20 Definite Integral and Area

NITTY-GRITTY

The integral is an important concept in mathematics. Integration is one of the two main operations in calculus, with its inverse, differentiation, being the other. Given a function f of a real variable x and an interval $[a, b]$ of the real line, the definite integral is defined informally as the signed area of the region in the xy-plane that is bounded by the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$. The area above the x-axis adds to the total and that below the x-axis subtracts from the total.

If the underlying theory of integration is not important, dx can be seen as strictly a notation indicating that x is a dummy variable of integration; if the integral is seen as a Riemann integral, dx indicates that the sum is over subintervals in the domain of x; in a Riemann–Stieltjes integral, it indicates the weight applied to a subinterval in the sum; in Lebesgue integration and its extensions, dx is a measure, a type of function which assigns sizes to sets; in non-standard analysis, it is an infinitesimal; and in the theory of differentiable manifolds, it is often a differential form, a quantity which assigns numbers to tangent vectors. An important connection is made in between a rate of change (e.g. rate of growth) and the total change (i.e. the net change resulting g from all the accumulation and loss over a time span). We show that such examples also involve the concept of integration, which, fundamentally, is a cumulative summation of infinitesimal changes.

Integrals appear in many practical situations. If a swimming pool is rectangular with a flat bottom, then from its length, width, and depth we can easily determine the volume of water it can contain (to fill it), the area of its surface (to cover it), and the length of its edge (to rope it). But if it is oval with a rounded bottom, all of these quantities call for integrals. Practical approximations may suffice for such trivial examples, but precision engineering (of any discipline) requires exact and rigorous values for these elements.

The actual definition of 'integral' is as a limit of sums, which might easily be viewed as having to do with area. One of the original issues integrals were intended to address was computation of area. So a definite integral is just the difference of two values of the function given by an indefinite integral.

Note

Integration is vital to many scientific areas. Many powerful mathematical tools are based on integration. Differential equations for instance are the direct consequence of the development of integration.

So what is integration? Integration stems from two different problems. The more immediate problem is to find the inverse transform of the derivative. This concept is known as finding the antiderivative. The other problem deals with areas and how to find them. The bridge between these two different problems is the Fundamental Theorem of Calculus.

What is the "**area problem**"? We want to find the area of a given region in the plane. It is not hard to see that the problem can be reduced to finding the area of the region bounded above by the graph of a positive function $f(x)$, bounded below by the x-axis, bounded to the left by the vertical line $x = a$, and to the right by the vertical line $x = b$.

Properties of definite integral

- $\int_a^b f(x)dx = \int_a^b f(z)dz$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$.

TIP: This property must be used for integrating functions of piecewise definition.

- $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
- $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if even, i.e., $f(-x) = f(x)$

  0 if $f(x)$ is odd, i.e., $f(-x) = -f(x)$

- $\int_0^{na} f(x)dx = n\int_0^a f(x)dx$ if $f(x)$ is a periodic function of the period a, i.e., $f(a+x)=f(x)$.
- $\int_{ma}^{na} f(x)dx = (n-m)\int_0^a f(x)dx$ if $f(x)$ is a periodic function of the period a.

Derivative of indefinite integral

- If $y=\int_a^x f(t)dt$ then $\frac{dy}{dx} = f(t)\Big]_{t=x} = f(x)$
- If $y=\int_a^{\phi(x)} f(t)dt$ then $\frac{dy}{d\phi} = f(t)\Big]_{t=\phi(x)} = f\{\phi(x)\}$

Inequalities in definite integrals

- If $f(x)\ge 0$ for all $x\in[a,b]$ then $\int_a^b f(x)dx \ge 0$, equality holding if $f(x)=0$ at all points of $[a, b]$.
- $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$
- If $f(x)\ge g(x)$ for all $x\in[a,b]$ then $\int_a^b f(x)dx \ge \int_a^b f(x)dx \ge \int_a^b g(x)dx$, equality holding if $f(x)=g(x)$ at all points of $[a, b]$.
- If minimum $f(x)=m$, maximum $f(x)=M$ in $[a,b]$ then $m(b-a)\le \int_a^b f(x)dx \le M(b-a)$, equality holding for constant functions.

