CHAPTER 08

Theory of Errors and Survey Adjustments

8.1 Introduction

- Measurement of lengths and angles done in various survey operations in field are accompanied with errors.
- It is almost impossible to have true measured value. Errors occur due to many reasons some of them are as follows:
  - (a) Errors due to imperfect instruments
  - (b) Errors due to environmental conditions or carelessness
  - (c) Errors due to human limitations
- Due the measurement operations, as far as possible, errors should be minimized.
- Errors which still occur must be adjusted or eliminated so as to counteract their effects.
- In triangulation, all the observed angles of the triangulation stations are required to be adjusted before using them in the computation of sides.
- The measured angles are so adjusted so as to satisfy the required geometrical conditions.

8.2 Types of Errors

(a) Gross errors or mistakes
(b) Systematic or cumulative errors
(c) Accidental or random errors

8.2.1 Gross Errors or Mistakes

- This mistake occurs on the part of survey personnel due to lack of experience or carelessness.

  For example: If a surveyor reads the tape reading as 29.5 m instead of 30 m, then it is a mistake or the gross error.
- Mistakes, if not detected, can lead to erroneous results thereby making the whole survey as faulty. Adequate check measurements are thus made to detect this type of error.

8.2.2 Systematic or Cumulative Errors

- These errors are called as systematic because they always follow a definite pattern or a mathematical/ physical law. These errors are of same magnitude and sign.

For example: Measuring a length with a steel tape and error involved due to temperature. This is a systematic error because it follows the physical law of expansion of solids on increasing the temperature.

- This type of error makes the result either too large or too small.

8.2.3 Accidental or Random Errors

- This type of error occurs due to human limitation in reading an observation.

  For example: While measuring an angle from a protector (say 30.6°), then it is quite possible that the observer may read 30.5° or 30.7° due to inability of human eye to judge the exact division.
- A good thing about accidental errors is that when a large number of observations are made, then they use to cancel out because there is equal probability of the error to be positive or negative. Thus this type of error is also called as **compensating error.**
- But compensating effect of accidental errors is not full proof and there always remains some accidental errors. This error cannot be eliminated altogether from the observations whatever precautions are taken but magnitude of this error is generally very small.
- Smaller the random error, more precise is the measurement. Thus random/accidental errors limit the level of precision while taking an observation.
- Accidental errors occur purely as a matter of chance and thus theory of probability is used to account for these types of errors.

NOTE: The theory of errors deals with accidental/random errors only with the presumption that all the systematic and gross errors have been eliminated from the measured values.

8.3 Terminologies in Theory of Errors

8.3.1 Observation

- It is the numerical value of an observed quantity in the field.
- When the quantity is directly measured then it is called as **direct observation** and when the quantity is found indirectly (from the direct observations of other quantities), then it is called as **indirect observation.**

8.3.2 Observed Value

- The observed value of a quantity is the value obtained from the observation after applying the corrections of systematic errors and gross errors.
- **Independent quantity:** When the value of observed quantity is independent of the values of other quantities then the observed quantity is called as **independent quantity.**
- **Dependent quantity:** When the value of observed quantity depends on other quantities then the observed quantity is called as **dependent quantity.**

8.3.3 True Value

- It is the value of the quantity which is free from all errors.
- Now because it is impossible to eliminate all the errors from the observed quantity, true value cannot be found.
- It is a purely hypothetical concept.

8.3.4 Most Probable Value

- It is the value of the quantity which has more chances of being true than any other value.
- Most probable value is thus very close to (but NOT equal to) the true value of an observed quantity.

8.3.5 True Error

- It is the difference between observed value of a quantity and its true value i.e.

  True error = Observed value – True value
- As the true value of a quantity can never be known and thus true error can never be determined.

8.3.6 Residual Error

- It is the difference between observed value and the most probable value of a quantity i.e.

  Residual error = Observed value – Most probable value

8.3.7 Observation Equation

- The relationship between the observed quantities is called as observation equation.

  For example:

  $$A + B = 90°$$

  It is an observation equation for the observed angles A and B.

8.3.8 Condition Equation

- It is the equation which expresses the relation between several dependent quantities.

  For example:

  $$A + B + C = 180°$$

  It is a condition equation for the dependent quantities.

8.4 Indices of Precision for Observations of Same Weight

8.4.1 Standard Deviation

It is a numerical value that indicates the amount of precision about a central value.

$$\sigma = \sqrt{\frac{\Sigma v^2}{n-1}} \quad ...(8.1)$$

Where, n = Number of observations made

v = residual / variation

8.4.2 Variance

- The square of standard deviation (σ) is called as **variance (V).**

  Thus, $$V = \sigma^2 \quad ...(8.2)$$
- It is used as a measure of dispersion or spread of the observations around a mean value.

8.4.3 Standard Error of Mean

- The standard deviation of the mean is called as the standard error of mean (σ_m).

  Thus, $$\sigma_m = \pm\sqrt{\frac{\Sigma v^2}{n(n-1)}} = \pm\frac{\sigma}{\sqrt{n}} \quad ...(8.3)$$
- It indicates the limits of error bound within which the true value of the mean lies.

