right)dx \quad \frac{x-2}{2}=t$

$= 4\int_{-\pi/2}^{\pi/2} t\sin t\,dt = 4\left[t(-\cos t)\Big|_{-\pi/2}^{\pi/2}+\int_{-\pi/2}^{\pi/2}\cos t\,dt\right]$

$= 4\left[\sin t\Big|_{-\pi/2}^{\pi/2}\right] = 8$

Exercise 2

Single Correct Choice Type

Sol 1: (C) $\int_0^2\left|(x+3)(x-1)\right|dx$

$= \int_0^1(x+3)(1-x)dx + \int_1^2(x+3)(x-1)dx$

$= -\int_1^1(x^2+2x-3)dx+\int_1^2(x^2+2x-3)dx$

$= -\left[\frac{x^3}{3}+x^2-3x\Big|_0^1\right]+\left[\frac{x^3}{3}+x^2-3x\Big|_1^2\right]$

$= -\left[\frac{1}{3}+1-3\right]+\left[\frac{8}{3}+4-6-\left(\frac{1}{3}+1-3\right)\right]$

$= \frac{5}{3}+\frac{2}{3}+\frac{5}{3}=4$

Sol 2: (B) $\int_0^{\pi/2}\left|\frac{1}{\sqrt{2}}\sin x-\frac{1}{\sqrt{2}}\cos x\right|dx$

$= \frac{1}{\sqrt{2}}\left[\int_0^{\pi/4}(\cos x-\sin x)dx+\int_{\pi/4}^{\pi/2}(\sin x-\cos x)dx\right]$

$= \frac{1}{\sqrt{2}}\left[\sin x+\cos x\Big|_0^{\pi/4}+(-\text{cox}-\sin x)\Big|_{\pi/4}^{\pi/2}\right]$

$= \frac{1}{\sqrt{2}}\left[\left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1\right]-\left[1-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\right]\right]$

$= \frac{1}{\sqrt{2}}\left[\sqrt{2}-1-1+\sqrt{2}\right] = \frac{2\sqrt{2}-2}{\sqrt{2}} = 2-\sqrt{2}$

Sol 3: (D) $2\int_0^{\pi/2}(\sin^4 x)dx = 2\times\left(\frac{(4-1)(4-3)}{4\times(4-2)}\right)\times\frac{\pi}{2} = \frac{3\pi}{8}$

Sol 4: (B) $\int_0^1 0dx = \int_1^{\sqrt{2}}1dx+\int_{\sqrt{2}}^{1.5}2dx$

$= \sqrt{2} - 1 + 2(1.5 - \sqrt{2}) = 2 - \sqrt{2}$

Sol 5: (A) $I=\int_0^{\pi}\frac{x}{a^2\cos^2x+b^2\sin^2x}dx$ ...(i)

Then $I=\int_0^{\pi}\frac{(\pi-x)}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}dx$

$\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$

Or $I=\int_0^{\pi}\frac{\pi-x}{a^2\cos^2x+b^2\sin^2x}dx$ ...(ii)

Adding (i) and (ii), we get

$$2I=\int_0^{\pi}\frac{x+\pi-x}{a^2\cos^2x+b^2\sin^2x}dx$$

$$=\pi\int_0^{\pi}\frac{1}{a^2\cos^2x+b^2\sin^2x}dx$$

$$=2\pi\int_0^{\pi/2}\frac{1}{a^2\cos^2x+b^2\sin^2x}dx$$

$\left[\because \int_0^{2a}f(x)\,dx=2\int_0^a f(x)\,dx,\ \text{if } f(2a-x)=f(x)\right]$

$$\therefore 2I=2\pi\int_0^{\pi/2}\frac{\sec^2x}{a^2+b^2\tan^2x}dx$$

[Dividing num. and denom. By $\cos^2 x$]

Or $I=\pi\int_0^{\pi/2}\frac{\sec^2x}{a^2+b^2\tan^2x}dx$

Let $\tan x=z$. Then, $\sec^2x\,dx=dz$

Also $x=0\Rightarrow z=\tan 0=0$ and $x\to\frac{\pi}{2}\Rightarrow z\to\tan\frac{\pi}{2}$ or $z\to\infty$

$$\therefore I=\pi\int_0^{\infty}\frac{dz}{a^2+b^2z^2}=\frac{\pi}{b^2}\int_0^{\infty}\frac{dz}{(a/b)^2+z^2}$$

$$=\frac{\pi}{b^2}\times\frac{1}{(a/b)}\left[\tan^{-1}\left(\frac{z}{a/b}\right)\right]_0^{\infty}$$

$$\Rightarrow I=\frac{\pi}{ab}\left[\tan^{-1}\left(\frac{bz}{a}\right)\right]_0^{\infty}=\frac{\pi}{ab}\left(\tan^{-1}\infty\ \tan^{-1}0\right)$$

$$=\frac{\pi}{ab}\left(\frac{\pi}{2}-0\right)=\frac{\pi^2}{2ab}$$

Sol 6: (A) $\int_0^{\pi/4}\frac{\sec x}{1+2\sin^2x}dx$

$$=\int_0^{\pi/4}\frac{dx}{\cos x+2\sin^2x\cos x}$$

$$=\int_0^{\pi/4}\frac{\cos x\ \ dx}{\cos^2x+2\sin^2x\cos^2x}$$

$$=\int_0^{\pi/4}\frac{\cos x dx}{(1-\sin^2x)(1+2\sin^2x)}$$

$$=\int_0^{1/\sqrt{2}}\frac{dt}{(1-t^2)(1+2t^2)}$$

$$=\frac{1}{3}\int_0^{1/\sqrt{2}}\left(\frac{1}{1-t^2}+\frac{2}{1+2t^2}\right)dt$$