Average value of a function over an interval

- The average value of $f(x)$ over the interval $[a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx$

  For example: The average value of $y=\sin x$ over the interval $[0,\pi] = \frac{1}{\pi-0}\int_0^\pi \sin x dx$

  $= \frac{1}{\pi}[-\cos x]_0^\pi = \frac{1}{\pi}(1+1) = \frac{2}{\pi}$.

Definition of definite integral

- The definite integral of $f(x)$ over the interval $[a, b]$, denoted by $\int_a^b f(x)dx$, is defined as the limit of a sum as follows :

  $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{r=1}^{n} hf(a+rh)$ or $\lim_{n\to\infty}\sum_{r=0}^{n-1} hf(a+rh)$, where $nh=b-a$

  $f(x)$ is said to be integrable over $[a, b]$ if the above two limits exist and are equal.

Fundamental theorem of definite integration

- $\lim_{n\to\infty}\sum_{r=1}^{n} hf(a+rh)$ *or* $\lim_{n\to\infty}\sum_{r=0}^{n-1} hf(a+rh) = F(b)-f(a)$

  where $nh=b-a$ and $F'(x)=f(x)$.

  Consequently, $\int_a^b f(x)dx = F(b)-F(a)$, i.e., $[F(x)]_{x=a}^{x=b}$

  Where $F'(x)=f(x)$, i.e, $F(x)$ is the primitive function of $f(x)$.

  TIP: In computing a definite integral $\int_a^b f(x)dx$ where $f(a)=f(b)$, it is convenient to break the interval in two parts and use $\int_a^b f(x)dx = \int_a^{\frac{a+b}{2}} f(x)dx + \int_{\frac{a+b}{2}}^b f(x)dx$.

Evaluation of integral from the first principle

- To find the value of $\int_a^b f(x)dx$ from the first principle i.e., definition, obtain $\lim_{n\to\infty}\sum_{r=1}^{n} hf(a+rh)$ or $\lim_{n\to\infty}\sum_{r=0}^{n-1} hf(a+rh)$ where $nh=b-a$.
- $\int_0^1 f(x)dx = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}f\left(\frac{r}{n}\right)$ or $\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n}f\left(\frac{r}{n}\right)$

Limit of a sum as definite integral

- $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{b-a}{n}f\left(a+r\cdot\frac{b-a}{n}\right)$

or $\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{b-a}{n}f\left(a+r\cdot\frac{b-a}{n}\right) = \int_a^b f(x)dx$

- $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}f\left(\frac{r}{n}\right)$ or $\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n}f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$
- $\lim_{n\to\infty}\sum_{r=1}^{pn}\frac{1}{n}f\left(\frac{r}{n}\right) = \int_\alpha^\beta f(x)dx$

  where $\alpha = \lim_{n\to\infty}\frac{r}{n}$ when $r=1$,

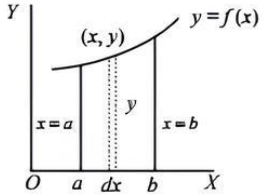
  $\beta = \lim_{n\to\infty}\frac{r}{n}$ when $r=pn$

Standard areas

- The area bounded by the curve $y = f(x)$, the x-axis, ($i.e., y = 0$) the ordinates $x = a, x = b (b > a)$ is given by

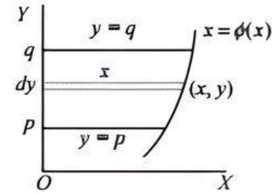
  area = $\int_a^b y dx$

  where y is to be expressed in terms of x from the equation of the curve $y = f(x)$



- The area bounded by the curve $x = \phi(y)$, the y-axis ($i.e., x = 0$), the lines (abscissae) $y = p, y = q, (q > p)$ is given by area = $\int_p^q x dy$

  where x is to be expressed in terms of y from the equation of the curve $x = \phi(y)$.

