CHAPTER 11

PROBABILITY DISTRIBUTIONS

Exercise 11.1

KEY POINTS

1. Random variable

A random variable X is a function defined on a sample space S into the real numbers R such that the inverse image of points of subset or interval of R is an event in S, for which probability is assigned.

Type I: Find the values of the random variable X and number of points in its inverse images.

Q.No: 1, 2, 3, 4, 5, Example 11.1, 11.2, 11.3, 11.4

1. **Suppose X is the number of tails occurred when three fair coins are tossed once simultaneously. Find the values of the random variable X and number of points in its inverse images.**

Three coins are tossed

$$\therefore \text{ sample space } S = \left\{ \begin{matrix} HHH, HHT, HTH, THH, \\ HTT, THT, TTH, TTT \end{matrix} \right\}$$

$$n(S) = 8$$

Given X be the random variable denotes number of tails.

Here X takes the values 0, 1, 2, 3

(0 tail), $X(HHH) = 0$

(1 tail), $X(HHT) = 1, X(HTH) = 1, X(THH) = 1$

(2 tail), $X(HTT) = 2, X(THT) = 2, X(TTH) = 2$

(3 tail), $X(TTT) = 3$

$\therefore X(\omega)$ denotes the number of tails

$$X(\omega) = \begin{cases} 0, & \text{if } \omega = HHH \\ 1, & \text{if } \omega = HHT, HTH, THH \\ 2, & \text{if } \omega = HTT, THT, TTH \\ 3, & \text{if } \omega = TTT, \end{cases}$$

Values of the Random Variable X	0	1	2	3	Total
Number of elements in inverse image	1	3	3	1	8

Example 11.1

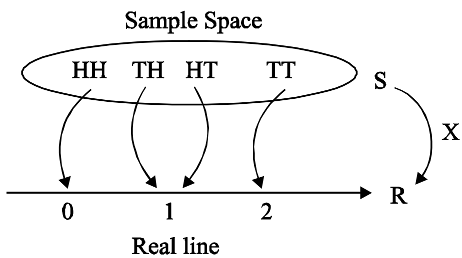
Suppose two coins are tossed once. If X denotes the number of tails, (i) write down the sample space (ii) find the inverse image of 1 (iii) the values of the random variable and number of elements in its inverse images.

Solution

(i) The sample space $S = \{ H, T \} \times \{ H, T \}$

That is $S = \{ TT, TH, HT, HH \}$

(ii) Let $X : S \to R$ be the number of tails



Then $X(TT) = 2$ (2 Tails)

$X(TH) = 1$ (1 Tail)

$X(HT) = 1$ (1 Tail)

and $X(HH) = 0$ (0 Tails)

Then X is a random variable that takes on the values 0, 1 and 2.

Let $X(\omega)$ denotes the number of tails, this gives

$$X(\omega) = \begin{cases} 2 & \text{if } \omega = TT \\ 1 & \text{if } \omega = HT, TH \\ 0 & \text{if } \omega = HH \end{cases}$$

The inverse images of 1 is $\{ TH, HT \}$. That is $X^{-1}(\{ 1 \}) = \{ TH, HT \}$

(iii) Number of elements in inverse images are shown in the table.

Values of the Random Variable	0	1	2	Total
Number of elements in inverse image	1	2	1	4

2. In a pack of 52 playing cards, two cards are drawn at random simultaneously. If the number of black cards drawn in a random variable, find the values of the random variable and number of points in its inverse images.

Total cards $= 52$

Black $-$ 26 cards

Others $-$ 26 cards

Here $n(s) = 52C_2$

$$= \frac{52 \times 51}{1 \times 2} = 1326$$

Given X be the random variable denotes number of black cards

$\therefore$ X takes the values 0, 1, 2

$X(\omega)$ denote number of black cards

$$X(\omega) = \begin{Bmatrix} 0, & \text{if no black card} \\ 1, & \text{if one black card} \\ 2, & \text{if two black card} \end{Bmatrix}$$

X (No black card) $= 26C_0 \times 26C_2$

$$= 1 \times \frac{26 \times 25}{1 \times 2}$$

$$= 325$$

X (one black card) $= 26C_1 \times 26C_1$

$$= 26 \times 26$$

$$= 576$$

X (two black card) $= 26C_2 \times 26C_0$

$$= \frac{26 \times 25}{1 \times 2} \times 1$$

$$= 325$$

Value of Random variable 'X'	0	1	2	Total
No. of elements in inverse image	325	676	325	1326

3. An urn contains 5 Mangoes and 4 apples three fruits are taken at random. If the number of apples taken is a random variable, then find the values of the random variable and numbers of points in its inverse image.

Given

$$\begin{aligned} \text{Total fruits} &= 5\ (\text{mango}) + 4\ (\text{Apple}) \\ &= 9 \end{aligned}$$

$$\text{Taken fruits} = 3$$

$$\begin{aligned} \therefore\ \text{Total}\ \ n(s) &= 9C_3 \\ &= \frac{9 \times 8 \times 7}{1 \times 2 \times 3} \\ &= 84 \end{aligned}$$

Given X be the random variable denotes number of apples taken

$\therefore X$ takes the values 0, 1, 2, 3.

$X(\omega)$ denote number of applies

$$X(\omega) = \left\{ \begin{array}{ll} 0\,, & \text{if no apple, 3 mangoes taken} \\ 1\,, & \text{if one apple, 2 mangoes taken} \\ 2\,, & \text{if two apples, 1 mango taken} \\ 3\,, & \text{if three apples, no mangoes taken} \end{array} \right\}$$

$$\begin{aligned} X\ (\text{No apple}) &= 4C_0 \times 5C_3 \\ &= 1 \times \frac{5 \times 4}{1 \times 2} \\ X(0) &= 10 \end{aligned}$$

$$\begin{aligned} X\ (\text{one apple}) &= 4C_1 \times 5C_2 \\ &= 4 \times \frac{5 \times 4}{1 \times 2} \\ X(1) &= 40 \end{aligned}$$

$$\begin{aligned} X\ (\text{two apple}) &= 4C_2 \times 5C_1 \\ &= \frac{4 \times 3}{2 \times 1} \times 5 \\ X(2) &= 30 \end{aligned}$$

$$X \text{ (three apple) } = 4C_3 \times 5C_0$$
$$= 4 \times 1$$
$$= 4$$

Value of Random variable 'X'	0	1	2	3	Total
Number of elements in inverse image	10	40	30	4	84

Example 11.3

An urn contains 2 white and 3 red balls. A sample of 3 balls chosen. If X denotes the number of red balls. Find the value of random variable X and its number of inverse images

Number of balls $= 2 \text{ white} + 5 \text{ red}$

$= 5 \text{ balls}$

Number of balls taken $= 3$

$\therefore$ Total $n(s) = 5C_3$

$$= \frac{5 \times 4}{1 \times 2}$$

$$= 10$$

Given X be the random variable denotes number of red balls.

Here X takes the values 1, 2, 3

[0 not possible because 2 white only given so 1 red ball must taken for 3 balls taken]

$$X \text{ (one red) } = 2C_2 \times 3C_1$$
$$= 1 \times 3$$
$$X(1) = 3$$

$$X \text{ (two red) } = 2C_1 \times 3C_2$$
$$= 2 \times 3$$
$$X(2) = 6$$

$$X \text{ (three red) } = 2C_0 \times 3C_3$$
$$= 1 \times 1$$
$$X(3) = 1$$

Value of Random variable 'X'	1	2	3	Total
Number of elements in inverse image	3	6	1	10

4. Two balls are chosen randomly from an urn containing 6 red and 8 black balls. Suppose that we win Rs.15 for each red ball selected and we lose Rs.10 for each black ball selected X denotes the winning amount, then find the values of X and number of points in its inverse images.