8.4.4 Standard Error of Single Observation

- The standard error of single observation is given by:

$$\sigma_1 = \pm\sqrt{\frac{\Sigma v^2}{(n-1)}} \quad ...(8.4)$$

- The standard error of single observation is the same as the standard deviation (σ). These two terms are often used synonymously.

8.4.5 Most Probable Error

It the error for which there is equal chances that the true error will be less than the probable error and equal chances that the true error will be more than the probable error i.e. each is having the probability of 50%.

$$\text{Most probable error} = \pm 0.6745\sqrt{\frac{\Sigma v^2}{n-1}} = \pm 0.6745\sigma \quad ...(8.5)$$

8.4.6 Most Probable Error of Mean

It is equal to 0.6745 times the standard error of mean i.e.,

$$\text{Probable error mean} = \pm 0.6745\sqrt{\frac{\Sigma v^2}{n(n-1)}} \quad ...(8.6)$$

8.4.7 Maximum Error

The maximum error of a quantity is almost impossible to determine absolutely. Thus, often 99.9% error is taken as the maximum error in surveying. This maximum error corresponds to $\pm 3.29\sigma$

8.5 The Law of Weights

8.5.1 Weight

- Weight of a quantity indicates the precision of the quantity within a set of observations or in other words it represents the trustworthiness of the quantity being measured.
- Greater the weight of an observation, greater is the precision with which the quantity is measured.
- Weights are expressed in terms of natural numbers with higher number representing higher weight thereby higher precision.

8.5.2 Allocation of Weights

(a) Weights are assigned as inverse proportion to **variance** or **square of standard deviations.**

For example: Let variance of set A observation = 80

Variance of set B observation = 120

Thus, $\frac{\text{Weight of set } A}{\text{Weight of set } B} = \frac{1/80}{1/120} = 1.5$

(b) Weights to the quantities measured in similar conditions are assigned in direct proportion to the number of times (say n) a quantity is measured.

For example: Let a quantity A is measured five times, then the weight of quantity A is 5.

(c) Many a times, weights are assigned based on personal perception based on field or other environmental conditions. Lower weights are generally allocated to quantities measured in difficult conditions and higher weights are assigned for quantities measured in relatively easy conditions.

(d) Weights are often assigned as in inverse proportion to the lengths of lines being measured.

8.5.3 Various Laws of Weight

(a) The weight of the weighted arithmetic mean is the sum of individual weights of the quantity.

For example: Let the length of a line is measured as:

25 m	weight 2
25.5 m	weight 4
24.8 m	weight 3

$$\therefore \quad \text{Arithmetic mean} = \frac{25\times 2 + 25.5\times 4 + 24.8\times 3}{2+4+3} = 25.16 \text{ m}$$

Thus the weight of arithmetic mean is (2 + 4 + 3) = 9

(b) Weight of algebraic sum of two or more quantities is equal to reciprocal of sum of reciprocal of individual weights.

For example: Let weight of quantity A is 3 and weight of quantity B is 5

$$\therefore \quad \text{Weight of quantity } (A+B) = \frac{1}{\frac{1}{3}+\frac{1}{5}} = 1.875$$

$$\text{Similarly, Weight of quantity } (A-B) = \frac{1}{\frac{1}{3}+\frac{1}{5}} = 1.875$$

(c) When a quantity of a given weight is multiplied by a factor, then the weight of resultant quantity is given by dividing the weight of the quantity by the square of the factor.

For example: Let weight of quantity A is 3

Then the weight of quantity $4A$ is $\frac{3}{4^2} = \frac{3}{16} = 0.1875$

(d) The weight of an equation remains unchanged when all the signs of the terms of equation are changed.

For example: If weight of equation $x + y = 79$ is 3

then the weight of equation $-x - y = -79$ is 3 only.

(e) The weight of an equation remains unchanged when it is added or subtracted from a constant.

For example: If weight of equation $x + y = 55$ is 5

then the weight of equation $10 + x + y = 65$ is 5 only.

Similarly, weight of equation $60 - x - y = 5$ is also 5.

8.6 Indices of Precision for Observations of Different Weights

8.6.1 Standard Deviation of Weighted Observations

The standard deviation of weighted observations is given by:

$$\sigma_w = \pm\sqrt{\frac{\Sigma wv^2}{n-1}} \quad ...(8.7)$$

Where, n = Number of observations made

v = residual / variation

8.6.2 Standard Error of Mean of Weighted Observations

The standard deviation of the mean is called as the **standard error of mean** $(\sigma_m)_w$.

Thus,
$$(\sigma_n)_w = \pm\sqrt{\frac{\Sigma wv^2}{(n-1)\Sigma w}} = \pm\frac{\sigma_w}{\sqrt{\Sigma w}} \quad ...(8.8)$$

It indicates the limits of error bound within which the true value of the mean lies.

