Differential Equation

Contents

JEE (Advanced) Syllabus

Differential Equation : Formation of ordinary differential equations, solution of homogeneous differential equations, separation of variables method, linear first order differential equation.

JEE (Main) Syllabus

Differential Equation : Ordinary differential equations, their order and degree. Formation of differential equation by the method of separation of variables, solution of homogeneous and linear differential equation.

DIFFERENTIAL EQUATION

1. INTRODUCTION :

An equation involving independent and dependent variables and the derivatives of the dependent variables is called a **differential equation**. There are two kinds of differential equation:

A differential equation is said to be **ordinary**, if the differential coefficients have reference to a single independent variable only e.g. $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + \cos x = 0$ and it is said to be **partial** if there are two or more independent variables. e.g. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ is a partial differential equation. We are concerned with ordinary differential equations only.

2. ORDER AND DEGREE OF A DIFFERENTIAL EQUATION:

ORDER OF DIFFERENTIAL EQUATION :

The order of a differential equation is the order of the highest differential coefficient occurring in it.

DEGREE OF DIFFERENTIAL EQUATION :

The exponent of the highest order differential coefficient, when the differential equation is expressed as a polynomial in all the differential coefficient.

Thus the differential equation :

$f(x, y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x, y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + ... = 0$ is of order m & degree p.

Note :

(i) The exponents of all the differential coefficient should be free from radicals and fraction.

(ii) The degree is always positive natural number.

(iii) The degree of differential equation may or may not exist.

DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE :

A differential equation of first order and first degree is of the type $\frac{dy}{dx}$ + f(x, y) = 0, which can also be written as : Mdx + Ndy = 0, where M and N are functions of x and y.

SOLVED EXAMPLE

Example 1 : The order and degree of the differential equation $\left(\frac{d^2s}{dt^2}\right)^2 + 3\left(\frac{ds}{dt}\right)^3 + 4 = 0$ are -

(A) 2 , 2 (B) 2 , 3 (C) 3 , 2 (D) none of these

Solution : Clearly order is 2 and degree is 2 (from the definition of order and degree of differential equations).

Example 2 : Find the order and degree of the following differential equations -

(i) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + y^4 = 0$ (ii) $\left(\frac{d^3y}{dx^3}\right)^2 + \frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = y$

(iii) $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$ (iv) $\left(\frac{dy}{dx}\right) + y = \frac{1}{\frac{dy}{dx}}$

Sol. **(i)** Order = 2, degree = 2

(ii) order 3, degree = 2

(iii) $\frac{dy}{dx} = \sin(x + y)$, order = 1 ,degree = 1

(iv) order = 1 ,degree = 2

Example 3 : Find the order and degree of the following differential equations -

(i) $e^{\frac{d^3y}{dx^3}} - x\frac{d^2y}{dx^2} + y = 0$ (ii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2} = x\frac{d^3y}{dx^3}$ (iii) $\frac{d^2y}{dx^2} = \sin\left(x+\frac{dy}{dx}\right)$

Solution : **(i)** equation can not be expressed as a polynomial in differential coefficients, so degree is not applicable but order is 3.

(ii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^5 = x^2\left(\frac{d^3y}{dx^3}\right)^2$ order = 3, degree = 2

(iii) equation can not be expressed as a polynomial in differential coefficients, so degree is not applicable but order is 2.

<u>Problems for Self Practice</u>-01

1. Find the order and degree of following differential equations $\left(y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}\right)$

(i) $[1 + (y')^2]^{1/2} = x^2 + y$ **(ii)** $(1 + y')^{1/2} = y''$ **(iii)** $y' = \sin y$

Answer : **1. (i)** one, two **(ii)** two, two **(iii)** one, one

3. FORMATION OF A DIFFERENTIAL EQUATION :

In order to obtain a differential equation whose solution is

$$f(x_1, y_1, c_1, c_2, c_3.........,c_n) = 0$$

where $c_1, c_2,.......c_n$ are 'n' arbitrary constants, we have to eliminate the 'n' constants for which we require (n+1) equations.

A differential equation is obtained as follows :

(a) Differentiate the given equation w.r.t the independent variable (say x) as many times as the number of independent arbitrary constants in it.

(b) Eliminate the arbitrary constants.

(c) The eliminant is the required differential equation.

Note :

(i) A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

(ii) For there being n differentiation, the resulting equation must contain a derivative of n^{th} order i.e. equal to number of independent arbitrary constant.

SOLVED EXAMPLE

Example 4 : Form a differential equation of family of straight lines passing through origin.

Solution : Family of straight lines passing through origin is y = mx where'm' is a parameter.

Differentiating w.r.t. x

$$\frac{dy}{dx} = m$$

Eliminating 'm' from both equations, we obtain

$\frac{dy}{dx} = \frac{y}{x}$ which is the required differential equation.

Example 5 : Find the differential equation of all parabolas whose axes is parallel to the x-axis and having latus rectum a.

Solution : Equation of parabola whose axes is parallel to x-axis and having latus rectum 'a' is $(y - \beta)^2 = a(x - \alpha)$

Differentiating both sides, we get $2(y - \beta)\frac{dy}{dx} = a$

Again differentiating, we get

$$\Rightarrow 2(y-\beta)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0 \Rightarrow \quad a\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 0.$$

Example 6 : Find the differential equation for all the straight lines which are at a unit distance from the origin.

Solution : Let equation of St. Line

$Y - y = m(X - x)$

Distance from origin $\Rightarrow \left|\frac{mx - y}{\sqrt{1+m^2}}\right| = 1$

$\therefore \quad (mx - y)^2 = 1 + m^2$

$$\left(y - \frac{dy}{dx}x\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

Problems for Self Practice-02

1. Find the differential equation whose solution represents the family : y = a cosθx + b sinθx, where θ = fixed constant
2. Find the differential equation whose solution represents the family : c (y + c)^2 = x^3
3. Form a differential equation of family of circles touching x-axis at the origin

Answer : 1. $\frac{d^2y}{dx^2} = -\theta^2 y$ 2. $\left[\frac{2x}{3}\left(\frac{dy}{dx}\right) - y\right]\left[\frac{2x}{3}\frac{dy}{dx}\right]^2 = x^3$ 3. $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$

4. SOLUTION OF A DIFFERENTIAL EQUATION:

The solution of the differential equation is a relation between the variables of the equation not containing the derivatives, but satisfying the given differential equation (i.e., from which the given differential equation can be derived).

Thus, the solution of $\frac{dy}{dx} = e^x$ could be obtained by simply integrating both sides, i.e., y = e^x + c and that of, $\frac{dy}{dx}$ = px + q is y = $\frac{px^2}{2}$ + qx + c, where c is arbitrary constant.

(i) **A general solution** or an integral of a differential equation is a relation between the variables (not involving the derivatives) which contains the same number of the arbitrary constants as the order of the differential equation.

For example, a general solution of the differential equation $\frac{d^2x}{dt^2}$ = –4x is x = A cos2t + B sin2t where A and B are the arbitrary constants.

(ii) **Particular solution or particular integral** is that solution of the differential equation which is obtained from the general solution by assigning particular values to the arbitrary constant in the general solution.

For example, x = 10 cos2t + 5 sin2t is a particular solution of differential equation $\frac{d^2x}{dt^2} = -4x$.

(iii) **Singular Solution is that solution of** the differential equation which is not obtainable from general solution. Geometrically, **General solution** acts as an envelope to **singular solution**.

Note :

(i) The general solution of a differential equation can be expressed in different (but equivalent) forms. For example

$$\log x - \log (y + 2) = k \quad \text{....(i)}$$

where k is an arbitrary constant is the general solution of the differential equation xy' = y + 2.
The solution given by equation (i) can also be re-written as

$$\log\left(\frac{x}{y+2}\right) = k \text{ or } \frac{x}{y+2} = e^k = c_1 \quad \text{...(ii)}$$

or $x = c_1 \ (y + 2)$ ...(iii)

where $c_1 = e^k$ is another arbitrary constant. The solution (iii) can also be written as

$$y + 2 = c_2x$$

where $c_2 = 1/c_1$ is another arbitrary constant.

(ii) All differential equations that we come across have unique solutions or a family of solutions.

For example, the differential equation $\left|\frac{dy}{dx}\right| + |y| = 0$ has only the trivial solution, i.e. y = 0.

The differential equation $\left|\frac{dy}{dx}\right| + |y| + c = 0,\ c > 0$ has no solution.

(iii) The solution of the differential equation is also called its primitive, because the differential equation can be regarded as a relation derived from it.

5. SOLUTION METHODS OF FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS :

5.1 VARIABLES SEPARABLE :

Some differential equations can be solved by the method of separation of variables (or "variable separable"). This method is only possible, if we can express the differential equation in the form A(x)dx + B(y) dy = 0 where A(x) is a function of 'x' only and B(y) is a function of 'y' only.

A general solution of this is given by,

$$\int A(x)\,dx + \int B(y)dy = c$$

where 'c' is the arbitrary constant.

SOLVED EXAMPLE

Example 7 : Solve the differential equation $(1 + x)\, y\, dx = (y - 1)\, x\, dy$

Solution The equation can be written as -

$$\left(\frac{1+x}{x}\right)dx = \left(\frac{y-1}{y}\right)dy \quad \Rightarrow \quad \int\left(\frac{1}{x}+1\right)dx = \int\left(1-\frac{1}{y}\right)dy$$

$$\ell n\, x + x = y - \ell n y + c \quad \Rightarrow \quad \ell n y + \ell n x = y - x + c$$

$$xy = ce^{y-x}$$

Example 8 : Solve : $\frac{dy}{dx} = (e^x + 1)(1 + y^2)$

Solution. The equation can be written as $\frac{dy}{1+y^2} = (e^x + 1)dx$

Integrating both sides,

$\tan^{-1} y = e^x + x + c.$

Example 9 : Solve : $\frac{dy}{dx}$ = (x – 3) (y + 1)$^{2/3}$

Solution : Case-I : y = –1 is one solution of differential equation

Case-II : If $y \neq -1$, then

$$\frac{dy}{dx} = (x-3)(y+1)^{2/3}$$

$$\int \frac{dy}{(y+1)^{2/3}} = \int (x-3)dx$$

Integrate and solve for y : $3(y + 1)^{1/3} = \frac{1}{2}(x - 3)^2 + C$

$(y + 1)^{1/3} = \frac{1}{6}(x - 3)^2 + C_0 \Rightarrow y+1 = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3$

y = –1 or $y = \left(\frac{1}{6}(x-3)^2 + C_0\right)^3 - 1$

Example 10 : Solve the differential equation $(x^3 - y^2x^3)\frac{dy}{dx} + y^3 + x^2y^3 = 0$.

Solution : The given equation $(x^3 - y^2x^3)\frac{dy}{dx} + y^3 + x^2y^3 = 0$

Case-I : y = 0 is one solution of differential equation

Case-II : If $y \neq 0$, then

$$\Rightarrow \frac{1-y^2}{y^3}dy + \frac{1+x^2}{x^3}dx = 0 \quad \Rightarrow \int\left(\frac{1}{y^3} - \frac{1}{y}\right)dy + \int\left(\frac{1}{x^3} + \frac{1}{x}\right)dx = 0$$

$$\Rightarrow \log\left(\frac{x}{y}\right) = \frac{1}{2}\left(\frac{1}{y^2} + \frac{1}{x^2}\right) + c$$

y = 0 or $\log\left(\frac{x}{y}\right) = \frac{1}{2}\left(\frac{1}{y^2} + \frac{1}{x^2}\right) + c$

5.1.1 Equations Reducible to the Variables Separable form :

If a differential equation can be reduced into a variables separable form by a proper substitution, then it is said to be

"Reducible to the variables separable type". Its general form is $\frac{dy}{dx}$ = f(ax + by + c) a, b $\neq$ 0. To solve this, put ax + by + c = t.

SOLVED EXAMPLE

Example 11 : Solve : $y' = (x + y + 1)^2$

Solution : $y' = (x + y + 1)^2$ ....(i)

Let t = x + y + 1

$$\frac{dt}{dx} = 1 + \frac{dy}{dx}$$

Substituting in equation (i) we get

$$\frac{dt}{dx} = t^2 + 1 \Rightarrow \int \frac{dt}{1+t^2} = \int dx \Rightarrow \tan^{-1} t = x + C \Rightarrow t = \tan(x + C)$$

$x + y + 1 = \tan(x + C) \Rightarrow y = \tan(x + C) - x - 1$

Example 12 : Solve $\frac{dy}{dx}$ = cos (x + y) – sin (x + y).

Solution : $\frac{dy}{dx}$ = cos (x + y) – sin (x + y)

Substituting, x + y = t, we get $\frac{dy}{dx} = \frac{dt}{dx} - 1$

Therefore $\frac{dt}{dx} - 1 = \cos t - \sin t$

$$\Rightarrow \int \frac{dt}{1+\cos t - \sin t} = \int dx \Rightarrow \int \frac{\sec^2 \frac{t}{2} dt}{2\left(1 - \tan \frac{t}{2}\right)} = \int dx \Rightarrow -\ell n\left|1 - \tan \frac{x+y}{2}\right| = x + c.$$

Problems for Self Practice-03

1. Solve the differential equation $xy \frac{dy}{dx} = \frac{1+y^2}{1+x^2}(1 + x + x^2)$.

2. Solve : $\frac{dy}{dx} = e^{x+y} + x^2e^y$

3. Solve : $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

4. Solve $\frac{dy}{dx} = 1 + e^{x-y}$

5. Solve $\frac{dy}{dx} = (4x + y + 1)^2$

6. Solve $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Answer :

1. $\sqrt{1+y^2} = cxe^{\tan^{-1}x}$.

2. $-\frac{1}{e^y} = e^x + \frac{x^3}{3} + c$

3. cy = (x + a) (1 – ay) Where 'c' is an arbitrary constant

4. $e^{y-x} = x + c$

5. $\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$]

6. $\sin(x + y) = x\cos(x + y) + c\cos(x + y) + 1$

5.2 HOMOGENEOUS EQUATION:

A function $f(x,y)$ is said to be a homogeneous function of degree n, if the substitution

$x = \lambda x,\ y = \lambda y,\ \lambda > 0$ produces the equality

$f(\lambda x, \lambda y) = \lambda^n f(x,y)$

The degree of homogeneity 'n' can be any real number.

SOLVED EXAMPLE

Example 13 : Find the degree of homogeneity of function

(i) $f(x,y) = x^2 + y^2$ (ii) $f(x,y) = (x^{3/2} + y^{3/2})/(x + y)$ (iii) $f(x,y) = \sin\left(\frac{x}{y}\right)$

Solution :

(i) $f(\lambda x, \lambda y) = \lambda^2x^2 + \lambda^2y^2 = \lambda^2(x^2 + y^2) = \lambda^2 f(x,y)$

degree of homogeneity $\rightarrow 2$

(ii) $f(\lambda x, \lambda y) = \frac{\lambda^{3/2}x^{3/2} + \lambda^{3/2}y^{3/2}}{\lambda x + \lambda y}$

$f(\lambda x, \lambda y) = \lambda^{1/2} f(x,y)$

degree of homogeneity $\rightarrow 1/2$

(iii) $f(\lambda x, \lambda y) = \sin\left(\frac{\lambda x}{\lambda y}\right) = \lambda^\circ \sin\left(\frac{x}{y}\right) = \lambda^\circ f(x,y)$

degree of homogeneity $\rightarrow 0$

Example 14 : Determine whether or not each of the following functions is homogeneous.

(i) $f(x,y) = x^2 - xy$ (ii) $f(x,y) = \dfrac{xy}{x+y^2}$ (iii) $f(x,y) = \sin xy$

Solution : (i) $f(\lambda x, \lambda y) = \lambda^2x^2 - \lambda^2xy = \lambda^2(x^2 - xy) = \lambda^2 f(x,y)$ homogeneous.

(ii) $f(\lambda x,\lambda y) = \dfrac{\lambda^2 xy}{\lambda x + \lambda^2 y^2} \neq \lambda^n f(x,y)$ not homogeneous.

(iii) $f(\lambda x, \lambda y) = \sin(\lambda^2 xy) \neq \lambda^n f(x,y)$ not homogeneous.

5.2.1 Homogeneous first order differential equation

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}$

where $f(x,y)$ and $g(x,y)$ are homogeneous functions of x,y and of the same degree, is said to be homogeneous. Such equations can be solved by substituting

$$y = vx,$$

so that the dependent variable y is changed to another variable v.