Total balls $= 6 \text{ red} + 8 \text{ black}$

$= 14$

Taken balls $= 2$

$\therefore$ Total $n(s) = 14C_2$

$$= \frac{14 \times 13}{1 \times 2}$$

$= 91$

Let X denotes the winning amount $\therefore$ X takes the values

X (two black ball) $= -10 \times 2 = -20$

X (one black one red) $= -10 + 15 = 5$

X (two red balls) $= 15 + 15 = 30$

$$X(\omega) = \begin{cases} -20 & \text{if two black balls} \\ 5 & \text{if one black one red} \\ 30 & \text{if two red balls} \end{cases}$$

X (two black ball) $= 6C_0 \times 8C_2$

$$= 1 \times \frac{8 \times 7}{1 \times 2}$$

$X(-20) = 28$

X (1 black, 1 red) $= 6C_1 \times 8C_1$

$= 6 \times 8$

$= X(5) = 48$

X (2 red balls) $= 6C_2 \times 8C_0$

$$= \frac{6 \times 5}{1 \times 2} \times 1$$

$X(30) = 15$

Value of Random variable X	-20	5	30	Total
No. of points in inverse image	28	48	15	91

Example 11.14

Two balls are chosen randomly from an urn containing 6 white, 4 black balls. Suppose that we win Rs.30 for each black ball selected and we lose Rs.20 for each white ball selected. If X denotes the winning amount, then find the values of X and number of points in its inverse images.

$$\text{Total balls} = 6 \text{ white} + 4 \text{ black}$$

$$= 10$$

$$\text{Taken balls} = 2$$

$$\therefore \text{ Total } n(s) = 10C_2$$

$$= \frac{10 \times 9}{2}$$

$$= 45$$

Let X denote the winning amount

$\therefore X$ takes the values

X (two black balls) $= 2\,(30) = 60$

X (one black, one white) $= 30 - 20 = 10$

X (both white) $= 2\,(-20) = -40$

$$X(\omega) = \begin{cases} 60, & \text{if two black} \\ 10, & \text{if one black, one white} \\ -40, & \text{two white} \end{cases}$$

X (two black) $= 6C_0 \times 4C_2$

$$= 1 \times \frac{4 \times 3}{2}$$

$$X(60) = 6$$

X (one black, one white) $= 6C_1 \times 4C_1$

$$= 6 \times 4$$

$$X(10) = 24$$

$$X \text{ (two white) } = 6C_2 \times 4C_0$$

$$= \frac{6 \times 5}{2} \times 1$$

$$= 15$$

Values of the Random Variable 'X'	60	10	– 40	Total
Number of elements in inverse images	6	24	15	45

5. **A six sided die is marked '2' on one face, '3' on two of its faces, and '4' on remaining three faces. The die is thrown twice. If *X* denotes the total score in two throws, find the values of the random variable and number of points in its inverse images.**

Sample space = { 2, 3, 4 } × { 2, 3, 4 }

$$S = \begin{Bmatrix} (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), \\ (3, 4), (4, 2), (4, 3), (4, 4) \end{Bmatrix}$$

$$n(s) = 9$$

X is assigned to each point (α, β) the sum of the numbers on the dice

$$X(\alpha + \beta) = \alpha + \beta$$

Here

$$X(2, 2) = 2 + 2 = 4$$

$$X(2, 3) = X(3, 2) = 5$$

$$X(2, 4) = X(3, 3) = 6$$

$$X(3, 4) = X(4, 3) = 7$$

$$X(4, 4) = 8$$

$\therefore X$ takes the values 4, 5, 6, 7, 8

Values of random variable X	4	5	6	7	8	Total
Number of elements in inverse image	1	2	3	2	1	9

Example 11.2

Suppose a pair of unbiased dice is rolled once. If X denotes the total score of two dice, write down (i) the sample space (ii) the values taken by the random variable X, (iii) the inverse image of 10, and (iv) the number of elements in inverse image of X.

Solution:

(i) The sample space

$$S = \{\ 1, 2, 3, 4, 5, 6\ \} \times \{\ 1, 2, 3, 4, 5, 6\ \},$$

consists of 36 ordered pairs (α, β) where α and β can take any integer value between 1 and 6 as shown. X is assigned to each point (α, β) the sum of the numbers on the dice.

$$S = \begin{Bmatrix} (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6) \\ (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6) \\ (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6) \\ (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6) \\ (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6) \\ (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6) \end{Bmatrix}$$

That is $X(\alpha, \beta) = \alpha + \beta$.

Therefore

$X(1, 1) = 1 + 1 = 2$

$X(1, 2) = X(2, 1) = 3$

$X(1, 3) = X(2, 2) = X(3, 1) = 4$

$X(1, 4) = X(2, 3) = X(3, 2) = X(4, 1) = 5$

$X(1, 5) = X(2, 4) = X(3, 3) = X(4, 2) = X(5, 1) = 6$

$X(1, 6) = X(2, 5) = X(3, 4) = X(4, 3) = X(5, 2) = X(6, 1) = 7$

$X(2, 6) = X(3, 5) = X(4, 4) = X(5, 3) = X(6, 2) = 8$

$X(3, 6) = X(4, 5) = X(5, 4) = X(6, 3) = 9$

$X(4, 6) = X(5, 5) = X(6, 4) = 10$

$X(5, 6) = X(6, 5) = 11$

$X(6, 6) = 12$

(ii) Here X takes the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

(iii) Inverse images of 10 is { (4, 6), (5, 5), (6, 4) }

(iv)

Values of the random variable	2	3	4	5	6	7	8	9	10	11	12	Total
Number of elements in inverse image	1	2	3	4	5	6	5	4	3	2	1	36

Exercise 11.2

KEY POINTS

Definition (Discrete Random Variable)

A random variable X is defined on a sample space S into the real number R is called discrete random variable if the range of X is countable, that is, it can assume only a finite or countably infinite number of values, where every value in the set S has positive probability with total one.

Definition (Probability mass function)

If X is a discrete random variable with discrete values $x_1, x_2, x_3, \ldots x_n \ldots$, then the function by $f(\cdot)$ or $p(\cdot)$ and defined by

$$f(x_k) = P(X = x_k), \quad \text{for } k = 1, 2, 3, \ldots n, \ldots$$

is called the probability mass function of X

Theorem (Without proof)

The function $f(x)$ is a probability mass function if and only if it satisfies the following properties for the set of real values $x_1, x_2, x_3, \ldots x_n \ldots$

(i) $f(x_k) \geq 0$ for $k = 1, 2, 3, \ldots n, \ldots$ and (ii) $\boxed{\sum_k f(x_k) = 1}$

Definition: (Cumulative distribution function)

The **cumulative distribution function** $F(x)$ of a discrete random variable X, taking the values $x_1, x_2, x_3, \ldots$ such that $x_1 < x_2 < x_3 < \ldots$ with probability mass function $f(x_1)$ is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i), \quad x \in R$$

Note:

Suppose X is a discrete random variable taking the values $x_1, x_2, x_3, \ldots$ such that $x_1 < x_2 < x_3 \ldots$ and $F(x_i)$ is the distribution function. Then the probability mass function $f(x_i)$ is given by

$$f(x_i) = F(x_i) - F(x_{i-1}), \; i = 1, 2, 3, \ldots$$

Type I: Discrete Random variables based sums, probability mass function, cumulative distributive function

Q.No: 1, 2, 3, Example 11.5, 11.6, 11.7, 11.8

1. **Three fair coins are tossed simultaneously find the probability mass function for number of heads occurred.**

Three coins are tossed

$\therefore S = \{ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT \}$

$n(s) = 8$

Let X be the random variable denotes number of heads.

$\therefore X$ takes the values 0, 1, 2, 3

Values of the Random Variable X	0	1	2	3	Total
Number of elements in inverse image	1	3	3	1	8

$$\therefore P(X = 0) = \frac{1}{8}$$

$$P(X = 1) = \frac{3}{8}$$

$$P(X = 2) = \frac{3}{8}$$

$$P(X = 3) = \frac{1}{8}$$

Probability mass function is

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Example 11.5

Two fair cons are tossed simultaneously (equivalent to a fair coin is tossed twice). Find the probability mass function for number of heads occurred.

Solution

The sample space $S = \{ H, T \} \times \{ H, T \}$

That is $S = \{ TT, TH, HT, HH \}$

Let X be the random variable denoting the number of heads.

Therefore

$$X(TT) = 0, \qquad X(TH) = 1,$$
$$X(HT) = 1, \text{ and } \qquad X(HH) = 2$$

Then the random variable X takes on the values 0, 1 and 2

Values of the Random Variable	0	1	2	Total
Number of elements in inverse images	1	2	1	4

The probabilities are given by

$$f(0) = P(X = 0) = \frac{1}{4},$$

$$f(1) = P(X = 1) = \frac{1}{2}$$

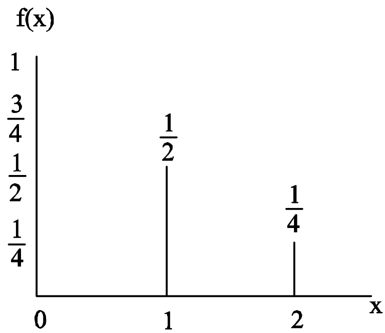
$$\text{and } f(2) = P(X = 2) = \frac{1}{4}$$

The function $f(x)$ satisfies the conditions

(i) $f(x) \geq 0$, for $x = 0, 1, 2$

(ii) $$\sum_{S} f(x) = \sum_{x=0}^{x=2} f(x) = f(0) + f(1) + f(2)$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$$



Probability mass function of f(x)

Therefore $f(x)$ is a probability mass function.

The probability mass function is given by

x	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(or)

$$f(x) = \begin{cases} \frac{1}{4} & \text{for } x = 0 \\ \frac{1}{2} & \text{for } x = 1 \\ \frac{1}{4} & \text{for } x = 2 \end{cases}$$

2. A six sides die is marked '1' on one face '3' on two of its faces and 5 on remaining three faces, the die is thrown twice. If X denotes the total score in two throws, Find

(i) The probability mass function.

