Solution of Triangle

Contents

JEE (Advanced) Syllabus

Solution of Triangle : Relations between sides and angles of a triangle, sine rule, cosine rule, half-angle formula and the area of a triangle.

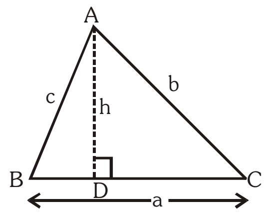
SOLUTION OF TRIANGLE

The process of calculating the sides and angles of triangle using given information is called solution of triangle. In a ΔABC, in general the angles are denoted by capital letters A, B and C and the length of the sides opposite these angle are denoted by small letter a, b and c respectively.

1. SINE FORMULAE :

In any triangle ABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$



where R is circumradius and Δ is area of triangle.

SOLVED EXAMPLE

Example 1 : Angles of a triangle are in 4 : 1 : 1 ratio, then find the ratio between its greatest side and perimeter.

Solution : Angles are in ratio 4 : 1 : 1.

$\Rightarrow$ angles are 120°, 30°, 30°.

If sides opposite to these angles are a, b, c respectively, then a will be the greatest side. Now from

sine formula $\frac{a}{\sin 120^\circ}=\frac{b}{\sin 30^\circ}=\frac{c}{\sin 30^\circ}$

$\Rightarrow \quad \frac{a}{\sqrt{3}/2}=\frac{b}{1/2}=\frac{c}{1/2} \qquad \Rightarrow \quad \frac{a}{\sqrt{3}}=\frac{b}{1}=\frac{c}{1}=k$ (say)

then a = $\sqrt{3}k$, perimeter = $(2+\sqrt{3})k$

$\therefore$ required ratio = $\frac{\sqrt{3}k}{(2+\sqrt{3})k}=\frac{\sqrt{3}}{2+\sqrt{3}}$

Example 2 : In triangle ABC, if b = 3, c = 4 and $\angle$B = π/3, then find the number of such triangles.

Solution : Using sine formulae $\frac{\sin B}{b}=\frac{\sin C}{c}$

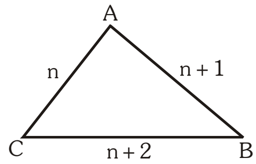
$\Rightarrow \quad \frac{\sin \pi/3}{3}=\frac{\sin C}{4} \Rightarrow \frac{\sqrt{3}}{6}=\frac{\sin C}{4} \Rightarrow$ sin C = $\frac{2}{\sqrt{3}}$ > 1 which is not possible.

Hence there exist no triangle with given elements.

Example 3 : The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

Solution : Let the sides be n, n + 1, n + 2 cms.

i.e. AC = n, AB = n + 1, BC = n + 2



Smallest angle is B and largest one is A.

Here, $\angle A = 2\angle B$

Also, $\angle A + \angle B + \angle C = 180°$

$\Rightarrow$ $3\angle B + \angle C = 180°$ $\Rightarrow$ $\angle C = 180° - 3\angle B$

We have, sine law as,

$$\frac{\sin A}{n+2} = \frac{\sin B}{n} = \frac{\sin C}{n+1} \Rightarrow \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin(180-3B)}{n+1}$$

$$\Rightarrow \underset{\text{(i)}}{\frac{\sin 2B}{n+2}} = \underset{\text{(ii)}}{\frac{\sin B}{n}} = \underset{\text{(iii)}}{\frac{\sin 3B}{n+1}}$$

from (i) and (ii);

$$\frac{2\sin B\cos B}{n+2} = \frac{\sin B}{n} \Rightarrow \cos B = \frac{n+2}{2n} \quad \text{......... (iv)}$$

and from (ii) and (iii);

$$\frac{\sin B}{n} = \frac{3\sin B - 4\sin^3 B}{n+1} \Rightarrow \frac{\sin B}{n} = \frac{\sin B(3-4\sin^2 B)}{n+1}$$

$$\Rightarrow \frac{n+1}{n} = 3 - 4(1-\cos^2 B) \quad \text{.......... (v)}$$

from (iv) and (v), we get

$$\frac{n+1}{n} = -1 + 4\left(\frac{n+2}{2n}\right)^2 \Rightarrow \frac{n+1}{n} + 1 = \left(\frac{n^2+4n+4}{n^2}\right)$$

$$\Rightarrow \frac{2n+1}{n} = \frac{n^2+4n+4}{n^2} \Rightarrow 2n^2 + n = n^2 + 4n + 4$$

$\Rightarrow$ $n^2 - 3n - 4 = 0$ $\Rightarrow$ $(n-4)(n+1) = 0$

$n = 4$ or -1

where $n \neq -1$

$\therefore$ n = 4. Hence the sides are 4, 5, 6

Example 4 : In any ΔABC, prove that $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$.