Since $f(x,y)$ and $g(x,y)$ are homogeneous functions of the same degree say, n, they can be written as

$$f(x,y) = x^n f_1\left(\frac{y}{x}\right) \quad \text{and} \quad g(x,y) = x^n g_1\left(\frac{y}{x}\right).$$

As $y = vx$, we have $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

The given differential equation, therefore, becomes

$$v + x\frac{dv}{dx} = \frac{f_1(v)}{g_1(v)} \quad \Rightarrow \frac{g_1(v)dv}{f_1(v) - vg_1(v)} = \frac{dx}{x},$$

so that the variables v and x are now separable.

Note : Sometimes homogeneous equation can be solved by substituting $x = vy$ or by using polar coordinate substitution.

SOLVED EXAMPLE

Example 15 : Solve : $(x^2 - y^2)\,dx + 2xydy = 0$ given that $y = 1$ when $x = 1$

Solution. $\dfrac{dy}{dx} = -\dfrac{x^2 - y^2}{2xy}$ $y = vx$

$\dfrac{dy}{dx} = v + \dfrac{dv}{dx}$ $\therefore$ $v + x\dfrac{dv}{dx} = -\dfrac{1-v^2}{2v}$ $\displaystyle\int \frac{2v}{1+v^2}\,dv = -\int \frac{dx}{x}$

$\ell n(1 + v^2) = -\ell nx + c$

at $x = 1, y = 1$ $\therefore$ $v = 1$ $\ell n\, 2 = c$

$\therefore$ $\ell n \left\{\left(1+\frac{y^2}{x^2}\right).x\right\} = \ell n2$ $x^2 + y^2 = 2x$

5.2.2 Equations Reducible to the Homogeneous form

Case I : The equation of the form $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

can be reduced to homgeneous form by changing the variable x, y to X,Y respectively

$x = X + h$, $y = Y + k$

where h,k are the constants to be chosen so as to make the given equation homgeneous.

We have

$$\frac{dy}{dx} = \frac{dY}{dX}$$

$\therefore$ The equation becomes, $\frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)}$

Let h and k be chosen so as to satisfy the equation

$a_1h + b_1k + c_1 = 0$ ...(i)

$a_2h + b_2k + c_2 = 0$ ...(ii)

Solve for h and k from (i) and (ii)

Now $\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$

is a homgeneous equation and can be solved by substituting Y = XV.

Special case

Case II : The equation of the form $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ where $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ then

Let $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda$ then $a_1 = \lambda a_2$.....(i) ; $b_1 = \lambda b_2$ ......(ii)

from (i) and (ii), differential equation becomes

$$\frac{dy}{dx} = \frac{\lambda a_2x + \lambda b_2y + c_1}{a_2x + b_2y + c_2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\lambda(a_2x + b_2y) + c_1}{a_2x + b_2y + c_2}$$

or we can say, $\frac{dy}{dx} = f(a_2x + b_2y)$

which can be solved by substituting $t = a_2x + b_2y$

SOLVED EXAMPLE

Example 16 : Solve $\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$

Solution : Put x = X + h, y = Y + k

We have $\frac{dY}{dX} = \frac{X+2Y+(h+2k+3)}{2X+3Y+(2h+3k+4)}$

To determine h and k, we write

h + 2k + 3 = 0, 2h + 3k + 4 = 0 $\Rightarrow$ h = 1, k = –2

So that $\frac{dY}{dX} = \frac{X+2Y}{2X+3Y}$ Putting Y = VX, we get

$V + X\frac{dV}{dX} = \frac{1+2V}{2+3V} \Rightarrow \frac{2+3V}{3V^2-1}dV = -\frac{dX}{X}$

$\Rightarrow \left[\frac{2+\sqrt{3}}{2(\sqrt{3}V-1)} - \frac{2-\sqrt{3}}{2(\sqrt{3}V+1)}\right]dV = -\frac{dX}{X}$

$\Rightarrow \frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V-1) - \frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V+1) = (-\log X + c)$

$\frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y-X) - \frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y+X) = A$ where A is another constant and X = x – 1, Y = y + 2.

Example 17 : Solve $\frac{dy}{dx} = \frac{2x+3y-1}{4x+6y-5}$

Solution. Putting u = 2x + 3y

$\frac{du}{dx} = 2 + 3\,.\,\frac{dy}{dx} \quad \Rightarrow \frac{1}{3}\left(\frac{du}{dx}-2\right) = \frac{u-1}{2u-5} \Rightarrow \frac{du}{dx} = \frac{3u-3+4u-10}{2u-5}$

$\int\frac{2u-5}{7u-13}\,dx = \int dx$

$\Rightarrow \frac{2}{7}\int 1.du - \frac{9}{7}\int\frac{1}{7u-13}\,.\,du = x + c \qquad \Rightarrow \frac{2}{7}u - \frac{9}{7}\,.\,\frac{1}{7}\,\ell n\,(7u-13) = x + c$

$\Rightarrow 4x + 6y - \frac{9}{7}\ell n\,(14x+21y-13) = 7x + 7c \qquad \Rightarrow -3x + 6y - \frac{9}{7}\ell n\,(14x+21y-13) = c'$

Example 18 : Solve the differential equation $\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$

Solution : Here, $$\frac{dy}{dx} = \frac{\sin y + x}{\sin 2y - x\cos y}$$

$$\Rightarrow \quad \cos y \frac{dy}{dx} = \frac{\sin y + x}{2\sin y - x}, \qquad \text{(put } \sin y = t\text{)}$$

$$\Rightarrow \quad \frac{dt}{dx} = \frac{t + x}{2t - x} \qquad \text{(put } t = vx\text{)}$$

$$\frac{xdv}{dx} + v = \frac{vx + x}{2vx - x} = \frac{v+1}{2v-1} \qquad \therefore \quad x\frac{dv}{dx} = \frac{v+1}{2v-1} - v = \frac{v+1-2v^2+v}{2v-1}$$

or $$\frac{2v-1}{-2v^2+2v+1}\,dv = \frac{dx}{x}$$ on solving, we get

$$\sin^2 y = x \sin y + \frac{x^2}{2} + c.$$

<u>Problems for Self Practice</u>-04

1. Solve the differential equation $(1 + 2e^{x/y})\,dx + 2e^{x/y}(1 - x/y)\,dy = 0$.

2. Solve $2\frac{y}{x} + \left(\left(\frac{y}{x}\right)^2 - 1\right)\frac{dy}{dx}$

3. Solve the differential equation $\frac{dy}{dx} = \frac{x + 2y - 5}{2x + y - 4}$

4. Solve : $(x + y)dx + (3x + 3y - 4)\,dy = 0$

Answer :
1. $x + 2\,ye^{x/y} = c$ **2.** $x^2 + y^2 = yc'$ where $c' = e^c$
3. $x + y - 3 = c'\,(x - y + 1)^3]$ **4.** $x + 3y + 2\ell n|(2 - x - y)| = c$

5.3 Polar coordinates transformations :

Sometimes transformation to the polar co-ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials:

(a) If $x = r\cos\theta$; $y = r\sin\theta$ then,

(i) $x\,dx + y\,dy = r\,dr$ (ii) $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$ (iii) $x\,dy - y\,dx = r^2 d\theta$

(b) If $x = r\sec\theta$ & $y = r\tan\theta$ then

(i) $x\,dx - y\,dy = r\,dr$ (ii) $x\,dy - y\,dx = r^2 \sec\theta\,d\theta$.

SOLVED EXAMPLE

Example 19 : Solve the differential equation xdx + ydy = x (xdy – ydx)

Solution. Taking $x = r\cos\theta,\ y = r\sin\theta$

$x^2 + y^2 = r^2$

$2x\,dx + 2y\,dy = 2r\,dr$

$x\,dx + y\,dy = r\,dr$ ....(i)

$$\frac{y}{x} = \tan\theta \quad \Rightarrow \quad \frac{x\dfrac{dy}{dx} - y}{x^2} = \sec^2\theta \cdot \frac{d\theta}{dx}$$

$x\,dy - y\,dx = x^2\sec^2\theta \cdot d\theta$

$x\,dy - y\,dx = r^2\,d\theta$ ....(ii)

Using (i) & (ii) in the given differential equation then it becomes

$r\,dr = r\cos\theta \cdot r^2\,d\theta$

$$\frac{dr}{r^2} = \cos\theta\,d\theta \quad \Rightarrow -\frac{1}{r} = \sin\theta + \lambda \quad \Rightarrow -\frac{1}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} + \lambda$$

$$\frac{y+1}{\sqrt{x^2+y^2}} = c \quad \text{where } -\lambda' = c \quad \Rightarrow \quad (y+1)^2 = c(x^2+y^2)$$

Example 20 : Solve : $\dfrac{x\,dx - y\,dy}{x\,dy - y\,dx} = \sqrt{\dfrac{1+x^2-y^2}{x^2-y^2}}$

Solution : $\dfrac{xdx - ydy}{xdy - ydx} = \sqrt{\dfrac{1+x^2-y^2}{x^2-y^2}}$

$x = r\sec\theta$ $y = r\tan\theta$ $x^2 - y^2 = r^2$

$dx = r\sec\theta\tan\theta\,d\theta + \sec\theta\,dr$ $dy = r\sec^2\theta\,d\theta + \tan\theta\,dr$

$$\Rightarrow \quad \frac{r\,dr}{r^2\sec\theta\,d\theta} = \sqrt{\frac{1+r^2}{r^2}} \quad \Rightarrow \quad \int\frac{dr}{\sqrt{1+r^2}} = \int\sec\theta\,d\theta$$

on solving $\sqrt{x^2-y^2} + \sqrt{1+x^2-y^2} = \dfrac{c(x+y)}{\sqrt{x^2-y^2}}$

5.4 EXACT DIFFERENTIAL EQUATION :

The differential equation $M + N\dfrac{dy}{dx} = 0$ ....(1)

Where M and N are functions of x and y is said to be exact if it can be derived by direct differentiation (without any subsequent multiplication, elimination etc.) of an equation of the form f(x, y) = c e.g. $y^2\,dy + x\,dx + \frac{dx}{x} = 0$ is an exact differential equation.

NOTE : (i) The necessary condition for (1) to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

(ii) For finding the solution of exact differential equation, following results on exact differentials should be remembered :

(a) $xdy + y\,dx = d(xy)$ (b) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$ (c) $2(x\,dx + y\,dy) = d\,(x^2 + y^2)$

(d) $\frac{xdy - ydx}{xy} = d\left(\ln\frac{y}{x}\right)$ (e) $\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$ (f) $\frac{xdy + ydx}{xy} = d(\ln xy)$

(g) $\frac{xdy + ydx}{x^2\,y^2} = d\left(-\frac{1}{xy}\right)$

SOLVED EXAMPLE

Example 21 : Solve : $y\,dx + x\,dy = \frac{xdy - ydx}{x^2 + y^2}$

Solution : $ydx + xdy = \frac{xdy - ydx}{x^2 + y^2}$ $d\,(xy) = d\,(\tan^{-1} y/x)$

Integrating both sides $xy = \tan^{-1} y/x + c$

Example 22 : Solve $\frac{y + \sin x \cos^2(xy)}{\cos^2(xy)}dx + \left(\frac{x}{\cos^2(xy)} + \sin y\right)dy = 0$.

Solution : The given differential equation can be written as;

$\frac{y\,dx + x\,dy}{\cos^2(xy)} + \sin x\,dx + \sin y\,dy = 0.$

$\Rightarrow \sec^2(xy)\,d\,(xy) + \sin x\,dx + \sin y\,dy = 0$

$\Rightarrow d\,(\tan(xy)) + d\,(-\cos x) + d\,(-\cos y) = 0$

$\Rightarrow \tan(xy) - \cos x - \cos y = c.$

<u>Problems for Self Practice</u>-05

1. Solve : $xdy + ydx + xy\,e^y\,dy = 0$ **2.** Solve : $x\,dy - ydx - (1 - x^2)dx = 0$.

3. Solve : $xdx + ydy = xdy - ydx$ **4.** Solve : $\frac{dy}{dx} = \frac{x - 2y + 1}{2x + 2y + 3}$

Answer : 1. $\ell n\,(xy) + e^y = c$ 2. $y + x^2 + 1 = Cx$ 3. $\ell n(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right) + c$

4. $y^2 + 3y = \frac{x^2}{2} + x - 2xy + c$

5.5 LINEAR DIFFERENTIAL EQUATION :

A linear differential equation has the following characteristics :

(i) Dependent variable and its derivative in first degree only and are not multiplied together

(ii) All the derivatives should be in a polynomial form

(iii) The order may be more than one

The m^{th} order linear differential equation is of the form.

$$P_0(x)\frac{d^my}{dx^m} + P_1(x)\frac{d^{m-1}y}{dx^{m-1}} + \ldots\ldots\ldots\ldots + P_{m-1}(x)\frac{dy}{dx} + P_m(x)\,y = \phi(x),$$

where $P_0(x), P_1(x)$ ....$P_m(x)$ are called the coefficients of the differential equation.

NOTE : $\frac{dy}{dx} + y^2 \sin x = \ln x$ is not a Linear differential equation.

5.5.1 Linear differential equations of first order :

The most general form of a linear differential equation of first order is $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x.

To solve such an equation multiply both sides by $e^{\int Pdx}$.

So that we get $\quad e^{\int Pdx}\left[\frac{dy}{dx} + Py\right] = Qe^{\int Pdx}$ ....(i)

$$\frac{d}{dx}\left(e^{\int Pdx}.y\right) = Qe^{\int Pdx} \qquad \text{...(ii)}$$

On integrating equation (ii), we get $\quad ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c$

This is the required general solution.

Note :

(i) The factor $e^{\int Pdx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x & y, is called integrating factor of the differential equation popularly abbreviated as I.F.

(ii) Sometimes a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable.

$\frac{dx}{dy} + P_1 x = Q_1$ where P_1 and Q_1 are functions of y. The I.F. now is $e^{\int P_1 dy}$

SOLVED EXAMPLE

Example 23 : Solve $\frac{dy}{dx} + \frac{3x^2}{1+x^3}\,y = \frac{\sin^2 x}{1+x^3}$

Solution : $\frac{dy}{dx} + Py = Q$

$$P = \frac{3x^2}{1+x^3}$$

IF = $e^{\int P.dx} = e^{\int \frac{3x^2}{1+x^3}dx} = e^{\ell n(1+x^3)} = 1 + x^3$

$\therefore$ General solution is

$y(IF) = \int Q(IF).dx + c$

$y\,(1 + x^3) = \int \frac{\sin^2 x}{1+x^3}\,(1 + x^3)\,dx + c$

$y(1 + x^3) = \int \frac{1-\cos 2x}{2}\,dx + c$

$y(1 + x^3) = \frac{1}{2}\,x - \frac{\sin 2x}{4} + c$

Example 24 : Find the solution of differential equation $(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2-1}$

Solution : The given differential equation is

$$(x^2-1)\frac{dy}{dx} + 2\ xy = \frac{1}{x^2-1}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{1}{(x^2-1)^2} \qquad \text{...(i)}$$

This is linear differential equation of the form

$\frac{dy}{dx} + Py = Q,$ where P = $\frac{2x}{x^2-1}$ and Q = $\frac{1}{(x^2-1)^2}$

$\therefore$ I.F. = $e^{\int P\,dx} = e^{\int 2x/(x^2-1)dx} = e^{\log(x^2-1)} = (x^2-1)$

multiplying both sides of (i) by I.F. = $(x^2 - 1)$, we get

$(x^2 - 1)\frac{dy}{dx} + 2\ xy = \frac{1}{x^2-1}$

integrating both sides we get

$y(x^2-1) = \int \frac{1}{x^2-1}dx + C$ [Using : y (I.F.) = $\int Q.(I.F.)\ dx + C$]

$\Rightarrow$ y $(x^2 - 1) = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$.

This is the required solution.

Example 25 : Solve the differential equation

$t(1 + t^2)\, dx = (x + xt^2 - t^2)\, dt$ and it given that $x = -\pi/4$ at $t = 1$

Solution : $t(1 + t^2)\, dx = [x(1 + t^2) - t^2]\, dt$

$$\frac{dx}{dt} = \frac{x}{t} - \frac{t}{(1+t^2)}$$

$$\frac{dx}{dt} - \frac{x}{t} = -\frac{t}{1+t^2}$$

which is linear in $\frac{dx}{dt}$

Here, $P = -\frac{1}{t}$, $Q = -\frac{t}{1+t^2}$ IF $= e^{-\int \frac{1}{t} dt} = e^{-\ell nt} = \frac{1}{t}$

$\therefore$ General solution is -

$$x \cdot \frac{1}{t} = \int \frac{1}{t} \cdot \left(-\frac{t}{1+t^2}\right) dt + c \qquad \frac{x}{t} = -\tan^{-1} t + c$$

putting $x = -\pi/4$, $t = 1$

$-\pi/4 = -\pi/4 + c \Rightarrow c = 0$

$\therefore$ $x = -t \tan^{-1} t$

5.5.2 Equations reducible to linear form

(I) By change of variable.