(ii) The cumulative distribution function.

(iii) $P(4 \le X < 10)$

(iv) $P(X \ge 6)$.

I/II	1	3	3	5	5	5
1	2	4	4	6	6	6
3	4	6	8	8	8	8
3	4	6	6	8	8	8
5	6	8	8	10	10	10
5	6	8	8	10	10	10
5	6	8	8	10	10	10

Let X denote the total score in two throws.

$\therefore$ X takes the values 2, 4, 6, 8, 10

Values of the Random Variable X	2	4	6	8	10	Total
Number of elements in inverse image	1	4	10	12	9	36

$$P(X=2)=\frac{1}{36}$$

$$P(X=4)=\frac{4}{36}$$

$$P(X=6)=\frac{10}{36}$$

$$P(X=8)=\frac{12}{36}$$

$$P(X=10)=\frac{9}{36}$$

(i) The probability mass function

X	2	4	6	8	10
$f(x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$

(ii) The cumulative distribution function

X	2	4	6	8	10
$F(x)$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{15}{36}$	$\frac{27}{36}$

(iii) $P(4 \le X < 10) = P(X=4) + P(X=6) + P(X=8)$

$$=\frac{4}{36}+\frac{10}{36}+\frac{12}{36}$$

$$=\frac{26}{36}$$

$$=\frac{13}{18}$$

(iv) $P(X \ge 6) = P(X=6) + P(X=8) + P(X=10)$

$$=\frac{10}{36}+\frac{12}{36}+\frac{9}{36}$$

$$=\frac{31}{36}$$

Note: Cumulative distribution function

$$F(2) = \frac{1}{36}$$

$$F(4) = \frac{1}{36} + \frac{4}{36} = \frac{5}{36}$$

$$F(6) = \frac{1}{36} + \frac{4}{36} + \frac{10}{36} = \frac{15}{36}$$

$$F(8) = \frac{1}{36} + \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{27}{36}$$

$$F(10) = \frac{1}{36} + \frac{4}{36} + \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = \frac{36}{36} = 1$$

Example 11.6

A pair of fair dice is rolled once. Find the probability mass function to get the number of fours.

✍ **Solution**

$$S = \begin{Bmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{Bmatrix}$$

Let X be a random variable whose values x are the number of fours.

Th sample space S is given in the table.

It can also be written as

$S = \{ (i, j) \}$, where $i = 1, 2, 3 \ldots, 6$ and $j = 1, 2, 3, \ldots 6$

Therefore X takes on the values of 0, 1, and 2.

We observe that

(i) $X = 0$, if (i, j) for $i \neq 4, j \neq 4$,

(ii) $X = 1$, if (1,4), (2,4), (3,4), (5,4), (6,4), (4,1), (4,2), (4,3), (4,5), (4,6)

(iii) $X = 2$, if (4,4).

Therefore,

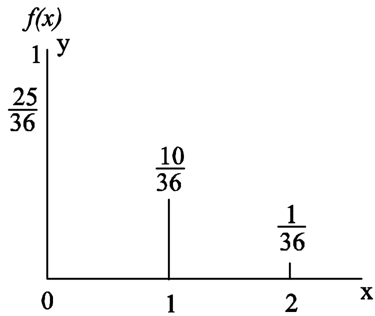
Values of the Random Variable X	0	1	2	Total
Number of elements in inverse image	25	10	1	36

The probabilities are

$$f(0) = P(X = 0) = \frac{25}{36},$$

$$f(1) = P(X = 1) = \frac{10}{36}$$

$$\text{and } f(2) = P(X = 2) = \frac{1}{36}$$



Probability mass function of *f(x)*

Clearly the function $f(x)$ satisfies the conditions

(i) $f(x) \geq 0,$ for $x = 0, 1, 2$

(ii) $$\sum_{s} f(x) = \sum_{x=0}^{x=2} f(x) = f(0) + f(1) + f(2)$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$$

The probability mass function is presented as

x	0	1	2
$f(x)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

(or)

$$f(x) = \begin{cases} \frac{25}{36} & \text{for } x = 0 \\ \frac{10}{36} & \text{for } x = 1 \\ \frac{1}{36} & \text{for } x = 2 \end{cases}$$

Example 11.7

The probability mass function $f(x)$ of a random variable X is

x	1	2	3	4
$f(x)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

find (i) its cumulative distribution function, hence find (ii) $P(X \le 3)$ and, (iii) $P(X \ge 2)$

Solution

(i) By definition the cumulative distribution function for discrete random variable is

$$F(x) = P(X \le x) = \sum_{x_i \le x} P(X = x_i)$$

$P(X < 1) = 0$ for $-\infty < X < 1$.

$$F(1) = P(X \le 1) = \sum_{x_i \le x} P(X = x_i) = \sum_{\infty}^{1} P(X = x) = P(X < 1)$$

$$+ P(X = 1) = 0 + \frac{1}{12} = \frac{1}{12}.$$

$$F(2) = P(X \le 2) = \sum_{-\infty}^{2} P(X = x) = P(X < 1) + P(X = 1) + P(X = 2)$$

$$= 0 + \frac{1}{12} + \frac{5}{12} = \frac{1}{2}$$

$$F(3) = P(X \le 3) = \sum_{-\infty}^{3} P(X = x) = P(X < 1)$$

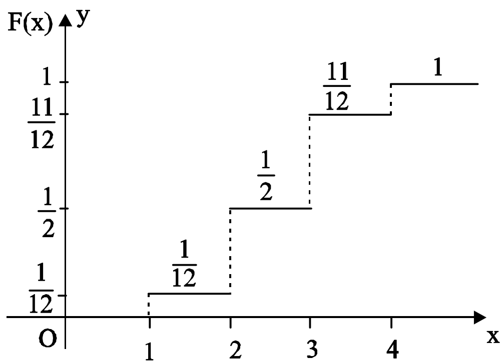
$$+ P(X = 1) + P(X = 2) + P(X = 3)$$

$$= 0 + \frac{1}{12} + \frac{5}{12} + \frac{5}{12} = \frac{11}{12}$$

$$F(4) = P(X \leq 4) = \sum_{-\infty}^{4} P(X = x) = P(X < 1) + P(X = 1)$$

$$+ P(X = 2) + P(X = 3) + P(X = 4)$$

$$= 0 + \frac{1}{12} + \frac{5}{12} + \frac{5}{12} = \frac{1}{12} = 1$$



Cumulative distribution function F(x)

Therefore the cumulative distribution function is

$$F(x) = \begin{cases} 0, & -\infty < x < 1 \\ \frac{1}{12}, & 1 \leq x < 2 \\ \frac{1}{2}, & 2 \leq x < 3 \\ \frac{11}{12}, & 3 \leq x < 4 \\ 1, & 4 \leq x < \infty \end{cases}$$

(ii) $P(X \leq 3) = F(3) = \frac{11}{12}$.

(iii) $P(X \geq 2) = 1 - P(X < 2) = 1 - P(X \leq 1)$

$$= 1 - F(1) = 1 - \frac{1}{12} = \frac{11}{12}$$

Type II: Find the value of 'k' given probability mass function

Q.No: 6, Example 11.10, 4

6. A random variable X has the following probability mass function.

x	1	2	3	4	5
$f(x)$	k^2	$2k^2$	$3k^2$	$2k$	$3k$

(i) Find the value of k (ii) $P(2 \le X < 5)$ (iii) $P(3 < X)$

Given $f(x)$ is a probability mass function

$$\therefore \Sigma f(x) = 1$$

$$k^2 + 2k^2 + 3k^2 + 2k + 3k = 1$$

$$6k^2 + 5k - 1 = 0$$

$$(k+1)\left(k - \frac{1}{6}\right) = 0$$

-6

$6 \quad -1$

$\frac{6}{6} \quad \frac{-1}{6}$

$1, \quad \frac{-1}{6}$

$k + 1 = 0$ $k = -1$ not possible	$k - \frac{1}{6} = 0$ $\boldsymbol{k = \frac{1}{6}}$

$\therefore$ Probability mass function is

x	1	2	3	4	5
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{6}$	$\frac{3}{6}$

(ii) $P(2 \le X < 5) = P(X = 2) + P(X = 3) + P(X = 4)$

$$= \frac{2}{36} + \frac{3}{36} + \frac{2 \times 6}{6 \times 6}$$

$$= \frac{2 + 3 + 12}{36}$$

$$= \frac{17}{36}$$

(iii) $P(3 < X) = P(X > 3)$

$$= P(X = 4) + P(X = 5)$$

$$= \frac{2}{6} + \frac{3}{6}$$

$$= \frac{5}{6}$$

Example 11.10

A random variable X has the following probability mass function.

x	1	2	3	4	5	6
$f(x)$	k	$2k$	$6k$	$5k$	$6k$	$10k$

Find (i) $P(2 < X < 6)$ (ii) $P(2 \le X < 5)$ (iii) $P(X \le 4)$ (iv) $P(3 < X)$

Solution

Since the given function is a probability mass function, the total probability is one. That is $\sum_x f(x) = 1$.

