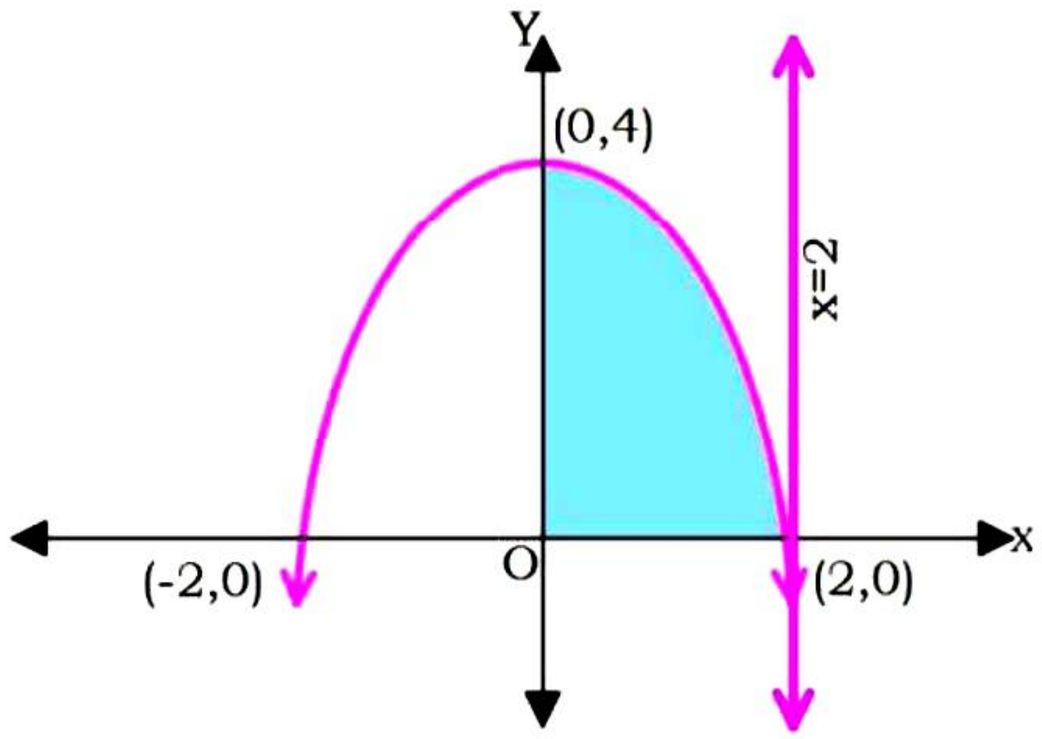
8 APPLICATION OF INTEGRALS

MODEL QUESTIONS

Question :

Find the area of the region bounded by the curves $y = 4 - x^2$, $x-$axis and the lines $x = 0$ and $x = 2$.



Solution :

