DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

DEFINITIONS:

1. An equation that involves independent (usually '*x*' in maths) and '*t*' in physics and dependent variables (usually the 1st letter of the physical quantity) and the derivatives of the dependent variables is called a **DIFFERENTIAL EQUATION.**

2. A differential equation is said to be ordinary, if the differential coefficients have reference to a single independent variable only and it is said to be **PARTIAL** if there are two or more independent variables. We are concerned with ordinary differential equations only. While an ordinary differential equation containing two or more dependent variables with their differential coefficients w.r.t. to a single independent variable is called a <u>total differential equation</u>.

eg. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$ is an ordinary diffrential equation

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$; $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 + y$ are partial differential equation.

3. Finding the unknown function is called **SOLVING OR INTEGRATING** the differential equation. The solution of the differential equation is also called its **PRIMITIVE** , because the differential equation can be regarded as a relation derived from it .

4. **<u>Order and Degree of Diffrential Equation</u>**

<u>The order of a differential equation</u> is the order of the highest differential coefficient occuring in it.

<u>The degree of a differential equation</u> which is expressed or can be expressed as a polynomial in the derivatives is the degree of the highest order derivative occuring in it, after it has been expressed in a form free from radicals & fractions so far as derivatives are concerned, thus the differential equation :

5. **Formation Of A Differential Equation : (Geometric origin)**

If an equation in independent and dependent variables having some arbitrary constant is given , then a differential equation is obtained as follows :

☞ Differentiate the given equation say $f(x,y,c_1) = 0$ w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it.

☞ Eliminate the arbitrary constants.

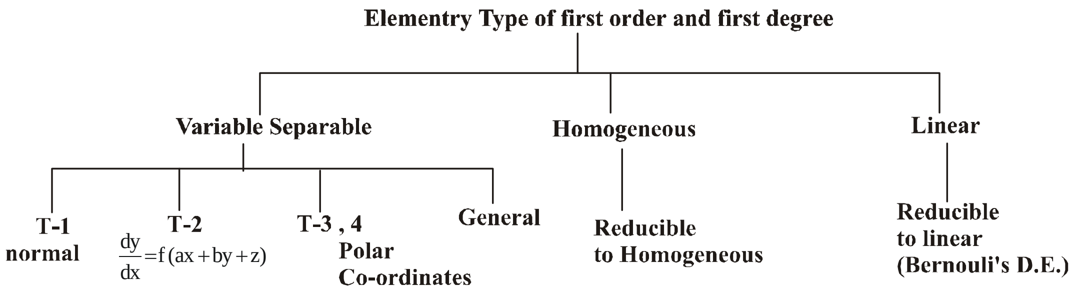
The eliminant is the required differential equation.

Note: A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

6. **General And Particular Solutions:**

The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the **GENERAL SOLUTION (OR COMPLETE INTEGRAL OR COMPLETE PRIMITIVE)**. A solution obtainable from the general solution by giving particular values to the constants is called a **PARTICULAR SOLUTION**.

7. Elementary Types Of First Order & First Degree Differential Equations.



VARIABLES SEPARABLE:

If the differential equation can be expressed as; $f(x)\,dx + g(y)\,dy = 0$ then this is said to be variable – separable type.

TYPE–1: A general solution of this is given by $\int f(x)\,dx + \int g(y)\,dy = c$; where c is the arbitrary constant.

TYPE–2: $\frac{dy}{dx} = f(ax + by + c)$, $b \neq 0$. (If $b = 0$ this is directly variable separable)

To solve this , substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved.

TYPE–3: $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ where $b_1 + a_2 = 0$

TYPE–4: Polar Coordinates

Sometimes transformation to the polar co–ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials. If $x = r\cos\theta$; $y = r\sin\theta$ where r and θ both are variable.

(i) $x\,dx + y\,dy = r\,dr$ **(ii)** $x\,dy - y\,dx = r^2\,d\theta$

If $x = r\sec\theta$ & $y = r\tan\theta$ then (i) $x\,dx - y\,dy = r\,dr$ and (ii) $x\,dy - y\,dx = r^2\sec\theta\,d\theta$.