Solution : Since $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (let)

$\Rightarrow$ $a = k \sin A$, $b = k \sin B$ and $c = k \sin C$

$\therefore$ L.H.S. $= \frac{a+b}{c} = \frac{k(\sin A + \sin B)}{k \sin C}$

$$= \frac{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}\cos\frac{C}{2}} = \frac{\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}\cos\frac{C}{2}} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}} = \text{R.H.S.}$$

Hence L.H.S. = R.H.S.

Example 5 : In any ΔABC, prove that

$(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$

Solution : Since $a = k \sin A$, $b = k \sin B$ and $c = k \sin C$

$\therefore$ $(b^2 - c^2) \cot A = k^2 (\sin^2 B - \sin^2 C) \cot A = k^2 \sin (B + C) \sin (B - C) \cot A$

$\therefore$ $= k^2 \sin A \sin (B - C) \frac{\cos A}{\sin A}$

$= - k^2 \sin (B - C) \cos (B + C)$ $\quad$ ($\because$ $\cos A = - \cos (B + C)$)

$= - \frac{k^2}{2} [2\sin (B - C) \cos (B + C)]$

$= - \frac{k^2}{2} [\sin 2B - \sin 2C]$ ....(i)

Similarly $(c^2 - a^2) \cot B = - \frac{k^2}{2} [\sin 2C - \sin 2A]$ ....(ii)

and $(a^2 - b^2) \cot C = - \frac{k^2}{2} [\sin 2A - \sin 2B]$ ....(iii)

adding equations (i), (ii) and (iii), we get

$(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$

Problems for Self Practice-1 :

(1) In any ΔABC, prove that

(i) $a \sin\left(\frac{A}{2}+B\right) = (b + c) \sin\left(\frac{A}{2}\right)$.

(ii) $\frac{a^2 \sin(B-C)}{\sin B + \sin C} + \frac{b^2 \sin(C-A)}{\sin C + \sin A} + \frac{c^2 \sin(A-B)}{\sin A + \sin B} = 0$

(iii) $\frac{c}{a-b} = \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{\tan\frac{A}{2} - \tan\frac{B}{2}}$.

(2) If in a ΔABC, $\angle A = \frac{\pi}{6}$ and b : c = 2 : $\sqrt{3}$, find $\angle B$.

(3) If in a ΔABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, show that a^2, b^2, c^2 are in A.P.

Answers : (2) 90º

2. COSINE FORMULAE :

(a) $\cos A = \frac{b^2+c^2-a^2}{2bc}$ (b) $\cos B = \frac{c^2+a^2-b^2}{2ca}$ (c) $\cos C = \frac{a^2+b^2-c^2}{2ab}$

or $a^2 = b^2 + c^2 - 2bc \cos A$

SOLVED EXAMPLE

Example 6 : In a triangle ABC, if B = 30° and c = $\sqrt{3}$ b, then find angle A (where A > 45º)

Solution : We have cos B = $\frac{c^2+a^2-b^2}{2ca} \Rightarrow \frac{\sqrt{3}}{2} = \frac{3b^2+a^2-b^2}{2\times\sqrt{3}b\times a}$

$\Rightarrow$ $a^2 - 3ab + 2b^2 = 0 \Rightarrow (a - 2b)(a - b) = 0$

$\Rightarrow$ Either a = b $\Rightarrow$ A = 30°

or a = 2b $\Rightarrow$ $a^2 = 4b^2 = b^2 + c^2$

$\Rightarrow$ A = 90°.

Example 7 : In a triangle ABC, find the value of $(a^2 - b^2 - c^2)\tan A + (a^2 - b^2 + c^2)\tan B$.

Solution : Using cosine law :

The given expression is equal to $-2bc\cos A\tan A + 2ac\cos B\tan B$

$$= 2abc\left(-\frac{\sin A}{a} + \frac{\sin B}{b}\right) = 0$$

Example 8 : In a triangle ABC if a = 13, b = 8 and c = 7, then find sin A.