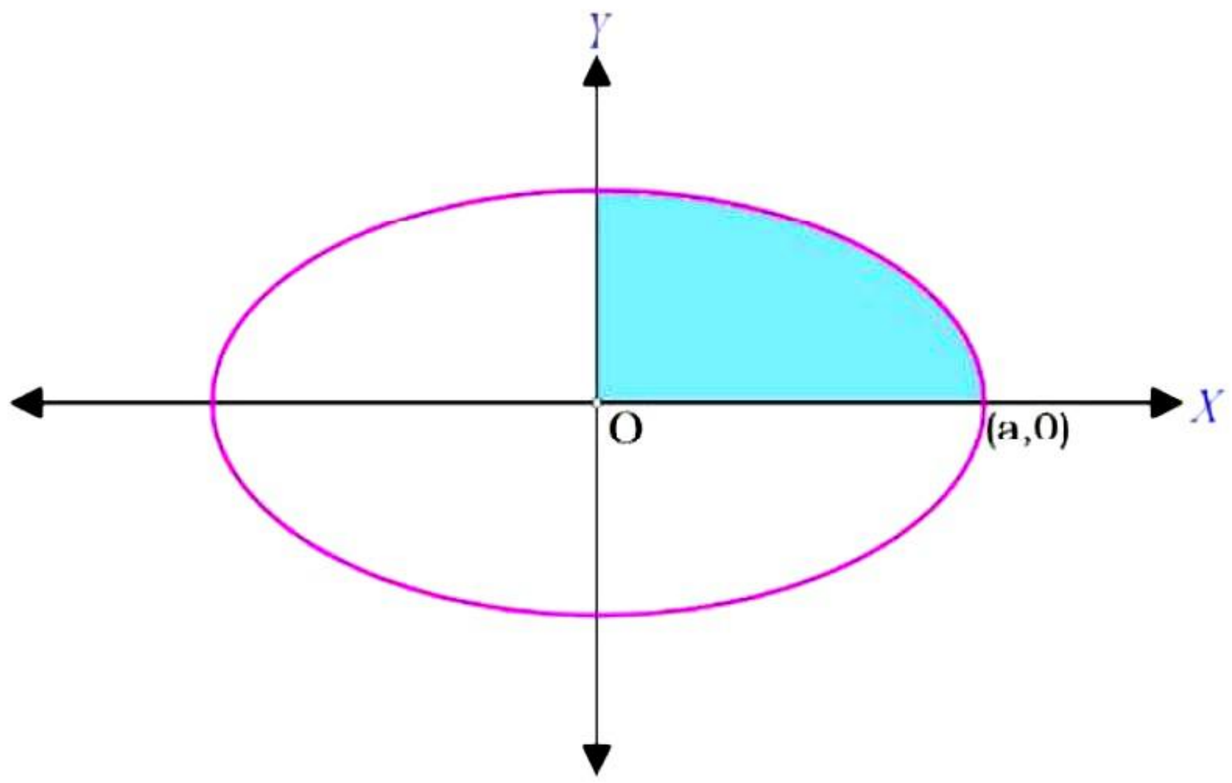
We have $y = 4 - x^2$ or $x^2 = -(y-4)$, which represents a parabola

$$I = \int_0^2 y\,dx$$

$$I = \int_0^2 \left(4 - x^2\right)dx = \left[4x - \frac{x^3}{3}\right]_0^2 = 8 - \frac{8}{3} = \underline{\frac{16}{3}} \text{ units}$$

Question :

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Solution :

We have $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2\left(1-\frac{x^2}{a^2}\right)$$

$$y^2=b^2\left(\frac{a^2-x^2}{a^2}\right)$$

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$\text{Area} = 4\times\text{Area of quadrant} = 4\times I$$

$$I=\int_0^a y\,dx$$

$$I=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}$$

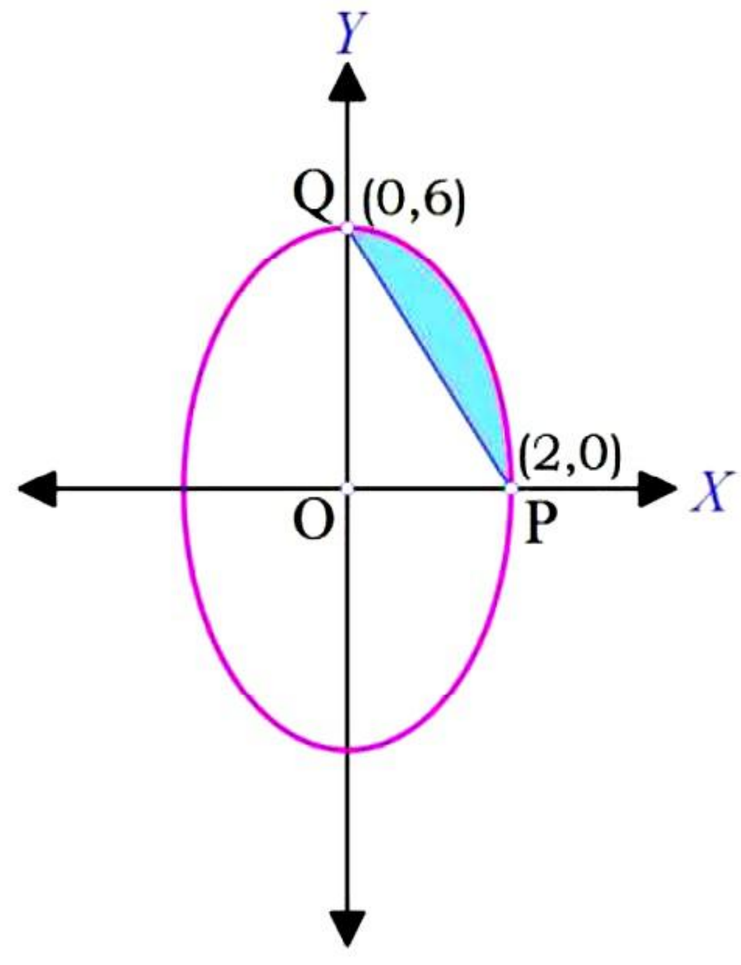
$$I=\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=\frac{b}{a}\times\frac{a^2}{2}\sin^{-1}\frac{a}{a}=\frac{ba}{2}\sin^{-1}1=\frac{\pi ab}{4}$$