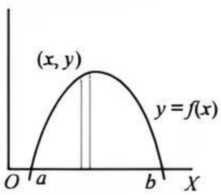


- The area bounded by a curve $y = f(x)$ and the x-axis is given by

  area = $\int_a^b y dy = \int_a^b (y)_{y=f(x)} dx$

  $\int_a^b f(x) dx$

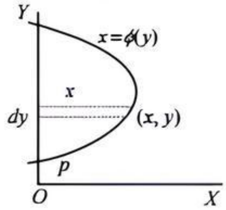
  where $y = f(x)$ cuts the x-axis at (a, 0) and (b, 0)



- The area bounded by a curve $x = \phi(y)$ and the y-axis is given by

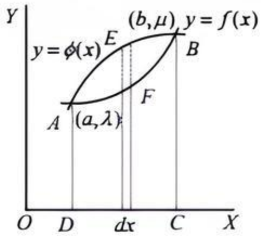
  area = $\int_p^q x dy = \int_p^q (x)_{x=\phi(y)} dy = \int_p^q \phi(y) dy$

  where $x = \phi(y)$ cuts the y-axis at (0, p) and (0, q)



Nonstandard areas

An area may be bounded by two curves, or a curve and a line, or two curves and a line etc. such areas are not standard areas. In order to compute such areas, do the following :



- Divide the whole area into a number of standard areas whose algebraic sum (i.e., addition/substraction/or both) gives the required area. Some nonstandard areas are given below. Observe their computation.

  The area bounded by the curve $y = f(x)$ and $y = \phi(x)$ is given by

  area = $\int_a^b \{f(x) - \phi(x)\} dx$

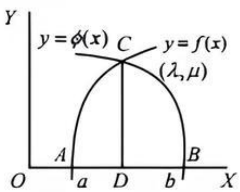
  = ar($AEBCDA$) − ar($AFBCDA$),

  where the x-coordinates of the points of intersection of the curves $y = f(x)$ and $y = \phi(x)$ are a and b.

  The area bounded by the curves $y = f(x)$

and $y = \phi(x)$, and the x-axis is given by

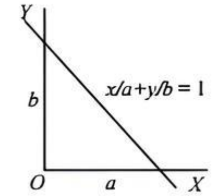
area = $\int_a^\lambda f(x) dx + \int_\lambda^b \phi(x) dx$ ar($ACDA$) + ar($CDBC$),

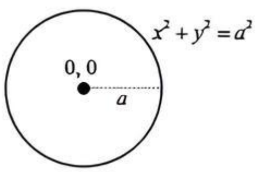


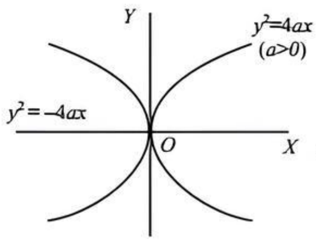
where the points of intersection of the curve with the x-axis are (a, 0) and (b, 0) respectively and the two curves intersect at (λ, μ).

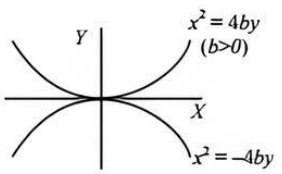
Sketches of curves: Clearly, in order to decide the limits of integration and sum or difference of standard areas to compute nonstandard areas, it is essential to have a rough idea about the shape and orientation of the given curves.