$$=\frac{1}{3}\int_0^{1/\sqrt{2}}\frac{1}{1-t^2}dt+\frac{1}{3}\int_0^{1/\sqrt{2}}\frac{1}{\frac{1}{2}+t^2}dt$$

$$=\frac{1}{6}\ell n\frac{1+t}{1-t}\Bigg|_0^{1/\sqrt{2}}+\frac{1}{3}\times\frac{1}{1/\sqrt{2}}\tan^{-1}\sqrt{2}t\Bigg|_0^{1/\sqrt{2}}$$

$$=\frac{1}{6}\ell n\frac{\sqrt{2}+1}{\sqrt{2}-1}+\frac{\sqrt{2}}{3}\times\frac{\pi}{4}=\frac{1}{3}\left[\ell n(\sqrt{2}+1)+\frac{\pi}{2\sqrt{2}}\right]$$

Sol 7: (C) $\int_0^1 e^{x^2}(x-\alpha)dx=0$

For this integral to be zero

If $\alpha<0$ then $x-\alpha$ when $x\in(0,1)>0$

$\therefore$ It is not possible that integral reduce to zero

If $2>\alpha>1$ then $x-a$

when $x\in(0,1)<0$ function gives negative value and so cannot reduced zero.

$\therefore$ If $0<\alpha<1$, f^n can take both positive and negative valuce and it is possible that integral reduced to zero

Sol 8: (A) $\int_0^{\pi/2}\{x-[\sin x]\}dx$

[ ] $\to$ greatest integer function

$[\sin x]=0$ $\sin x\in[0,1)$ i.e. $x\in\left[0,\frac{\pi}{2}\right)$

$$=\int_0^{\pi/2}(x-0)dx=\frac{x^2}{2}\Bigg|_0^{\pi/2}=\frac{\pi^2}{8}$$

Sol 9: (B) $\int_0^{100} \sin(x[x])\pi dx$

Since x – [x] has a period of 1

$\therefore I = 100\int_0^1 \sin \pi x dx = \frac{100}{\pi}\left(-\cos \pi x\Big|_0^1\right)$

$= \frac{100}{\pi}\left(-(-1-1)\right) = \frac{200}{\pi}$

Sol 10: (B) $\int_0^{\infty} \frac{x\log x}{(1+x^2)^2}dx$

$x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$I = \int_0^{\pi/2} \frac{\tan\theta\log(\tan\theta)\sec^2\theta d\theta}{\sec^4\theta}$

$= \int_0^{\pi/2} \tan\theta\log(\tan\theta)\cos^2\theta d\theta$

$= \int_0^{\pi/2} \sin\theta\cos\theta\log(\tan\theta)d\theta$

$= \int_0^{\pi/2} \cos\theta\sin\theta\log\cot\theta d\theta$

$\therefore 2I = \int_0^{\pi/2} \sin\theta\cos\theta\left[\log\tan\theta + \log\cot\theta\right]d\theta$

$= \int_0^{\pi/2} \sin\theta\cos\theta\left[\log\tan - \log\tan\theta\right]d\theta = 0$

Sol 11: (B) $\log I = \lim_{n\to\infty}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)\times\frac{1}{n}$

$\log I = \int_0^1 \log(1 + x)dx$

$\log I = x\log(1+x)\Big|_0^1 - \int_0^1 \frac{x}{1+x}dx$

$= \log 2 - \int_0^1 1 - \frac{1}{1+x}dx = \log 2 - [1] + \log(1+x)_0^1$

$= 2\log 2 - \log e = \log\frac{4}{e} \Rightarrow I = \frac{4}{e}$

Sol 12: (A) $\int_{\log 2}^{x} \frac{1}{\sqrt{e^x - 1}}dx$

Put $e^x - 1 = t^2 \Rightarrow e^x dx = 2tdt$ or $dx = \frac{2t}{1+t^2}dt$

$\int_1^{\sqrt{e^x-1}} \frac{2}{1+t^2}dt = 2\tan^{-1}t\Big|_1^{\sqrt{e^x-1}} = \frac{\pi}{6}$

$2\left[\tan^{-1}t - \frac{\pi}{4}\right] = \frac{\pi}{6}$

$\therefore \tan^{-1}t = \frac{\pi}{3} \Rightarrow \therefore t = \tan\frac{\pi}{3} = \sqrt{3}$

$\therefore e^x - 1 = 3 \Rightarrow x = \log 4$

Sol 13: (A) $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{r^3+n^3} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\left(\frac{r}{n}\right)^2}{\left(\frac{r}{n}\right)^3+1}\times\frac{1}{n}$

$= \int_0^1 \frac{x^2}{x^3+1}dx = \frac{1}{3}\ell n(x^3+1)\Big|_0^1 = \frac{\ell n 2}{3}$

Sol 14: (B) $\lim_{n\to\infty}\left[\left(\sum_{r=1}^{n}\frac{n^2}{(n^2+r^2)^{3/2}}\right) - \frac{1}{2n}\right]$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\left(1+\left(\frac{r}{n}\right)^2\right)^{3/2}}\times\frac{1}{n} = \int_0^1 \frac{1}{\left(1+x^2\right)^{3/2}}dx$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$\frac{\pi}{4}\int\frac{\sec^2\theta d\theta}{\sec^3\theta} = \int_0^{\pi/4}\cos\theta d\theta = \sin\theta\Big|_0^{\pi/4} = \frac{1}{\sqrt{2}}$

Sol 15: (A) $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{[n+4(r-1)]^3}} - \frac{\sqrt{n}}{\sqrt{(n+4n)^3}}$