$$\text{Area}=4\times\frac{\pi ab}{4}=\underline{\pi ab}\text{ units}$$

Question :

Find the area between the arc PQ and chord PQ of the ellipse $\frac{x^2}{4}+\frac{y^2}{36}=1$



Solution :

The equation of the chord is

$$y-0=\frac{6-0}{0-2}\times(x-2)$$

$$y=-3(x-2)$$

$$y=-3x+2$$

Also the equation of the ellipse is

$$\frac{x^2}{4}+\frac{y^2}{36}=1$$

$$\frac{y^2}{36}=1-\frac{x^2}{4} \quad \text{or} \quad y^2=36\left(1-\frac{x^2}{4}\right) \quad \text{or} \quad y^2=36\left(\frac{4-x^2}{4}\right)$$

$$\therefore \qquad y=3\sqrt{4-x^2}$$

$$\text{Area}=3\int_0^2\sqrt{4-x^2}\,dx-\int_0^2(6-3x)dx$$

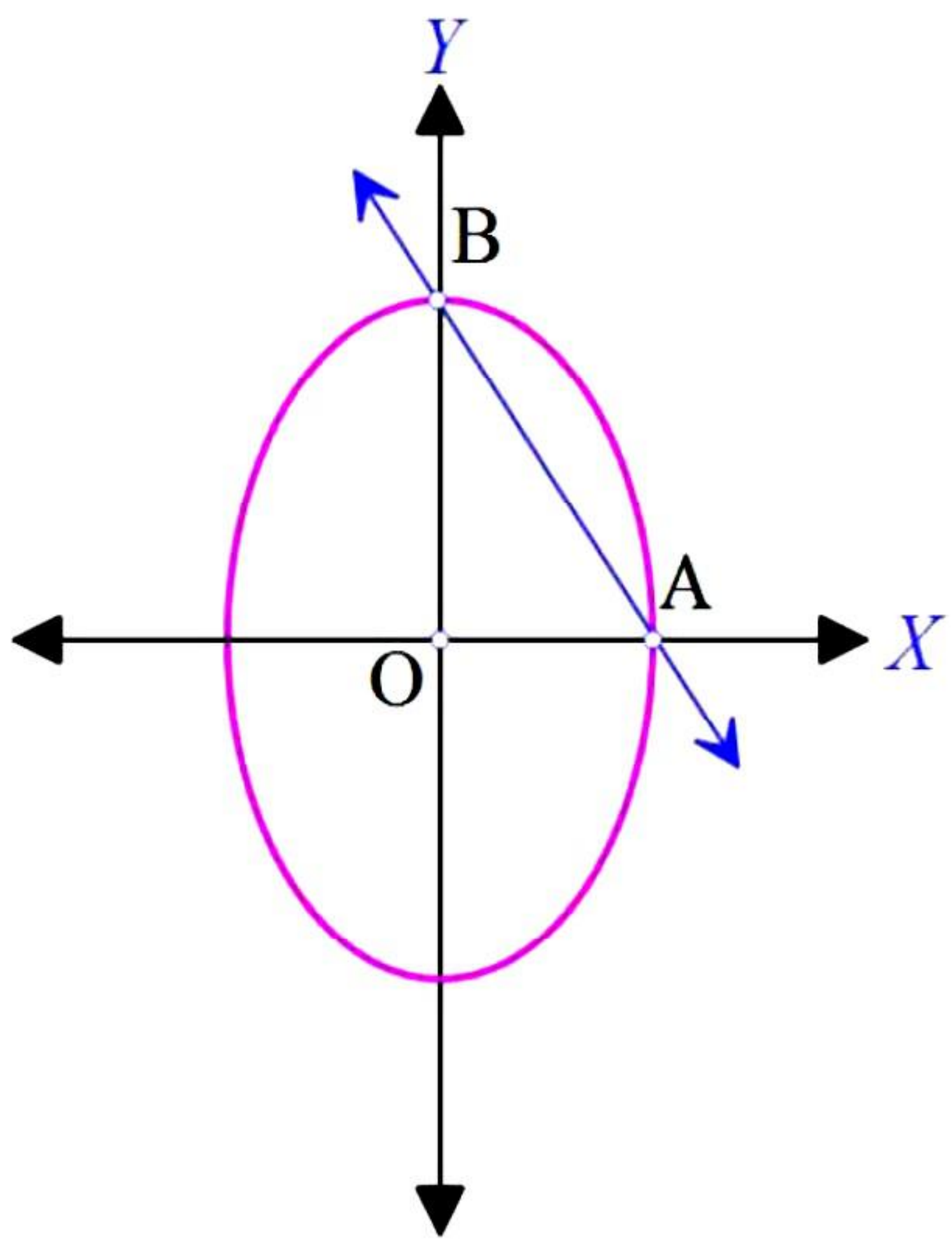
$$\text{Area}=3\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2-\left[6x-\frac{3x^2}{2}\right]_0^2$$