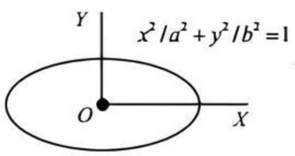
- **Sketches of standard curves**

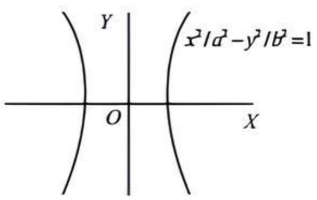


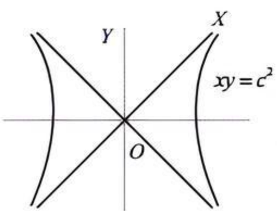


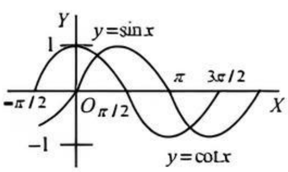


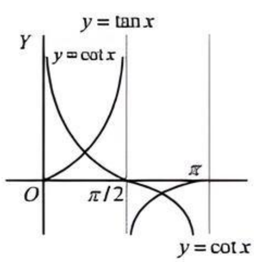


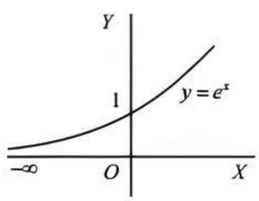


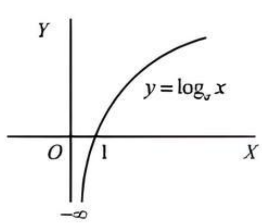








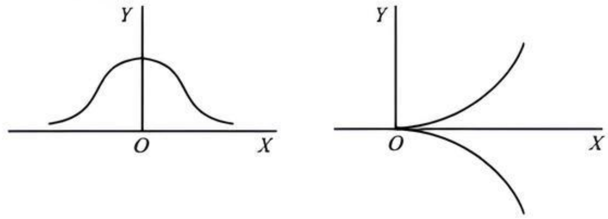




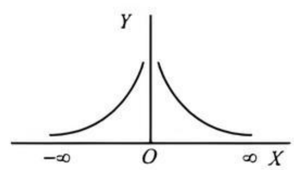
- **Sketches of non-standard curves**

When the curve represented by the equation is not a known curve we take the following steps to get a rough sketch of the curve.

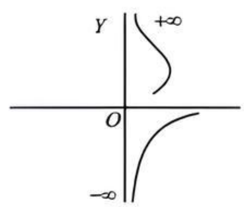
(a) Symmetry about the axes– If the equation of the curve contains only even powers of x then the curve is symmetrical about the y-axis, Again, if only even powers of y appear in the equation then the curve is symmetrical about the x-axis.



(b) Points of intersection with the axes– Solve the equation of the curve and $y = 0$ to find the points of intersection of the curve with the x-axis. If we get only one value of x, say a, the curve will cut the x-axis at one point only. If $x \to \infty$ when $y = 0$ then the curve will meet the x-axis at infinity. This fact is shown in the sketch by drawing the end part of the sketch parallel to the positive side of the x-axis. Similarly on the other side of $x \to \infty$ when $y = 0$.

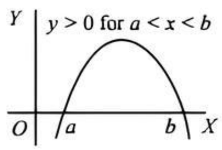


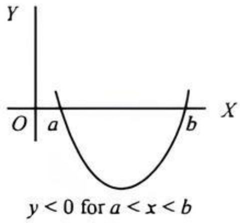
Solve the equation of the curve and $x = 0$ to find the points of intersection of the curve with the y-axis. If $y \to \infty$ (or $y \to -\infty$) where $x = 0$ then the curve will meet the y-axis at infinity.



(c) Trend of values– Observe the effect on values of y when x changes. See the sign of y for values of x in different intervals. If $y > 0$ for $a < x < b$, the curve will be above the x-axis between the points $(a, f(a))$ and $(b, f(b))$. Instead, if $y < 0$ then the curve will be below the x-axis.

If y goes on increasing when $a<x<b$, then portion of the graph in $[a, b]$ will be rising lower to higher. But if y goes on decreasing in $a<x<b$, the graph will be falling higher to lower.





Observe the effect on values of x when y changes and make similar conclusions.

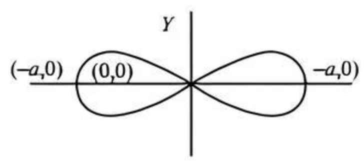
For example: Rough sketch of the curve $y^2(a^2+x^2)=x^2(a^2-x^2)$.

Here only even powers of x as well as y appear. So the curve is symmetrical about both the axes.

Put $y=0$ in the equation of the curve.

We get $0=x^2(a^2-x^2)$

i.e., $x=0, a, -a$.



So, the curve cuts the x-axis at (0, 0), $(a, 0)$ and $(-a, 0)$.

Similarly, putting $x=0$ we get $y=0$;

so, the curve cuts the y-axis only at (0, 0).

As $y^2=\dfrac{x^2(a^2-x^2)}{a^2+x^2}$, we get, for $x>a$ or $x<-a$ the value of y^2 is negative and so y will not be real. Thus, we find the curve cannot go the right of $x=a$ or to the left of $x=-a$. Hence, the rough sketch of the curve is as given below:

Illustrations

Illustration 1: $\int_0^a \frac{dx}{a\sqrt{a^2-x^2}}$ is equal to?