Often differential equation can be reduced to linear form by appropriate substitution of the non-linear term

(II) BERNOULL'S EQUATION :

The equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x,

is called **Bernoulli's equation.**

On dividing by y^n, we get $y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q$

Let $y^{-n+1} = t$, so that $(-n + 1)y^{-n}\frac{dy}{dx} = \frac{dt}{dx}$

then equation becomes $\frac{dt}{dx} + P(1-n)t = Q(1-n)$

which is linear with t as a dependent variable.

SOLVED EXAMPLE

Example 26 : Solve : $y \sin x \dfrac{dy}{dx} = \cos x(\sin x - y^2)$

Solution : The given differential equation can be reduced to linear form by change of variable by a suitable subtitution.

Substituting $y^2 = z$

$$2y\frac{dy}{dx} = \frac{dz}{dx}$$

differential equation becomes

$$\frac{\sin x}{2}\frac{dz}{dx} + \cos x.z = \sin x \cos x$$

$$\frac{dz}{dx} + 2\cot x \,.\, z = 2\cos x \quad \text{which is linear in } \frac{dz}{dx}$$

$$\text{IF} = e^{\int 2\cot x\,dx} = e^{2\ell n \sin x} = \sin^2 x$$

$\therefore$ General solution is -

$$z.\sin^2 x = \int 2\cos x.\sin^2 x.dx + c \qquad\qquad y^2\sin^2 x = \frac{2}{3}\sin^3 x + c$$

Example 27 : Solve $\dfrac{dy}{dx} = \tan y \cot x - \sec y \cos x.$

Solution : $\dfrac{dy}{dx} = \tan y \cot x - \sec y \cos x.$

Rearrange it : $(\sin x - \sin y)\cos x\,dx + \sin x \cos y\,dy = 0.$

Put $u = \sin y$, So, $du = \cos y\,dy$:

Substituting, we get

$$(\sin x - u)\cos x\,dx + \sin x\,du = 0, \quad \frac{du}{dx} - u\frac{\cos x}{\sin x} = -\cos x$$

The equation is first-order linear in u.

The integrating factor is

$$I = \exp\int -\frac{\cos x}{\sin x}dx = \exp\{-\ln(\sin x)\} = \frac{1}{\sin x}.$$

Hence, $u\dfrac{1}{\sin x} = -\int \dfrac{\cos x}{\sin x}dx = -\ln|\sin x| + C$.

Solve for u : u = –sin x ln |sin x|+ C sin x.

Put y back : siny = – sin x ln |sin x|+ C sin x.

Example 28 : Solve the differential equation $x\dfrac{dy}{dx} + y = x^3y^6$.

Solution : The given differential equation can be written as

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{xy^5} = x^2$$

Putting $y^{-5} = v$ so that

$$-5\,y^{-6}\frac{dy}{dx} = \frac{dv}{dx} \text{ or } y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dv}{dx} \quad \text{we get}$$

$$-\frac{1}{5}\frac{dv}{dx} + \frac{1}{x}v = x^2 \Rightarrow \frac{dv}{dx} - \frac{5}{x}v = -5x^2 \quad ......(i)$$

This is the standard form of the linear deferential equation having integrating factor

$$\text{I.F} = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = \frac{1}{x^5}$$

Multiplying both sides of (i) by I.F. and integrating w.r.t. x

We get $v\,.\,\dfrac{1}{x^5} = \int -5x^2\,.\,\dfrac{1}{x^5}dx \Rightarrow \dfrac{v}{x^5} = \dfrac{5}{2}x^{-2} + c$

$\Rightarrow y^{-5}x^{-5} = \dfrac{5}{2}x^{-2} + c$ which is the required solution.

Problems for Self Practice-06

1. Solve : $x\,\ell nx\dfrac{dy}{dx} + y = 2\,\ell n\,x$

2. Solve $(1 + y^2) + (x - e^{\tan^{-1}y})\dfrac{dy}{dx} = 0$.

3. Solve $\dfrac{dy}{dx} - y\tan x = -y^2\sec x$.

4. Solve $xy^2\left(\dfrac{dy}{dx}\right) - 2y^3 = 2x^3$ given y = 1 at x = 1

Answer :
1. $y(\ell n\, x) = (\ell n\, x)^2 + c$
2. $2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + c$.
3. $y^{-1}|\sec x| = \tan x + c$
4. $y^3 + 2x^3 = 3x^6$

6. HIGHER ORDER AND HIGHER DEGREE DIFFERENTIAL EQUATIONS :

The differential equation

$y = mx + f(m)$, ...(1), where $m = \frac{dy}{dx}$

is known as Clairaut's Equation.

To solve (10), differentiate it w.r.t. x, which gives

$$\frac{dy}{dx} = m + x\frac{dm}{dx} + \frac{d\,f(m)}{dm}\frac{dm}{dx}$$

$$x\frac{dm}{dx} + \frac{d\,f(m)}{dm}\frac{dm}{dx} = 0$$

either $\frac{dm}{dx} = 0 \Rightarrow m = c$ ....(2)

or $x + f'(m) = 0$ ....(3)

NOTE : (i) If m is eliminated between (1) and (2), the solution obtained is a general solution of (1)

(ii) If m is eliminated between (1) and (3), then solution obtained does not contain any arbitrary constant and is not particular solution of (1). This solution is called singular solution of (1).

SOLVED EXAMPLE

Example 29 : Solve : $y = mx + m - m^3$ where, $m = \frac{dy}{dx}$

Solution : $y = mx + m - m^3$ ..... (i)

The given equation is in clairaut's form.

Now, differentiating wrt. x -

$$\frac{dy}{dx} = m + x\frac{dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx} \qquad m = m + x\frac{dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx}$$

$$\frac{dm}{dx}(x + 1 - 3m^2) = 0$$

$\frac{dm}{dx} = 0 \Rightarrow m = c$ ..... (ii)

or $x + 1 - 3m^2 = 0 \Rightarrow m^2 = \frac{x+1}{3}$ ..... (iii)

Eliminating 'm' between (i) & (ii) is called the general solution of the given equation.

$y = cx + c - c^3$ where, 'c' is an arbitrary constant.

Again, eliminating 'm' between (i) & (iii) is called singular solution of the given equation.

$y = m(x + 1 - m^2)$

$$y = \pm\left(\frac{x+1}{3}\right)^{1/2}\left(x+1-\frac{x+1}{3}\right) \quad\Rightarrow\quad y = \pm\left(\frac{x+1}{3}\right)^{1/2}\frac{2}{3}(x+1)$$

$$y = \pm\ 2\left(\frac{x+1}{3}\right)^{3/2} \quad\Rightarrow\quad y^2 = \frac{4}{27}(x+1)^3$$

$$27y^2 = 4(x+1)^3$$

<u>Problems for Self Practice</u>-07

1. Solve the differential equation $Y = mx + 2/m$ where, $m = \frac{dy}{dx}$

2. Solve : $\sin px \cos y = \cos px \sin y + p$ where $p = \frac{dy}{dx}$

Answers:

1. General solution : $y = cx + 2/c$ where c is an arbitrary constant

   Singular solution : $y^2 = 8x$

2. General solution : $y = cx - \sin^{-1}(c)$ where c is an arbitrary constant.

   Singular solution : $y = \sqrt{x^2-1} - \sin^{-1}\sqrt{\frac{x^2-1}{x^2}}$

7 APPLICATION OF DIFFERENTIAL EQUATIONS

7.1 GEOMETRICAL APPLICATIONS :

Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$, then slope of the tangent at point P is $\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}$

(i) The equation of the tangent at P is $y - y_1 = \frac{dy}{dx}(x - x_1)$

x-intercept of the tangent = $x_1 - y_1\left(\frac{dx}{dy}\right)$

y-intercept of the tangent = $y_1 - x_1\left(\frac{dy}{dx}\right)$

(ii) The equation of normal at P is $y - y_1 = -\frac{1}{(dy/dx)}(x - x_1)$

x and y-intercepts of normal are ; $x_1 + y_1\frac{dy}{dx}$ and $y_1 + x_1\frac{dx}{dy}$

(iii) Length of tangent $= PT = |y_1|\sqrt{1+(dx/dy)^2_{(x_1,\,y_1)}}$

(iv) Length of normal $= PN = |y_1|\sqrt{1+(dy/dx)^2_{(x_1,\,y_1)}}$

(v) Length of sub-tangent = $ST = \left|y_1\left(\dfrac{dx}{dy}\right)_{(x_1,\,y_1)}\right|$

(vi) Length of sub-normal = $SN = \left|y_1\left(\dfrac{dy}{dx}\right)_{(x_1,\,y_1)}\right|$

SOLVED EXAMPLE

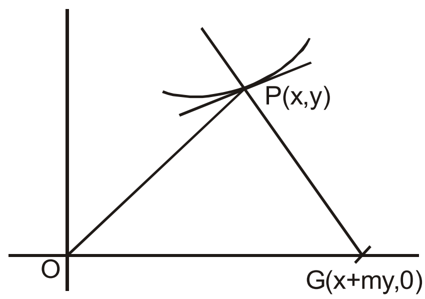
Example 30 : Find the nature of the curve for which the length of the normal at a point 'P' is equal to the radius vector of the point 'P'.

Solution : Let the equation of the curve be y = f(x). P(x, y) be any point on the curve.

Slope of the tanget at P(x, y) is $\dfrac{dy}{dx} = m$

$\therefore$ Slope of the normal at P is

$m' = -\dfrac{1}{m}$



Equation of the normal at 'P'

$Y - y = -\dfrac{1}{m}(X - x)$

Co-ordinates of G (x + my, 0)

Now, $OP^2 = PG^2 \Rightarrow x^2 + y^2 = m^2y^2 + y^2 \quad \Rightarrow m = \pm\dfrac{x}{y} \Rightarrow \dfrac{dy}{dx} = \pm\dfrac{x}{y}$

Taking as the sign

$\dfrac{dy}{dx} = \dfrac{x}{y}$

$\Rightarrow \quad y\,.\,dy = x\,.\,dx$

$\Rightarrow \quad \dfrac{y^2}{2} = \dfrac{x^2}{2} + \lambda$

$\Rightarrow \quad x^2 - y^2 = -2\lambda$

$x^2 - y^2 = c$ (Rectangular hyperbola)

Again taking as –ve sign

$$\frac{dy}{dx} = -\frac{x}{y} \quad \Rightarrow \quad y\,dy = -x\,dx$$

$$\Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + \lambda'$$

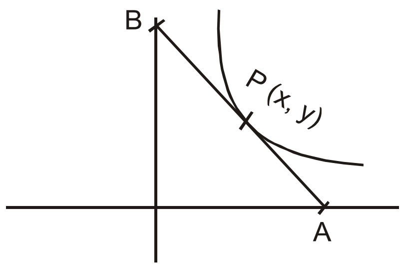
$$\Rightarrow \quad x^2 + y^2 = 2\lambda'$$

$x^2 + y^2 = c'$ (circle)

Example 31 : Find the curves for which the portion of the tangent included between the co-ordinate axes is bisected at the point of contact.

Solution : Let P (x, y) be any point on the curve.

Equation of tangent at P (x, y) is -



$Y - y = m(X - x)$ where $m = \frac{dy}{dx}$

is slope of the tangent at P (x, y).

Co-ordinates of $A\left(\frac{mx-y}{m}, 0\right)$ & B (0, y–mx)

P is the middle point of A & B

$$\therefore \quad \frac{mx-y}{m} = 2x \quad \Rightarrow \quad mx - y = 2mx \quad \Rightarrow \quad mx = -y \quad \Rightarrow \quad \frac{dy}{dx}x = -y$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} = 0 \quad \Rightarrow \quad \ell nx + \ell ny = \ell nc \quad \therefore \quad xy = c$$

Example 32 : The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through (1, 1)

Solution : Let P (x, y) be any point on the curve

Equation of tangent at 'P' is -

$Y - y = m(X - x)$ $\qquad mX - Y + y - mx = 0$

$$\text{Now} \quad \left(\frac{y-mx}{\sqrt{1+m^2}}\right) = x$$

$$y^2 + m^2x^2 - 2mxy = x^2(1 + m^2)$$

$\frac{y^2 - x^2}{2xy} = \frac{dy}{dx}$ which is homogeneous equation

Putting $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore \quad v + x\frac{dv}{dx} = \frac{v^2-1}{2v}$

$x\frac{dv}{dx} = \frac{v^2-1-2v^2}{2v}$

$\Rightarrow \quad \int \frac{2v}{v^2+1}dv = -\int \frac{dx}{x} \qquad \Rightarrow \ell n\,(v^2+1) = -\ell n\,x + \ell n\,c$

$x\left(\frac{y^2}{x^2}+1\right) = c$

Curve is passing through (1, 1)

$\therefore \quad c = 2 \quad \Rightarrow x^2 + y^2 - 2x = 0$

Problems for Self Practice-08

1. At each point (x,y) of a curve the intercept of the tangent on the y-axis is equal to $2xy^2$. Find the curve.

2. Find the equation of the curve for which the normal at any point (x,y) passes through the origin.

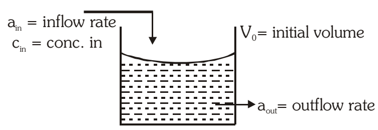
Answer : 1. $\frac{x}{y} = x^2 + C$ 2. $x^2 + y^2 = C$

7.2 MIXING PROBLEMS

A chemical in a liquid solution with given concentration c_{in} gm/lit. (or dispersed in a gas) runs into a container with a rate of a_{in} lit/min. holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate (a_{out} litre/min.). In this process it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula.

$$\begin{matrix}\text{Rate of change}\\ \text{of amount}\\ \text{in container}\end{matrix} = \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{arrives}\end{pmatrix} - \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{departs}\end{pmatrix} \qquad ....(i)$$

Arrival rate = (conc. in) × (inflow rate) = $c_{in} \times a_{in}$



If y(t) denotes the amount of substance in the tank at time t & V(t) denotes the amount of mixture in tank at that time

Departure rate = $\begin{pmatrix}\text{concentration in}\\ \text{container at time t}\end{pmatrix}$. (outflow rate) $= \frac{y(t)}{V(t)}\,.\,(a_{out})$

where volume of mixture at time t, V(t) = initial volume + (inflow rate – outflow rate) × t

$$= V_0 + (a_{in} - a_{out})t$$

Accordingly, Equation (i) becomes

$$\frac{dy(t)}{dt} = \text{(chemical's given arrival rate)} - \frac{y(t)}{V(t)} \cdot \text{(out flow rate)} \qquad \text{.......(ii)}$$

$$\frac{d(y(t))}{dt} = c_{in}a_{in} - \frac{y(t)}{V_0 + (a_{in} - a_{out})t} . a_{out}$$

This leads to a first order linear D.E. which can be solved to obtain y(t) i.e. amount of chemical at time 't'.

SOLVED EXAMPLE

Example 33 : A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour ?

Solution : Let y(t) be the amount of salt after t min.

Given y(0) = 20 kg

$$\text{rate in} = \left(\frac{0.03\text{kg}}{\text{L}}\right)\left(\frac{25\text{L}}{\text{min.}}\right) = \frac{0.75\text{kg}}{\text{min.}}$$

As $a_{in} = a_{out}$, so the tank always contains 5000 L of liquid so the conc. at time 't' is $\left(\frac{y(t)}{5000}\right)\frac{\text{kg}}{\text{L}}$

$$\text{so} \qquad \text{rate out} = \left(\frac{y(t)}{5000}\frac{\text{kg}}{\text{L}}\right)\left(\frac{25\text{L}}{\text{min}}\right) = \frac{y(t)}{200}\frac{\text{kg}}{\text{min}}$$

$$\frac{dy(t)}{dt} = 0.75 - \frac{y(t)}{200}$$

by solving as linear D.E. or variable separable and using initial condition, we get

$y(t) = 150 - 130\, e^{-t/200}$

The amount of salt after 30 min is $y(30) = 150 - 130\, e^{-30/100} = 38.1$ kg

<u>Problems for Self Practice</u>-09

1. A tank initially holds 10 lit. of fresh water. At t = 0, a brine solution containing $\frac{1}{2}$kg of salt per lit. is poured into the tank at a rate of 2 lit/min. while the well-stirred mixture leaves the tank at the same rate. Find

(a) the amount and

(b) the concentration of salt in the tank at any time t.