$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{\left(1+\frac{4r}{n}\right)^3}}\times\frac{1}{n} - \frac{1}{(5)^{3/2}\times n^{1/2}}$

$= \int_0^1 \frac{1}{\sqrt{(1+4x)^3}}dx = \int_0^1 (1+4x)^{-3/2}dx$

$= \frac{(1+4x)^{-1/2}}{-1/2}\times\frac{1}{4}\Bigg|_0^1 = -\frac{1}{2}\left(\frac{1}{1+4x}\right)^{1/2}\Bigg|_0^1$

$= -\frac{1}{2}\left[\frac{1}{\sqrt{5}} - 1\right] = \frac{1}{10}\left(5-\sqrt{5}\right)$

Sol 16: (B) $\log I = \lim_{n\to\infty}\left(\sum_{r=1}^{n}\log\tan\left(\frac{\pi r}{2n}\right)\right)\frac{1}{n}$

$= \int_0^1 \log\tan\left(\frac{\pi}{2}x\right)dx$

$\frac{\pi}{2}x = t \Rightarrow dx = \frac{2}{\pi}dt$

$\Rightarrow \int_0^{\pi/2}\log\tan t\,dt = 0$

$\therefore I = e^0 = 1$

Previous Years' Questions

Sol 1: (A) $\int_{-1/2}^{1/2}\left([x]+\log\left(\frac{1+x}{1-x}\right)\right)dx$

$= \int_{-1/2}^{1/2}[x]dx + \int_{-1/2}^{1/2}\log\left(\frac{1+x}{1-x}\right)dx$

$= \int_{-1/2}^{1/2}[x]dx + 0$

$\left[\because \log\left(\frac{1+x}{1-x}\right) \text{is an odd function}\right]$

$= \int_{-1/2}^{0}[x]dx + \int_0^{1/2}[x]dx$

$= \int_{-1/2}^{0}(-1)dx + \int_0^{1/2}(0)dx$

$= [x]_{-1/2}^{0}$

$= -\left(0+\frac{1}{2}\right) = -\frac{1}{2}$

Sol 2: (A) Here, $I(m, n) = \int_0^1 t^m(1+t)^n dt$ reduce into $I(m + 1, n - 1)$ [we apply integration by parts taking $(1 + t)^n$ as first and t^m as second function]

$\therefore I(m,n) = \left[(1+t)^n.\frac{t^{m+1}}{m+1}\right]_0^1$

$-\int_0^1 n(1+t)^{(n-1)}.\frac{t^{m+1}}{m+1}dt$

$= \frac{2^n}{m+1} - \frac{n}{m+1}\int_0^1 (1+t)^{(n-1)}.\ t^{m+1}dt$

$\therefore I(m,n) = \frac{2^n}{m+1} - \frac{n}{m+1}.\ I(m+1, n-1)$

Sol 3: (C) Given $\int_0^x \sqrt{1-(f'(t))^2}dt$

$= \int_0^x f(t)dt,\ 0 \le x \le 1$

Differentiating both sides w.r.t.x by using Leibnitz rule, we get

$\sqrt{1-(f'(x))^2} = f(x)$

$\Rightarrow f'(x) = \pm\sqrt{1-(f(x))^2}$

$\Rightarrow \int \frac{f'(x)}{\sqrt{1-(f(x))^2}}dx = \pm\int dx$

$\Rightarrow \sin^{-1}(f(x)) = \pm x + c$

Put $x = 0$

$\Rightarrow \sin^{-1}(f(0)) = c$

$\Rightarrow c = \sin^{-1}(0) = 0 \quad (\because f(0) = 0)$

$\therefore f(x) = \pm \sin x$

but $f(x) \ge 0,\ \forall x \in [0, 1]$

$\therefore f(x) = \sin x$

As we know that,

$\sin x < x\ \forall x > 0$

$\therefore \sin\left(\frac{1}{2}\right) < \frac{1}{2}$ and $\sin\left(\frac{1}{3}\right) < \frac{1}{3}$

$\Rightarrow f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

Sol 4: (A) $x^2 = t \Rightarrow 2x\,dx = dt$

$\therefore I = \frac{1}{2}\int_{\log 2}^{\log 3}\frac{\sin t}{\sin t + \sin(\log 6 - t)}dt$

$I = \frac{1}{2}\int_{\log 2}^{\log 3}\frac{\sin(\log 6 - t)}{\sin(\log 6 - t) + \sin t}dt$

$2I = \frac{1}{2}\int_{\log 2}^{\log 3} 1dt \Rightarrow I\ \frac{1}{4}\log\frac{3}{2}$

Sol 5: (A, D) Gives, $S_n = \sum_{k=0}^{n} \frac{n}{n^2+kn+k^2}$

$$= \sum_{k=0}^{n} \frac{1}{n}\left(\frac{1}{1+\frac{k}{n}+\frac{k^2}{n^2}}\right) < \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{n}$$

$$\left(\frac{1}{1+\frac{k}{n}+\left(\frac{k}{n}\right)^2}\right)$$

$$= \int_0^1 \frac{1}{1+x+x^2}dx = \left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}.\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$$

i.e. $S_n < \frac{\pi}{3\sqrt{3}}$

Similarly, $T_n > \frac{\pi}{3\sqrt{3}}$

Sol 6: (A, B, C) Given $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx$ ... (i)

Using $\int_a^b f(x)dx = \int_a^b f(b+a-x)dx$

we get $I_n = \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(1+\pi^x)\sin x}dx$ ... (ii)

Adding Eqs. (i) and (ii), we get

$$2I_n = \int_{-\pi}^{\pi}\frac{\sin nx}{\sin x}dx = 2\int_0^{\pi}\frac{\sin nx}{\sin x}dx$$