$$=3\left(\frac{2}{2}\sqrt{4-2^2}+\frac{4}{2}\sin^{-1}\frac{2}{2}\right)-\left(6\times2-\frac{3\times2^2}{2}\right)$$

$$=3\left(0+2\frac{\pi}{2}\right)-12+6=\underline{3\pi-6}\ \text{units}$$

Question :

Find the area between the arc PQ and line $\frac{x}{3}+\frac{y}{2}=1$ of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$



Solution :

The equation of the line is

$$\frac{x}{3}+\frac{y}{2}=1$$

$$y=2\left(1-\frac{x}{3}\right)$$

$$y=\frac{2}{3}(3-x)$$

When $x=3,\ y=2$

When $x=0,\ y=2$

Also the equation of the ellipse is

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

$$\frac{y^2}{4}=1-\frac{x^2}{9} \quad \text{or} \quad y^2=4\left(1-\frac{x^2}{9}\right) \quad \text{or} \quad y^2=4\left(\frac{9-x^2}{9}\right)$$

$$\therefore \qquad y=\frac{2}{3}\sqrt{9-x^2}$$

$$\text{Area}=\frac{2}{3}\int_0^3\sqrt{9-x^2}\,dx-\frac{2}{3}\int_0^2(3-x)dx$$

$$\text{Area}=\frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3-\frac{2}{3}\left[3x-\frac{x^2}{2}\right]_0^3$$

$$=\frac{2}{3}\left(\frac{3}{2}\sqrt{9-3^2}+\frac{9}{2}\sin^{-1}\frac{3}{3}\right)-\frac{2}{3}\left(3\times3-\frac{3^2}{2}\right)$$

$$=\frac{2}{3}\left(0+\frac{9}{2}\times\frac{\pi}{2}\right)-\frac{2}{3}\left(9-\frac{9}{2}\right)$$

$$=\frac{2}{3}\left(\frac{9\pi}{4}\right)-\frac{2}{3}\left(\frac{9}{2}\right)$$

$$=\frac{2}{3}\left(\frac{9\pi}{4}-\frac{18}{4}\right)$$

$$=\frac{2}{3}\times\frac{9}{4}(\pi-2)=\underline{\underline{\frac{3}{2}(\pi-2)\ \text{units}}}$$

HOME WORK QUESTIONS

Question :(Imp2017)

(a) Area below the curve $y = -2x + 3$ in the first quadrant.