Answer : 1. (a) $-5e^{-0.2t} + 5$ kg (b) $\frac{1}{2}(-e^{0.2t} + 1)$ kg/ℓ

7.3 EXPONENTIAL GROWTH AND DECAY :

In general, if y(t) is the value of quantity y at time t and if the rate of change of y with respect to t is proportional to its value y(t) at that time, then

$$\frac{dy(t)}{dt} = ky(t), \text{ where k is a constant} \qquad ....(i)$$

$$\int \frac{dy(t)}{y(t)} = \int kdt$$

Solving, we get $y(t) = Ae^{kt}$

equation (i) is sometimes called the law of natural growth (if $k > 0$) or law of natural decay (if $k < 0$).

In the context of population growth, we can write

$$\frac{dP}{dt} = kP \qquad \text{or} \qquad \frac{1}{P}\frac{dP}{dt} = k$$

where k is growth rate divided by the population size; it is called **the relative growth rate.**

SOLVED EXAMPLE

Example 34 : A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is 50 kg of the material present and after two hours it is observed that the material has lost 10 percent of its original mass, find **(a)** an expression for the mass of the material remaining at any time t, **(b)** the mass of the material after four hours, and **(c)** the time at which the material has decayed to one half of its initial mass.

Solution : **(a)** Let N denote the amount of material present at time t.

So, $\frac{dN}{dt} - kN = 0$

This differential equation is separable and linear, its solution is

$N = ce^{kt}$ ....(i)

At t=0, we are given that N = 50. Therefore, from (i), $50 = ce^{k(0)}$ or c = 50. Thus,

$N = 50e^{kt}$ ....(ii)

At t = 2, 10 percent of the original mass of 50kg or 5kg has decayed. Hence, at t = 2, N = 50 – 5 = 45. Substituting these values into (ii) and solving for k, we have

$45 = 50e^{2k}$ or $k = \frac{1}{2}\ell n\frac{45}{50}$

Substituting this value into (ii), we obtain the amount of mass present at any time t as

$$N = 50e^{\frac{1}{2}(\ell n 0.9)t} \qquad .....(iii)$$

where t is measured in hours.

(b) We require N at t = 4. Substituting t = 4 into (iii) and then solving for N, we find

$N = 50e^{-2\,\ell n\,(0.9)}$ kg

(c) We require when N = 50/2 = 25. Substituting N = 25 into (iii) and solving for t, we find

$$25 = 50e^{\frac{1}{2}(\ell n\,0.9)t} \Rightarrow t = \ell n\left(\frac{1}{2}\right) / \left[\frac{1}{2}\ell n\,(0.9)\right] \text{hours}$$

MISCELLANEOUS EXAMPLES

Example 35 : For a certain curve y = f(x) satisfying $\frac{d^2y}{dx^2} = 6x - 4$, f(x) has a local minimum value 5 when x = 1. Find the equation of the curve and also the global maximum and global minimum values of f(x) given that $0 \le x \le 2$.

Solution : Integrating $\frac{d^2y}{dx^2} = 6x - 4$, we get $\frac{dy}{dx} = 3x^2 - 4x + A$

When x = 1, $\frac{dy}{dx} = 0$, so that A = 1. Hence

$$\frac{dy}{dx} = 3x^2 - 4x + 1 \qquad ...(i)$$

Integrating, we get $y = x^3 - 2x^2 + x + B$

When x = 1, y = 5, so that B = 5.

Thus we have $y = x^3 - 2x^2 + x + 5$.

From (i), we get the critical points x = 1/3, x = 1

At the critical point $x = \frac{1}{3}$, $\frac{d^2y}{dx^2}$ is negative.

Therefore at x = 1/3, y has a local maximum.

At x = 1, $\frac{d^2y}{dx^2}$ is positive.

Therefore at x = 1, y has a local minimum.

Also f(1) = 5, $f\left(\frac{1}{3}\right) = \frac{139}{27}$. f(0) = 5, f(2) = 7

Hence the global maximum value = 7, and the global minimum value = 5.

Example 36 : Solve the equation $x\int_0^x y(t)dt = (x+1)\int_0^x t\,y(t)dt, x > 0$

Solution : Differentiating the equation with respect to x, we get

$$xy(x) + 1.\int_0^x y(t)dt = (x+1)xy(x) + 1.\int_0^x ty(t)dt$$

i.e., $\int_0^x y(t)dt = x^2y(x) + \int_0^x ty(t)dt$

Differentiating again with respect to x, we get $y(x) = x^2 y'(x) + 2xy(x) + xy(x)$

i.e., $(1 - 3x)y(x) = \dfrac{x^2dy(x)}{dx}$

i.e., $\dfrac{(1-3x)dx}{x^2} = \dfrac{dy(x)}{y(x)}$,

integrating we get

i.e., $y = \dfrac{c}{x^3}e^{-1/x}$

Exercise # 1

PART-I : SUBJECTIVE QUESTIONS

Section (A) : Degree & Order, Differential equation formation

A-1. Find the order & degree of following differential equations.

(i) $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^6\right]^{1/4}$ (ii) $y = e^{\left(\frac{dy}{dx} + \frac{d^2y}{dx^2}\right)}$

(iii) $\sin\left(\frac{dy}{dx} + \frac{d^2y}{dx^2}\right) = y$ (iv) $e^{y'''} - xy'' + y = 0$

A-2 Identify the order of the following equations, (where a, b, c, d are parameters)

(i) $(\sin a)\, x + (\cos a)\, y = \pi$

(ii) $y^2 = 4a\, e^{x+b}$

(iii) $\ell n\, (ay) = be^x + c$

(iv) $y = \tan\left(\frac{\pi}{4} + ax\right) \tan\left(\frac{\pi}{4} - ax\right) + c\, e^{bx+d}$

A-3. Eliminate the arbitrary constants (a, b, c) and obtain the differential equation satisfied by it.

(i) $y = 2x + ce^x$ **(ii)** $y = \left(\frac{a}{x^2}\right) + bx$ **(iii)** $y = ae^{2x} + be^{-2x} + c$

(vi) $ax^2 + by^2 = c$

A-4 (i) Form a differential equation for the family of curves represented by $ax^2 + by^2 = 1$, where a & b are arbitrary constants.

(ii) Obtain the differential equation of the family of circles $x^2 + y^2 + 2gx + 2fy + c = 0$; where g , f & c are arbitrary constants.

(iii) Obtain the differential equation associated with the primitive , $y = c_1e^{3x} + c_2e^{2x} + c_3e^x$, where c_1 , c_2 , c_3 are arbitrary constants.

Section (B) : Variable separable, Homogeneous equation, polar substitution

B-1. Solve the following differential equations :

(i) $\frac{2dy}{dx} = \frac{y(x+1)}{x}$ **(ii)** $\sqrt{1+4x^2}\,dy = y^3x\,dx$

(iii) $(\tan y)\frac{dy}{dx} = \sin(x + y) + \sin(x - y)$ **(iv)** $\frac{\ell n(\sec x + \tan x)}{\cos x}dx = \frac{\ell n(\sec y + \tan y)}{\cos y}dy$

(v) $(1 - x^2)(1 - y)\, dx = xy\, (1 + y)\, dy$

(v) $\frac{dy}{dx} + \frac{\sqrt{(x^2-1)(y^2-1)}}{xy} = 0$

B-2. Solve the following differential equations :

(i) $\frac{dy}{dx} = (y - 4x)^2$ **(ii)** $\tan^2(x + y)dx - dy = 0$

B-3. Solve :

(i) dy/dx = y / x + tan y /x

(ii) $(1 + 2e^{x/y})\, dx + 2e^{x/y}\,(1 - x/y)\, dy = 0$

(iii) $\left[x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right] y - \left[y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right] x\ \frac{dy}{dx} = 0$

B-4 Find the equation of the curve satisfying $\frac{dy}{dx} = \frac{(y+x)(y-x)-2xy}{(x+y)(x-y)+2xy}$ and passing through (1, –1).

B-5. Solve :

(i) $(2x - y + 1)\, dx + (2y - x - 1)\, dy = 0$ (ii) $\frac{dy}{dx} = \frac{4x+6y+5}{3y+2x+4}$

(iii) $(2x + 3y - 5)\, dy + (3x + 2y - 5)\, dx = 0$

B-6. Solve the differential equation $(x - y)\, dy = (x + y + 1)\, dx$

B-7. Solve :

(i) $\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = x^2 + 2y^2 + \frac{y^4}{x^2}$ (ii) $\frac{xdx + ydy}{\sqrt{x^2+y^2}} = \frac{xdy - ydx}{x^2}$

Section (C) : Linear diff. eq. & bernaullis diff. eq.

C-1. Solve :

(i) $\sin x \frac{dy}{dx} + 3y = \cos x$ (ii) $(1 - x^2)\frac{dy}{dx} + 2xy = x\,(1 - x^2)^{1/2}$

(iii) $x\frac{dy}{dx} - y = 2x^2 \operatorname{cosec} 2x$ (iv) $\frac{dy}{dx} + \frac{x}{1+x^2}y = \frac{1}{2x(1+x^2)}$

C-2. Solve :

(i) $2\frac{dy}{dx} - y \sec x = y^3 \tan x$ (ii) $\frac{dy}{dx} = \frac{y^2 - x}{2y\,(x+1)}$

(iii) $(1 + y^2)\, dx = (\tan^{-1} y - x)dy$ (iv) $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1 + x)\, e^x \sec y$

C-3. (a) Find the integrating factor of the following equations

(i) $(x \log x)\frac{dy}{dx} + y = 2 \log x$ (ii) $\frac{dy}{dx} = y \tan x - y^2 \sec x$, is

(b) If the integrating factor of $x(1 - x^2)\, dy + (2x^2 y - y - ax^3)\, dx = 0$ is $e^{\int p\,.\,dx}$, then P is equal to

Section (D) : Exact diff. eq. , Higher degree differential equation and clairut's form

D-1. Solve :

(i) $x\,dx + y\,dy + 4y^3(x^2 + y^2)dy = 0$.

(ii) $(2x\,\ell ny)\,dx + \left(\frac{x^2}{y} + 3y^2\right)dy = 0$

(iii) $ye^{-x/y}\,dx - (xe^{-x/y} + y^3)\,dy = 0$

D-2. Solve :

(i) $y(x^2y + e^x)\,dx = e^x\,dy$ (ii) $2y \sin x \frac{dy}{dx} + y^2 \cos x + 2x = 0$

D-3. Solve

(i) $\frac{d^2y}{dx^2} = \frac{dy}{dx}$ (ii) $\frac{d^3y}{dx^3} = 8\frac{d^2y}{dx^2}$ satisfying $y(0) = \frac{1}{8}$, $y_1(0) = 0$ and $y_2(0) = 1$.

D-4. Solve

(i) $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$ (ii) $\left(y - x\frac{dy}{dx}\right)\left(\frac{dy}{dx} - 1\right) = \frac{dy}{dx}$ (iii) $y + x.\frac{dy}{dx} = x^4\left(\frac{dy}{dx}\right)^2$

Section (E) : Geometrical and physical problems

E–1 The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through (1 , 1).

E–2. Find the curve for which any tangent intersects the y-axis at the point equidistant from the point of tangency and the origin.

E-3. (i) Find the curve such that the area of the trapezium formed by the co-ordinate axes, ordinate of an arbitrary point & the tangent at this point equals half the square of its abscissa.

(ii) A curve in the first quadrant is such that the area of the triangle formed in the first quadrant by the x-axis, a tangent to the curve at any of its point P and radius vector of the point P is 2sq. units. If the curve passes through (2,1), find the equation of the curve.

E-4 Find the curve such that the ordinate of any of its points is the geometric mean between the abscissa and the sum of the abscissa and subnormal at the point.

E-5. (i) The temperature T of a cooling object drops at a rate which is proportional to the difference $T - S$, where S is constant temperature of the surrounding medium.

Thus, $\frac{dT}{dt} = -k(T - S)$, where $k > 0$ is a constant and t is the time. Solve the differential equation if it is given that $T(0) = 150$.

(ii) A tank contains 100 litres of fresh water. A solution containing 1 gm/litre of soluble lawn fertilizer runs into the tank at the rate of 1 lit/min and the mixture in pumped out of the tank at the rate of 3 litres/min. Find the time when the amount of fertilizer in the tank is maximum.

(iii) The slope of the tangent at any point of a curve is λ times the slope of the straight line joining the point of contact to the origin. Formulate the differential equation representing the problem and hence find the equation of the curve.

E-6 The population P of a town decreases at a rate proportional to the number by which the population exceeds 1000, proportionality constant being $k > 0$. Find

(i) Population at any time t, given initial population of the town being 2500.

(ii) If 10 years later the population has fallen to 1900, find the time when the population will be 1500.

(iii) Predict about the population of the town in the long run.

PART-II : OBJECTIVE QUESTIONS

Section (A) : Degree & Order, Differential equation formation

A-1. The order and degree of the differential equation $r\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$ are respectively

(A) 2, 2 (B) 2, 3 (C) 2, 1 (D) none of these

A-2 The order of the differential equation whose general solution is given by

$y = (C_1 + C_2) \sin (x + C_3) - C_4\, e^{x+C_5}$ is

(A) 5 (B) 4 (C) 2 (D) 3

A-3 Number of values of $m \in N$ for which $y = e^{mx}$ is a solution of the differential equation $D^3y - 3D^2y - 4Dy + 12y = 0$, is

(A) 0 (B) 1 (C) 2 (D) more than 2

A-4. If p and q are order and degree of differential equation $y^2\left(\frac{d^2y}{dx^2}\right)^2 + 3x\left(\frac{dy}{dx}\right)^{1/3} + x^2y^2 = \sin x$, then :

(A) $p > q$ (B) $\frac{p}{q} = \frac{1}{2}$ (C) $p = q$ (D) $p < q$

A-5 Family $y = Ax + A^3$ of curve represented by the differential equation of degree

(A) three (B) two (C) one (D) four

A-6 The differential equation whose solution is $(x - h)^2 + (y - k)^2 = a^2$ is (a is a constant)

(A) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\frac{d^2y}{dx^2}$ (B) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$

(C) $\left[1+\left(\frac{dy}{dx}\right)\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$ (D) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^2 = a^2\left(\frac{d^2y}{dx^2}\right)^3$

A-7 The differential equations of all conics whose centre lie at the origin is of order :

(A) 2 (B) 3 (C) 4 (D) none of these

Section (B) : Variable separable, Homogeneous equation, polar substitution

B-1. If $\frac{dy}{dx} = e^{-2y}$ and $y = 0$ when $x = 5$, the value of x for $y = 3$ is

(A) e^5 (B) $e^6 + 1$ (C) $\frac{e^6+9}{2}$ (D) $\log_e 6$

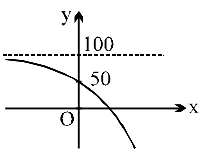
B-2 If $y = \dfrac{x}{\ln|cx|}$ (where c is an arbitrary constant) is the general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x} + \phi\left(\dfrac{x}{y}\right)$ then the function $\phi\left(\dfrac{x}{y}\right)$ is :

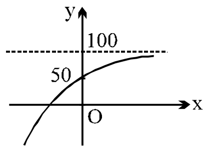
(A) $\dfrac{x^2}{y^2}$ (B) $-\dfrac{x^2}{y^2}$ (C) $\dfrac{y^2}{x^2}$ (D) $-\dfrac{y^2}{x^2}$

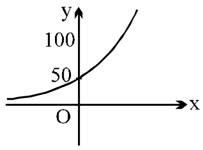
B-3. If $\dfrac{dy}{dx} = 1 + x + y + xy$ and $y(-1) = 0$, then function y is

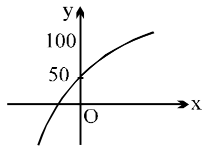
(A) $e^{(1-x)^2/2}$ (B) $e^{(1+x)^2/2} - 1$ (C) $\log_e(1 + x) - 1$ (D) $1 + x$

B-4 Which one of the following curves represents the solution of the initial value problem
$Dy = 100 - y$, where $y(0) = 50$

(A) 

(B) 

(C) 

(D) 

B-5. The solution of $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$ is

(A) $\sin^{-1}x \sin^{-1}y = C$ (B) $\sin^{-1}x = C\sin^{-1}y$ (C) $\sin^{-1}x - \sin^{-1}y = C$ (D) $\sin^{-1}x + \sin^{-1}y = C$

B-6 Integral curve satisfying $y' = \dfrac{x^2+y^2}{x^2-y^2}$, $y(1) = 2$, has the slope at the point (1, 2) of the curve, equal to

(A) $-\dfrac{5}{3}$ (B) -1 (C) 1 (D) $\dfrac{5}{3}$

B-7 The real value of m for which the substitution, $y = u^m$ will transform the differential equation, $2x^4y\dfrac{dy}{dx} + y^4 = 4x^6$ into a homogeneous equation is :