8.6.3 Standard Error of Single Observation of Weight w_i

The standard error of single observation is given by:

$$(\sigma_1)_w = \pm\sqrt{\frac{\Sigma wv^2}{w_i(n-1)}} = \pm\frac{\sigma_w}{\sqrt{w_i}} \quad ...(8.9)$$

8.6.4 Most Probable Error of Single Observation of Weight w_i

If the error for which there is equal chances that the true error will be less than the probable error and equal chances that the true error will be more than the probable error i.e. each is having the probability of 50%.

$$\text{Most probable error} = \pm 0.6745\sqrt{\frac{\Sigma wv^2}{w_i(n-1)}} = \pm 0.6745(\sigma_1)_w \quad ...(8.10)$$

8.6.5 Most Probable Error of Mean

It is given by,

$$\text{Most Probable error of mean} = \pm 0.6745\sqrt{\frac{\Sigma wv^2}{(n-1)\Sigma w}} \quad ...(8.11)$$

8.7 Corrections to be Applied to Field Measurements for Closing Error

- It is almost quite impossible to have a totally error free observation in the field. Some error do crept in while taking the field observations.
- Once the error is known, correction can be applied.
- The sign of correction is opposite to that of error.

The following **general principles** are used for correcting the closing error:

(a) If all the observations are made with same weights then error is distributed equally to all the observations.

(b) The correction applied to an observation is inversely proportional to the weight of the observation.

(c) The correction applied to an observation is directly proportional to the square of the standard error (deviation).

(d) In case of line of levels, the correction to be applied is proportional to the length.

8.8 Theory of Least Squares

- The theory of least squares is useful in estimating the most probable value of a quantity.
- According to this theory, the most probable value of a quantity is the one which makes the sum of the product of weights and square of the residuals to a minimum i.e.

$$\Sigma wr^2 = \text{a minimum}$$

Here, w = Weight of the quantity measured

r = Residual error

- The adjustments to the observations as per this theory i.e. the least square adjustments can be done by either of the following two methods:
  - (a) Method of normal equations/observation equations method
  - (b) Method of correlates/condition equation method

8.8.1 The Method of Normal Equations

Let A be the Most Probable Value (MPV) of the measurements viz. $A_1, A_2, A_3, A_4, \ldots. A_n$.

Therefore, the residuals are,

$$r_1 = A_1 - A$$
$$r_2 = A_2 - A$$
$$r_3 = A_3 - A$$
$$\vdots$$
$$r_n = A_n - A$$

Thus, from the theory of least squares,

$$\frac{d}{dA}(\Sigma r^2) = 0$$

$$\frac{d[(A_1 - A)^2 + (A_2 - A)^2 + \ldots(A_n - A)^2]}{dA} = 0$$

$$\Rightarrow \quad (A_1 - A) + (A_2 - A) + \ldots (A_n - A) = 0$$

$$\Rightarrow \quad nA = A_1 + A_2 + \ldots + A_n$$

$$\Rightarrow \quad A = \frac{A_1 + A_2 + \ldots + A_n}{n} \quad ...(8.12)$$

- Eq. (8.12) is called as **normal equation**. The solution of this equation gives the most probable value. The normal equations are solved simultaneously to make Σr^2 minimum.
- In order to form normal equations for each of the unknown quantities, it is required to multiply each of the observation equations by the product of algebraic coefficients of that unknown quantity in that equation and the weight of that observation and add the results.

  Let error in the following equations be

$$e_1 = a_1x + b_1y + c_1z + d \qquad \text{weight } w_1$$
$$e_2 = a_2x + b_2y + c_2z + d \qquad \text{weight } w_2$$
$$e_3 = a_3x + b_3y + c_3z + d \qquad \text{weight } w_3$$

As per the theory of least squares,

$$\Sigma e^2 = \text{a minimum}$$

i.e., $\Sigma w(ax + by + cz + d)^2 = \text{a minimum}$

Differentiating the above relation with respect to x, y and z respectively will yield the following equations which are called as normal equations with respect to x, y and z:

$$\Sigma wa\,(ax + by + cz + d) = 0 \qquad \text{(Normal equation for } x) \ldots\text{(i)}$$

i.e.

$$a_1w_1(a_1x + b_1y + c_1z + d_1) + a_2w_2(a_2x + b_2y + c_2z + d_2) + a_3w_3(a_3x + b_3y + c_3z + d_3) = 0$$

Similarly $\Sigma wb\,(ax + by + cz + d) = 0$ (Normal equation for y) ...(ii)

i.e.

$$b_1w_1(a_1x + b_1y + c_1z + d_1) + b_2w_2(a_2x + b_2y + c_2z + d_2) + b_3w_3(a_3x + b_3y + c_3z + d_3) = 0$$

and $\Sigma wc\,(ax + by + cz + d) = 0$ (Normal equation for z) ...(iii)

i.e.

$$c_1w_1(a_1x + b_1y + c_1z + d_1) + c_2w_2(a_2x + b_2y + c_2z + d_2) + c_3w_3(a_3x + b_3y + c_3z + d_3) = 0$$

The above equations (i), (ii) and (iii) are the required normal equations for x, y and z respectively.

8.9 Most Probable Values (MPV) of Directly Observed Quantities

Let the observations $A_1, A_2, A_3, \ldots A_n$ are made with respective weights as $w_1, w_2, w_3, \ldots w_n$.