From the given data $k + 2k + 6k + 5k + 6k + 10k = 1$

$$30k = 1 \Rightarrow k = \frac{1}{30}$$

Therefore the probability mass function is

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{30}$	$\frac{2}{30}$	$\frac{6}{30}$	$\frac{5}{30}$	$\frac{6}{30}$	$\frac{10}{30}$

(i) $P(2 < X < 6) = f(3) + f(4) + f(5) = \frac{6}{30} + \frac{5}{30} + \frac{6}{30} = \frac{17}{30}$

(ii) $P(2 \le X < 5) = f(2) + f(3) + f(4) = \frac{2}{30} + \frac{6}{30} + \frac{5}{30} = \frac{13}{30}$

(iii) $P(X \le 4) = f(1) + f(2) + f(3) + f(4) = \frac{1}{30} + \frac{2}{30} + \frac{6}{30} + \frac{5}{30} = \frac{14}{30}$

(iv) $P(3 < X) = f(4) + f(5) + f(6) = \frac{5}{30} + \frac{6}{30} + \frac{10}{30} = \frac{21}{30}$

4. Suppose a discrete random variable can only take the values 0, 1 and 2 the probability mass function is defined by

$$f(x) = \begin{cases} \dfrac{x^2+1}{k} & \text{for } x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) The value of k (ii) Cumulative distributive function (iii) $P(X \geq 1)$

Given $x = 0, 1, 2$ and $f(x) = \dfrac{x^2+1}{k}$

Probability mass function

X	0	1	2
$f(x)$	$\dfrac{0^2+1}{k}$ $= \dfrac{1}{k}$	$\dfrac{1^2+1}{k}$ $= \dfrac{2}{k}$	$\dfrac{2^2+1}{k}$ $= \dfrac{5}{k}$

Given function Probability mass function

$$\therefore \ \Sigma f(x) = 1$$

$$\frac{1}{k} + \frac{2}{k} + \frac{5}{k} = 1$$

$$\frac{8}{k} = 1$$

$$\boxed{k = 8}$$

$$\therefore f(x) = \begin{cases} \dfrac{x^2+1}{8} & \text{for } x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Probability mass function

x	0	1	2
$f(x)$	$\dfrac{1}{8}$	$\dfrac{2}{8}$	$\dfrac{5}{8}$

(ii) Cumulative function

x	0	1	2
$F(x)$	$\frac{1}{8}$	$\frac{3}{8}$	1

Note

$$F(0) = \frac{1}{8}$$

$$F(1) = \frac{1}{8} + \frac{2}{8} = \frac{3}{8}$$

$$F(2) = \frac{1}{8} + \frac{2}{8} + \frac{5}{8} = \frac{8}{8} = 1$$

(iii) $P(X \geq 1) = P(X = 1) + P(X = 2)$

$$= \frac{2}{8} + \frac{5}{8}$$

$$= \frac{7}{8}$$

Type III: Given cumulative distribution function. Find probability mass function

Q.No: 5, 7, Example 11.9

5. The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0 & -\infty < x < -1 \\ 0.15 & -1 \leq x < 0 \\ 0.35 & 0 \leq x < 1 \\ 0.60 & 1 \leq x < 2 \\ 0.85 & 2 \leq x < 3 \\ 1 & 3 \leq x < \infty \end{cases}$$

Find (i) the probability mass function (ii) $P(X < 1)$ and $P(X \geq 2)$.

$$f(-1) = P(X = -1) = 0.15$$

$$f(0) = P(X = 0)$$

$$= F(0) - F(-1)$$

$$= 0.35 - 0.15$$

$$= 0.20$$

$$f(1) = f(X = 1)$$
$$= F(1) - F(0)$$
$$= 0.60 - 0.35$$
$$= 0.25$$
$$f(2) = P(X = 2)$$
$$= F(2) - F(1)$$
$$= 0.85 - 0.60$$
$$= 0.25$$
$$f(3) = P(X = 3)$$
$$= F(3) - F(2)$$
$$= 1 - 0.85$$
$$= 0.15$$

x	$F(x)$	$f(x)$	
-1	0.15	0.15	0.15
0	0.35	0.20	$0.35 - 0.15 = 0.20$
1	0.60	0.25	$0.60 - 0.35 = 0.25$
2	0.85	0.25	$0.85 - 0.60 = 0.25$
3	1	0.15	$1 - 0.85 = 0.15$

(i) Probability mass function

X	-1	0	1	2	3
$f(x)$	0.15	0.20	0.25	0.25	0.15

(ii) $P(X < 1) = P(X = -1) + P(X = 0)$
$$= 0.15 + 0.20$$
$$= 0.35$$

(iii) $P(X \geq 2) = P(X = 2) + P(X = 3)$
$$= 0.25 + 0.15$$
$$= 0.40$$

Exercise 11.4

KEY POINTS

Definition (Mean)

Suppose X is a random variable with probability mass (or) density function $f(x)$. **The expected value or mean or mathematical expectation of** X, denoted by $E(X)$ or μ is

$$E(X) = \begin{cases} \sum\limits_{x} xf(x) & \text{if } X \text{ is discrete} \\ \int\limits_{-\infty}^{\infty} xf(x)\,dx & \text{if } X \text{ is continuous} \end{cases}$$

Theorem (Without proof)

Suppose X is a random variable with probability mass (or) density function $f(x)$. **The expected value of the function** $g(X)$, a new random variable is

$$E(g(X)) = \begin{cases} \sum\limits_{x} g(x)f(x) & \text{if } g(x) \text{ is discrete} \\ \int\limits_{-\infty}^{\infty} g(x)f(x)\,dx & \text{if } g(x) \text{ is continuous} \end{cases}$$

Definition (Variance)

The **variance** of a random variable X denoted by $\text{Var}(X)$ or $V(X)$ or σ^2 (or σ_x^2) is

$$V(X) = E(X - E(X))^2 = E(X - \mu)^2$$

Properties of Mathematical expectation and variance

(i) $\boldsymbol{E(aX + b) = aE(X) + b}$**, where** a **and** b **are constants**

Proof

Let X be a discrete random variable

$$E(aX+b) = \sum_{i=1}^{\infty} (ax_i + b) f(x_i) \qquad \text{(by definition)}$$

$$= \sum_{i=1}^{\infty} (ax_i f(x_i) + bf(x_i))$$

$$= a \sum_{i=1}^{\infty} x_i f(x_i) + b \sum_{i=1}^{\infty} f(x_i)$$

$$= aE(X) + b(1) \qquad \left(\because \sum_{i=1}^{\infty} f(x_i) = 1 \right)$$

$$E(aX+b) = aE(X) + b$$

Similarly, when X is a continuous random variable, we can prove it, by replacing summation by integration.

Corollary 1:	$E(aX) = aE(X)$	(when $b = 0$)
Corollary 2:	$E(b) = b$	(when $a = 0$)

(ii) Var $(X) = E(X^2) - E(X)^2$

Proof

We know $E(x) = \mu$

$$\text{Var}(X) = E(X-\mu)^2$$

$$= E(X^2 - 2X\mu + \mu^2)$$

$$= E(X^2) - 2\mu E(X) + \mu^2 \qquad \text{(since } \mu \text{ is a constant)}$$

$$= E(X^2) - 2\mu\mu + \mu^2 = E(X^2) - \mu^2$$

$$\boxed{\textbf{Var}(X) = E(X^2) - (E(X))^2}$$

(iii) Var $(aX+b) = a^2$ Var (X) where a and b are constants

Proof

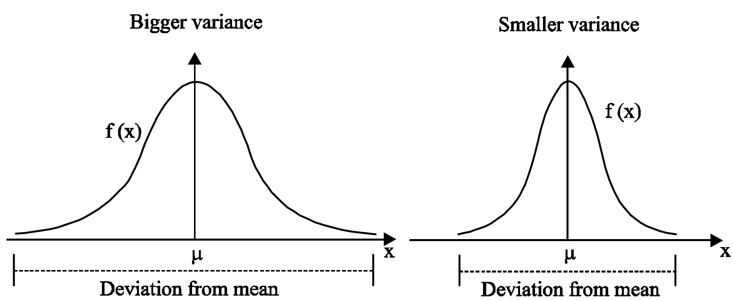
$$\text{Var}(aX+b) = E((aX+b) - E(aX+b))^2$$

$$= E\,(aX + b - aE\,(X) - b))^2$$
$$= E\,(aX - aE\,(X))^2$$
$$= E\,(a^2\,(X - E\,(X)))^2$$
$$= a^2\,E\,(X - E\,(X))^2$$

Hence $\text{Var}\,(aX + b) = a^2\,\text{Var}\,(X)$

Corollary 3:	$V\,(aX) = a^2\,V\,(X)$	(when $b = 0$)
Corollary 4:	$V\,(b) = 0$	(when $a = 0$)

Variance gives information about the deviation of the values of the random variable about the mean μ. A smaller σ^2 implies that the random values are more clustered about the mean, similarly, a bigger σ^2 implies that the random values are more scattered from the mean.