Solution: Let $I=\int_0^a \frac{dx}{a+\sqrt{a^2-x^2}}=\int_0^{\pi/2}\frac{a\cos\theta\, d\theta}{a+a\cos\theta}$

[Putting $x=a\sin\theta \Rightarrow dx=a\cos\theta\, d\theta$]

$$=\int_0^{\pi/2}\frac{\cos\theta}{1+\cos\theta}d\theta=\int_0^{\pi/2}\left(1-\frac{1}{1+\cos\theta}\right)d\theta$$

$$=\int_0^{\pi/2}1\,d\theta-\frac{1}{2}\int_0^{\pi/2}\sec^2\frac{\theta}{2}\,d\theta$$

$$=\left[\theta-\tan\frac{\theta}{2}\right]^{\pi/2}$$

$$=\frac{\pi}{2}-\tan\frac{\pi}{4}=\left(\frac{\pi}{2}-1\right)$$

Illustration 2: $\int_0^{\pi/2}(\tan x+\cot x)dx$ is equal to?

Solution: Let $I=\int_0^{\pi/2}\log(\tan x+\cot x)\,dx$

$$=\int_0^{\pi/2}\log\left(\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right)dx$$

$$=\int_0^{\pi/2}\log\left(\frac{2}{\sin 2x}\right)dx$$

$$=\log 2\int_0^{\pi/2}1\,dx-\int_0^{\pi/2}\log\sin 2x\,dx$$

$$=\frac{\pi}{2}\log 2-\frac{1}{2}.\int_0^{\pi}\log\sin z\,dz$$

$$\left(\text{Putting } 2x=z\Rightarrow dx=\frac{1}{2}dz\right)$$

$$=\frac{\pi}{2}\log 2-\frac{1}{2}.2\int_0^{\pi/2}\log\sin z\,dz=\frac{\pi}{2}\log 2+\frac{\pi}{2}\log 2=\pi\log 2$$

Illustration 3: The value of the integral $\int_0^{\pi}\frac{\sin 2kx}{\sin x}dx$, where $k\in I$, is?

Solution: Let $I=\int_0^{\pi}\frac{\sin 2kx}{\sin x}\,dx=\int_0^{\pi}\frac{\sin 2k(\pi-x)}{\sin(\pi-x)}dx$

$$=\int_0^{\pi}\frac{\sin(2k\pi-2kx)}{\sin x}\,dx=-\int_0^{\pi}\frac{\sin 2kx}{\sin x}dx=-I$$

$\therefore\quad 2I=0\Rightarrow I=0$

Illustration 4: The value of the integral $\int_c^{\pi/2}\frac{1+2\cos x}{(2+\cos x)^2}dx$ is?

Solution: Let $P=\frac{\sin x}{2+\cos x}$

$$\Rightarrow\quad \frac{dP}{dx}=\frac{(2+\cos x)\cdot\cos x-\sin x\cdot(-\sin x)}{(2+\cos x)^2}$$

$$=\frac{2\cos x+1}{(2+\cos x)^2}$$

Integrating both sides with respect to x between the limits 0 and $\frac{\pi}{2}$, we get

$$(P)_0^{\pi/2} = \int_0^{\pi/2} \frac{2\cos x + 1}{(2+\cos x)^2} dx$$

i.e., $$\int_0^{\pi/2} \frac{2\cos x + 1}{(2+\cos x)^2} dx = \left(\frac{\sin x}{2+\cos x}\right)_0^{\pi/2} = \left(\frac{1}{2} - 0\right)$$

$$= \frac{1}{2}$$

Illustration 5: The value of $\int_a^b (x-a)^3(b-x)^4$ is?

Solution: Put $x = a\cos^2\theta + b\sin^2\theta$

$\Rightarrow \quad dx = 2(b-a)\sin\theta\cos\theta d\theta$

$\therefore \quad \int_a^b (x-a)^3(b-x)^4\, dx$

$$= 2(b-a)\int_0^{\pi/2} (a\cos^2\theta + b\sin^2\theta - a)^3$$

$$(b - a\cos^2\theta - b\sin^2\theta)^4 \sin\theta\cos\theta d\theta$$

$$= 2(b-a)^8 \int_0^{\pi/2} \sin^7\theta\cos^9\theta d\theta$$

$$= 2(b-a)^8 \frac{6\cdot4\cdot2\cdot8\cdot6\cdot4\cdot2}{16\cdot14\cdot12\cdot10\cdot8\cdot6\cdot4\cdot2} = \frac{(b-a)^8}{280}$$

Illustration 6: $\int_0^{\pi} |\sin x + \cos x|\, dx$ is equal to?