$\left(\because f(x) = \frac{\sin nx}{\sin x}\text{ is an even function}\right)$

$$\Rightarrow I_n = \int_0^{\pi}\frac{\sin nx}{\sin x}dx$$

Now, $I_{n+2} - I_n = \int_0^{\pi}\frac{\sin(n+2)x - \sin nx}{\sin x}dx$

$$= \int_0^{\pi}\frac{2\cos(n+1)x.\sin x}{\sin x}dx$$

$$= 2\int_0^{\pi}\cos(n+1)x\ dx$$

$$= 2\left[\frac{\sin(n+1)x}{(n+1)}\right]_0^{\pi} = 0$$

$\therefore I_{n+2} = I_n$ . ... (iii)

Since, $I_n = \int_0^{\pi}\frac{\sin nx}{\sin x}dx \Rightarrow I_1 = \pi$ and $I_2 = 0$

$\therefore$ From Eq. (iii) $I_1 = I_3 = I_5 = ... = p$

and $I_2 = I_4 = I_6 = ... = 0$

$$\Rightarrow \sum_{m=1}^{10} I_{2m+1} = 10\pi \text{ and } \sum_{m=1}^{10} I_{2m} = 0$$

$\therefore$ Correct options are A, B, C.

Sol 7: (A) Let $I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$

$$= \int_0^1 \frac{(x^4-1)(1-x)^4 + (1-x)^4}{(1+x^2)}dx$$

$$= \int_0^1 (x^2-1)(1-x)^4 dx + \int_0^1 \frac{(1+x^2-2x)^2}{(1+x^2)}dx$$

$$= \int_0^1 \left\{(x^2-1)(1-x)^4 + (1+x^2) - 4x + 4 - \frac{4x^2}{(1+x^2)}\right\}dx$$

$$= \int_0^1 \left((x^2-1)(1-x)^4 + (1+x^2) - 4x + 4 - \frac{4}{1-x^2}\right)dx$$

$$= \int_0^1 \left(x^6 - 4x_1^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\right)dx$$

$$= \int_0^1 \left[\frac{x^7}{7} - \frac{4x^6}{6} + \frac{5x^5}{5} - \frac{4x^3}{3} + 4x - 4\tan^{-1}x\right]_0^1$$

$$= \frac{1}{7} - \frac{4}{6} + \frac{5}{5} - \frac{4}{3} + 4 - 4\left(\frac{\pi}{4} - 0\right) = \frac{22}{7} - \pi$$

= (cosec x.cot x + $\sec^2$x – cos x)

. ($\cos^3$x – cos x) . cos x

$$= -\left[\frac{\sin^2 x + \cos^3 x - \cos^3 x.\sin^2 x}{\sin^2 x.\cos^2 x}\right]$$

. $\cos^2$x . $\sin^2$x

= – $\sin^2$x – $\cos^3$x (1 – $\sin^2$x)

= – $\sin^2$x – $\cos^5$x

$$\therefore \int_0^{\pi/2} f(x)dx = -\int_0^{\pi/2}(\sin^2 x + \cos^5 x)dx$$

$$\left[\because \int_0^{\pi/2}\sin^m x.\cos^n x\,dx = \frac{\sqrt{\frac{m+1}{2}}\sqrt{\frac{n+2}{2}}}{2\sqrt{\frac{m+n+2}{2}}}\right]$$

$$\therefore \int_0^{\pi/2} f(x)dx = -\left\{\frac{\sqrt{\frac{3}{2}}.\sqrt{\frac{1}{2}}}{2\sqrt{2}} + \frac{\sqrt{\frac{6}{2}}.\sqrt{\frac{1}{2}}}{2\sqrt{\frac{7}{2}}}\right\}$$

Sol 8: (C) $\int_0^{\pi/2}\sin x\,dx$

$$= \frac{\frac{\pi}{2}-0}{4}\left(\sin 0 + \sin\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{0+\frac{\pi}{2}}{2}\right)\right)$$

$$= \frac{\pi}{8}(1+\sqrt{2})$$

Sol 9: (A) $F'(c) = (b-a)\,f'(c) + f(a) - f(b)$

$F''(c) = f''(c)\,(b-a) < 0$

$\Rightarrow F'(c) = 0$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

Sol 10: (B) Given, $\lim\limits_{t\to a}\dfrac{\int_a^t f(x)dx - \frac{(t-a)}{2}\{f(t)+f(a)\}}{(t-a)^3} = 0$

Using L' Hospital's rule

$$\Rightarrow \lim_{h\to 0}\frac{\int_a^{a+h} f(x)dx - \frac{h}{2}\{f(a+h)+f(a)\}}{h^3} = 0$$

$$\Rightarrow \lim_{h\to 0}\frac{f(a+h) - \frac{1}{2}\{f(a+h)+f(a)\} - \frac{h}{2}\{f'(a+h)\}}{3h^2} = 0$$

Again, using L'Hospital's rule

$$\Rightarrow \lim_{h\to 0}\frac{f'(a+h) - \frac{1}{2}f'(a+h) - \frac{1}{2}f'(a+h) - \frac{h}{2}f''(a+h)}{6h} = 0$$

$$\Rightarrow \lim_{h\to 0}\frac{-\frac{h}{2}f''(a+h)}{6h} = 0$$

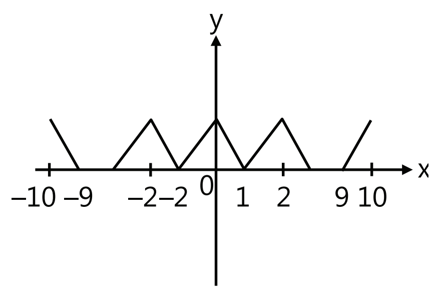
$\Rightarrow f''(a) = 0, \forall\ a \Rightarrow R$