From the theory of least squares,

$$\Sigma wr^2 = \text{a minimum}$$

i.e., $w_1(A_1 - A)^2 + w_2(A_2 - A)^2 + \ldots + w_n(A_n - A)^2 = \text{a minimum}$

Differentiating the above equation with respect to A and equating it to zero, we have,

$$w_1(A_1 - A) + w_2(A_2 - A) + w_3(A_3 - A) + \ldots + w_n(A_n - A) = 0$$

$$A = \frac{w_1A_1 + w_2A_2 + \ldots + w_nA_n}{w_1 + w_2 + \ldots + w_n} \qquad \ldots(8.13)$$

8.10 Most Probable Values (MPV) of Indirectly Observed Quantities

- In case where the unknown quantities are independent of each other, then their most probable values can be determined by forming the normal equations of each of the unknown quantity and then solving the simultaneous equations so formed.

8.11 The Method of Differences

- Solution of simultaneous normal equations is an easy task when the number of such equations is small (say two or three).
- However, when the number of normal equations is too large, then simultaneously solving all the equations is quite tedious. In such cases, we go for the method of differences.

The method of differences is used to simplify the normal equations as per the following procedure:

1. Assume corrections as $c_1, c_2, c_3, \ldots c_n$.
2. Subtract the observed values from the assumed values to express the discrepancy.
3. Form normal equations in $c_1, c_2, c_3, \ldots c_n$.
4. Solve simultaneously the normal equations for $c_1, c_2, c_3, \ldots c_n$.
5. Add all the corrections algebraically to the observed values to obtain the most probable values.

8.12 The Method of Correlates

- This method is also known as the **method of condition equations** or **method of Lagrange multiplier.**
- When there are a large number of condition equations, then this method is more suitable than the method of normal equations.
- After forming all the condition equations, additional equation from the theory of least squares is also applied.
- Then condition equation is multiplied by an unknown multiplier called as the **correlate** or the **Lagrange multiplier (λ).**
- The resultant condition equations are then combined with the condition of least squares, which on differentiation is expressed as a linear function of correlates. These equations are then solved to find the values of the correlates.

8.13 Adjustment of Two Connected Triangles

The **Fig. 8.1** shows two connected triangles which share a common side i.e. diagonal of the quadrilateral formed by the two triangles. ΔABC and ΔBCD form the $\square ABCD$ with BC as diagonal. Firstly the station adjustments are done and then the corrected values of the angles A, B_1, B_2, D, C_1 and C_2 are determined.

Now there are a total of eight angles viz. A, B_1, B_2, D, C_1, C_2, $B\,(= B_1 + B_2)$ and $C\,(= C_1 + C_2)$. These eight angles must satisfy the following geometric conditions:

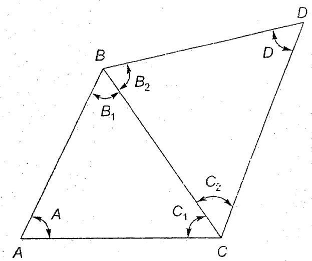

Fig. 8.1 *Adjustment of two connected triangles*

$$\angle A + \angle B_1 + \angle C_1 = 180°$$
$$\angle D + \angle B_2 + \angle C_2 = 180°$$
$$\angle C_1 + \angle C_2 = \angle C$$
$$\angle B_1 + \angle B_2 = \angle B$$

Now there are eight unknowns and four equations. If we assume that angles B_1, B_2, C_1 and C_2 are independent unknowns (or variables) and angles A, B, C and D are dependent unknowns (or variables). Then,

$$\angle A = 180° - (\angle B_1 + \angle C_1)$$
$$\angle D = 180° - (\angle B_2 + \angle C_2)$$
$$\angle C = \angle C_1 + \angle C_2$$
$$\angle B = \angle B_1 + \angle B_2$$

From these observation equations, normal equations can be formed and values of angles B_1, B_2, C_1 and C_2 can be determined.

8.14 Adjustment of a Braced Quadrilateral

As shown in **Fig. 8.2** both the diagonals AC and BD are measured and there is no station at the point of intersection of these diagonals. Thus theodolite is set up at any of the four stations A, B, C or D.

Let $\theta_1, \theta_2, \ldots \theta_8$ are the eight corner angles measured at *A*, *B*, *C* and *D*.

In the above quadrilateral, angles θ_1, θ_3, θ_5 and θ_7 are called as left angles and θ_2, θ_4, θ_6 and θ_8 are called as right angles.

In order to adjust the above braced quadrilateral, the following conditions must be satisfied:

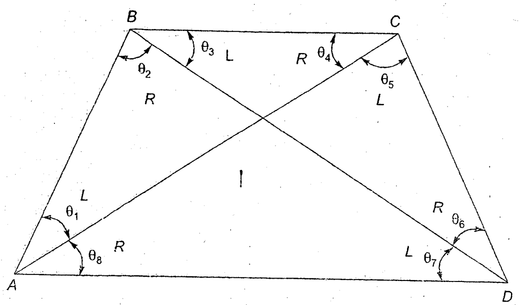

Fig. 8.2 *Adjustment of braced quadrilateral*

Equations Related to Angles

(a) $\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \theta_7 + \theta_8 = 360°$

(b) $\theta_1 + \theta_2 = \theta_5 + \theta_6$

(c) $\theta_3 + \theta_4 = \theta_7 + \theta_8$

Equations Related to Sides

The angles of the braced quadrilateral must also satisfy the side equations so that the figure is closed. Even if the above three angle equations are satisfied, the quadrilateral may not be closed if side equations are not satisfied.