Solution: $\int_0^{\pi} |\sin x + \cos x|\, dx$

$$= \int_0^{3\pi/4} |\sin x + \cos x|\, dx + \int_{3\pi/4}^{\pi} |\sin x + \cos x|\, dx$$

$$\left[\begin{array}{l} \therefore \sin x + \cos x = 0 \Rightarrow \cos x = -\sin x \\ \quad \Rightarrow \tan x - 1 \Rightarrow x = \frac{3\pi}{4} \in (0,\pi) \end{array}\right]$$

$$= \int_0^{3\pi/4} (\sin x + \cos x)\, dx - \int_{3\pi/4}^{\pi} (\sin x + \cos x)\, dx$$

$$= (-\cos x + \sin x)\big|_0^{3\pi/4} - (-\cos x + \cos x)\big|_{3\pi/4}^{\pi}$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + 1\right] - \left[1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right] = 2\sqrt{2}.$$

Illustration 7: $\int_{\pi}^{100\pi} (x - [x])dx$ is equal to?

Solution: Since $x - [x]$ is a periodic function of period 1

$$\therefore \quad \int_0^{100} [x-(x)]\,dx = 100\int_0^1 [x-(x)]\,dx$$

$$= 100\int_0^1 x\,dx - 100\int_0^1 (x)\,dx$$

$$= 100\left(\frac{x^2}{2}\right)\Big|_0^1 - 100\int_0^1 0\,dx = 50$$

Illustration 8: $\lim_{x\to0} \frac{1}{x^3}\int_0^{x^2} \sin\sqrt{t}\, dt$ is equal to?

Solution: $\lim_{x\to0} \frac{1}{x^3}\int_0^{x^2} \sin\sqrt{t}\, dt \qquad = \lim_{x\to0} \frac{\int_0^{x^2} \sin\sqrt{t}\, dt}{x^3}$

$$\left(\frac{0}{0}\text{form}\right)\left(\therefore \int_0^{x^2} \sin\sqrt{t}\, dt = 0 \text{ at } x = 0\right)$$

$$= \lim_{x\to0} \frac{[\sin\sqrt{t}]_{t=x^2}\cdot\frac{d}{dx}(x^2)}{3x^2} \text{ (Using L' Hospital's Rule)}$$

$$= \lim_{x\to0} \frac{(\sin x)(2x)}{3x^2} = \frac{2}{3}$$

Illustration 9: $\int_0^{10\pi} (|\sin x| + |\cos x|)\, dx$ is equal to?

Solution: $$\int_0^{10\pi} (|\sin x| + |\cos x|)\,dx = \int_0^{20\left(\frac{\pi}{2}\right)} (|\sin x| + |\cos x|)\,dx$$

$$= 20\int_0^{\pi/2} (|\sin x| + |\cos x|)\,dx$$

$$\left(\therefore |\sin x| + |\cos x| \text{ is a periodic function of period } \frac{\pi}{2}\right)$$

$$= 20\int_0^{\pi/2} (\sin x + \cos x)dx$$

$$\left[\therefore \text{in}\left(0, \frac{\pi}{2}\right) \text{both } \sin x \text{ and } \cos x \text{ are positive}\right]$$

$$= 20(-\cos x + \sin x)_0^{\pi/2} = 20(1+1) = 40.$$

Illustration 10: If $f(x) = a + bx + cx^2$, then $\int_0^1 f(x)dx$ equals?

Solution: $f(0) = a, f(1) = a+b+c, f\left(\frac{1}{2}\right) = a + \frac{b}{2} + \frac{c}{4}.$

$$\therefore \quad I = \int_0^1 f(x)\,dx = \int_0^1 (a + bx + cx^2)\, dx$$

$$= \left(ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right)_0^1 = \frac{1}{6}(6a + 3b + 2c)$$

$$= \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$$

Illustration 11: The value of the integral $\int_0^{2[x]} (x-[x])\,dx$ is?