Different variance with same mean

The above figure shows the probability density functions of two continuous random variables whose curves are bell-shaped with same mean but different variances.

Type I: Mean and variance (without integration)

Q.No: 1. (i) (ii) 2, 3, 4, 8 Example 11.16, 11.17

1. For the random variable X with the given probability mass function as below, find the mean and variance.

1. $f(x) = \begin{cases} \dfrac{1}{10} & x = 2, 5 \\ \dfrac{1}{5} & x = 0, 1, 3, 4 \end{cases}$

x	0	1	2	3	4	5
$f(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$
$xf(x)$	0	$\frac{1}{5}$	$\frac{2}{10}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{10}$
$x^2 f(x)$	0	$\frac{1}{5}$	$\frac{4}{10}$	$\frac{9}{5}$	$\frac{16}{5}$	$\frac{25}{10}$

$$\text{Mean} = \sum_x xf(x) = 0 + \frac{1}{5} + \frac{2}{10} + \frac{3}{5} + \frac{4}{5} + \frac{5}{10}$$

$$E(X) = \frac{8}{5} + \frac{7}{10} = \frac{16+7}{10} = \frac{23}{10} = 2.3$$

$$E(X^2) = \sum_x x^2 f(x) = 0 + \frac{1}{5} + \frac{4}{10} + \frac{9}{5} + \frac{16}{5} + \frac{25}{10}$$

$$= \frac{29}{10} + \frac{26}{5} = \frac{29+52}{10} = \frac{81}{10} = 8.1$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= 8.1 - 2.3^2$$

$$= 8.1 - 5.29$$

$$= 2.81$$

Mean $= 2.3$

Variance $= 2.81$

(ii) $f(x) = \begin{cases} \dfrac{4-x}{6} & x = 1, 2, 3 \end{cases}$

x	1	2	3
$f(x)$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

$xf(x)$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{3}{6}$
$x^2 f(x)$	$\frac{3}{6}$	$\frac{8}{6}$	$\frac{9}{6}$

$$E(X) = \Sigma\, xf(x) = \frac{3}{6} + \frac{4}{6} + \frac{3}{6} = \frac{5}{3} = 1.667$$

$$E(X^2) = \Sigma\, x^2 f(x) = \frac{3}{6} + \frac{8}{6} + \frac{9}{6} = \frac{20}{6}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{20}{6} - \left(\frac{5}{3}\right)^2 = \frac{20}{6} - \frac{25}{9} = \frac{60-50}{18} = \frac{5}{9}$$

$$= 0.56$$

$$\boxed{\textbf{Mean} = \textbf{1.67},\ \textbf{Variance} = \textbf{0.56}}$$

(iii) $f(x) = \begin{cases} 2(x-1) & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

$$E(X) = \int_1^2 xf(x)\, dx$$

$$= 2\int_1^2 x(x-1)\, dx = 2\int_1^2 (x^2 - x)\, dx$$

$$= 2\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2 = 2\left[\left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$$

$$= 2\left[\frac{8}{3} - \frac{4}{2} - \frac{1}{3} + \frac{1}{2}\right] = 2\left[\frac{7}{3} - \frac{3}{2}\right] = 2\left[\frac{14-9}{6}\right]$$

$$E(X) = \frac{5}{3}$$

$$E(X^2) = \int_1^2 x^2 f(x)\, dx$$

$$= \int_1^2 x^2\, 2(x-1)\, dx = 2\int_1^2 (x^3 - x^2)\, dx$$

$$= 2\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2 = 2\left[\left(\frac{16}{4} - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right]$$

$$= 2\left[\frac{16}{4} - \frac{8}{3} - \frac{1}{4} + \frac{1}{3}\right] = 2\left[\frac{15}{4} - \frac{7}{3}\right] = 2\left[\frac{45-28}{6}\right]$$

$$E(X^2) = \frac{17}{6}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{17}{6} - \frac{25}{9} = \frac{51-50}{18} = \frac{1}{18}$$

$$\boxed{\textbf{Mean} = \frac{5}{3},\ \textbf{Variance} = \frac{1}{18}}$$

(iv) $f(x) = \begin{cases} \frac{1}{2} e^{-x/2} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$

$$E(X) = \int_0^\infty x f(x)\, dx$$

$$= \int_0^\infty x \frac{1}{2} e^{-x/2}\, dx \qquad\qquad \int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}$$

$$= \frac{1}{2} \frac{1!}{(1/2)^{1+1}} = \frac{1}{2} 2^2$$

$n = 1$
$a = 1/2$

$$E(X) = 2$$

$$E(X^2) = \int_0^\infty x^2 f(x)\, dx = \int_0^\infty x^2 \frac{1}{2} e^{-x/2}\, dx$$

$$= \frac{1}{2} \frac{2!}{(1/2)^{2+1}} = \frac{1}{2}\, 2 \times 2^3$$

$n = 2 \quad a = 1/2$

$$E(X^2) = 8$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= 8 - 2^2 = 8 - 4$$

$$= 4$$

mean $= 2$

variance $= 4$

2. Two balls are drawn in succession without replacement from an urn containing four red balls and 3 black balls Let *X* be the possible outcomes drawing red balls. Find the probability mass function and mean of *X*.

RED	BLACK	TOTAL
4	3	7

Let X be the random variable denotes the number of red balls.

$$X = 0, 1, 2$$

$$f(0) = P(X = 0) = \frac{4C_0 \times 3C_2}{7C_2} = \frac{1 \times 3}{\frac{7 \times 6}{1 \times 2}} = \frac{3}{21} = \frac{1}{7}$$

$$f(1) = P(X = 1) = \frac{4C_1 \times 3C_1}{7C_2} = \frac{4 \times 3}{21} = \frac{4}{7}$$

$$f(2) = P(X = 2) = \frac{4C_2 \times 3C_0}{7C_2} = \frac{26 \times 1}{21} = \frac{2}{7}$$

Probability mass function is

x	0	1	2
$P(X = x)$ $f(x)$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{2}{7}$

$$E(X) = \sum_x xf(x)$$

$$= 0\left(\frac{1}{7}\right) + 1\left(\frac{4}{7}\right) + 2\left(\frac{2}{7}\right)$$

$$= 0 + \frac{4}{7} + \frac{4}{7}$$

$$\text{mean} = \frac{8}{7}$$

3. If μ and σ^2 are the mean and variance of the discrete random variable X and $E(X+3) = 10$ and $E(X+3)^2 = 116$ find μ and σ^2

$$E(X+3) = 10 \qquad \boxed{\boldsymbol{E(aX+b) = aE(X)+b}}$$

$$E(X) + 3 = 10$$

$$E(X) = 10 - 3$$

$$\boldsymbol{E(X) = 7} \qquad \boxed{\therefore\ \mu = 7}$$

$$E(X+3)^2 = 116$$

$$E(X^2 + 6X + 9) = 116$$

$$E(X^2) + 6E(X) + 9 = 116$$

$$E(X^2) + 6(7) + 9 = 116$$

$$E(X^2) = 116 - 51$$

$$E(X^2) = 65$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = 65 - 7^2 = 65 - 49 = 16$$

$$\boxed{\begin{array}{l}\boldsymbol{\mu = 7}\\ \boldsymbol{\sigma^2 = 16}\end{array}}$$

4. Four fair coins are tossed once, Find the probability mass function, mean and variance for number of heads occurred.

$$n(s) = 2^4 = 16$$

Let X be the random variable denotes number of heads

$$X = 0, 1, 2, 3, 4$$

$$P\,(X=0)=\frac{4C_0}{16}=\frac{1}{16},\quad P\,(X=1)=\frac{4C_1}{16}=\frac{4}{16}$$

$$P\,(X=2)=\frac{4C_2}{16}=\frac{6}{16},\quad P\,(X=3)=\frac{4C_3}{16}=\frac{4}{16},$$

$$P\,(X=4)=\frac{4C_4}{16}=\frac{1}{16}$$

Probability is

$\boldsymbol{x}$	0	1	2	3	4
$\boldsymbol{f(x)}$ or $P\,(X=x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$
$\boldsymbol{xf(x)}$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$
$\boldsymbol{x^2 f(x)}$	0	$\frac{4}{16}$	$\frac{24}{16}$	$\frac{36}{16}$	$\frac{16}{16}$

$$E\,(X)=\sum_x xf\,(x)=0+\frac{4}{16}+\frac{12}{16}+\frac{12}{16}+\frac{4}{16}=\frac{32}{16}=2$$

$$E\,(X^2)=\sum_x x^2 f\,(x)=0+\frac{4}{16}+\frac{24}{16}+\frac{36}{16}+\frac{16}{16}$$

$$=\frac{80}{16}=5$$

$$\text{Var}\,(X)=E\,(X^2)-(E\,(X))^2=5-2^2=5-4=1$$

Mean $=2$

Variance $=1$

8. **A lottery with 600 tickets gives one prize of Rs.200 four prizes of Rs.100 and six prizes of Rs.50. If the ticket costs is Rs.2. Find the expected winning amount of ticket.**

$$n(s) = 600$$

Let X be the random variable denote the amount of winning.