(A) m = 0 (B) m = 1 (C) m = 3/2 (D) no value of m

B-8. The solution of differential equation $\dfrac{dy}{dx} = \dfrac{y}{x} + 2\dfrac{\phi(y/x)}{\phi'(y/x)}$ is

(A) $x^2\phi(y/x) = k$ (B) $y^2\phi(y/x) = k$ (C) $\phi(y/x) = kx^2$ (D) $\phi(y/x) = ky^2$

B-9 Solution of differential equation $\dfrac{xdx - ydy}{x^2-y^2} = \dfrac{xydy - y^2dx}{x^3}$ is

(A) $x^2 - y^2 = k\,e^{\left(\frac{y}{x}\right)^2}$ (B) $x^2 - y^2 = k\,e^{-\left(\frac{y}{x}\right)^2}$ (C) $x^2 - y^2 = k\,e^{\frac{y}{x}}$ (D) $x^2 - y^2 = k\,e^{-\frac{y}{x}}$

Section (C) : Linear differential equation

C-1. The solution of the differential equation $\frac{dy}{dx} - ky = 0, y(0) = 1$, approaches zero when $x \to \infty$, if

(A) $k = 0$ (B) $k > 0$ (C) $k < 0$ (D) k may be any real value

C-2 The solution of $\frac{dv}{dt} + \frac{k}{m}v = -g$ is

(A) $v = ce^{-\frac{k}{m}t} - \frac{mg}{k}$ (B) $v = c - \frac{mg}{k}e^{-\frac{k}{m}t}$ (C) $v\,e^{-\frac{k}{m}t} = c - \frac{mg}{k}$ (D) $v\,e^{\frac{k}{m}t} = c - \frac{mg}{k}$

C-3 Solution of the differential equation $x\frac{dy}{dx} + y = x^2y^4$ is

(A) $\frac{1}{y^3} = 2x^2 - kx^3$ (B) $\frac{1}{y^3} = 4x^3 + kx$

(C) $\frac{1}{y^3} = 3x^2 + kx$ (D) $\frac{1}{y^3} = 3x^2 + kx^3$

Section (D) : Exact diff. eq. , Higher degree differential equation and clairut's form

D-1_. Solution of differential equation $(1 + x\sqrt{x^2+y^2})\,dx + y(-1 + \sqrt{x^2+y^2})\,dy = 0$ is

(A) $x - \frac{y^2}{2} + \frac{1}{3}(x^2 + y^2)^{3/2} + c = 0$ (B) $x + \frac{y^2}{2} + \frac{1}{3}(x^2 + y^2)^{3/2} + c = 0$

(C) $x - \frac{y^2}{2} - \frac{1}{3}(x^2 + y^2)^{3/2} + c = 0$ (D) $x + \frac{y^2}{2} - \frac{1}{3}(x^2 + y^2)^{3/2} + c = 0$

D-2 The equation of the curve satisfying the differential equation $y_2(x^2 + 1) = 2xy_1$ passing through the point (0, 1) and having slope of tangnet at x = 0 as 3, is

(A) $y = x^2 + 3x + 2$ (B) $y^2 = x^2 + 3x + 1$ (C) $y = x^3 + 3x + 1$ (D) none of these

D-3 Solution of $\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0$ is

(A) $y = 3x^2 + 9$ (B) $y = 3x + 9$ (C) $y = \frac{4}{3}x^2$ (D) $y = 9x + 3$

Section (E) : Geometrical and physical problems

E-1. The equation of the curve whose subnormal is constant a is

(A) $y = ax + b$ (B) $y^2 = 2ax + b$ (C) $ay^2 - x^2 = a$ (D) $ay^2 + x^2 = a$

E-2 A curve C passes through origin and has the property that at each point (x, y) on it the normal line at that point passes through (1, 0). The equation of a common tangent to the curve C and the parabola $y^2 = 4x$ is

(A) $x = 0$ (B) $y = 0$ (C) $y = x + 1$ (D) $x + y + 1 = 0$

E-3. The x-intercept of the tangent to a curve is equal to the ordinate of the point of contact. The equation of the curve through the point (1, 1) is

(A) $ye^{\frac{x}{y}} = e$ (B) $xe^{\frac{x}{y}} = e$ (C) $xe^{\frac{y}{x}} = e$ (D) $ye^{\frac{y}{x}} = e$

E–4 Spherical rain drop evaporates at a rate proportional to its surface area. The differential equation corresponding to the rate of change of the radius of the rain drop if the constant of proportionality is K > 0, is

(A) $\frac{dr}{dt} + K = 0$ (B) $\frac{dr}{dt} - K = 0$ (C) $\frac{dr}{dt} = Kr$ (D) none

PART-III : MATCH THE COLUMN

1 Match the following

Column - I	Column - II
(A) Solution of $y - \frac{xdy}{dx} = y^2 + \frac{dy}{dx}$ is	(p) $xy^2 = 2y^5 + c$
(B) Solution of $(2x - 10y^3)\frac{dy}{dx} + y = 0$ is	(q) $\sec y = x + 1 + ce^x$
(C) Solution of $\sec^2 y\, dy + \tan y\, dx = dx$ is	(r) $(x + 1)(1 - y) = cy$
(D) Solution of $\sin y \frac{dy}{dx} = \cos y (1 - x \cos y)$ is	(s) $\tan y = 1 + ce^{-x}$

2. Match the following

Column - I	Column - II
(A) $xdy = y(dx + ydy)$, $y(1) = 1$ and $y(x_0) = -3$, then $x_0 =$	(p) $\frac{1}{4}$
(B) If y(t) is solution of $(t + 1)\frac{dy}{dt} - ty = 1$, $y(0) = -1$, then $y(1) =$	(q) -15
(C) $(x^2 + y^2)\, dy = xydx$ and $y(1) = 1$ and $y(x_0) = e$, then $x_0 =$	(r) $-\frac{1}{2}$
(D) $\frac{dy}{dx} + \frac{2y}{x} = 0$, $y(1) = 1$, then $y(2) =$	(s) $\sqrt{3}\, e$

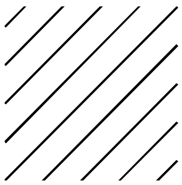
Exercise # 2

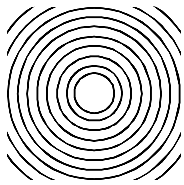
PART-I : ONLY ONE OPTION CORRECT TYPE

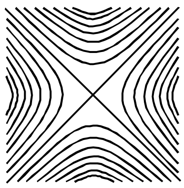
1. The differential equation of all parabola having their axis of symmetry coinciding with the x-axis is

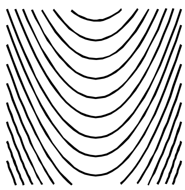
(A) $y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=0$ (B) $y\dfrac{d^2x}{dy^2}+\left(\dfrac{dx}{dy}\right)^2=0$ (C) $y\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=0$ (D) none of these

2✍ The general solution of the differential equation $\dfrac{dy}{dx}=\dfrac{1-x}{y}$ is a family of curves which looks most like which of the following?

(A) 

(B) 

(C) 

(D) 

3. The solution y = y(x) of the differential equation $(x^2 + y^2)\, dy = xy\, dx$ statisfies the conditions $y(1) = 1$ and $y(x_0) = e$, then value of x_0 is –

(A) $\sqrt{3}\,e$ (B) $\sqrt{2(e^2-1)}$ (C) $\sqrt{2(e^2+1)}$ (D) $\sqrt{(e^2+1)/2}$

4✍ If $y_1(x)$ is a solution of the differential equation $\dfrac{dy}{dx} + f(x)\, y = 0$, then a solution of differential equation $\dfrac{dy}{dx} + f(x)\, y = r(x)$ is

(A) $\dfrac{1}{y(x)}\displaystyle\int y_1(x)\,dx$

(B) $y_1(x)\displaystyle\int \dfrac{r(x)}{y_1(x)}\,dx$

(C) $\displaystyle\int r(x)y_1(x)\,dx$

(D) none of these

5. If $y_1(x)$ and $y_2(x)$ are two solutions of $\dfrac{dy}{dx} + f(x)\, y = r(x)$ then $y_1(x) + y_2(x)$ is solution of :

(A) $\dfrac{dy}{dx} + f(x)\, y = 0$

(B) $\dfrac{dy}{dx} + 2f(x)\, y = r(x)$

(C) $\dfrac{dy}{dx} + f(x)\, y = 2\, r(x)$

(D) $\dfrac{dy}{dx} + 2f(x)\, y = 2r(x)$

6✍ Solution of differential equation $\frac{dy}{dx} = (\sin x - \sin y)\frac{\cos x}{\cos y}$ is

(A) $\sin y = \sin x - 1 + c\,e^{\sin x}$ (B) $\sin y = \sin x - 1 + c\,e^{-\cos x}$
(C) $\sin y = \sin x - 1 + c\,e^{-\sin x}$ (D) $\sin y = \cos x - 1 + c\,e^{-\sin x}$

7✍ The solution of $y\,dx - x\,dy + 3x^2 y^2 e^{x^3}\,dx = 0$ is

(A) $\frac{x}{y} + e^{x^3} = C$ (B) $\frac{x}{y} - e^{x^3} = 0$ (C) $-\frac{x}{y} + e^{x^3} = C$ (D) $\frac{y}{x} + e^{x^3} = c$

8✍ The solution of the differential equation $(x^2 \sin^3 y - y^2 \cos x)\,dx + (x^3 \cos y \sin^2 y - 2y \sin x)\,dy = 0$ is

(A) $x^3 \sin^3 y = 3y^2 \sin x + C$ (B) $x^3 \sin^3 y + 3y^2 \sin x = C$
(C) $x^2 \sin^3 y + y^3 \sin x = C$ (D) $2x^2 \sin y + y^2 \sin x = C$

9. The equation of the curve which is such that the portion of the axis of x cut off between the origin and tangent at any point is proportional to the ordinate of that point is

(A) $x = y\,(b - a \log y)$ (B) $\log x = by^2 + a$
(C) $x^2 = y\,(a - b \log y)$ (D) $y = x\,(b - a \log x)$
(a is constant of proportionality)

10. A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x-axis. Then equation of the curve is.

(A) $x^2 + y^2 = 2x$ (B) $2x^2 + y^2 = 3x$ (C) $x^2 + 2y^2 = 3x$ (D) $x^2 - y^2 = x - 1$

11✍ Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of water depth y, where the constant of proportionality k > 0 depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and $k = \frac{1}{15}$ then the time to drain the tank if the water is 4 meter deep to start with is

(A) 30 min (B) 45 min (C) 60 min (D) 80 min

12. A tank consists of 50 litres of fresh water. Two litres of brine each litre containing 5 gms of dissolved salt are run into tank per minute; the mixture is kept uniform by stirring, and runs out at the rate of one litre per minute then the amount of salt present after 10 minutes is.

(A) 90 gms (B) 80 gms (C) $81\frac{2}{3}$ gms (D) $91\frac{2}{3}$ gms

13. f(x) is a continuous and differentiable function defined in $x \in [0, \infty)$. If f(0) = 1 and $f'(x) > 3f(x)\ \forall\, x \geq 0$ then

(A) $f(x) \leq e^{3x}\ \forall x \geq 0$ (B) $f(x) \leq e^{-3x}\ \forall x \geq 0$ (C) $f(x) > e^{3x}\ \forall x \geq 0$ (D) $f(x) \geq e^{3x}\ \forall x \geq 0$

14✍ Let f(x) be differentiable on the interval $(0, \infty)$ such that f(1) = 1 and $\lim_{t\to x}\frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each x > 0. Then f(x) is

(A) $\frac{1}{3x} + \frac{2x^2}{3}$ (B) $\frac{-1}{3x} + \frac{4x^2}{3}$ (C) $\frac{-1}{x} + \frac{2}{x^2}$ (D) $\frac{1}{x}$

15. Let f be a non-negative function defined on the interval [0, 1]. If $\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)\,dt$, $0 \le x \le 1$ and $f(0) = 0$, then

(A) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(B) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(C) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

(D) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

PART-II : NUMERICAL QUESTIONS

1. If differential equations of the curves $c(y + c)^2 = x^3$, where 'c' is any arbitrary constant is $12y(y')^2 + ax = bx(y')^3$ then $\frac{a}{b}$ is equal to

2 The value of $\lim_{x\to\infty} y(x)$ obtained from the differential equation $\frac{dy}{dx} = y - y^2$, where $y(0) = 2$ is

3 By putting $X = x^m - h$ and $Y = y^n - k$ (where m, n, h, k are real numbers) in the differential equation $(2x^2 + 3y^2 - 7)x\,dx - (3x^2 + 2y^2 - 8)y\,dy = 0$ it become homogenious differential equation then the value of $(m + n + h + k)$ is equal to

4. If y(x) satisfies the differential equation ; $\cos^2 x\,(dy/dx) - (\tan 2x)\,y = \cos^4 x$, $|x| < \frac{\pi}{4}$, and $y(0) = 0$ then $\frac{54y\left(\frac{\pi}{6}\right)}{\sqrt{3}}$ is equal to

5 Let y_1 and y_2 are two different solutions of the equation $y' + P(x)\,.\,y = Q(x)$.
Such that the linear combination $\alpha y_1 + \beta y_2$ is also solution of given differential equation . Then value of $\alpha + \beta$ is

6. If solution of the differential equation $\frac{dy}{dx} = \frac{1}{x\cos y + \sin 2y}$ is $x = ce^{\sin y} - k(1 + \sin y)$, then k is equal to

7. If y(x) satisfies the equation $y'(x) = y(x) + \int_0^1 y\,dx$ & $y(0) = 1$ then value of y(1) is (take e = 2.71)

8 If the equation of curve passing through (3, 4) and satisfying the differential equation
$y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0$ is $Ax + By + 2 = 0$ then value of $\frac{90A}{B^4}$ is

9 Let the curve $y = f(x)$ passes through (4, –2) satisfy the differential equation,
$y(x + y^3)\,dx = x(y^3 - x)\,dy$ & let $y = g(x) = \int_{1/8}^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_{1/8}^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$, $0 \le x \le \frac{\pi}{2}$. If the area of the region bounded by curves $y = f(x)$, $y = g(x)$ and $x = 0$ is $\left(\frac{3^a \pi^4}{2^b}\right)$ where $a, b \in N$ then b/a is equal to

10✍ Let c_1 and c_2 be two integral curves of the differential equation $\frac{dy}{dx} = \frac{x^2 - y^2}{x^2 + y^2}$. A line passing through origin meets c_1 at $P(x_1, y_1)$ and c_2 at $Q(x_2, y_2)$. If $c_1 : y = f(x)$ and $c_2 : y = g(x)$ then the value of $\frac{f'(x_1)}{g'(x_2)}$ is

11. A curve passing through point (1, 2) possessing the following property; the segment of the tangent between the point of tangency & the x–axis is bisected at the point of intersection with the y–axis. If A is area bounded by the curve & line x = 1 then $\frac{100}{A}$ is equal to

12✍ The curve which passes through the point (2, 0) such that the segment of the tangent between the point of tangency & the y–axis has a constant length equal to 2 is given by $y = \pm\left[\sqrt{a-x^2} + b\ell n\frac{2-\sqrt{a-x^2}}{x}\right]$ $(a, b \in N)$ then a^3/b is equal to

13. The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. If equation of tangent to the curve at (1, 3) is ax + by + 5 = 0 then value of b^4/a is equal to

14. A & B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlet are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservior at that time. One hour after the water is released, the quantity of water in reservoir A is 1.5 times the quantity of water in reservior B. If after $\log_a 2$ hours do both the reservoirs have the same quantity of water then $\frac{1}{a}$ is equal to

15✍ It is known that the decay rate of radium is directly proportional to its quantity at each given instant. If the law of variation of a mass of radium as a function of time is given by $m = m_0 e^{\frac{1}{t_0} \ell n\left(1-\frac{\alpha}{\lambda}\right)t}$ where m_0 was the mass of the radius at t = 0 and α % of the original mass of radium decay during time t_0 then $1/\lambda$ is equal to

PART - III : ONE OR MORE THAN ONE CORRECT QUESTION

1. Which of the following statements is/are correct ?

(A) The differential equation of all conics whose axes coincide with the axes of co-ordinates is of order 2.

(B) The differential equation of all staright lines which are at a fixed distance p from origin is of degree 2.

(C) The differential equation of all parabola each of which has a latus rectum 4a & whose axes are parallel to y-axis is of order 2.

(D) The differential equation of all parabolas of given vertex, is of order 3.