Solution : $\because \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{64 + 49 - 169}{2.8.7} \Rightarrow \cos A = -\frac{1}{2} \Rightarrow A = \frac{2\pi}{3}$

$\therefore \quad \sin A = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

Example 9 : In a ΔABC, prove that $a(b\cos C - c\cos B) = b^2 - c^2$

Solution : Since $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ & $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

$\therefore \quad \text{L.H.S.} = a\left\{b\left(\frac{a^2 + b^2 - c^2}{2ab}\right) - c\left(\frac{a^2 + c^2 - b^2}{2ac}\right)\right\}$

$= \frac{a^2 + b^2 - c^2}{2} - \frac{(a^2 + c^2 - b^2)}{2} = (b^2 - c^2) = \text{R.H.S.}$

Hence L.H.S. = R.H.S.

Example 10 : If in a ΔABC, $\angle A = 60°$, then find the value of $\left(1 + \frac{a}{c} + \frac{b}{c}\right)\left(1 + \frac{c}{b} - \frac{a}{b}\right)$.

Solution : $\left(1 + \frac{a}{c} + \frac{b}{c}\right)\left(1 + \frac{c}{b} - \frac{a}{b}\right) = \left(\frac{c + a + b}{c}\right)\left(\frac{b + c - a}{b}\right) = \frac{(b + c)^2 - a^2}{bc} = \frac{(b^2 + c^2 - a^2) + 2bc}{bc}$

$= \frac{b^2 + c^2 - a^2}{bc} + 2 = 2\left(\frac{b^2 + c^2 - a^2}{2bc}\right) + 2$

$= 2\cos A + 2 = 3 \qquad \{\because \angle A = 60°\}$

$\therefore \quad \left(1 + \frac{a}{c} + \frac{b}{c}\right)\left(1 + \frac{c}{b} - \frac{a}{b}\right) = 3$

Problems for Self Practice-2 :

(1) The sides of a triangle ABC are a, b, $\sqrt{a^2+ab+b^2}$, then prove that the greatest angle is 120°.

(2) In a triangle ABC, prove that $a(\cos B + \cos C) = 2(b + c)\sin^2\frac{A}{2}$.

(3) If a : b : c = 4 : 5 : 6, then show that $\angle C = 2\angle A$.

(4) In any ΔABC, prove that

(i) $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$

(ii) $\frac{b^2}{a}\cos A+\frac{c^2}{b}\cos B+\frac{a^2}{c}\cos C=\frac{a^4+b^4+c^4}{2abc}$

3. PROJECTION FORMULAE :

(a) b cos C + c cos B = a (b) c cos A + a cos C = b (c) a cos B + b cos A = c

SOLVED EXAMPLE

Example 11 : In a ΔABC, $c\cos^2\frac{A}{2}+a\cos^2\frac{C}{2}=\frac{3b}{2}$, then show a, b, c are in A.P.

Solution : Here, $\frac{c}{2}(1+\cos A)+\frac{a}{2}(1+\cos C)=\frac{3b}{2}$

$\Rightarrow$ a + c + (c cos A + a cos C) = 3b

$\Rightarrow$ a + c + b = 3b {using projection formula}

$\Rightarrow$ a + c = 2b

which shows a, b, c are in A.P.

Example 12 : In a triangle ABC, prove that

(i) $a(b\cos C - c\cos B) = b^2 - c^2$

(ii) (b + c) cos A + (c + a) cos B + (a + b) cos C = a + b + c.

Solution : **(i)** $\because$ L.H.S. = a (b cosC – c cosB)

= b (a cosC) – c (a cosB) ....(i)

$\because$ From **projection rule**, we know that

b = a cosC + c cosA $\Rightarrow$ a cosC = b – c cosA

& c = a cosB + b cosA $\Rightarrow$ a cosB = c – b cosA

Put values of $a \cos C$ and $a \cos B$ in equation (i), we get

L.H.S. $= b(b - c\cos A) - c(c - b\cos A)$

$= b^2 - bc\cos A - c^2 + bc\cos A$

$= b^2 - c^2$

= R.H.S.

Hence L.H.S. = R.H.S.

Note: We have also proved $a(b\cos C - c\cos B) = b^2 - c^2$ by using **cosine – rule** in solved ***Example.**

(ii) $\because$ L.H.S. $= (b + c)\cos A + (c + a)\cos B + (a + b)\cos C$

$= b\cos A + c\cos A + c\cos B + a\cos B + a\cos C + b\cos C$

$= (b\cos A + a\cos B) + (c\cos A + a\cos C) + (c\cos B + b\cos C)$

$= a + b + c$

= R.H.S.