$$X = 200, 100, 50, 0$$

$$P(X=200) = \frac{1}{600} \quad P(X=100) = \frac{4}{600} \quad P(X=50) = \frac{6}{600}$$

$$P(X=0) = \frac{589}{600}$$

x	200	100	50	0
$P(X=x)$ or $f(x)$	$\frac{1}{600}$	$\frac{4}{600}$	$\frac{6}{600}$	$\frac{589}{600}$

$$E(X) = \sum_x xf(x)$$

$$= 200\left(\frac{1}{600}\right) + 100 \times \frac{4}{600} + 50 \times \frac{6}{600} + 0\left(\frac{589}{600}\right)$$

$$= \frac{200+400+300}{600} = \frac{900}{600} = \frac{3}{2} = 1.50$$

Expected amount winning = Rs.1.50

One Ticket cost = Rs.2.00

Profit (difference) $= 1.50 - 200 = -\,0.50$

i.e. Loss $= 0.50$ Rupees

Example 11.16

Suppose $f(x)$ given below represents a probability mass function,

x	1	2	3	4	5	6
$f(x)$	c^2	$2c^2$	$3c^2$	$4c^2$	c	$2c$

Find (i) the value of c (ii) Mean and variance.

✍ **Solution:**

(i) Since $f(x)$ is a probability mass function, $f(x) \geq 0$ for all x and

$$\sum_{s} f(x) = 1.$$

Thus, $$\sum_{x} f(x) = 1$$

$$c^2 + 2c^2 + 3c^2 + 4c^2 + c + 2c = 1$$

$$10c^2 + 3c = 1$$

$$10c^2 + 3\,c - 1 = 0$$

$$(c - 1/5)\,(c + 1/2) = 0$$

$$c = \frac{1}{5} \text{ or } c = \frac{-1}{2}$$

$$c = \frac{1}{5} \text{ or } -\frac{1}{2}$$

Since $f(x) \geq 0$ for all x, the possible value of c is $\frac{1}{5}$.
Hence, the probability mass function is

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{25}$	$\frac{2}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{1}{5}$	$\frac{2}{5}$

(ii) To find mean and variance, let us use the following table

x	$f(x)$	$xf(x)$	$x^2 f(x)$
1	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$
2	$\frac{2}{25}$	$\frac{4}{25}$	$\frac{8}{25}$
3	$\frac{3}{25}$	$\frac{9}{25}$	$\frac{27}{25}$
4	$\frac{4}{25}$	$\frac{16}{25}$	$\frac{64}{25}$
5	$\frac{1}{5}$	$\frac{5}{5}$	$\frac{25}{5}$
6	$\frac{2}{5}$	$\frac{12}{5}$	$\frac{72}{5}$
	$\Sigma f(x) = 1$	$\Sigma f(x) = \frac{115}{25}$	$\Sigma x^2 f(x) = \frac{585}{25}$

Mean: $E(X) = \Sigma\, xf(x) = \dfrac{115}{25} = 4.6$

Variance: $V(X) = E(X^2) - (E(X))^2 = \Sigma\, x^2 f(x) - (\Sigma\, xf(x))^2$

$$= \frac{585}{25} - \left(\frac{115}{25}\right)^2 = 23.40 - 21.16 = 2.24$$

Therefore the mean and variance are 4.6 and 2.24 respectively.

Example 11.17

Two balls are chosen randomly from an urn containing 8 white and 4 black balls. Suppose that we win Rs.20 for each black ball selected and we lose Rs.10 for each white ball selected. Find the expected winning amount and variance.

✍ **Solution**

Let X denote the winning amount. The possible events of selection are (i) both balls are black, or (ii) one white and one black or (iii) both are white. Therefore X is a random variable that can be defined as

X (both are black balls) $=$ Rs.2 (20) $=$ Rs.40

X (one black and one white ball) $=$ Rs.20 $-$ Rs.10 $=$ Rs.10

X (both are white balls) $=$ Rs.($-$ 20) $= -$ Rs.20

Therefore X takes on the values 40, 10 and -20

Total number of ways of selecting 2 balls is $12c_2$

$$\left(\frac{12 \times 11}{1 \times 2}\right) = 66$$

Number of ways of selecting 2 black balls $= 4c_2 = 6$

Number of ways of selecting one black ball and one white ball

$$8c_1 \times 4c_1 = 8 \times 4 = 32$$

Number of ways of selecting 2 white balls $= 8c_2 = 28$

Values of Random Variable X	40	10	-20	Total
Number of elements in inverse images	6	32	28	66

Probability mass function is

X	40	10	-20	Total
$f(x)$	$\frac{6}{66}$	$\frac{32}{66}$	$\frac{28}{66}$	1

Mean:

$$E(X) = \Sigma\, xf(x) = 40\left(\frac{6}{66}\right) + 10\left(\frac{32}{66}\right) + (-20)\left(\frac{28}{66}\right) = 0$$

That is expected winning amount is 0,

Variance:

$$E(X^2) = \Sigma\, x^2 f(x) = 40^2 \cdot \left(\frac{6}{66}\right) + 10^2 \cdot \left(\frac{32}{66}\right) + (-20)^2 \cdot \left(\frac{28}{66}\right) = \frac{4000}{11}$$

$$(E(X))^2 = 0^2 = 0$$

This gives $V(X) = E(X^2) - (E(X))^2 = \frac{4000}{11} - 0 = \frac{4000}{11}$

Therefore $E(X) = 0$ and $V(x) = \frac{4000}{11}$

Type II: Mean and variance (with integration)
Q.NO. 1 (iii) (iv), 5, 6, 7 Example 11.8

5. A commuter train punctually at a station every half hour. Each morning, a student leaves his house to the train station. Let X denote the amount of time, in minutes that the student waits for the train from the time he reaches the train station. It is known that the probability density function is

$$f(x) = \begin{cases} \frac{1}{30} & 0 < x < 30 \\ 0 & \text{elsewhere} \end{cases}$$

Obtain and interpret the expected value of the random variable X.

Given X be the random variable denotes the waiting time X is continuous on (0, 30).

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{0}^{30} x \frac{1}{30}\, dx$$

$$= \frac{1}{30}\left[\frac{x^2}{2}\right]_0^{30} = \frac{1}{30 \times 2}[30^2 - 0]$$

$$= \frac{30 \times 30}{30 \times 2} = 15$$

$$E(X) = 15$$

expected value of waiting time = 15 minutes

6. **The time to failure in thousands of hours of an electronic equipment used in a manufactured computer has the density function** $f(x) = \begin{cases} 3e^{-3x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$

Find the expected life of this electronic equipment.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{0}^{\infty} x\, 3e^{-3x}\, dx$$

$$= 3\,\frac{1!}{3^{1+1}} = \frac{3}{9}$$

$n = 1$
$a = 3$

$$E(X) = \frac{1}{3}$$

$$\int_{0}^{\infty} x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{0}^{\infty} x^2\, 3e^{-x}\, dx$$

$n = 2$
$a = 3$

$$= 3\,\frac{2!}{3^{2+1}} = 3 \times \frac{2}{3} = \frac{2}{9}$$

7. The probability density function of the random variable X is given by $f(x) = \begin{cases} 16xe^{-4x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$ **find the mean and variance of X**

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\boxed{\int_{0}^{\infty} x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}}$$

$$= \int_{0}^{\infty} x\, 16xe^{-4x}\, dx = 16 \int_{0}^{\infty} x^2 e^{-4x}\, dx \qquad \boxed{n = 2 \quad a = 4}$$

$$= 16 \frac{2!}{4^{2+1}} = 16 \times \frac{2}{4 \times 4 \times 4} = \frac{1}{2}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = 16 \int_{0}^{\infty} x^3 e^{-4x}\, dx \qquad \boxed{\begin{matrix} n = 3 \\ a = 4 \end{matrix}}$$

$$= 16 \frac{3!}{4^4} = 16 \times \frac{6}{16 \times 16} = \frac{3}{8}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{3}{8} - \left(\frac{1}{2}\right)^2 = \frac{3}{8} - \frac{1}{4} = \frac{3-2}{8} = \frac{1}{8}$$