$\Rightarrow$ f(x) must have maximum degree 1

Sol 11: Given, $f(x) = \begin{cases} x-[x] & \text{if } [x] \text{ is odd} \\ 1+[x]-x & \text{if } [x] \text{ is even}\end{cases}$

f(x) and cos πx both are periodic with period 2 and both are even.

$$\therefore \int_{-10}^{10} f(x)\cos\pi x dx = 2\int_0^{10} f(x)\cos\pi x dx$$



$$= 10\int_0^3 f(x)\cos\pi x\,dx$$

Now, $\int_0^1 f(x)\cos\pi x\,dx$

$$= \int_0^1 (1-x)\cos\pi x\,dx = -\int_0^1 u\cos\pi u\,du \text{ and}$$

$$\int_1^2 f(x)\cos\pi x\,dx = \int_1^2 (x-1)\cos\pi x\,dx = -\int_0^1 u\cos\pi u\,du$$

$$\therefore \int_{-10}^{10} f(x)\cos\pi x\,dx = -20\int_0^1 u\cos\pi u\,du = \frac{40}{\pi^2}$$

$$\Rightarrow \frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos\pi x\,dx = 4$$

Sol 12: $f(x) = \int_1^x \frac{\ln t}{1+t}dt$ for $x > 0$ (given)

Now, $f(1/x) = \int_1^{1/x}\frac{\ln t}{1+t}dt$

Put t = 1/u

$\Rightarrow dt = (-1/u^2)du$

$$\therefore f(1/x) = \int_1^x \frac{\ln(1/u)}{1+1/u}.\frac{(-1)}{u^2}du$$

$$= \int_1^x \frac{\ln u}{u(u+1)}du = \int_1^x \frac{\ln t}{t(1+t)}dt$$

Now, $f(x)+f\left(\frac{1}{x}\right)=\int_1^x \frac{\log t}{(1+t)}dt+\int_1^x \frac{\log t}{(1+t)}dt$

$$=\int_1^x \frac{(1+t)\log t}{t(1+t)}dt+\int_1^x \frac{x\log t}{t}dt$$

$$=\frac{1}{2}\left[(\log t)^2\right]_1^x=\frac{1}{2}(\log x)^2$$

Put x = e

$$\therefore\ f(e)+f\left(\frac{1}{e}\right)=\frac{1}{2}(\log e)^2=\frac{1}{2}$$

Hence proved.

Sol 13: Let $I=\int_0^{\pi/2} f(\cos 2x)\cos x\,dx$ ... (i)

$$I=\int_0^{\pi/2} f\left(\cos 2\left(\frac{\pi}{2}-x\right)\right)\cos\left(\frac{\pi}{2}-x\right)dx$$

$$\left[\text{using}\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$I=\int_0^{\pi/2} f(\cos 2x)\sin x\,dx$ ... (ii)

On adding Eqs. (i) and (ii), we get

$$2I=\int_0^{\pi/2} f(\cos 2x)(\sin x+\cos x)dx$$

$$=\sqrt{2}\int_0^{\pi/2} f(\cos 2x)[\cos(x-\pi/4)]dx$$

Put $-x+\frac{\pi}{4}=t\Rightarrow -dx=dt$

$$\therefore\ 2I=-\sqrt{2}\int_{\pi/4}^{-\pi/4} f\left(\cos\left(\frac{\pi}{2}-2t\right)\right)\cos t\,dt$$

$$\therefore\ 2I=\sqrt{2}\int_{-\pi/4}^{\pi/4} f(\sin 2t)\cot t\,dt$$

$$\therefore\ I=\sqrt{2}\int_0^{\pi/4} f(\sin 2t)\cos t\,dt$$

Sol 14:

Let $I=\int_{-\pi/3}^{\pi/3}\frac{\pi\,dx}{2-\cos\left(|x|+\frac{\pi}{3}\right)}+4\int_{-\pi/3}^{\pi/3}\frac{x^3\,dx}{2-\cos\left(|x|+\frac{\pi}{3}\right)}$

Using $\int_{-a}^{a} f(x)dx=\begin{cases}0, & f(-x)=-f(x)\\ 2\int_0^a f(x)dx, & f(-x)=f(x)\end{cases}$

$$\therefore\ I=2\int_0^{\pi/3}\frac{\pi\,dx}{2-\cos\left(|x|+\frac{\pi}{3}\right)}+0$$

$$\left[\frac{x^3dx}{2-\cos\left(|x|+\frac{\pi}{3}\right)}\text{ is odd}\right]$$

$$I=2\pi\int_0^{\pi/3}\frac{dx}{2-\cos(x+\pi/3)}$$

Put $x+\frac{\pi}{3}=t\Rightarrow dx=dt$

$$\therefore\ I=2\pi\int_{\pi/3}^{2\pi/3}\frac{dt}{2-\cos t}=2\pi\int_{\pi/3}^{2\pi/3}\frac{\sec^2\frac{t}{2}dt}{1+3\tan^2\frac{t}{2}}$$

Put $\tan\frac{t}{2}=u\Rightarrow \sec^2\frac{t}{2}dt=2du$

$$\Rightarrow I=2\pi\int_{1/\sqrt{3}}^{\sqrt{3}}\frac{2\,du}{1+3u^2}=\frac{4\pi}{3}[\sqrt{3}\tan^{-1}\sqrt{3}u]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$=\frac{4\pi}{\sqrt{3}}(\tan^{-1}3-\tan^{-1}1)=\frac{4\pi}{\sqrt{3}}\tan^{-1}\left(\frac{1}{2}\right)$$

$$\therefore\int_{-\pi/3}^{\pi/3}\frac{\pi+4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)}dx=\frac{4\pi}{\sqrt{3}}\tan^{-1}\left(\frac{1}{2}\right)$$