The side equation is expressed as:

$$\Sigma \log \sin\theta_L = \Sigma \log \sin\theta_R \quad \ldots(8.14)$$

where θ_L and θ_R are respectively the left and right angles as defined above.

Thus,

$$\Sigma \log \sin\theta_1 + \Sigma \log \sin\theta_3 + \Sigma \log \sin\theta_5 + \Sigma \log \sin\theta_7 = \Sigma \log \sin\theta_2 + \Sigma \log \sin\theta_4 + \Sigma \log \sin\theta_6 + \Sigma \log \sin\theta_8$$

Derivation of side equations

From ΔABC, $\dfrac{AB}{\sin\theta_4} = \dfrac{BC}{\sin\theta_1}$

$$AB = BC\frac{\sin\theta_4}{\sin\theta_1} \quad \ldots(8.15)$$

Similarly from ΔBCD, $BC = CD\dfrac{\sin\theta_6}{\sin\theta_3}$ ...(8.16)

From Eqs. (8.15) and (8.16), $AB = CD\dfrac{\sin\theta_4}{\sin\theta_1}\cdot\dfrac{\sin\theta_6}{\sin\theta_3}$ ...(8.17)

From ΔCDA, $CD = AD\dfrac{\sin\theta_8}{\sin\theta_5}$ ...(8.18)

Substituting this value of *CD* from Eq. (8.18) in Eq. (8.17)

$$AB = AD\frac{\sin\theta_8}{\sin\theta_5}\cdot\frac{\sin\theta_4}{\sin\theta_1}\cdot\frac{\sin\theta_6}{\sin\theta_3} \quad \ldots(8.19)$$

From ΔDAB, $AD = AB\dfrac{\sin\theta_2}{\sin\theta_7}$ ...(8.20)

Substituting this value of *AD* from Eq. (8.20) and Eq. (8.19)

$$AB = AB\frac{\sin\theta_2}{\sin\theta_1}\cdot\frac{\sin\theta_4}{\sin\theta_3}\cdot\frac{\sin\theta_6}{\sin\theta_5}\cdot\frac{\sin\theta_8}{\sin\theta_7}$$

$$\Rightarrow \quad \sin\theta_1\cdot\sin\theta_3\cdot\sin\theta_5\cdot\sin\theta_7 = \sin\theta_2\cdot\sin\theta_4\cdot\sin\theta_6\cdot\sin\theta_8 \quad \ldots(8.21)$$

$$\Rightarrow \log\sin\theta_1 + \log\sin\theta_3 + \log\sin\theta_5 + \log\sin\theta_7 = \log\sin\theta_2 + \log\sin\theta_4 + \log\sin\theta_6 + \log\sin\theta_8$$

$$\Sigma\log\sin\theta_L = \Sigma\log\sin\theta_R$$

Illustrative Examples

Example 8.1 Find the most probable error and the most probable value of the area of a circle of radius (10.05 ± 0.02) m.

Solution:

Radius of circle, $r = (10.05 \pm 0.02)\text{m}$

Area of circle, $A = \pi r^2 = \pi(10.05)^2 = 317.31\text{ m}^2$

$\therefore \quad \dfrac{\partial A}{\partial r} = 2\pi r$

$\therefore \quad$ Error in area, $e_A = \dfrac{\partial A}{\partial r}\cdot e_r = \pm(2\pi r)e_r = 2\pi(10.05)0.02 = 1.263\text{ m}^2$

$\therefore \quad$ Most probable area $= (317.31 \pm 1.263)\text{ m}^2$

Example 8.2 Determine the probable error in circumference of a circle of radius 16.5 m with probable error in radius of ±0.35 m.

Solution:

Circumference, $C = 2\pi r = 2\pi(16.5) = 103.67$ m

Error in circumference, $(e_c) = 2\pi \times$ error in $r(e_r)$

$= 2\pi e_r$

$= 2\pi(\pm 0.35)$

$= \pm 2.199$ m

Example 8.3 For the following system of equations, form the normal equations in α, β and γ.