$\therefore$ **Mean** $= \dfrac{1}{2}$

Variance $= \dfrac{1}{8}$

Type I: Binomial distribution

Q.No: 1. (i) (ii) (iii) 2, 4, 5, 6 Example 11.19, 11.20, 11.21, 11.22

1. **Compute $P\,(X = k)$ for the binomial distribution $B\,(n, p)$ where**

(i) $n = 6, p = \frac{1}{3}, k = 3$

$$P\,(X = x) = nC_x\, p^x\, (1 - p)^{n - x}$$

$$P\,(X = k) = 6C_k \left(\frac{1}{3}\right)^k \left(1 - \frac{1}{3}\right)^{6 - k}$$

$k = 3$

$$P\,(X = 3) = 6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{6 - 3}$$

$$= \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 = \frac{20 \times 2^3}{3^6}$$

$$P\,(X = 3) = \frac{20 \times 8}{9 \times 9 \times 9} = \frac{160}{729}$$

(ii) $n = 9, p = \frac{1}{2}, k = 7$

$$P\,(X = k) = nC_k\, p^k\, (1 - p)^{n - k}$$

$$P\,(X = 4) = 10C_4 \left(\frac{1}{5}\right)^4 \left(1 - \frac{1}{5}\right)^{10 - 4}$$

$$= \frac{10 \times 3 \times 7}{1 \times 2 \times 3 \times 4} \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^6$$

$$P\,(X = 4) = 210 \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^6$$

(iii) $n = 9, p = \frac{1}{2}, k = 7$

$$P\,(X = k) = nC_k\, p^k\, (1 - p)^{n - k}$$

$$P\,(X = 7) = 9C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^2$$

$$= \frac{9 \times 8^4}{1 \times 2} \left(\frac{1}{2^9}\right) = \frac{9}{27} = \frac{9}{128}$$

2. The probability that Mr.Q hits a target at any trial is 1/4 suppose he tries at the target 10 times. Find the probability that he hits the target (i) exactly 4 times (ii) atleast one time.

$$n = 10$$

Let X be the random variable denote number of hits

$$X = 0, 1, 2, \ldots 10$$

X follows the binomial distribution

$P =$ probability of success $= \dfrac{1}{4}$

$q = 1 - p = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

$$f(x) \text{ or } P(X = x) = nC_x \, p^x \, q^{n-x} \qquad x = 0, 1, \ldots 10$$

$$P(X = x) \text{ or } f(x) = 10C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{10-x}$$

(i) exactly 4 times

$$f(4) = 10C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{10-4}$$

$$= 10C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^6$$

(ii) atleast one time $= P(X \geq 1)$

$$= 1 - P(X < 1)$$

$$= 1 - P(X = 0)$$

$$= 1 - 10C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{10-0}$$

$$= 1 - \left(\frac{3}{4}\right)^{10}$$

4. The probability that a certain kind of component will service a electrical test is $\frac{3}{4}$. Find the probability that exactly 3 of the 5 components tested survive.

$$n = 5$$

$$p = \frac{3}{4}$$

Let X be the random variable denotes number of survive components

$$X = 0, 1, 2, 3, 4, 5$$

$$X \sim \left(5, \frac{3}{4}\right)$$

$P =$ probability of a components service after test

$$P = \frac{3}{4} \quad q = 1 - p \quad q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$P\,(X = x) = nC_x\, p^x\, q^{n-x}, \quad x = 0, 1, \ldots 5$$

$$P\,(X = x) = 5C_x \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{5-x}$$

P (exactly 3 survive) $= P\,(X = 3)$

$$= 5C_3 \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^{5-3}$$

$$= \frac{5 \times 4}{1 \times 2} \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^2$$

$$= 10\, \frac{3^3}{4^3}\, \frac{1}{4^2} = \frac{270}{1024}$$

5. A retailer purchases a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective device rate is 5%
The inspect of the retailer randomly picks 10 items from a shipment, what is the probability that there will be (i) atleast one defect item (ii) exactly 2 defective items.

$$n = 10$$

X be the random variable denotes number of defective items.

$$X \sim (10, 0.05)$$

$P =$ probability of a defective item

$P = 5\%$

$P = 0.05$

$$q = 1 - P = 0.95$$

$$P(X = x) = nC_x p^x q^{n-x}$$

$$\boxed{\mathbf{P(X = x) = 10C_x (0.05)^x (0.95)^{10-x}}}$$

(i) atleast one defective

$$= P(X \geq 1)$$

$$= 1 - P(X < 1)$$

$$= 1 - P(X = 0)$$

$$= 1 + 10C_0 (0.05)^0 (0.95)^{10-0}$$

$$= 1 - (0.95)^{10}$$

(ii) exactly 2 defective

$$= P(X = 2)$$

$$= 10C_2 (0.05)^2 (0.95)^{10-2}$$

$$= 10C_2 (0.05)^2 (0.95)^8$$

6. If the probability that a fluorescent light has a useful life of atleast 600 hours is 0.9, find the probability that amount 12 such lights

(i) exactly 10 will have a useful life of atleast 600 hours.

(ii) atleast 11 will have a useful life of atleast 600 hours.

(iii) atleast 2 will not have a useful life of atleast 600 hours.

$n = 12$ $p =$ probability of useful life

$p = 0.9$

$q = 1 - p = 1 - 0.9 = 0.1$

X be the random variable denotes useful life of atleast 600 hours of a light.

$X \sim B\,(12, 0.9)$

$P\,(X = x) = nC_x\, p^x\, q^{n-x}$

$$\boxed{\boldsymbol{P\,(X = x) = 12C_x\,(0.9)^x\,(0.1)^{12-x}}} \qquad x = 0, 1, \ldots 12$$

(i) exactly 10

$P\,(X = 10) = 12C_{10}\,(0.9)^{10}\,(0.1)^{12-10}$

$= 12C_{10}\,(0.9)^{10}\,(0.1)^2$

(ii) atleast 11

$P\,(X \geq 11) = P\,(X = 11) + P\,(X = 12)$

$= 12C_{11}\,(0.9)^{11}\,(0.1)^{12-11} + 12C_{12}\,(0.9)^{12}\,(0.1)^{12-12}$

$= 12\,(0.9)^{11}\,(0.1) + (0.9)^{12}\,(1)$

$= (0.9)^{11}\,[12 \times 0.1 + 0.9]$

$= (0.9)^{11}\,[1.2 + 0.9]$

$P\,(X \geq 11) = 2.1\,(0.9)^{11}$

(iii) atleast 2 will not have a useful life of atleast 600 hours

$= P\,(X < 11)$

P (atleast 2 will not) $= 1 - P\,(X \geq 11)$

$= 1 - (2.1)\,(0.9)^{11}$

[each probability having 2 or more defectives]

Example 11.19

Find the binomial distribution function for each of the following.

(i) Five fair coins are tossed once and X denotes the number of heads.
(ii) A fair die is rolled 10 times and X denotes the number of times 4 appeared.

Solution

(i) Given that five fair coins are tossed once. Since the coins are fair coins the probability of getting an head in a single coin is $p = \frac{1}{2}$ and $q = 1 - p = \frac{1}{2}$

Let X denote the number of heads that appear in five coins. X is binomial random variable that takes on the values 0, 1, 2, 3, 4 and 5 with $n = 5$ and $p = \frac{1}{2}$. That is $X \sim B\left(5, \frac{1}{2}\right)$

Therefore the binomial distribution is

$$f(x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

becomes

$$f(x) = \binom{5}{x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{n-x}, \quad x = 0, 1, 2, \ldots, 5$$

That is $$f(x) = 5c_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

(ii) A fair die is rolled ten times and X denotes the number of times 4 appeared. X is binomial random variable that takes on the values 0, 1, 2, 3, … 10, with $n = 10$ and $p = \frac{1}{6}$. That is $X \sim B\left(10, \frac{1}{6}\right)$

Probability of getting a four in a die is $p = \frac{1}{6}$ and $q = 1 - p = \frac{5}{6}$.

Therefore the binomial distribution is

$$f(x) = 10c_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{10-x}, \quad x = 0, 1, 2, \ldots, 10$$

Example 11.20

A multiple choice examination has ten questions, each question has four distractors with exactly one correct answer. Suppose a student answers by guessing and if X denotes the number of correct answers, find (i) binomial distribution (ii) probability that the student will get seven correct answers (iii) the probability of getting at least one correct answer

✍ Solution

(i) Since X denotes the number of success, X can take the values 0, 1, 2, …, 10

The probability for success is $p = \frac{1}{4}$ and for failure $q = 1 - p = \frac{3}{4}$, and $n = 10$.

Therefore X follows a binomial distribution denoted by $X \sim B\left(10, \frac{1}{4}\right)$.

This gives, $f(x) = 10c_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{10-x}$, $x = 0, 1, 2, \ldots, 10$

(ii) Probability for seven correct answers is

$$P(X = 7) = f(7) = 10c_7 \left(\frac{1}{4}\right)^7 \left(\frac{3}{4}\right)^{10-7} = 120\left(\frac{3^3}{4^{10}}\right)$$

Probability that the student will get seven correct answers is $120\left(\frac{3^3}{4^{10}}\right)$.