Sol 15: Let

$$I=\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\,dx$$

$$\Rightarrow I=\int_0^{\pi} e^{|\cos x|}.\sin x.2\sin\left(\frac{1}{2}\cos x\right)dx$$

$+\int_0^{\pi} e^{|\cos x|}.3\cos\left(\frac{1}{2}\cos x\right).\sin x\,dx$ ... (i)

$\Rightarrow I = I_1 + I_2$

$$\left(\text{using }\int_0^{2a} f(x)dx=\begin{cases}0, & f(2a-x)=-f(x)\\ 2\int_0^a f(x)dx, & f(2a-x)=+f(x)\end{cases}\right)$$

where $I_1 = 0$ [$\because f(\pi-x)=-f(x)$] ...(ii)

and

$$I_2=6\int_0^{\pi/2} e^{\cos x}.\sin x.\cos\left(\frac{1}{2}\cos x\right)dx$$

Now, $I_2 = 6\int_0^1 e^t.\cos\left(\frac{t}{2}\right)dt$

(Put cos x = t ⇒ – sin x dx = dt)

$= 6\left[e^t\cos\left(\frac{t}{2}\right) + \frac{1}{2}\int e^t \sin\frac{t}{2}dt\right]_0^1$

$= 6\left[e^t\cos\left(\frac{t}{2}\right) + \frac{1}{2}\left(e^t\sin\frac{t}{2} - \int\frac{e^t}{2}\cos\frac{t}{2}dt\right)\right]_0^1$

$= 6\left[e^t\cos\frac{t}{2} + \frac{1}{2}e^t\sin\frac{t}{2}\right]_0^1 - \frac{I_2}{4}$

$= \frac{24}{5}\left(e\cos\left(\frac{1}{2}\right) + \frac{e}{2}\sin\left(\frac{1}{2}\right) - 1\right)$ ... (iii)

From Eqs. (i), we get

$I = \frac{24}{5}\left(e\cos\left(\frac{1}{2}\right) + \frac{e}{2}\sin\left(\frac{1}{2}\right) - 1\right)$

Sol 16: Let $I_2 = \int_0^1 (1-x^{50})^{101}dx,$

using integration by parts

$= \left[(1-x^{50})^{101}.x\right]_0^1$

$+\int_0^1 (1-x^{50})^{100}\ 50.x^{49}.x\,dx$

$= 0 - \int_0^1 (50)(101)(1-x^{50})^{100}(-x^{50})dx$

$= -50(101)\int_0^1 (1-x^{50})^{101}dx + (50)(101)\int_0^1 (1-x^{50})^{100}dx$

$= 5050I_2 + 5050I_1$

$\therefore I_2 + 5050I_2 = 5050I_1$

$\therefore \frac{(5050)I_1}{I_2} = 5051$

Sol 17: (B) $g(x) = \frac{f'(e^x)e^x}{1+e^{2x}}$

Hence positive for $(0,\infty)$ and negative for $(-\infty,0)$

Consider the line

$L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}, L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$

Sol 18: (D) Hence $\int_{-1}^{1} g'(x) = g(1) - g(-1) = 2g(1)$

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F, respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

Further, it is given that the origin and the centre of C are on the same side of the line PQ.

Sol 19: (A) Let $f(x) = \int_0^x \frac{t^2}{1+t^4}dt = 2x - 1$

$f(x) = \frac{x^2 1 + x^4 - 2}{} \Rightarrow \frac{-2x^4 + x - 2}{x^4 + 1} < \forall x \in R$

$f(0) > 0,\ f(1) < 0$

∴ One solution in (0, 1)

Sol 20: $f(x) = \int_0^x f(t)dt \Rightarrow f(0) = 0$

Also $f(x) = f(x), x > 0 \Rightarrow f(x) = ke^x, x > 0$

$\because f(0) = 0$ and $f(x)$ is continuous

$\Rightarrow f(x) = 0 \forall x > 0$

$\therefore f(\text{In}5) = 0$

Sol 21: (C) $f' = \pm\sqrt{1-f^2}$

$\Rightarrow f(x) = \sin x \text{ or } f(x) = -\sin x$ (not possible)

$\Rightarrow f(x) = \sin x$

Also $x > \sin x \forall > 0$.

Sol 22: A → p, q, s; B → p, t; C → p, q, r, t; D → s

(A) $(x-3)^2\frac{dy}{dx} + y = 0$

$\int\frac{dx}{(x-3)^2} = \int\frac{dy}{y} \Rightarrow \frac{1}{x-3} = \text{In}|y| + c$

So domain is $R - \{3\}$.

(B) Put x = t + 3

$\int^2 (t+2)(t+1)t(t-1)(t-2)dt = \int^2 t(t^2-1)(t^2-4)dt = 0$

(C) $f(x) = \frac{5}{4} - \left(\sin x - \frac{1}{2}\right)^2$

Maximum value occurs when $\sin x = \frac{1}{2}$

(D) $f(x) > 0$ if $\cos > \sin x$

Sol 23: (A) Let the line be $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ intersects the lines

$\Rightarrow$ S.D = 0 $\Rightarrow a+3b+5c=0$ and

$3a+b-5c=0 \Rightarrow a:b:c::5r:-5r:2r$

on solving with given lines we get points of intersection

$P\equiv(5,-5,2)$ and $Q\equiv\left(\frac{10}{3},-\frac{10}{3},\frac{8}{3}\right)\Rightarrow PQ^2=d^2=6$

(B) (p, r)