$2\alpha + 3\beta + 4\gamma + 7 = 0$ weight = 3

$6\alpha + 2\beta + \gamma + 5 = 0$ weight = 4

$\alpha + 5\beta + 4\gamma + 3 = 0$ weight = 2

Solution:

The given system of equations are:

$2\alpha + 3\beta + 4\gamma + 7 = 0$ weight = 3

$6\alpha + 2\beta + \gamma + 5 = 0$ weight = 4

$\alpha + 5\beta + 4\gamma + 3 = 0$ weight = 2

Normal equation of α

$$6(2\alpha + 3\beta + 4\gamma + 7) + 24(6\alpha + 2\beta + \gamma + 5) + 2(\alpha + 5\beta + 4\gamma + 3) = 0$$

$\Rightarrow (6\alpha + 9\beta + 12\gamma + 21) + (72\alpha + 24\beta + 12\gamma + 60) + (\alpha + 5\beta + 4\gamma + 3) = 0$

$\Rightarrow 79\alpha + 38\beta + 28\gamma + 84 = 0$ ...(i)

Normal equation of β

$9(2\alpha + 3\beta + 4\gamma + 7) + 8(6\alpha + 2\beta + \gamma + 5) + 10(\alpha + 5\beta + 4\gamma + 3) = 0$

$\Rightarrow 76\alpha + 93\beta + 84\gamma + 133 = 0$ ...(ii)

Normal equation of γ

$12(2\alpha + 3\beta + 4\gamma + 7) + 4(6\alpha + 2\beta + \gamma + 5) + 8(\alpha + 5\beta + 4\gamma + 3) = 0$

$\Rightarrow 14\alpha + 21\beta + 21\gamma + 32 = 0$ ...(iii)

This equations (i), (ii) and (iii) are normal equations of α, β and γ respectively. Which can be solved to arrive at the most probable values of α, β and γ.

Example 8.4 The following observations were made using a current meter:

The rating formula is $V = aN + b$

Find the values of a and b.

No. of revolutions (N)	Flow velocity (m/s)
0.5	0.8
1.1	1.6
1.4	2.4
2.3	2.9

Solution:

The observation equation will be,

$$0.8 = 0.5\,a + b$$
$$1.6 = 1.1\,a + b$$
$$2.4 = 1.4\,a + b$$
$$2.9 = 2.3\,a + b$$

Thus error of observation will be,

$$e_1 = 0.8 - 0.5\,a - b$$
$$e_2 = 1.6 - 1.1\,a - b$$
$$e_3 = 2.4 - 1.4\,a - b$$
$$e_4 = 2.9 - 2.3\,a - b$$

From the theory of least squares,

$$e_1^2 + e_2^2 + e_3^2 + e_4^2 = \text{minimum}$$

$\Rightarrow (0.8 - 0.5\,a - b)^2 + (1.6 - 1.1\,a - b)^2 + (2.4\,a - 1.4\,a - b)^2 + (2.9 - 2.3\,a - b)^2 = \text{minimum}$ ...(i)

Differentiating (i) w.r.t. 'a',

$2(0.8 - 0.5a - b)(-0.5) + 2(1.6 - 1.1a - b)(-1.1) + \{2(2.4 - 1.4a - b)\}(-1.4) + 2(2.9 - 2.3a - b)(-2.3) = 0$

$(0.8 - 0.5a - b) + (3.52 - 2.42a - 2.2b) + (6.72 - 3.92a - 2.8b) + (13.34 - 10.58a - 4.6b) = 0$

$\Rightarrow -17.42\,a - 10.6b + 24.38 = 0$

$\Rightarrow 1.6434\,a + b - 2.3 = 0$ ...(ii)

Differentiating (i) w.r.t. 'b'.

$-2(0.8 - 0.5a - b) - 2(1.6 - 1.1a - b) - 2(2.4 - 1.4a - b) - 2(2.9 - 2.3a - b) = 0$

$\Rightarrow (0.8 - 0.5a - b) + (1.6 - 1.1a - b) + (2.4 - 1.4a - b) + (2.9\ \ 2.3a - b) = 0$

$\Rightarrow 1.325\,a + b - 1.925 = 0$ ...(iii)

Solving (ii) and (iii),

$$a = 1.1778$$
$$b = 0.3644$$

Hence, $v = 1.1778\,N + 0.3644$

Example 8.5 The following three angles A, B and C were observed from a station P:

$$\angle A = 83°\ 15'\ 12'' \pm 4''$$
$$\angle B = 126°\ 13'\ 17'' \pm 3''$$
$$\angle C = 150°\ 30'\ 19'' \pm 5''$$

Find the corrected angles.

Solution:

Sum of the angles $= \angle A + \angle B + \angle C$
$= 83°\ 15'\ 12''$
$+ 126°\ 13'\ 17''$
$+ 150°\ 30'\ 19''$
$= 359°\ 58'\ 48''$
$\neq 360°$

Thus closing error $= 359°\ 58'\ 48'' - 360°$
$= 359°\ 58'\ 48' - 359°\ 60''$
$= -1'\ 12''$
$= -72''$

∴ Total correction $(c) = +72''$

Let C_1, C_2, C_3 be the corrections applied to angles A, B and C respectively.

∴ $C_1 : C_2 : C_3 = 4^2 : 3^2 : 5^2$

$\Rightarrow C_1 : C_2 : C_3 = 16 : 9 : 25$

∴ $C_1 = \frac{16}{50} \times 72 = 23.04''$

$C_2 = \frac{9}{50} \times 72 = 12.96''$

$C_3 = \frac{25}{50} \times 72 = 36''$

∴ Corrected angles are:

$\angle A = 83°\ 15'\ 12'' + 23.04'' = 83°\ 15'\ 35.04''$

$\angle B = 126°\ 13'\ 17'' + 12.96'' = 126°\ 13'\ 29.96''$

$\angle C = 150°\ 30'\ 19'' + 36'' = 150°\ 30'\ 55''$

360° (O.K.)