(iii) Probability for at least one correct answer is

$$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0)$$

$$= 1 - 10_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{10} = 1 - \left(\frac{3}{4}\right)^{10}$$

Probability that the student will get for at least one correct answer is $1 - \left(\frac{3}{4}\right)^{10}$

Example 11.21

The mean and variance of a binomial variate X are respectively 2 and 1.5. Find (i) $P(X = 0)$ (ii) $P(X = 1)$ (iii) $P(X \geq 1)$

Solution

To find the probabilities, the values of the parameters n and p must be known.

Given that

$$\text{Mean } = np = 2 \text{ and variance} = npq = 1.5$$

This gives $\dfrac{npq}{np} = \dfrac{1.5}{2} = \dfrac{3}{4}$

$$q = \frac{3}{4} \text{ and } p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$np = 2 \text{ gives, } n = \frac{2}{p} = 8$$

Therefore $X \sim B\left(8, \dfrac{1}{4}\right)$

Therefore probability distribution is

$$P(X = x) = f(x) = 8c_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{8-x} \qquad x = 0, 1, 2, \ldots 8$$

(i) $P(X = 0) = f(0) = 8c_0 \left(\dfrac{1}{4}\right)^0 \left(\dfrac{3}{4}\right)^{8-0} = \left(\dfrac{3}{4}\right)^8$

(ii) $P(X = 1) = f(1) = 8c_1 \left(\dfrac{1}{4}\right)^1 \left(\dfrac{3}{4}\right)^{8-1} = 2\left(\dfrac{3}{4}\right)^7$

(iii) $P(X \geq 1 - P(X < 1) = 1 - P(X = 0) = 1 - \left(\dfrac{3}{4}\right)^8$

Example 11.22

On the average, 20% of the products manufactured by ABC Company are found to be defective. If we select 6 of these products at random and X denote the number of defective products find the probability (i) two products are defective (ii) at most one product is defective (iii) atleast two products are defective.

Solution

Given that $n = 6$

Probability for selecting a defective product is $\frac{20}{100}$, that is $p = \frac{1}{5}$.

Since X denotes the number defective products, X can be on the values $0, 1, 2, \ldots, 6$

The probability for defective (success) is $p = \frac{1}{5}$ and for failure $q = 1 - p = \frac{4}{5}$, and $n = 6$

Therefore X follows a binomial distribution denoted by $X \sim B\left(6, \frac{1}{5}\right)$.

This gives

$$f(x) = 6c_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{6-x}, \quad x = 0, 1, 2, \ldots, 6$$

(i) Probability for two defective product is

$$P(X = 2) = f(2) = \left(6c_2\right)\left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{6-2} = 15\left(\frac{4^4}{5^6}\right)$$

(ii) Probability for at most one defective product is

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$= 6c_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{6-0} + 6c_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^{6-1}$$

$$= \left(\frac{4}{5}\right)^6 + (6)\left(\frac{4^5}{5^6}\right) = 2\left(\frac{4}{5}\right)^5$$

Probability for at most one defective product is $2\left(\frac{4}{5}\right)^5$.

(iii) Probability for at least two defective product is

$$P(X \geq 2) = 1 - P(X < 2) = 1 - P(X \leq 1) = 1 - 2\left(\frac{4}{5}\right)^5$$

Probability for at least two defective product is $\mathbf{1 - 2\left(\frac{4}{5}\right)^5}$.

Type II: Mean (*np*), variance (*npq*) based sums
Q.No: 3, 7, 8, 9

3. Using binomial distribution find the mean and variance of X for the following experiments.

(i) A fair coin is tossed 100 times, and X denote the number of heads.

$$n = 100$$

$$P = \text{Probability of getting head}$$

$$P = \frac{1}{2} \quad q = 1 - p = \frac{1}{2}$$

$$\text{mean } = np = 50 \times \frac{1}{2} = 50$$

$$\text{variance } = npq = 50 \cdot \frac{1}{2} = 25$$

(ii) A fair die is tossed 240 times and X denotes the number of times that 4 appeared.

$$n = 240$$

$$P = \text{probability of getting 4}$$

$$P = \frac{1}{6} \quad q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

$$\text{mean } = np = 40 \times \frac{1}{6} = 40$$

$$\text{variance } = npq = 40 \times \frac{1}{6} \times \frac{5}{6} = \frac{200}{6} = \frac{100}{3}$$

7. The mean and standard deviation of a binomial variate X are respectively 6 and 2. Find (i) The probability mass function (ii) $P(X = 3)$ (iii) $P(X \geq 2)$

mean $= 6$ | S.D $= 2$ | $\text{S.D}^2 = 4$

$np = 6 \ \ldots (1)$ | variance $= 4$

$npq = 4 \ \ldots (2)$

$$\frac{(2)}{(1)} \Rightarrow \frac{npq}{np} = \frac{4}{6} \qquad \boldsymbol{q = \frac{2}{3}} \qquad p = 1 - q = 1 - \frac{2}{3}$$

$$\boldsymbol{p = \frac{1}{3}}$$

$$(1) \Rightarrow n\frac{1}{3} = 6$$

$$\boxed{\boldsymbol{n = 18}}$$

(i) Probability mass function

$$P\,(X = x) = nC_x\, p^x\, q^{n-x}$$

$$\boxed{\boldsymbol{P\,(X = x = 18C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{18-x}}} \qquad x = 0, 12 \ldots 18$$

(ii) $P\,(X = 3) = 18C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{18-3}$

$$= 18C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{15}$$

(iii) $P\,(X \geq 2) = 1 - P\,(X < 2)$

$$= 1 - [P\,(X = 0) + P\,(X = 1)]$$

$$= 1 - \left[18C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{18-0} + 18C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{18-1}\right]$$

$$= 1 - \left[1\,(1) \left(\frac{2}{3}\right)^{18} + 18\,\frac{1}{3} \left(\frac{2}{3}\right)^{17}\right]$$

$$= 1 - \left(\frac{2}{3}\right)^{17} \left[\frac{2}{3} + 6\right]$$

$$= 1 - \frac{20}{3} \left(\frac{2}{3}\right)^{17}$$

8. If $X \sim B\,(n, p)$ such that $4P\,(X = 4) = P\,(X = 2)$ and $n = 6$. Find the distribution, mean and S.D.

$$n = 6 \qquad 4P\,(X = 4) = P\,(X = 2)$$

$$\boxed{\boldsymbol{P\,(X = x) = nC_x\, p^x\, q^{n-x}}}$$

$$4\,(6C_4\,(P)^4\,(q)^{6-4}) = 6C_2\,P^2\,(q)^{6-2}$$

$$4 \times 6C_2\,P^4\,q^2 = 6C_2\,P^2\,q^4$$

$$4\frac{P^4}{P^2} = \frac{q^4}{q^2}$$

$$4\,(P^2) = q^2$$

$$4P^2 = (1-P)^2$$

$$4P^2 = 1 + P^2 - 2P$$

$$4P^2 - 1 - P^2 + 2P = 0$$

$$3P^2 + 2P - 1 = 0$$

$$(P+1)\,(3P-1) = 0$$

$p = -1$ $\qquad\qquad$ $3p = 1$

$$\boldsymbol{p = \frac{1}{3}} \qquad q = 1 - p = 1 - \frac{1}{3}$$

$$\boldsymbol{q = \frac{2}{3}}$$

The distribution is

$$P\,(X = x) = nC_x\, p^x\, q^{n-x}$$

$$= 6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x} \qquad x = 0, 1, \ldots 6$$

mean $= np = 2 \times \frac{1}{3} = 2$

variance $= npq = 6 \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{3}$

standard deviation $= \sqrt{npq} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$

9. In a Binomial distribution consisting of 5 independent trails, the probability of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the mean and variance.

$$n = 5$$

given $P\,(X = 1) = 0.4096$

$$P\ (X = 2) = 0.2048$$

$$P\ (X = x) = nC_x\, p^x\, q^{n-x}$$

$$P\ (X = 1) = 5C_1\, p^1\, q^{5-1} = 0.4096 \;\Rightarrow\; 5pq^4 = 0.4096 \qquad ...(1)$$

$$P\ (X = 2) = 5C_1\, p^2\, q^{5-2} = 0.2048 \;\Rightarrow\; 10p^2\, q^3 = 0.2048 \qquad ...(2)$$

$$(1) \div (2) \Rightarrow \frac{5pq^4}{10p^2\, q^3} = \frac{0.4096}{0.2048}$$

$$\frac{q}{2p} = 2$$

$$1 - p = 4p$$

$$1 = 5p$$

$$p = \frac{1}{5}$$

$$q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$$

$$\text{mean } = np = 5 \times \frac{1}{5} = 1$$

$$\text{variance } = npq = 1 \times \frac{4}{5} = \frac{4}{5}$$

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