2. The differential equation of all circles in a plane must be $\left(y_1 = \frac{dy}{dx}, y_2 = \frac{d^2y}{dx^2}, \text{.........etc.}\right)$

(A) $y_3\left(1+y_1^2\right) - 3y_1y_2^2 = 0$ (B) of order 3 and degree 1

(C) of order 3 and degree 2 (D) $y_3^2\left(1-y_1^2\right) - 3y_1y_2^2 = 0$

3✍ If $y = e^{-x}\cos x$ and $y_n + k_n y = 0$, where $y_n = \frac{d^n y}{dx^n}$ and k_n, $n \in N$ are constants.

(A) $k_4 = 4$ (B) $k_8 = -16$ (C) $k_{12} = 20$ (D) $k_{16} = -24$

4. Solution of the differential equation $\frac{dy}{dx} + \frac{1+y^2}{\sqrt{1-x^2}} = 0$ is

(A) $\tan^{-1} y + \sin^{-1} x = c$ (B) $\tan^{-1} x + \sin^{-1} y = c$

(C) $\tan^{-1} y \cdot \sin^{-1} x = c$ (D) $\cot^{-1} \frac{1}{y} + \cos^{-1} \sqrt{1-x^2} = c$

5✍ The solution of $\frac{dx}{dy} + y = ye^{(n-1)x}$, $(n \neq 1)$

(A) $\frac{1}{n-1}\ell n\left(\frac{e^{(n-1)x}-1}{e^{(n-1)x}}\right) = \frac{y^2}{2} + C$ (B) $e^{(1-n)x} = 1 + ce^{(n-1)\frac{y^2}{2}}$

(C) $\ell n\,(1 + ce^{(n-1)\frac{y^2}{2}}) + nx + 1 = 0$ (D) $e^{(n-1)x} = ce^{(n-1)+\frac{(n-1)y^2}{2}} + 1$

6. The solution of $(x + y + 1)\, dy = dx$ are

(A) $x + y + 2 = Ce^y$ (B) $x + y + 4 = = C \log y$

(C) $\log (x + y + 2) = Cy$ (D) $\log (x + y + 2) = C + y$

7✍ Which of the following statements is/are correct ?

(A) $f(x, y) = x^2e^{\frac{x}{y}} + \frac{y^3}{x} + y^2\ell n\left(\frac{y}{x}\right)$ is a homogenous function of degree two.

(B) $f(x, y) = \frac{\sin y + x}{\sin 2y + x\cos y}$ is homogenous function of degree one.

(C) $x\sin\left(\frac{y}{x}\right)dy + \left(y\sin\frac{y}{x} - x\right)dx = 0$ is a homogenous differential equation.

(D) $f(x, y) = e^{\frac{y}{x}} + \tan\frac{y}{x}$ is homogenous function of degree zero.

8. Consider the differential equation $\frac{dy}{dx} + y\tan x = x\tan x + 1$. Then

(A) The integral curves satisfying the differential equation and given by $y = x + c\cos x$.

(B) The angle at which the integral curves cut the y-axis is $\frac{\pi}{4}$.

(C) Tangents to all the integral curves at their point of intersection with y-axis are parallel.

(D) none of these

9✍ Solution of differential equation $f(x)\frac{dy}{dx} = f^2(x) + f(x)\,y + f'(x)\,y$ is

(A) $y = f(x) + ce^x$ (B) $y = -f(x) + ce^x$ (C) $y = -f(x) + ce^x f(x)$ (D) $y = cf(x) + e^x$

10. Consider $g(x)=\begin{cases}\sin x & 0\le x<\dfrac{\pi}{2}\\ \cos x & x\ge\dfrac{\pi}{2}\end{cases}$ and a continuous function y = f(x) satisfies $\dfrac{5dy}{dx}+5y=g(x)$, $f(0) = 0$, then-

(A) $f\left(\frac{\pi}{4}\right)=\frac{e^{-\pi/4}}{10}$ (B) $f\left(\frac{\pi}{4}\right)=\frac{e^{-\pi/4}-1}{10}$ (C) $f\left(\frac{\pi}{2}\right)=\frac{e^{-\pi/2}+1}{10}$ (D) $f\left(\frac{\pi}{2}\right)=e^{-\pi/2}$

11✍ A solution of the differential equation $y_1\, y_3 = 3y_2^{\,2}$ can be

(A) $x = A_1y^2 + A_2\, y + A_3$ (B) $x = A_1\, y + A_2$

(C) $x = A_1\, y^2 + A_2\, y$ (D) $y = A_1\, x^2 + A_2x + A_3$

12. Which of the following can be the solutions of differential equation $\dfrac{d^2y}{dx^2} + 6y\left(\dfrac{dy}{dx}\right)^2 = 0$

(A) $x = y^3 + y + 1$ (B) $x = y^3 - 1$ (C) $y = x^{1/3}$ (D) $y = (x + 8)^{1/3}$

13. The solution of differential equation $\left(\dfrac{dy}{dx}\right)^2-\dfrac{dy}{dx}(e^x + e^{-x}) + 1 = 0$ is

(A) $y = e^x + c$ (B) $y = -e^{-x} + c$ (C) $y = 2e^x + 3e^{-x} + c$ (D) $ye^x + 1 = ce^x$

14✍ The differential equation $\dfrac{d^2y}{dx^2} + y + \cot^2 x = 0$ must be satisfied by

(A) $y = 2 + c_1 \cos x + \sqrt{c_2}\, \sin x$ (B) $y = \cos x\, .\, \ell n\left(\tan\dfrac{x}{2}\right) + 2$

(C) $y = 2 + c_1 \cos x + c_2 \sin x + \cos x \log\left(\tan\dfrac{x}{2}\right)$

(D) all the above

15✍ The function f(x) satisfying the equation, $f^2(x) + 4 f'(x)\, .\, f(x) + [f'(x)]^2 = 0$.

(A) $f(x) = c\, .\, e^{(2-\sqrt{3})x}$ (B) $f(x) = c\, .\, e^{(2+\sqrt{3})x}$

(C) $f(x) = c\, .\, e^{(\sqrt{3}-2)x}$ (D) $f(x) = c\, .\, e^{-(2+\sqrt{3})x}$

16. Let f(x) is a continuous function which takes positive values for $x \ge 0$ and satisfy $\int_0^x f(t)\,dt = x\sqrt{f(x)}$ with $f(1) = \dfrac{1}{2}$ then

(A) $f(x) = \dfrac{1}{\left[1+\left(1-\sqrt{2}\right)x\right]^2}$

(B) $f\left(\cot\dfrac{\pi}{8}\right) = \dfrac{1}{4}$

(C) Area bounded by f(x) and x-axis between x = 0 to x = $\sqrt{2}$ + 1 is $\dfrac{1}{2\left(\sqrt{2}-1\right)}$ square units.

(D) $f\left(\sin\dfrac{\pi}{4}\right) = 2$

17 Let f(x), $x \geq 0$ be a non negative continuous function & let $F(x) = \int_0^x f(t)\,dt$, $x \geq 0$. If for some $c > 0$, $f(x) \leq c\,F(x)$ for all $x \geq 0$ then
(A) $f(0) = 0 \ \forall \ x \geq 0$
(B) $f(0) = 0$
(C) $e^{-cx} F(x)$ is a non-increasing function on $[0, \infty)$
(D) $F(x) \leq 0 \ \forall \ x \leq 0$

18. A differentiable function satisfies equation $f(x) = \int_0^x \left(f(t)\cos t - \cos(t - x)\right)dt$ then

(A) $f''\left(\frac{\pi}{2}\right) = e$ (B) $\lim_{x \to -\infty} f(x) = 1$
(C) f(x) has minimum value $1 - e^{-1}$ (D) $f'(0) = -1$

PART - IV : COMPREHENSION

Comprehension # 1

Differential equations are solved by reducing them to the exact differential of an expression in x & y i.e., they are reduced to the form d(f(x, y)) = 0
e.g. :

$$\frac{xdx + ydy}{\sqrt{x^2 + y^2}} = \frac{ydx - xdy}{x^2}$$

$$\Rightarrow \quad \frac{1}{2}\frac{2xdx + 2ydy}{\sqrt{x^2 + y^2}} = -\frac{xdy - ydx}{x^2}$$

$$\Rightarrow \quad d\left(\sqrt{x^2 + y^2}\right) = -\,d\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad d\left(\sqrt{x^2 + y^2} + \frac{y}{x}\right) = 0$$

$\therefore$ solution is $\sqrt{x^2 + y^2} + \frac{y}{x} = c$.

Use the above method to answer the following question (3 to 5)

1 The general solution of $(2x^3 - xy^2)\,dx + (2y^3 - x^2y)\,dy = 0$ is
(A) $x^4 + x^2y^2 - y^4 = c$ (B) $x^4 - x^2y^2 + y^4 = c$ (C) $x^4 - x^2y^2 - y^4 = c$ (D) $x^4 + x^2y^2 + y^4 = c$

2. General solution of the differential equation $\frac{xdy}{x^2 + y^2} + \left(1 - \frac{y}{x^2 + y^2}\right)dx = 0$ is

(A) $x + \tan^{-1}\left(\frac{y}{x}\right) = c$ (B) $x + \tan^{-1}\frac{x}{y} = c$ (C) $x - \tan^{-1}\left(\frac{y}{x}\right) = c$ (D) none of these

3. General solution of the differential equation $e^y\,dx + (xe^y - 2y)\,dy = 0$ is
(A) $xe^y - y^2 = c$ (B) $ye^x - x^2 = c$ (C) $ye^y + x = c$ (D) $xe^y - 1 = cy^2$

Comprehension # 2

In order to solve the differential equation of the form $a_0 \dfrac{d^n y}{dx^n} + a_1 \dfrac{d^{n-1}y}{dx^{n-1}} + ... + a_n y = 0$, where a_0, a_1, a_2 are constants.

We take the auxiliary equation $a_0 D^n + a_1 D^{n-1} + ... + a_n = 0$

Find the roots of this equation and then solution of the given differential equation will be as given in the following table.

Roots of the auxiliary equation	Corresponding complementary function
1. One real root α_1	$c_1 e^{\alpha_1 x}$
2. Two real and different roots α_1 and α_2	$c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}$
3. Two real and equal roots α_1 and α_1	$(c_1 + c_2 x)\ e^{\alpha_1 x}$
4. Three real and equal roots $\alpha_1, \alpha_1, \alpha_1$	$(c_1 + c_2 x + c_3 x^2)\ e^{\alpha_1 x}$
5. One pair of imaginary roots $\alpha \pm i\beta$	$(c_1 \cos \beta x + c_2 \sin \beta x)\ e^{\alpha x}$
6. Two pair of equal imaginary roots $\alpha \pm i\beta$ and $\alpha \pm i\beta$	$[(c_1 + c_2 x) \cos \beta x + (c_1 + c_2 x) \sin \beta x]\ e^{\alpha x}$

Solution of the given differential equation will be y = sum of all the corresponding parts of the complementary functions.

4. Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$.

(A) $y = (c_1 + c_2 x)e^x$ (B) $y = (c_1 e^x + c_2 e^x)$ (C) $y = (c_1 x)e^x$ (D) none of these

5. Solve $\dfrac{d^2y}{dx^2} + a^2 y = 0$.

(A) $y = (c_1 \cos ax + c_2 \sin ax)e^{ax}$
(B) $y = c_1 \cos ax + c_2 \sin ax$
(C) $y = c_1 e^{ax} + c_2 e^{-ax}$
(D) none of these

6. Solve $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$

(A) $y = (c_1 + c_2 x + c_3 x^2)e^x$
(B) $y = x\,(c_1 e^x + c_2 e^{2x} + c_3 e^{3x})$
(C) $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$
(D) none of these

Comprehension # 3 (Q.No. 7 to 8)

Let f(x) be a differentiable function, satisfying $f(0) = 2$, $f'(0) = 3$ and $f''(x) = f(x)$

7 Graph of y = f(x) cuts x -axis at

(A) $x = -\dfrac{1}{2}\ell n5$ (B) $x = \dfrac{1}{2}\ell n5$ (C) $x = -\ell n5$ (D) $x = \ell n5$

8 Area enclosed by y = f(x) in the second quadrant is

(A) $3 + \dfrac{1}{2}\ell n\sqrt{5}$ (B) $2 + \dfrac{1}{2}\ell n5$ (C) $3 - \sqrt{5}$ (D) 3

Comprehension # 4 (Q.No. 9 to 10)

ORTHOGONAL TRAJECTORY :

An orthogonal trajectory of a given system of curves is defined to be a curve which cuts every member of a given family of curve at right angle.

Steps to find orthogonal trajectory :

(i) Let f (x, y, c) = 0 be the equation of the given family of curves, where 'c' is an arbitrary constant.

(ii) Differentiate the given equation w.r.t. x and then eliminate c.

(iii) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the equation obtained in (ii).

(iv) Solve the differential equation obtained in (iii).

Hence solution obtained in (iv) is the required orthogonal trajectory.

9. Find the orthogonal trajectrories of the circles $x^2 + y^2 - ay = 0$ (where a is parameter).

(A) $y^2 + x^2 = cx$ (B) $y^2 - x^2 = cx$ (C) $x^2 - y^2 = cy$ (D) $x^2 + y^2 = cxy$

10. Which of the following family of curves is self othrogonal ?

(A) $y^2 = 4a(x + a)$ (B) $x^2 + y^2 = a$ (C) $yx^2 = a$ (D) $y = ae^{-x}$

Exercise # 3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

*** Marked Questions may have more than one correct option.**

1. Let f be a real-valued differentiable function on **R** (the set of all real numbers) such that f(1) = 1. If the y-intercept of the tangent at any point P(x, y) on the curve y = f(x) is equal to the cube of the abscissa of P, then the value of f(–3) is equal to **[IIT-JEE 2010, Paper-1, (3, 0), 84]**

2. Let $f : [1, \infty) \to [2, \infty)$ be a differentiable function such that f(1) = 2. If $6\int_1^x f(t)dt = 3xf(x) - x^3$ for all $x \geq 1$, then the value of f(2) is **[IIT-JEE 2011, Paper-1, (4, 0), 80]**

3. Let $y'(x) + y(x)\,g'(x) = g(x)\,g'(x)$, $y(0) = 0$, $x \in \mathbf{R}$, where $f'(x)$ denotes $\frac{df(x)}{dx}$ and g(x) is a given non-constant differentiable function on R with g(0) = g(2) = 0. Then the value of y(2) is **[IIT-JEE 2011, Paper-2, (4, 0), 80]**

4.* If y(x) satisfies the differential equation $y' - y\tan x = 2x\sec x$ and y(0) = 0, then **[IIT-JEE 2012, Paper-1, (4, 0), 70]**

(A) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (B) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(C) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$ (D) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

5. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right)$, x > 0. Then the equation of the curve is **[JEE (Advanced) 2013, Paper-1, (2, 0), 60]**

(A) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$ (B) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x + 2$

(C) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (D) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

6. The function y = f(x) is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$ in (–1, 1) satisfying f(0) = 0. Then $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx$ is **[JEE (Advanced) 2014, Paper-2, (3, –1),60]**

(A) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ (B) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$ (C) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (D) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

7. Let $f : [0, 2] \to R$ be a function which is continuous on [0, 2] and is differentiable on (0, 2) with f(0) = 1. Let $F(x) = \int_0^{x^2} f(\sqrt{t})dt$ for $x \in [0, 2]$. If F'(x) = f'(x) for all $x \in (0, 2)$, then F(2) equals **[JEE (Advanced) 2014, Paper-2, (3, –1),60]**

(A) $e^2 - 1$ (B) $e^4 - 1$ (C) $e - 1$ (D) e^4

8*. Let y(x) be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If y(0) = 2, then which of the following statements is (are) true ? **[JEE (Advanced) 2015, P-1 (4, –2), 88]**

(A) y(–4) = 0
(B) y(–2) = 0
(C) y(x) has a critical point in the interval (–1, 0)
(D) y(x) has no critical point in the interval (–1, 0)

9*✍ Consider the family of all circles whose centers lie on the straight line y = x. If this family of circles is represented by the differential equation Py'' + Qy' + 1 = 0, where P, Q are functions of x, y and y' (here $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$), then which of the following statements is (are) true?