Note: $x = r\cos\theta$; $y = r\sin\theta \Rightarrow x^2 + y^2 = r^2$ and $\tan\theta = y/x$

Hence $x\,dx + y\,dy = r\,dr$ and $x\,dy - y\,dx = x^2\sec^2\theta\,d\theta$

$x\,dy - y\,dx = r^2\,d\theta$

|||ly $x = r\sec\theta$ and $y = r\tan\theta$ $x^2 - y^2 = r^2$ and $y/x = \sin\theta$

Hence $x\,dx - y\,dy = r\,dr$ and $x\,dy - y\,dx = x^2\cos\theta\,d\theta = r^2\sec\theta\,d\theta$

8. HOMOGENEOUS EQUATIONS:

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$, where $f(x, y)$ & $\phi(x, y)$ are homogeneous functions of x & y , and of the same degree , is called **HOMOGENEOUS**.

This equation may also be reduced to the form $\frac{dy}{dx} = g\left(\frac{x}{y}\right) \Big/ g\left(\frac{y}{x}\right)$ & is solved by putting $y = vx$ so that the dependent variable y is changed to another variable v , where v is some unknown function, the differential equation is transformed to an equation with variables separable.

IMPORTANT NOTE:

(a) The function $f(x, y)$ is said to be a homogeneous function of degree n if for any real number $t (\neq 0)$, we have $f(tx, ty) = t^n f(x, y)$.
e.g. $f(x, y) = ax^{2/3} + hx^{1/3} \cdot y^{1/3} + by^{2/3}$ is a homogeneous function of degree 2/3.

(b) A differential equation of the form $\frac{dy}{dx} = f(x, y)$ is homogeneous if $f(x, y)$ is a homogeneous function of degree zero i.e. $f(tx, ty) = t^{\circ} f(x, y) = f(x, y)$. The function f does not depend on x & y separately but only on their ratio $\frac{y}{x}$ or $\frac{x}{y}$.

Equations Reducible To The Homogeneous Form :

If $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$; where $a_1 b_2 - a_2 b_1 \neq 0$ and $a_2 + b_1 \neq 0$, i.e. $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$
then the substitution $x = u + h$, $y = v + k$ transform this equation to a homogeneous type in the new variables u and v where h and k are arbitrary constants to be chosen so as to make the given equation homogeneous which can be solved by the method as given in Type – 3. If

(i) $a_1 b_2 - a_2 b_1 = 0$, then a substitution $u = a_1 x + b_1 y$ transforms the differential equation to an equation with variables separable. and

(ii) $b_1 + a_2 = 0$, then a simple cross multiplication and substituting $d(xy)$ for $x\,dy + y\,dx$ & integrating term by term yields the result easily.

Purpose of Crash Course :

1. **Finishing your unfinished sheet.**
2. **Quick Revision of *all topics* in chapter.**
3. ***Target is Problems* ; Not Good or Bad problems.**
4. **Giving mental training to keep time a prime focus.**
5. **Regaining Your Confidence.**
6. **Completing more than 1500 questions in just 40 - 45 days.**
7. **Covering all institutes Exercise 1 , 2 along with Past Year problems**

9. LINEAR DIFERENTIAL EQUATIONS :

A differential equation is said to be linear if the dependent variable & all its differential coefficients occur in degree one only and are never multiplied together.

The nth order linear differential equation is of the form ;

$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_n(x) \cdot y = \phi(x)$. Where $a_0(x)$, $a_1(x)$ ..... $a_n(x)$ are the coefficients of the differential equation.

Note that a linear differential equation is always of the first degree but every differential equation of the first degree need not be linear. e.g. the differential equation $\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$ is not linear , though its degree is 1.

Linear Differential Equations Of First Order :

The most general form of a linear differential equations of first order is $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x. (Independent variable)

[P and Q are functions of independent variable]

In physics it will look like $\frac{ds}{dt} + f(t).s = g(t)$

To solve such an equation multiply both sides by $e^{\int P dx}$ or $e^{\int f(t) dt}$

10. Equations Reducible To Linear Form : (Bernoulli's Equation)

The equation $\frac{dy}{dx} + py = Q \cdot y^n$ where P & Q functions of x , is reducible to the linear form by dividing it by y^n & then substituting $y^{-n+1} = Z$. Its solution can be obtained as in the normal case.