$\tan^{-1}(x+3)-\tan^{-1}(x-3)=\sin^{-1}(3/5)$

$$\Rightarrow \tan^{-1}\frac{(x+3)-(x-3)}{1+(x^2-9)}=\tan^{-1}\frac{3}{4}\Rightarrow\frac{6}{x^2-8}=\frac{3}{4}$$

$\therefore x^2-8=8$

Or $x=\pm4$

(C) (q, s)

As $\vec{a}=\mu\vec{b}+4\vec{c}\Rightarrow\mu(|\vec{b}|)=-4\vec{b}.\vec{c}$ and $|\vec{b}|^2=4\vec{a}.\vec{c}$

and $|\vec{b}|^2+\vec{b}.\vec{c}-\vec{d}.\vec{c}=0$

Again, as $2|\vec{b}+\vec{c}|=|\vec{b}-\vec{a}|$

Solving and eliminating $\vec{b}.\vec{c}$ and eliminating $|\vec{a}|^2$

We get $(2\mu^2-10\mu)|\vec{b}|^2=0\Rightarrow\mu=0$ and 5.

(D) $$I=\frac{2}{\pi}\int_{-x}^{x}\frac{\sin9(x/2)}{\sin(x/2)}dx=\frac{2}{\pi}\times2\int_0^{\pi}\frac{\sin9(x/2)}{\sin(x/2)}dx$$

$x/2=\theta\Rightarrow dx=2b\theta$

$x=\pi\theta=\pi/2$

$$=\frac{8}{\pi}\int_0^{\pi/2}\frac{\sin9\theta-\sin7\theta}{\sin\theta}+\frac{(\sin7\theta-\sin5\theta)}{\sin\theta}$$

$$+\frac{(\sin5\theta-\sin3\theta)}{\sin\theta}+\frac{(\sin3\theta-\sin\theta)}{\sin\theta}+\frac{\sin\theta}{\sin}d\theta=\frac{16}{\pi}$$

$$\int_0^{\pi/2}(\cos8\theta+\cos6\theta+\cos4\theta+\cos2\theta+1)+\frac{8}{\pi}\int_0^{\pi/2}d\theta$$

$$=\frac{16}{\pi}\left[\frac{\sin8\theta}{8}+\frac{\sin6\theta}{6}+\frac{\sin4\theta}{4}+\frac{\sin2\theta}{2}\right]$$

$$+\frac{8}{\pi}[\theta]^{\pi/2}=0\frac{8}{\pi}\times\left[\frac{\pi}{2}-0\right]=4$$

Sol 24: (B) $$\lim_{x\to0}\frac{x\ell n(1+x)}{(x^4+4)\times3x^2}=\lim_{x\to0}\frac{1}{4}\times\frac{1}{3}=\frac{1}{12}$$

Sol 25: (A)

$$\int_0^1\frac{x^4(1-x)^1}{1+x^2}=\int_0^1\frac{x^{4\left[\left(1+x^2\right)-2x\right]^2}}{1+x^2}=$$

$$\int_0^1\frac{x^4(1-x)^1}{1+x^2}=\int_0^1\frac{x^{4\left[\left(1+x^2\right)-2x\right]^2}}{1+x^2}$$

$$=\int_0^1x^4\left[(1+x^2)-4x+\frac{4x^2}{1+x^2}\right]dx$$

$$=\int\left[x^6+x^4-4x^5+\frac{4x^6}{1+x^2}\right]dx$$

Now on polynomial division of x^6 by $1+x^2$, we obtain

$$\int\left[x^6+x^4-4x^5+4\left[(x^4-x^2+1)-\frac{1}{1+x^2}\right]\right]$$

$$dx=\int\left[(x^6-4x^5+5x^4-4x^2+4)-\frac{4}{1+x^2}\right]dx$$

$$=\left[\frac{x^7}{7}-\frac{4x^6}{6}+\frac{5.x^5}{5}-\frac{4x^3}{3}+4x\right]_0^1-4\left[\tan^{-x}x\right]$$

$$=\left(\frac{1}{7}-\frac{4}{6}+1\frac{4}{3}+4\right)-4\left(\frac{\pi}{4}\right)=\left(\frac{1}{7}+3\right)-\pi=\frac{22}{7}-\pi$$

Sol 26: (A) $x^2=t\Rightarrow2xdx=dt$

$$I=\frac{1}{2}\int_{In2}^{In3}\frac{\sin t}{\sin t+\sin(In6-t)}dt$$

and $$I=\frac{1}{2}\int_{In2}^{In3}\frac{\sin(In6-t)}{\sin t(In6-t)+\sin t}dt$$

$x^2=t\Rightarrow2xdx=dt$

$$2I=\frac{1}{2}\int_{In2}^{In3}1dt\Rightarrow I=\frac{1}{4}In\frac{3}{2}$$

Sol 27: (B)

$$\int_{-\pi/2}^{\pi/2}\left(x^2+\ell n\left(\frac{\pi+x}{\pi-x}\right)\right)\cos xdx=2\int_0^{\pi/2}x^2\cos xdx+0\quad\left(\because\ell n\left(\frac{\pi+x}{\pi-x}\right)\text{is an odd function}\right)$$

$$=2\left[(x^2\sin x)_0^{\pi/2}\int_0^{\pi/2}2x\sin xdx\right]=2\left(\frac{\pi^2}{4}-0\right)-4\int_0^{\pi/2}x\sin xdx$$

$$=\frac{\pi^2}{2}-4\left[(-x\cos x)^{\pi/2}{}_0+\int_0^{\pi/2}\cos xdx\right]=\frac{\pi^2}{2}-4$$