Example 8.6 For a telescope fitted with stadia hairs, it is required to find the most probable values of constant C and K of a tacheometer. The staff reading was taken in the field with the staff held vertical and line of sight horizontal. Find the most probable values of C and K.

Distance of staff from tacheometer, D(m)	Staff Intercept (m)
160	1.595
215	2.050
275	3.155

Solution:

$$D = ks + C$$

Thus observation equation are,

$$160 = 1.595\,k + C$$

$$215 = 2.050\,k + C$$
$$275 = 3.155\,k + C$$

From errors of observation are,

$$e_1 = (160 - 1.595\,k - C)$$
$$e_2 = (215 - 2.050\,k - C)$$
$$e_3 = (275 - 3.155\,k - C)$$

From the theory of least squares,

$$e_1^2 + e_2^2 + e_3^2 = \text{Minimum}$$

$$\Rightarrow (160 - 1.595\,k - C)^2 + (215 - 2.050\,k - C)^2 + (275 - 3.155\,k - C)^2 = \text{Minimum} \quad \text{...(i)}$$

Differentiating (i) w.r.t. 'k'

$$2(160 - 1.595\,k - C)(-1.595) + 2(215 - 2.050\,k - C)(-2.050) + 2(275 - 3.155\,k - C)(-3.155) = 0$$
$$\Rightarrow (255.2 - 2.544025\,k - 1.595\,C) + (440.75 - 4.2025\,k - 2.050\,C) + (867.625 - 9.954025\,k - 3.155\,C) = 0$$
$$\Rightarrow 1563.575 - 16.70055\,k - 6.8\,C = 0$$
$$\Rightarrow 16.70055\,k + 6.8\,C = 1563.575$$
$$\Rightarrow 2.455963\,k + C = 229.9375 \quad \text{...(ii)}$$

Differentiating (i) w.r.t. 'c'

$$-2(160 - 1.595\,k - C) - 2(215 - 2.050\,k - C) - 2(275 - 3.155\,k - C) = 0$$
$$\Rightarrow 650 - 6.8\,k - 3C = 0$$
$$\Rightarrow 6.8\,k + 3C = 650$$
$$\Rightarrow 2.2667\,k + C = 216.667 \quad \text{...(iii)}$$

Solving (ii) and (iii),

$$k = 70.10301109$$
$$C = 57.76450475$$

Thus $D = 70.103(s) + 57.76$

Example 8.7 Angles were measured on a station and following readings were obtained

Angle	Value	Weight
A	35° 30′ 05″	3
B	43° 15′ 15″	1
A+B	78° 45′ 25″	2

Find the most probable values of angle A and B.

Solution:

Given,

$A = 35° 30′ 05″$ weight = 3

$B = 43° 15′ 15″$ weight = 1

$A + B = 78° 45′ 25″$ weight = 2

Normal equation for A

$$3 \times A = 3 \times 35° 30′ 05″ = 106° 30′ 15″ \quad \text{...(i)}$$
$$2 \times (A + B) = 2 \times 78° 45′ 25″ = 157° 30′ 50″ \quad \text{...(ii)}$$

Adding (i) and (ii)

$$5A + 2B = 264° 01′ 05″ \quad \text{...(iii)}$$

Normal equation for B

$$1 \times B = 43° 15′ 15″ \quad \text{...(iv)}$$
$$2 \times (A + B) = 157° 30′ 50″ \quad \text{...(v)}$$

Adding (iii) and (iv)

$$2A + 3B = 200° 46′ 05″ \quad \text{...(vi)}$$

Solving (iii) and (vi)

$$A = 35° 30′ 06″$$
$$B = 43° 15′ 17.7″$$
(Most probable values)

Example 8.8 **Given below are angles measured in the field for a triangular plot. Neglecting the spherical excess, adjust the angles of the triangle.**

$$\angle P = 50° 29′ 23″ \quad \text{weight} = 2$$
$$\angle Q = 78° 30′ 35″ \quad \text{weight} = 4$$
$$\angle R = 51° 20′ 05″ \quad \text{weight} = 3$$

Solution:

Sum of angles of triangle are

$$\angle P + \angle Q + \angle R = 180° 20′ 03″ \neq 180° 00′ 00″$$
$$\therefore \quad \text{Total error} = 20′ 03″$$
$$\therefore \quad \text{Total correction} = -\text{Error} = -20′ 03″$$

Correction will be applied to observed angles in inverse proportion of their weights. Let c_1, c_2 and c_3 are correction applied to angles P, Q and R respectively

$$\therefore \quad c_1 : c_2 : c_3 = \frac{1}{2} : \frac{1}{4} : \frac{1}{3} = 12 : 6 : 8$$
$$\therefore \quad c_1 + c_2 + c_3 = 26$$
$$\therefore \quad c_1 = \frac{12}{26} \times \text{Total correction} = \frac{12}{26}(-20′ 03″) = -09′ 15.23″$$
$$c_2 = \frac{6}{26} \times (-20′ 03″) = -04′ 37.62″$$
$$c_3 = \frac{8}{26} \times (-20′ 03″) = -06′ 10.15″$$

$\therefore$ Corrected angles are:

$$\angle P = 50° 29′ 23″ - 09′ 15.23″ = 50° 20′ 7.77″$$
$$\angle Q = 78° 30′ 35″ - 04′ 37.62″ = 78° 25′ 57.37″$$
$$\angle R = 51° 20′ 05″ - 06′ 10.15″ = 51° 13′ 54.85″$$
$$\therefore \quad \angle P + \angle Q + \angle R = 180° 00′ 00″$$
which is correct.