11. CHANGE OF VARIABLE BY A SUITABLE SUBSTITUTION :

(i) $y \sin x \frac{dy}{dx} = \cos x (\sin x - y^2)$ put $y^2 = z$ $\Rightarrow$ $2y\frac{dy}{dx} = \frac{dz}{dx}$

$\sin x \frac{dz}{dx} = 2 \cos x (\sin x - z)$

$\frac{dz}{dx} = 2 \cos x - 2 \cot x \, z$ or $\frac{dz}{dx} + 2 \cot x \, z = 2 \cos x$ which is linear.

(ii) $\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3 \cos^2 y$. [Hint: multiply by $\sec^2 y$ and put $\tan y = t$]

(iii) $\frac{dy}{dx} - \frac{\tan y}{x+1} = (1+x)e^x \sec y$ [Hint: multiply by $\cos y$ and put $\sin y = t$]

Note: FOLLOWING EXACT DIFFERENTIALS MUST BE REMEMBERED :

(i) $x\,dy + y\,dx = d(xy)$

(ii) $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$

(iv) $\dfrac{x\,dy + y\,dx}{xy} = d(\ln xy)$

(v) $\dfrac{dx + dy}{x + y} = d(\ln(x+y))$

(vi) $\dfrac{x\,dy - y\,dx}{xy} = d\left(\ln\dfrac{y}{x}\right)$

(vii) $\dfrac{y\,dx - x\,dy}{xy} = d\left(\ln\dfrac{x}{y}\right)$

(viii) $\dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(ix) $\dfrac{y\,dx - x\,dy}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$

(x) $\dfrac{x\,dx + y\,dy}{x^2 + y^2} = d\left[\ln\sqrt{x^2 + y^2}\right]$

(xi) $d\left(-\dfrac{1}{xy}\right) = \dfrac{x\,dy + y\,dx}{x^2y^2}$

(xii) $d\left(\dfrac{e^x}{y}\right) = \dfrac{y\,e^x\,dx - e^x\,dy}{y^2}$

(xiii) $d\left(\dfrac{e^y}{x}\right) = \dfrac{x\,e^y\,dy - e^y\,dx}{x^2}$

1. The order and degree of the differential equation

$\left(1+3\frac{dy}{dx}\right)^{\frac{2}{3}}=4\frac{d^3y}{dx^3}$ are

(A) $1, \frac{2}{3}$ (B) 3, 1

(C) 1, 2 (D) 3, 3

2. The order and degree of the differential equation

$\sqrt[3]{\frac{dy}{dx}}-4\frac{d^2y}{dx^2}-7x=0$ are a and b, then a + b is

(A) 3 (B) 4

(C) 5 (D) 6

3. The degree of the differential equation

$\left(\frac{d^3y}{dx^3}\right)^{2/3}+4-3\frac{d^2y}{dx^2}+5\frac{dy}{dx}=0$ is

(A) 1 (B) 2

(C) 3 (D) None

4. The degree and order of the differential equation of the family of all parabolas whose axis is x - axis, are respectively-

(A) 2 , 3 (B) 2, 1

(C) 1, 2 (D) 3, 2

5. If $y=e^{(K+1)x}$ is a solution of differential equation

$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$, then k equals

(A) –1 (B) 0

(C) 1 (D) 2

6. The differential equation, which represents the family of plane curves $y=e^{cx}$, is-

(A) y' = cy (B) xy' – log y = 0

(C) x log y = yy' (D) y log y = xy'