[JEE (Advanced) 2015, P-1 (4, –2), 88]

(A) P = y + x
(B) P = y – x
(C) $P + Q = 1 - x + y + y' + (y')^2$
(D) $P - Q = x + y - y' - (y')^2$

10. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0, x > 0$, passes through the point (1,3). The the solution curve- **[JEE(Advanced)-2016, Paper-1 (4,–2), 62]**

(A) intersects y = x + 2 exactly at one point
(B) intersects y = x + 2 exactly at two points
(C) intersects $y = (x + 2)^2$
(D) does NOT intersect $y = (x + 3)^2$

11✍ Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0,\infty)$ and $f(1) \neq 1$. Then **[JEE(Advanced)-2016, Paper-1 (3,–1), 62]**

(A) $\lim\limits_{x\to 0^+} f'\left(\frac{1}{x}\right) = 1$
(B) $\lim\limits_{x\to 0^+} xf\left(\frac{1}{x}\right) = 2$
(C) $\lim\limits_{x\to 0^+} x^2 f'(x) = 0$
(D) $|f(x)| \leq 2$ for all $x \in (0,2)$

12✍ If y = y(x) satisfies the differential equation $8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1}dx,\ x > 0$ and $y(0) = \sqrt{7}$, then y(256) = **[JEE(Advanced)-2017, Paper-2 (3, –1), 61]**

(A) 80
(B) 3
(C) 16
(D) 9

13✍ If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$, then **[JEE(Advanced)-2017, Paper-2 (4, –2), 61]**

(A) $f(x) > e^{2x}$ in $(0,\infty)$
(B) $f(x)$ is decreasing in $(0,\infty)$
(C) $f(x)$ is increasing in $(0,\infty)$
(D) $f'(x) < e^{2x}$ in $(0,\infty)$

14. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two non-constant differentiable functions. If $f'(x) = (e^{(f(x) - g(x))})g'(x)$ for all $x \in \mathbb{R}$, and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE ?

[JEE(Advanced)-2018, Paper-1, (4, –2), 60]

(A) $f(2) < 1 - \log_e 2$
(B) $f(2) > 1 - \log_e 2$
(C) $g(1) > 1 - \log_e 2$
(D) $g(1) < 1 - \log_e 2$

15 Let $f : (0, \pi) \rightarrow \mathbb{R}$ be a twice differentiable function such that $\lim\limits_{t \to x} \frac{f(x)\sin t - f(t)\sin x}{t - x} = \sin^2 x$ for all x $\in (0, \pi)$. If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE ?

[JEE(Advanced)-2018, Paper-2, (4, –2), 60]

(A) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$

(B) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(D) $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

16. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} = (2 + 5y)(5y - 2)$,

then the value of $\lim\limits_{x \to -\infty} f(x)$ is _____ . **[JEE(Advanced)-2018, Paper-2, (3, 0), 60]**

17 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function with $f(0) = 1$ and satisfying the equation

$$f(x + y) = f(x)f'(y) + f'(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

Then, then value of $\log_e(f(4))$ is _____ . **[JEE(Advanced)-2018, Paper-2, (3, 0), 60]**

PART - II : JEE(MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1 Solution of the differential equation cosx dy = y(sinx – y) dx, $0 < x < \frac{\pi}{2}$ is **[AIEEE 2010 (4, –1), 144]**

(1) y sec x = tan x + c (2) y tan x = sec x + c (3) tanx = (sec x + c)y (4) secx = (tanx + c) y

2. Let I be the purchase value of an equipment and V(t) be the value after it has been used for t years. The value V(t) depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T - t)$, where k > 0 is a constant and T is the total life in years of the equipment. Then the scrap value V(T) of the equipment is :

[AIEEE 2011, I, (4, –1), 120]

(1) $T^2 - \frac{1}{k}$ (2) $I - \frac{kT^2}{2}$ (3) $I - \frac{k(T-t)^2}{2}$ (4) e^{-kT}

3. If $\frac{dy}{dx} = y + 3 > 0$ and y(0) = 2, then y(ℓn2) is equal to : **[AIEEE 2011, I, (4, –1), 120]**

(1) 7 (2) 5 (3) 13 (4) –2

4✍ The curve that passes through the point (2, 3), and has the property that the segment of any tangent to it lying between the coordinate axes is bisected by the point of contact is given by: **[AIEEE 2011, II, (4, –1), 120]**

(1) $2y - 3x = 0$ (2) $y = \frac{6}{x}$ (3) $x^2 + y^2 = 13$ (4) $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 2$

5. Consider the differential equation $y^2dx + \left(x - \frac{1}{y}\right)dy = 0$. If y (1) = 1, then x is given by :

[AIEEE 2011, II, (4, –1), 120]

(1) $4 - \frac{2}{y} - \frac{e^{\frac{1}{y}}}{e}$ (2) $3 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$ (3) $1 + \frac{1}{y} - \frac{e^{\frac{1}{y}}}{e}$ (4) $1 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$

6. The population p(t) at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt} = 0.5\,p(t) - 450$. If p(0) = 850, then the time at which the population becomes zero is :

[AIEEE-2012, (4, –1)/120]

(1) $2\,\ell n\,18$ (2) $\ell n\,9$ (3) $\frac{1}{2}\,\ell n\,18$ (4) $\ell n\,18$

7✍ At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production P w.r.t. additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is **[AIEEE - 2013, (4, –¼),360]**

(1) 2500 (2) 3000 (3) 3500 (4) 4500

8✍ Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}\,p(t) - 200$. If p(0) = 100, then p(t) equals : **[JEE(Main) 2014, (4, – 1), 120]**

(1) $600 - 500\,e^{t/2}$ (2) $400 - 300\,e^{-t/2}$ (3) $400 - 300\,e^{t/2}$ (4) $300 - 200\,e^{-t/2}$

9. Let y(x) be the solution of the differential equal $(x \log x)\frac{dy}{dx} + y = 2x \log x$, $(x \geq 1)$. Then y(e) is equal to

[JEE(Main) 2015, (4, –1), 120]

(1) e (2) 0 (3) 2 (4) 2e

10✍ If a curve y = f(x) passes through the point (1, –1) and satisfies the differential equation, y(1 + xy) dx = x dy, then $f\left(-\frac{1}{2}\right)$ is equal to : **[JEE(Main)-2016, (4, –1), 120]**

(1) $\frac{4}{5}$ (2) $-\frac{2}{5}$ (3) $-\frac{4}{5}$ (4) $\frac{2}{5}$

11. If $(2 + \sin x)\frac{dy}{dx} + (y+1)\cos x = 0$ and y(0) = 1, then $y\left(\frac{\pi}{2}\right)$ is equal to :- **[JEE(Main)-2017, (4, –1), 120]**

(1) $\frac{4}{3}$ (2) $\frac{1}{3}$ (3) $-\frac{2}{3}$ (4) $-\frac{1}{3}$

12. Let y = y(x) be the solution of the differential equation $\sin x \frac{dy}{dx} + y \cos x = 4x$, $x \in (0, \pi)$. If $y\left(\frac{\pi}{2}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to : **[JEE(Main)-2018, (4, –1), 120]**

(1) $\frac{-8}{9\sqrt{3}}\pi^2$ (2) $-\frac{8}{9}\pi^2$ (3) $-\frac{4}{9}\pi^2$ (4) $\frac{4}{9\sqrt{3}}\pi^2$

13. If y = y(x) is the solution of the differential equation, $x\frac{dy}{dx} + 2y = x^2$ satisfying y(1) = 1, then $y\left(\frac{1}{2}\right)$ is equal to : **[JEE(Main) 2019, Online (09-01-19) Shift-1 (4, –1), 120]**

(1) $\frac{7}{64}$ (2) $\frac{13}{16}$ (3) $\frac{49}{16}$ (4) $\frac{1}{4}$

14. Let f:[0,1]→R be such that $f(xy) = f(x).f(y)$ for all x,y∈[0,1], and $f(0)\neq 0$. If y = y(x) satisfies the differential equation, $\frac{dy}{dx} = f(x)$ with y(0) = 1, then $y\left(\frac{1}{4}\right) + y\left(\frac{3}{4}\right)$ is equal to

[JEE(Main) 2019, Online (09-01-19) Shift-2 (4, –1), 120]

(1) 4 (2) 3 (3) 5 (4) 2

15. If y = y(x) is the solution of the differential equation, $e^y\left(\frac{dy}{dx} - 1\right) = e^x$ such that y(0) = 0, then y(1) is equal to : **[JEE(Main) 2020, Online (09-01-20) Shift-1 (4, –1), 100]**

(1) $2 + \log_e 2$ (2) 2e (3) $\log_e 2$ (4) $1 + \log_e 2$

16. Let y = y(x) be the solution curve of the differential equation, $(y^2 - x)\frac{dy}{dx} = 1$, satisfying y(0) = 1. This curve intersects the x-axis at a point whose abscissa is :

[JEE(Main) 2020, Online (07-01-20) Shift-2 (4, –1), 100]

(1) 2 + e (2) 2 (3) 2 – e (4) –e

Answers

Exercise # 1

PART-I

Section (A)

A-1. (i) order = 2, degree = 4
(ii) order = 2, degree = 1
(iii) order = 2, degree = 1
(iv) Order = 3, degree is not defined

A-2 (i) 1 (ii) 1 (iii) 2
(iv) 2

A-3. (i) $y' - y = 2(1 - x)$ (ii) $x^2y'' + 2xy' - 2y = 0$
(iii) $y''' = 4y'$
(iv) $xyy'' + x(y')^2 - yy' = 0$

A-4 (i) $xy\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx}$
(ii) $[1 + (y')^2].\ y''' - 3y'(y'')^2 = 0$
(iii) $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$

Section (B)

B-1. (i) $\ell ny^2 = x + \ell n|x| + k$
(ii) $-\dfrac{1}{2y^2} = \dfrac{1}{4}\sqrt{1+4x^2} + k$
(iii) $\sec y = -2\cos x + C$
(iv) $\ln^2(\sec x + \tan x) - \ln^2(\sec y + \tan y) = c$
(v) $\ln x\,(1 - y)^2 = c - \dfrac{1}{2}y^2 - 2y + \dfrac{1}{2}x^2$
(vi) $\sqrt{x^2 - 1} - \sec^{-1} x + \sqrt{y^2 - 1} = c$

B-2. (i) $\dfrac{y - 4x + 2}{y - 4x - 2} = ce^{-4x}$
(ii) $2(x - y) = c + \sin 2(x + y)$

B-3. (i) $x = C' \sin (y/x)$ (ii) $x + 2ye^{x/y} = c$
(iii) $xy \cos\left(\dfrac{y}{x}\right) = c$

B-4. $x + y = 0$

B-5. (i) $x^2 + y^2 - xy + x - y = c$
(ii) $y - 2x + \dfrac{3}{8}\ \ell n\,(24y + 16x + 23) = c$
(iii) $4xy + 3(x^2 + y^2) - 10(x + y) = c$

B-6. $\arctan \dfrac{2y+1}{2x+1} = \ln c\sqrt{x^2 + y^2 + x + y + \dfrac{1}{2}}$

B-7. (i) $\dfrac{1}{2(x^2+y^2)} = \dfrac{y}{x} + K$
(ii) $\sqrt{x^2+y^2} = \dfrac{y}{x} + c$

Section (C)

C-1. (i) $\left(\dfrac{1}{3} + y\right)\tan^3\dfrac{x}{2} = c + 2\tan\dfrac{x}{2} - x$
(ii) $y = c(1 - x^2) + \sqrt{1 - x^2}$
(iii) $y = cx + x\ln\tan x$
(iv) $y\sqrt{1+x^2} = c + \dfrac{1}{2}\ln\left[\tan\dfrac{1}{2}\arctan x\right]$.

Another form is $y\sqrt{1+x^2} = c + \dfrac{1}{2}\ln\dfrac{\sqrt{1+x^2} - 1}{x}$

C-2. (i) $\dfrac{1}{y^2} = -1 + (c + x)\cot\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)$
(ii) $y^2 = -1 + (x + 1)\ln\dfrac{c}{x+1}$ or $x + (x + 1)\ln\dfrac{c}{x+}$
(iii) $x = ce^{-\arctan y} + \arctan y - 1$
(iv) $\sin y = (e^x + c)(1 + x)$

C-3. (a) (i) $|\ell n\, x|$ (ii) $|\sec x|$

(b) $\dfrac{(2x^2-1)}{x(1-x^2)}$

Section (D)

D-1. (i) $\frac{1}{2}\ \ell n(x^2 + y^2) + y^4 = C$

(ii) $x^2 \ell ny + y^3 = c$

(iii) $2e^{-x/y} + y^2 = c$

D-2. (i) $\dfrac{1}{y}\ e^x = -\dfrac{x^3}{3} + c$ (ii) $y^2 \sin x = -x^2 + c$

D-3. (i) $c_1e^x + c_2$ (ii) $64y = (e^{8x} - 8x) + 7$

D-4. (i) $(y - cx^2)(yx^3 - c) = 0$

(ii) General solution: $y = cx + \dfrac{c}{c-1}$, Singular solution : $y = (\sqrt{x} \pm 1)^2$

(iii) General solution : $xy + c = c^2x$, Singular solution : $4x^2y + 1 = 0$

Section (E)

E–1. $x^2 + y^2 - 2x = 0$

E–2. $x^2 + y^2 = cx$

E-3. (i) $y = cx^2 \pm x$; (ii) $xy = 2$

E-4. $y^2 = \dfrac{x^4+c^4}{2x^2}$ or $y^2 + 2x^2\ell nx = cx^2$

E-5. (i) $\dfrac{T-S}{150-S} = e^{-kt}$ (ii) $27\frac{7}{9}$ minutes

(iii) $y = kx^{\lambda}$ where, k is some constant

E-6. (i) $P = 1000 + 1500e^{-kt}$ where $k = \dfrac{1}{10}\ln\left(\dfrac{5}{3}\right)$

(ii) $T = 10 \log_{5/3}(3)$; (iii) $P = 1000$ as $t\to\infty$

PART-II

Section (A)

A-1.	(A)	**A-2.**	(D)	**A-3.**	(C)
A-4.	(D)	**A-5.**	(A)	**A-6.**	(B)
A-7.	(B)				

Section (B)

B-1.	(C)	**B-2.**	(D)	**B-3.**	(B)
B-4.	(B)	**B-5.**	(D)	**B-6.**	(A)
B-7.	(C)	**B-8.**	(C)	**B-9.**	(A)

Section (C)

C-1.	(C)	**C-2.**	(A)	**C-3.**	(D)

Section (D)

D-1.	(A)	**D-2.**	(C)	**D-3.**	(B)

Section (E)

E-1.	(B)	**E-2.**	(A)	**E-3.**	(A)
E–4.	(A)				

PART-III

1. (A) – (r), (B) – (p), (C) – (s), (D) – (q)

2. (A) – (q), (B) – (r), (C) – (s), (D) – (p)

Exercise # 2

PART-I

1.	(A)	**2.**	(B)	**3.**	(A)
4.	(B)	**5.**	(C)	**6.**	(C)
7.	(A)	**8.**	(A)	**9.**	(A)
10.	(A)	**11.**	(C)	**12.**	(D)
13.	(D)	**14.**	(A)	**15.**	(C)

PART-II

1.	3.37 or 3.38			**2.**	1
3.	5	**4.**	20.25	**5.**	01.00
6.	02.00	**7.**	12.79	**8.**	11.25
9.	04.75	**10.**	01.00	**11.**	37.50
12.	32.00	**13.**	20.25	**14.**	00.75
15.	00.01				

PART - III

1.	(A,C)	**2.**	(A,B)	**3.**	(A,B)
4.	(A,D)	**5.**	(A,B,D)	**6.**	(A,D)
7.	(A,C,D)	**8.**	(A,B,C)	**9.**	(C)
10.	(A,C)	**11.**	(A,B,C)	**12.**	(A,B,C,D)
13.	(A,B,D)	**14.**	(B,C)	**15.**	(C,D)
16.	(B,C)	**17.**	(A,B,C)	**18.**	(A,D)

PART - IV

1.	(B)	**2.**	(A)	**3.**	(A)
4.	(A)	**5.**	(B)	**6.**	(C)
7.	(A)	**8.**	(C)	9.	(A)
10.	(A)				

Exercise # 3

PART - I

1. 9

2. Bonus (Taking $x = 1$, the integral becomes zero, whereas the right side of the equation gives 5. Therefore, the function f does not exist.)

3.	0	**4.**	(A,D)	**5.**	(A)
6.	(B)	**7.**	(B)	**8.**	(A)
9.	(B,C)	**10.**	(A,D)	**11.**	(A)
12.	(B)	**13.**	(A,C)	**14.**	(B,C)
15.	(B,C,D)	**16.**	0.4	**17.**	2

PART - II

1.	(4)	**2.**	(2)	**3.**	(1)
4.	(2)	**5.**	(3)	**6.**	(1)
7.	(3)	**8.**	(3)	**9.**	(3)
10.	(1)	**11.**	(2)	**12.**	(2)
13.	(3)	**14.**	(2)	**15.**	(4)
16.	(3)				

SUBJECTIVE QUESTIONS

1. Find the integral curve of the differential equation $x(1 - x\ell ny)\dfrac{dy}{dx} + y = 0$ which passes through (1, 1/e).