Solution: $\int_0^{2(x)} [x-(x)]dx = \int_0^{2(x).1} [x-(x)]dx = 2(x)\int_0^1 [x-(x)]\,dx$

$[\because x-(x)$ is a periodic function of period 1]

$$= 2(x)\left[\frac{x^2}{2}\bigg|_0^1 - \int_0^1 (x)\,dx\right] = 2(x)\left(\frac{1}{2}-0\right) = (x)$$

Illustration 12: The value of the

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2-1}}+\frac{1}{\sqrt{n^2-2^2}}+...+\frac{n^2}{\sqrt{n^2-(n-1)^2}}\right] \text{ is?}$$

Solution:

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2-1}}+\frac{1}{\sqrt{n^2-2^2}}+...+\frac{n^2}{\sqrt{n^2-(n-1)^2}}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2-0^2}}+\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+...+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}} = \lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n}\cdot\frac{1}{\sqrt{1-r^2/n^2}}$$

$$=\int_0^1\frac{dx}{\sqrt{1-x^2}} = (\sin^{-1}x)_0^1 = \frac{\pi}{2}$$

Illustration 13: $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{n^2}.\sec^2\frac{r^2}{n^2}$ is equal is?

Solution: $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{n^2}\cdot\sec^2\frac{r^2}{n^2}$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{r}{n}\cdot\sec^2\frac{r^2}{n^2}\right)\frac{1}{n} = \int_0^1 x\sec^2 x^2\,dx$$

$$=\frac{1}{2}\int_0^1\sec^2 t\,dt \qquad \text{(Putting } x^2 = t \Rightarrow 2x\,dx = dt\text{)}$$

$$=\frac{1}{2}[\tan t]_0^1 = \frac{1}{2}\tan 1$$

Illustration 14: $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)...\left(1+\frac{n}{n}\right)\right]^{1/n}$ is equal to?

Solution: Let $y = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)...\left(1+\frac{n}{n}\right)\right]^{1/n}$

$$\Rightarrow \quad \log y = \lim_{n\to\infty}\frac{1}{n}\times\log\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)...\left(1+\frac{n}{n}\right)\right]$$

$$=\lim_{n\to 0}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right) = \int_0^1\log(1+x)dx$$

$$=[x\log(1+x)]_0^1 - \int_0^1\frac{x}{1+x}dx$$

$$=\log 2 - \int_0^1\frac{(1+x)-1}{1+x}dx \qquad = \log 2 - \int_0^1\left(1-\frac{1}{1+x}\right)dx$$

$$=\log 2 - [x-\log(1+x)]_0^1 \qquad = \log 2 - [(1-\log 2)-0]$$

$$=2\log 2 - \log e = \log\frac{4}{e} \qquad \therefore\ y = \frac{4}{e}$$

Illustration 15: The area bounded by the curve $y = \sin^{-1}x$ and the lines $x = 0, |y| = \frac{\pi}{2}$ is?

Solution: The required area

$$= 2\int_0^{\pi/2} d\,dy \text{ where } y = \sin^{-1}x \text{ i.e. } x = \sin y$$

$$= 2\int_0^{\pi/2}\sin y\,dy = -2[\cos y]_0^{\pi/2} = 2$$

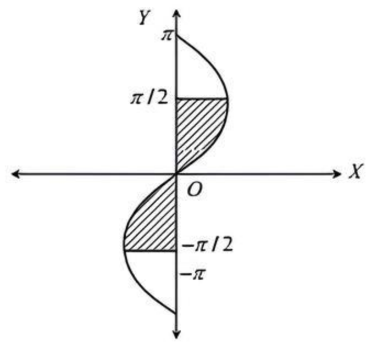


Illustration 16: The area of the smaller part bounded by the semi-circle $y = \sqrt{4-x^2}$, $y = x\sqrt{3}$ and x-axis is?

Solution: The required area $= \int_0^1\sqrt{3}x\,dx + \int_1^2\sqrt{4-x^2}\,dx$

$$=\frac{\sqrt{3}}{2}(x^2)_0^1 + \left(\frac{x}{2}\sqrt{4-x^2}+\frac{2^2}{2}\sin^{-1}\frac{x}{2}\right)_1^2$$

$$=\frac{\sqrt{3}}{2}+2\cdot\frac{\pi}{2}-\frac{\sqrt{3}}{2}-2\cdot\sin^{-1}\frac{1}{2} = \pi-\frac{\pi}{3} = \frac{2\pi}{3}$$