7. The solution of $\frac{xdy}{x^2+y^2}=\left(\frac{y}{x^2+y^2}-1\right)dx$ is

(A) $y=x\cot(c-x)$ (B) $\cos^{-1}y/x=-x+c$

(C) $y=x\tan(c-x)$ (D) $y^2/x^2=x\tan(c-x)$

8. The equation of the curve through the point (1, 0), whose slope is $\frac{y-1}{x^2+x}$, is-

(A) (y – 1) (x + 1)+2x = 0

(B) 2x (y – 1) + x + 1 = 0

(C) x (y – 1) (x + 1)+2 = 0

(D) x (y + 1) + y (x + 1) = 0

9. A curve passing through (2, 3) and satisfying the differential equation $\int_0^x ty(t)dt=x^2y(x),(x>0)$ is

(A) $x^2+y^2=13$ (B) $y^2=\frac{9}{2}x$

(C) $\frac{x^2}{8}+\frac{y^2}{18}=1$ (D) xy = 6

10. The solution of the equation

$x\frac{dy}{dx}=y-x\tan\left(\frac{y}{x}\right)$ is-

(A) $x\sin\left(\frac{x}{y}\right)+c=0$ (B) x sin y + c = 0

(C) $x\sin\left(\frac{y}{x}\right)=c$ (D) None of these

11. The solution of the differential equation

$x^2\frac{dy}{dx}=x^2+xy+y^2$ is-

(A) $\tan^{-1}\left(\frac{y}{x}\right)=\log x+c$

(B) $\tan^{-1}\left(\frac{y}{x}\right)=-\log x+c$

(C) $\sin^{-1}\left(\frac{y}{x}\right)=\log x+c$

(D) $\tan^{-1}\left(\frac{x}{y}\right)=\log x+c$

12. The equation of the curve passing through origin and satisfying the differential equation

$\frac{dy}{dx}=\sin(10x+6y)$ is

(A) $y=\frac{1}{3}\tan^{-1}\left(\frac{5\tan 4x}{4-3\tan 4x}\right)-\frac{5x}{3}$

(B) $y=\frac{1}{3}\tan^{-1}\left(\frac{5\tan 4x}{4+3\tan 4x}\right)-\frac{5x}{3}$

(C) $y=\frac{1}{3}\tan^{-1}\left(\frac{3+\tan 4x}{4-3\tan 4x}\right)-\frac{5x}{3}$

(D) none of these

13. The solution of $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$ is -

(A) $xe^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + c$

(B) $xe^{\tan^{-1} y} = (\tan^{-1} y + 1) - c$

(C) $xe^{\tan^{-1} y} = (\tan^{-1} y - 1) + c$

(D) None of these

14. The solution of the differential equation $(2x - 10y^3)\frac{dy}{dx} + y = 0$ is

(A) $x + y = ce^{2x}$ (B) $y^2 = 2x^3 + c$

(C) $xy^2 = 2y^5 + c$ (D) $x(y^2 + xy) = 0$

15. The solution of the differential equation

$(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$, is-

(A) $x\, e^{2\tan^{-1} y} = e^{\tan^{-1} y} + k$

(B) $(x - 2) = k\, e^{-\tan^{-1} y}$

(C) $2x\, e^{\tan^{-1} y} = e^{2\tan^{-1} y} + k$

(D) $x\, e^{\tan^{-1} y} = \tan^{-1} y + k$

16. The solution of the differential equation,

$x^2 \frac{dy}{dx} . \cos\frac{1}{x} - y \sin\frac{1}{x} = -1$, where $y \to -1$ as $x \to \infty$ is

(A) $y = \sin\frac{1}{x} - \cos\frac{1}{x}$ (B) $y = \frac{x+1}{x\sin\frac{1}{x}}$

(C) $y = \cos\frac{1}{x} + \sin\frac{1}{x}$ (D) $y = \frac{x+1}{x\cos\frac{1}{x}}$

17. Solution of $y \log y\, dx - x dy = 0$ is

(A) $y = e^{cx}$ (B) $y = e^{-cx}$

(C) $y = \log x$ (D) None

18. Solution of $x \cos y\, dy = (xe^x \log x + e^x)\, dx$ is

(A) $\sin y = \log x + c$ (B) $\sin y = e^{-x}\log x + c$

(C) $\sin y = e^x \log x + c$ (D) None

19. Solve the differential equation

$(2xy - 3x^2)dx + (x^2 - 2y)dy = 0$

(A) $x^2y - x^3 - y^2 = c$ (B) $x^2y + x^3 + y^2 =$ (C) $x^2y + x^3 - y^2 = c$ (D) None of these

20. Find the particular solution of $(\cos x - x \sin x + y^2)dx + 2xy\, dy = 0$

that satisfies the initial condition $y = 1$ when $x = \pi$

(A) $xy^2 + x\cos x = 0$ (B) $xy^2 - x\cos x = 0$

(C) $xy^2 + x\sin x = 0$ (D) $x^2y + x \sin x = 0$

21. Orthogonal trajectories of family of the curve $x^{2/3} + y^{2/3} = a^{2/3}$, where a is any arbitrary constant, is

(A) $x^{2/3} - y^{2/3} = c$ (B) $x^{4/3} - y^{4/3} = c$

(C) $x^{4/3} + y^{4/3} = c$ (D) $x^{1/3} - y^{1/3} = c$

22. The curve for which the normal at any point (x,y) and the line joining origin to that point form an isosceles triangle with the x-axis as base is