2. Solve the differential equation, $(x^2 + 4y^2 + 4xy)\, dy = (2x + 4y + 1)\, dx$.

3. Solve : $\dfrac{dy}{dx} + \left(\dfrac{2x}{1+x^2}\right)y = \dfrac{1}{(1+x^2)^2}$ given that y = 0, when x = 1.

4. Find the equation of the curve which passes through the origin and the tangent to which at every point (x, y) has slope equal to $\dfrac{x^2 + 2xy - 1}{1 + x^2}$.

5. Show that the integral curves of the equation $(1 - x^2)\dfrac{dy}{dx} + xy = ax$ are ellipses, with the centres at the point (0, a) and the axes parallel to the co-ordinate axes, each curve having one constant axis whose length is equal to 2.

6. Solve : $\dfrac{dy}{dx} - y\ell n2 = 2^{\sin x}(\cos x - 1)\ell n2$, y being bounded when $x \to +\infty$.

7. Let y_1 and y_2 are two different solutions of the equation $y' + P(x)\,.\,y = Q(x)$.
Prove that $y = y_1 + C(y_2 - y_1)$ is the general solution of the same equation (C is a constant)

8. Consider the differential equation, $\dfrac{dy}{dx} + P(x)y = Q(x)$

(i) If two particular solutions of given equation u(x) and v(x) are known, find the general solution of the same equation in terms of u(x) and v(x).

(ii) If α and β are constants such that the linear combinations $\alpha \cdot u(x) + \beta \cdot v(x)$ is a solution of the given equation, find the relation between α and β.

(iii) If w(x) is the third particular solution different from u(x) and v(x) then find the ratio $\dfrac{v(x) - u(x)}{w(x) - u(x)}$.

9. Solve the following differential equations.

(i) $3\dfrac{dy}{dx} + \dfrac{2y}{x+1} = \dfrac{x^3}{y^2}$ (ii) $x^2y - x^3\dfrac{dy}{dx} = y^4 \cos x$

10. Solve the following differential equations.

(i) $(x^2 + y^2 + a^2)\, y\dfrac{dy}{dx} + x\,(x^2 + y^2 - a^2) = 0$ (ii) $(1 + \tan y)\,(dx - dy) + 2x\, dy = 0$

11. If y_1 & y_2 be solutions of the differential equation $\dfrac{dy}{dx} + Py = Q$, where P & Q are functions of x alone, and $y_2 = y_1 z$, then prove that $z = 1 + a e^{-\int \frac{Q}{y_1} dx}$, 'a' being an arbitrary constant.

12. Solve the differential equation $\left\{\frac{y^2}{(x-y)^2}-\frac{1}{x}\right\}dx + \left\{\frac{1}{y}-\frac{x^2}{(x-y)^2}\right\}dy = 0$

13. $(1 + xy)\, y + (1 - xy)\, x\, \frac{dy}{dx} = 0$

14. Use the substitution $y^2 = a - x$ to reduce the equation $y^3\frac{dy}{dx}+x+y^2=0$ to homogeneous form and hence solve it. (where 'a' is variable)

15. Solve the difff. equation; $y\cos\frac{y}{x}(xdy-ydx)+x\sin\frac{y}{x}(xdy+ydx)=0$, when $y(1) = \frac{\pi}{2}$.

16. Let f(x) be a differentiable function and satisfy $f(0) = 2$, $f'(0) = 3$ and $f''(x) = f(x)$. Find
(a) the range of the function f(x)
(b) the value of the function when $x = \ell n2$
(c) the area enclosed by $y = f(x)$ in the 2nd quadrant

17. A curve $y = f(x)$ passes through the point p (1, 1). The normal to the curve at P is; $a(y - 1) + (x - 1) = 0$. If the slope of the tangent at any point on the curve is proportional to the ordinate of the point, determine the equation of the curve. Also obtain the area bounded by the y-axis, the curve & the normal to the curve at P.

18. Consider a curved mirror $y = f(x)$ passing through (8, 6) having the property that all rays emerging from origin after getting reflected from the mirror becomes parallel to x - axis. Find the equation of curve (s)

19. Find the curve for which sum of the lengths of the tangent and subtangent at any of its point is proportional to the product of the co-ordinates of the point of tangency, the proportionality factor is equal to k.

20. Find the curve $y = f(x)$ where $f(x) \geq 0$, $f(0) = 0$, bounding a curvilinear trapezoid with the base [0, x] whose area is proportional to $(n + 1)^{th}$ power of f(x). It is known that $f(1) = 1$

21. Find the nature of the curve for which the length of the normal at the point P is equal to the radius vector of the point P.

22. A curve passing through (1 , 0) such that the ratio of the square of the intercept cut by any tangent off the y–axis to the subnormal is equal to the ratio of the product of the co–ordinates of the point of tangency to the product of square of the slope of the tangent and the subtangent at the same point. Determine all such possible curves.

23. A curve is such that the length of the polar radius of any point on the curve is equal to the length of the tangent drawn at this point. Form the differential equation and solve it to find the equation of the curve.

24. Let $f(x, y, c_1) = 0$ and $f(x, y, c_2) = 0$ define two integral curves of a homogeneous first order differential equation. If P_1 and P_2 are respectively the points of intersection of these curves with an arbitrary line, $y = mx$ then prove that the slopes of these two curves at P_1 and P_2 are equal.

25. A country has a food deficit of 10 %. Its population grows continuously at a rate of 3 % per year. Its annual food production every year is 4 % more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self–sufficient in food after 'n' years, where 'n' is the smallest integer bigger than or equal to, $\frac{\ell n\ 10 - \ell n\ 9}{\ell n\,(1.04) - 0.03}$.

26. Let the function $\ell n\ f(x)$ is defined where f(x) exists for $x \geq 2$ & k is fixed positive real number, prove that if $\frac{d}{dx}(x \cdot f(x)) \leq -k f(x)$ then $f(x) \leq A x^{-1-k}$ where A is independent of x.

27. Find the differentiable function which satisfies the equation $f(x) = -\int_0^x f(t)\tan t\,dt + \int_0^x \tan(t-x)\,dt$ where $x \in (-\pi/2, \pi/2)$

Answers

1. $x(ey + \ell ny + 1) = 1$

2. $y = \ell n\left((x+2y)^2 + 4(x+2y)+2\right) - \frac{3}{2\sqrt{2}}\ell n\left(\frac{x+2y+2-\sqrt{2}}{x+2y+2+\sqrt{2}}\right) + c$

3. $y(1 + x^2) = \tan^{-1}x - \frac{\pi}{4}$

4. $y = (x - 2\tan^{-1} x)(1 + x^2)$

6. $y = 2^{\sin x}$

8. (i) $y = u(x) + K(u(x) - v(x))$, where K is any constant ; (ii) $\alpha + \beta = 1$; (iii) constant

9. (i) $y^3 (x + 1)^2 = \frac{x^6}{6} + \frac{2}{5} x^5 + \frac{1}{4} x^4 + c$ (ii) $x^3 y^{-3} = -3\sin x + c$

10. (i) $(x^2 + y^2)^2 + 2a^2 (y^2 - x^2) = c$ (ii) $x e^y (\cos y + \sin y) = e^y \sin y + C$

12. $\ell n\left(\frac{y}{x}\right) + \frac{xy}{y-x} = C$

13. $\ell n \frac{x}{y} - \frac{1}{xy} = c$

14. $\frac{1}{2}\ell n\,|x^2 + a^2| - \tan^{-1}\left(\frac{a}{x}\right) = c$ where, $a = x + y^2$

15. $xy \sin (y/x) = \frac{\pi}{2}$

16. (a) $(-\infty,\infty)$ (b) $\frac{19}{4}$ (c) $3-\sqrt{5}$

17. $e^{a(x-1)}$, $\frac{1}{a}\left[a - \frac{1}{2} + e^{-a}\right]$ sq. unit

18. $y^2 = 4(1 + x)$ or (या) $y^2 = 36(9 - x)$

19. $y = \pm\frac{1}{k}\ \ell n\,|c(k^2x^2 - 1)|$

20. $y = x^{1/n}$

21. Rectangular hyperbola or circle.

22. $x = e^{2\sqrt{y/x}}$; $x = e^{-2\sqrt{y/x}}$

23. $y = kx$ or $xy = c$

25. 19

27. $\cos x - 1$

Self Assessment Paper

JEE ADVANCED

Maximum Marks : 62 **Total Time : 1:00 Hr**

SECTION-1 : ONE OPTION CORRECT (Marks - 12)

1. A curve y = f(x) passes through the point P(1, 1), the equation of the normal at P(1, 1) to the curve y = f(x) is (x – 1) + a(y – 1) = 0 and the slope of tangent at any pont on the curve is proportional to ordinate of the point then equation of the curve is

(A) $x^2 + y^2 = 1$ (B) $y^2 = 2$ (C) $(y - 1)^2 = a(x - 1)$ (D) $y = e^{a(x-1)}$

2. Solution of differential equation $\frac{dy}{dx} = \frac{y^3 + y^2\sqrt{y^2 - x^2}}{x^3}$ is

(A) $y + \sqrt{y^2 - x^2} = kxy$ (B) $\sqrt{y^2 - x^2} = kxy$

(C) $y^2 + x^2 = kxy$ (D) $y^2 - x^2 = kx^2y^2$

3. The solution of differential equation $yy' = x\left\{\frac{y^2}{x^2} + \frac{f(y^2/x^2)}{f'(y^2/x^2)}\right\}$ is

(A) $f(y^2/x^2) = cx^2$ (B) $x^2 f(y^2/x^2) = c^2y^2$ (C) $x^2f(y^2/x^2) = c$ (D) $f(y^2/x^2) = cy/x$

4. Solution of differential equation $\{y + x\sqrt{xy}(x + y)\}\,dx + \{y\sqrt{xy}(x + y) - x\}\,dy = 0$ is

(A) $\frac{x^2+y^2}{2} + \tan^{-1}\sqrt{\frac{y}{x}} = c$ (B) $\frac{x^2+y^2}{2} + 2\tan^{-1}\sqrt{\frac{x}{y}} = c$

(C) $\frac{x^2+y^2}{2} + 2\cot^{-1}\sqrt{\frac{x}{y}} = c$ (D) $\frac{x^2+y^2}{2} + \cot^{-1}\sqrt{\frac{y}{x}} = c$

SECTION-2 : ONE OR MORE THAN ONE CORRECT (Marks - 32)

5. For differential equation of $y = c_1\left(e^{\sqrt{x} + c_2} + e^{-\sqrt{x} + c_2}\right)$:

(A) Order = 1 (B) Order = 2 (C) Degree = 1 (D) Degree = 2

6. Solution of $(1 - xy - x^3y^3)dx = x^2(x^2y^2 + 1)dy$ (where x > 0) is -

(A) $\ell nx = xy + \frac{(xy)^3}{3} + C$ (B) $\ell n\left(\frac{x}{e^{xy}}\right) = \frac{(xy)^3}{3} + C$

(C) $\ell ny = xy + \frac{(xy)^3}{3} + C$ (D) $\ell n\left(\frac{y}{e^{xy}}\right) = \frac{(xy)^3}{3} + C$

7. Let C be a curve such that normal at any point P on it meets x-axis and y-axis at A and B respectively. If BP : PA = 1 : 2 (internally) and curve passes through (0,4), then-

(A) The curve 'C' passes through $\left(\sqrt{10}, -6\right)$

(B) Equation of tangent at $\left(4, 4\sqrt{3}\right)$ is $\sqrt{3}x - 2y + 4\sqrt{3} = 0$

(C) Equation of tangent at $\left(4, 4\sqrt{3}\right)$ is $2x - \sqrt{3}y + 4 = 0$

(D) If point P is (8,12), then point A and point B are (24,0) and (0,18) respectively.

8. F is the family of curves in the xy–plane, given by $y = x + ce^{-x}$, where $c \in R$ is an arbitrary constant. The orthogonal trajectories of the curves of family F form the family F'. Then which of the following statements are true for F'. (If we replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in differential equation of given family of curve then we get differential equation of orthogonal trajectory of given family of curve)

(A) $xe^{y} = e^{y}(y-2)+c$, is general solution for F'

(B) $x = y-2$ is a particular solution for F'

(C) If F' passes through (4, 0), then F' is $x = (y-2)+6e^{y}$

(D) $xe = e(y-2)+c$, is general solution for F'

9. If $f(1) = e^2$ and $\frac{d}{dx}(f(x)) > 3x^2 f(x) \forall x \geq 1$ then f(x) cannot take values

(A) e (B) $\frac{9}{2}$ (C) 8 (D) 10

10. Which of the following curve(s) will have 2[nd] order differential equation

(A) $y = e^{Ax} + B$ (B) $y = A\sin(Bx + C)$

(C) $y = A\sin x + B\cos x$ (D) $y = Ae^{Bx+C}$

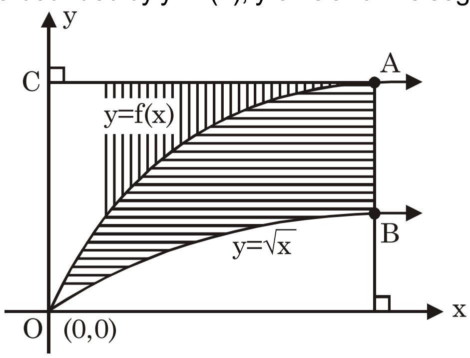
(where A,B,C are real paraemters)

11. Equation of curve passing through (3, 4) and satisfying the differential equation,

$$y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0 \text{ can be}$$

(A) $x^2 + y^2 = 25$ (B) $x^2 - y^2 + 7 = 0$ (C) $xy = 12$ (D) $x - y + 1 = 0$

12. Let f(x) is continuous function as shown in the figure. If the area bounded by the curve $y = f(x)$, $y = \sqrt{x}$ and line segment AB is equal to area bounded by $y = f(x)$, y-axis and line segment AC then which of the following is/are correct?



(A) If $\int_0^1 f(x)dx = \frac{1}{3}$ then $f\left(\frac{1}{4}\right) = \frac{1}{2}$

(B) If $\int_0^1 f(x)dx = \frac{1}{3}$ then $f\left(\frac{1}{4}\right) = 1$

(C) There is only one such function f

(D) There are infinitely many such function(s) f

SECTION-3 : NUMERICAL VALUE TYPE (Marks - 18)

13. If the solution of differential equation $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 12y$ is $y = Ax^m + Bx^{-n}$ then sum of all possible values of $|m + n|$ is

14. The real value of 'm' for which the substitution $y = u^m$ will transform the differential equation $2x^4y\frac{dy}{dx} + y^4 = 4x^6$ into homogeneous equation, is equal to

15. Let $f : [0, 1] \to [0, 1]$ be a continuous function such that

$x^2 + (f(x))^2 \le 1$ for all $x \in [0, 1]$ and $\int_0^1 f(x)dx = \frac{\pi}{4}$ Then $\frac{1}{\pi}\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{f(x)}{1-x^2}dx$ equals

16. Let there exists infinite number of vectors $\vec{x}$ such that $A\vec{X} = B\vec{X}$, where $A = \begin{bmatrix} f'(x) & f(x) \\ 1 & x \end{bmatrix}$

$B = \begin{bmatrix} \sin\left(\frac{f(x)}{x}\right) & 0 \\ 0 & 0 \end{bmatrix}$. If $\tan\left(\frac{f(x)}{kx}\right) = cx$, $c \in R$ and k is constant then $\frac{k^4}{3}$ is equal to

17. If the differential equation of a curve, passing through $\left(0, -\frac{\pi}{4}\right)$ and $\left(t, \frac{\pi}{4}\right)$ is

$\cos y\left(\frac{dy}{dx} + e^{-x}\right) + \sin y\left(e^{-x} - \frac{dy}{dx}\right) = e^{e^{-x}}$, then the value of $t.e^{e^{-t}}$ is equal to

18. A solid cylinder of height 6 meters has a conical portion of same height and radius $\frac{1}{3}$rd of height, removed from it. Rain water is falling in the conical hole at rate equal to π times the instantaneous radius of water surface inside hole, the time after which hole will fill up with water is.

Answers

1.	(D)	**2.**	(A)	**3.**	(A)	**4.**	(B)	**5.**	(A,C)
6.	(A,B)	**7.**	(A,C,D)	**8.**	(AB)	**9.**	(A,B)	**10.**	(A,C,D)
11.	(A,D)	**12.**	(A,D)	**13.**	14.00	**14.**	01.50	**15.**	00.08
16.	05.33	**17.**	01.41	**18.**	06.00				