(b) Draw a rough sketch of the curves

$$x^2 + y^2 = 4 \text{ and } (x - y)^2 + y^2 = 4$$

Also find the area between these two curves.

Answer : (a) $\frac{9}{4}$ (b) $\frac{8}{3}\pi - 2\sqrt{3}$

Question :(March2017)

(a) Area bounded by the curves $y = \cos x$, $x = \frac{\pi}{2}$, $x = 0$, $y = 0$ is

(b) Find the area between the curves

$$y^2 = 4ax \text{ and } x^2 = 4ay, a > 0$$

Answer : (a) 1 (b) $\frac{16}{3}a^2$

Question :(Imp2016)

(a) The area bounded by the curves $y = 2\cos x$, the x axis from $x = 0$ to $x = \frac{\pi}{2}$ is (0, 1, 2, –1)

(b) Find the area of the region bounded by the curves

$$y^2 = 4ax \text{ and } x^2 = 4ay, a > 0$$

Answer : (a) 2 (b) $\frac{16}{3}a^2$

Question :(March2016)

Find the area of the circle $x^2 + y^2 = 4$ using integration.

Answer : 4π

NCERT TEXT BOOK QUESTIONS

EXERCISE 8.1

1. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the $x-$axis. Ans: $\frac{14}{3}$

2. Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the $x-$axis in the first quadrant. Ans: $16 - 4\sqrt{2}$

3. Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the $y-$axis in the first quadrant. Ans: $\frac{32 - 84\sqrt{2}}{3}$

4. Find the area of the region bounded by the ellipse
$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$ Ans: 12π

5. Find the area of the region bounded by the ellipse
$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$ Ans: 6π

6. Find the area of the region in the first quadrant enclosed by $x-$axis, line $x = 3y$ and the circle $x^2 + y^2 = 4$. Ans: $\frac{\pi}{3}$

7. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \frac{a}{\sqrt{2}}$ Ans: $\frac{a^2}{2}\left(\frac{\pi}{2} - 1\right)$

8. The area between $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a. Ans: $4^{2/3}$

9. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$. Ans: $\frac{1}{3}$

10. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$ Ans: $\frac{9}{8}$

11. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$. Ans: $8\sqrt{3}$

EXERCISE 8.2

1. Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$. Ans: $\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

2. Find the area bounded by curves

   $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$ Ans: $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

3. Find the area of the region bounded by the curves

   $y = x^2 + 2$ and $y = x$, $y = 0$ and $x = 3$

   Ans: $\frac{21}{2}$

4. Using integration find the area of region bounded by the triangle whose vertices are $(-1,0),(1,3)$ and $(3,2)$

   Ans:4

5. Using integration find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

   Ans:8

EXERCISE

1. Find the area of the region bounded by the curves

   $y^2 = 9x$, $y = 3x$ Ans: $\frac{1}{2}$

2. Find the area of the region bounded by the parabola

   $y^2 = 2px$, $x^2 = 2py$ Ans: $\frac{4p^2}{3}$

3. Find the area of the region bounded by the curve

   $y = x^3$ and $y = x + 6$ and $x = 0$ Ans:10

4. Find the area of the region bounded by the curve

   $y^2 = 4x$, $x^2 = 4y$ Ans: $\frac{16}{3}$

5. Find the area of the region bounded by the parabola $y^2 = 2x$ and the straight line $x - y = 4$

   Ans:18

6. Find the area of the region bounded by the parabolas $y^2 = 6x$ and $x^2 = 6y$. Ans:12

7. Find the area enclosed by the curve
$x = 3\cos t,\ y = 2\sin t$. Ans:$6\pi$

8. Find the area of the region included between the parabola $y = \frac{3}{4}x^2$ and the line $3x - 2y + 12 = 0$ Ans:27