(A) an ellipse

(B) a rectangular hyperbola

(C) a circle (D) none of these

23. Which of the following transformation reduce the differential equation

$\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$ into the form

$\frac{du}{dx} + P(x)\, u = Q(x)$?

(A) $u = \log z$ (B) $u = e^z$

(C) $u = (\log z)^{-1}$ (D) $u = (\log z)^2$

24. Number of straight lines which satisfy the differential equation $\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 - y = 0$ is

(A) 1 (B) 2

(C) 3 (D) 4

25. The differential equation of the family of circles with fixed radius 5 units and centre on the line $y = 2$ is **[AIEEE 2008]**

(A) $(y - 2)\, y'^2 = 25 - (y - 2)^2$

(B) $(y - 2)^2\, y'^2 = 25 - (y - 2)^2$

(C) $(x - 2)^2\, y'^2 = 25 - (y - 2)^2$

(D) $(x - 2)\, y'^2 = 25 - (y - 2)^2$

26. The differential equation which represents the family of curves $y = c_1 e^{c_2 x}$ where c_1 and c_2 are arbitrary constants, is - **[AIEEE 2009]**

(A) $y' = y^2$ (B) $y'' = y' y$

(C) $yy'' = y'$ (D) $yy'' = (y')^2$

27. Solution of the differential equation cos x dy = y (sin x – y) dx, $0 < x < \frac{\pi}{2}$ is - **[AIEEE 2010]**

(A) sec x = (tan x + c) y

(B) y sec x = tan x + c

(C) y tan x = sec x + c

(D) tan x = (sec x + c) y

28. Let I be the purchase value of an equipment and V(t) be the value after it has been used for t years. The value V(t) depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k\,(T-t)$, where k>0 is a constant and T is the total life in years of the equipment. Then the scrap value V(T) of the equipment is : **[AIEEE 2011]**

(A) $T^2 - \frac{1}{k}$ (B) $I - \frac{kT^2}{2}$

(C) $I - \frac{k(T-t)^2}{2}$ (D) e^{-kT}

29 The population p(t) at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt} = 0.5\,p(t) - 450$. If p(0) =850, then the time at which the population becomes zero is: **[AIEEE 2012]**

(A) $\frac{1}{2}\ln 18$ (B) ln 18

(C) 2 ln 18 (D) ln 9

30. At present, a firm is manufacturing 2000 items. It is estimated that rate of change of production P w.r.t additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is : **[JEE Main 2013]**

(A) 3500 (B) 4500 (C) 2500 (D) 3000

31. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$. **[JEE Main 2014]**

If p(0) = 100, then p(t) equals :

(1) $400 - 300\,e^{t/2}$ (2) $300 - 200\,e^{-t/2}$

(3) $600 - 500\,e^{t/2}$ (4) $400 - 300\,e^{-t/2}$

32. Let y(x) be the solution of the differential equation $(x \log x)\frac{dy}{dx} + y = 2x \log x$, $(x \geq 1)$. Then y(e) is equal to : **[JEE Main 2015]**

(A) 2 (B) 2e (C) e (D) 0

33. If a curve y = f(x) passes through the point (1, –1) and satisfies the differential equation, y(1 + xy) dx = x dy, then $f\left(-\frac{1}{2}\right)$ is equal to: **[JEE Main 2016]**

(A) $-\frac{4}{5}$ (B) $\frac{2}{5}$ (C) $\frac{4}{5}$ (D) $-\frac{2}{5}$

34. If $(2 + \sin x)\frac{dy}{dx} + (y + 1)\cos x = 0$ and y(0) = 1, then $y\left(\frac{\pi}{2}\right)$ is equal to : **[JEE Main 2017]**

(A) $\frac{1}{3}$ (B) $-\frac{2}{3}$ (C) $-\frac{1}{3}$ (D) $\frac{4}{3}$