Example 8.9 The following are observed values of angle:

Angle	Weightage
45° 43′ 11″	2
45° 43′ 15″	3
45° 43′ 28″	4

Determine

(i) the probable error of single observation of unit weight

(ii) the probable error of weightage arithmetic mean

(iii) the probable error of single observation of weight 4

Solution:

It is observed that error is only in the seconds part and thus degree and minute part of the angle can be dispensed with.

Value	Weight (w)	v	v^2	wv^2
11″	2	−8.89″	79.0321	158.0642
15″	3	−4.89″	23.9121	71.7363
28″	4	+8.11″	65.7721	263.0884
	$\Sigma w = 9$			$\Sigma wv^2 = 492.8889$

Weighted AM of seconds part of observed angles

$$= \frac{11\times2+15\times3+28\times4}{2+3+4} = 19.89''$$

∴ Weight AM of observed angles is 45° 43′ 19.89′

∴ Residuals,

$$v_1 = 11'' - 19.89'' = -8.89''$$

$$v_2 = 15'' - 19.89'' = -4.89''$$

$$v_3 = 28'' - 19.89'' = +8.11''$$

(i) Probable error of single observation of unit weight

$$E_s = \pm0.6745\sqrt{\frac{\Sigma wv^2}{n-1}} = \pm0.6745\sqrt{\frac{492.8889}{3-1}} = \pm10.59''$$

(ii) Probable error of weight AM

$$E_m = \pm0.6745\sqrt{\frac{\Sigma wv^2}{\Sigma w(n-1)}} = \pm0.6745\sqrt{\frac{492.8889}{9(3-1)}} = \pm3.53''$$

(iii) Probable error of single observation of weight 4

$$E_w = \pm\frac{E_s}{\sqrt{w}} = \pm0.6745\sqrt{\frac{\Sigma wv^2}{w(n-1)}} = \pm\frac{10.59}{\sqrt{4}} = \pm5.3''$$

Objective Brain Teasers

Q.1 Error due to carelessness of an observer is called as:
(a) Mistake (b) Compensating error
(c) Systematic error (d) All of these

Q.2 Residual error is the difference of
(a) MPV and true value
(b) MPV and observed value
(c) Observed value and true value
(d) None of these

Q.3 Which of the following is not a method of solving the normal equations?
(a) Method of correlates
(b) Method of differences
(c) Direct method
(d) Method of least squares

Q.4 The weight of an angle θ is 3, then weight of angle 2θ is
(a) $\frac{3}{2^2}$ (b) 3×2^2
(c) $\frac{3}{2}$ (d) $\frac{3^2}{2^2}$

Q.5 The weight of an angle β is 3, then weight of angle β/2 is
(a) $\frac{3}{2^2}$ (b) 3×2^2
(c) $\frac{3}{2}$ (d) $\frac{3^2}{2^2}$

Q.6 Which of the following error has a cumulative effect?
(a) Mistakes
(b) Compensating errors
(c) Systematic errors
(d) All of these

Q.7 While running a line of levels along a highway, the weight varies as
(a) inversely proportional to length of survey line
(b) directly proportional to length of survey line
(c) square of length of survey line
(d) cube of length of survey line

Q.8 An equation of weight '*k*' is added to a constant '*c*'. The weight of resulting equation is
(a) *k* (b) (*k* + *c*)
(c) *kc* (d) *k*/*c*

Answers

1. (a) 2. (b) 3. (d) 4. (a) 5. (b)
6. (c) 7. (a) 8. (a)

Student's Assignments

Ex.1 An angle was measured four times and its standard error came out to be ±10.5″. Determine how many measurements are required to halve this area?

Ans. 16

Ex.2 An angle '*P*' was measured by the method of repetition and following values were obtained.

Observer No.	∠P	No. of measurements
1.	35° 40′	2
2.	35° 30′	3
3.	35° 25′	4

Find the MPV of ∠P.

Ans. 35° 30′

Ex.3 Compute the MPV and most possible error in computing the area of a circle of radius 13.09 ±0.02 m.

Ans. (538.306 ± 1.645) m^2 ± 1.645 m^2

Ex.4 From a central station *P*, three horizontal angles viz. α, β and γ were measured as 34° 10′ 20″ ±3″, 176° 40′ 32″ 4″ and 149° 09′ 04″ ±5″ respectively. Determine the corrected angles.

Ans. α = 34° 10′ 20.7″, β = 176° 40′ 33.3″, γ = 149° 09′ 06″

■■■■