Sol 28: (A) $\int_{\pi/4}^{\pi/2}(2\operatorname{cosec}x)^{17}dx$

Let

$e^{n}+e^{-n}=2\operatorname{cosec}x, x=\frac{\pi}{4}\Rightarrow u\log(1+\sqrt{2}), x=\frac{\pi}{2}\Rightarrow u=0$

$\Rightarrow \operatorname{cosec}x+\cot x=e^{n}$ and

$x-\cot x=e^{-n}\Rightarrow \cot x=\frac{e^{u}-e^{-u}}{2}$

$(e^{u}-e^{-u})dx=-2\operatorname{cosec}x\cot x dx$

$\Rightarrow -\int(e^{u}+e^{-u})^{17}\frac{(e^{u}-e^{-u})}{2\operatorname{cosec}x\cot x}du$

$=-2\int_{\log(1+\sqrt{2})}^{0}(e^{u}+e^{-u})du=\int^{\log(1+\sqrt{2})}2(e^{u}+e^{u})du$

Sol 29: (D) (p) $f(x)=ax^2+bx, \int_0^1 f(x)dx=1$

$\Rightarrow 2a+3b=6$

$\Rightarrow (a,b)\equiv(0,2)\text{and}(3,0)$

(q) $f(x)=\sqrt{2}\cos\left(x^2-\frac{\pi}{4}\right)$

$x^2-\frac{\pi}{4}=2n\pi \Rightarrow x^2=2n\pi+\frac{\pi}{4}$

$\Rightarrow x=\pm\sqrt{\frac{\pi}{4}},\pm\sqrt{\frac{9\pi}{4}}$ as $x\in\left[-\sqrt{3},\sqrt{13}\right]$

(r) $\int_0^2\left(\frac{3x^2}{1+e^{x}}+\frac{3x^2}{1+e^{-x}}\right)dx=\int_0^2 3x^2dx=8$

(s) $\int_{-1/2}^{1/2}\cos 2x\ln\left(\frac{1+x}{1-x}\right)dx=0$ as it is an odd function

Sol 30: (2) $\int_0^1 4x^3\frac{d^2}{dx^2}(1-x^2)dx$

$=\left[4x^3\frac{d}{dx}(1-x^2)^5\right]_0^1-\int_0^1 12x^2\frac{d}{dx}(1-x^2)dx$

$=\left[4x^3\times5(1-x^2)^4(-2x)\right]_0^1-12\left[\left[x^2(1-x^2)^5\right]_0^1-\int_0^1 2x(1-x)^5dx\right]$

$=0-0-12[0-0]+12\int_0^1 2x(1-x^2)dx$

$=12\times\left[-\frac{(1-x^2)^6}{6}\right]_0^1=12\left[0+\frac{1}{6}\right]=2$

Sol 31: $\alpha=\int_0^1\left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$

Put $9x+3\tan^{-1}x=t \Rightarrow \left(9+\frac{3}{1+x^2}\right)dx=dt$

$\Rightarrow \alpha=\int_0^4 e^{t}dt=e^{9+\frac{3\pi}{4}}-1 \Rightarrow \left(\log_e|1+\alpha|-\frac{3\pi}{4}\right)=9$

Sol 32: (A, C) Let $\int_0^{\pi}e^{t}((\sin^6 at+\cos^4 at))dt=A$

$I=\int_{\pi}^{2\pi}e^{t}(\sin^6 at+\cos^4 at)dt$

Put $t=\pi+x$

$dt=dx$

For a = 2 as well as a = 4

$I=e^{x}\int_0^{\pi}e^{x}(\sin^6 ax+\cos^4 at)dt=e^{2\pi}A \Rightarrow I=e^{\pi}A$

Similarly $\int_0^{\pi}e^{\pi}e^{\pi}(\sin^6 at+\cos^4 at)dt=e^{2\pi}A$

$L=\frac{A+e^{\pi}A+e^{2\pi}A+e^{3\pi}A}{A}=\frac{e^{4\pi}-1}{e^{\pi}-1}$ $\therefore$ For both a = 2, 4

Sol 33: (A, B, C) (A) $f'(x)=F(x)+xF'((x)$

$f(1)=F(1)+F'(1)$

$f'(1)=F(1)<0$

(B) $f(2)=2F(2)$

$F(x)$ is decreasing and F(1) = 0

Hence F(2) < 0

$\Rightarrow f'(2)<0$

(C) $f(x)=F(x)+xF'(x)$

$F(x)<0\forall x\in(1,0)$

$F'(x) < 0 \forall x \in (1,3)$

Hence $f(x) < 0 \forall x \in (1,3)$

Sol 34: $I = \int_{-1}^{0} \frac{x.0}{2+0} dx + \int_{0}^{1} \frac{x.0}{2+1} dx + \int_{1}^{\sqrt{2}} \frac{x.1}{2+0} dx + 0 = \frac{1}{4}$

$\Rightarrow 4I - 1 = 0$

Sol 35: (A) $I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1+e^x} dx$ ... (i)

$I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1+\frac{1}{e^x}} dx$ ... (ii)

$= \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x.e^x}{1+e^x} dx$

(i) and (ii)

$2I \int_{-\pi/2}^{\pi/2} x^2 \cos x \, dx$

$I = \int_{0}^{\pi/2} x^2 \cos x \, dx$ (even fn)

$= x^2.\sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x \, dx$

$= \frac{\pi^2}{4} - 2\left[(-x \cos x)_0^{\pi/2} - \int_0^{\pi/2} (-\cos x) \, dx\right]$

$= \frac{\pi^2}{4} - 2\left[0 + \sin x \Big|_0^{\pi/2}\right] = \frac{\pi^2}{4} - 2[1] = \frac{\pi^2}{4} - 2$