Hence L.H.S. = R.H.S.

Problems for Self Practice-3 :

(1) In a ΔABC, if $\angle A = \frac{\pi}{4}$, $\angle B = \frac{5\pi}{12}$, show that $a + c\sqrt{2} = 2b$.

(2) In a ΔABC, prove that :

(i) $b(a\cos C - c\cos A) = a^2 - c^2$

(ii) $2\left(b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}\right) = a + b + c$

(iii) $\frac{\cos B}{\cos C} = \frac{c - b\cos A}{b - c\cos A}$.

(iv) $\frac{\cos A}{c\cos B + b\cos C} + \frac{\cos B}{a\cos C + c\cos A} + \frac{\cos C}{a\cos B + b\cos A} = \frac{a^2 + b^2 + c^2}{2abc}$.

4. NAPIER'S ANALOGY (TANGENT RULE) :

(a) $\tan\left(\frac{B - C}{2}\right) = \frac{b - c}{b + c}\cot\frac{A}{2}$

(b) $\tan\left(\frac{C - A}{2}\right) = \frac{c - a}{c + a}\cot\frac{B}{2}$

(c) $\tan\left(\frac{A - B}{2}\right) = \frac{a - b}{a + b}\cot\frac{C}{2}$

SOLVED EXAMPLE

Example 13 : In a ΔABC, the tangent of half the difference of two angles is one-third the tangent of half the sum of the angles. Determine the ratio of the sides opposite to the angles.

Solution : Here, $\tan\left(\frac{A-B}{2}\right)=\frac{1}{3}\tan\left(\frac{A+B}{2}\right)$ .... (i)

using Napier's analogy, $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$ .... (ii)

from (i) & (ii) ;

$$\frac{1}{3}\tan\left(\frac{A+B}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right) \Rightarrow \frac{1}{3}\cot\left(\frac{C}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$$

{as $A + B + C = \pi$ $\therefore$ $\tan\left(\frac{B+C}{2}\right) = \tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\frac{C}{2}$}

$\Rightarrow$ $\frac{a-b}{a+b}=\frac{1}{3}$ or $3a - 3b = a + b$

$2a = 4b$ or $\frac{a}{b}=\frac{2}{1}\Rightarrow\frac{b}{a}=\frac{1}{2}$

Thus the ratio of the sides opposite to the angles is $b : a = 1 : 2$.

Example 14 : Find the unknown elements of the ΔABC in which $a = \sqrt{3} + 1$, $b = \sqrt{3} - 1$, $C = 60°$.

Solution : $\because$ $a = \sqrt{3} + 1$, $b = \sqrt{3} - 1$, $C = 60°$

$\because$ $A + B + C = 180°$

$\therefore$ $A + B = 120°$ ....(i)

$\because$ From law of tangent, we know that $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

$$= \frac{(\sqrt{3}+1)-(\sqrt{3}-1)}{(\sqrt{3}+1)+(\sqrt{3}-1)}\cot 30° = \frac{2}{2\sqrt{3}}\cot 30° \Rightarrow \tan\left(\frac{A-B}{2}\right) = 1$$

$\therefore$ $\frac{A-B}{2} = \frac{\pi}{4} = 45°$

$\Rightarrow$ $A - B = 90°$ ....(ii)

From equation (i) and (ii), we get

$A = 105°$ and $B = 15°$

Now,

$\because$ From **sine-rule,** we know that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\therefore$ $c = \frac{a\sin C}{\sin A} = \frac{(\sqrt{3}+1)\sin 60°}{\sin 105°} = \frac{(\sqrt{3}+1)\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$ $\because$ $\sin 105° = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\Rightarrow$ $c = \sqrt{6}$

$\therefore$ $c = \sqrt{6}$, A = 105°, B = 15°

<u>Problems for Self Practice</u>-4 :

(1) If ΔABC is right angled at C, prove that : (a) $\tan\frac{A}{2} = \sqrt{\frac{c-b}{c+b}}$ (b) $\sin(A-B) = \frac{a^2-b^2}{a^2+b^2}$

(2) In a ΔABC if b = 3, c = 5 and cos (B – C) = $\frac{7}{25}$, then find the value of $\tan\frac{A}{2}$.

(3) If in a ΔABC, we define $x = \tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2}$, $y = \tan\left(\frac{C-A}{2}\right)\tan\frac{B}{2}$ and $z = \tan\left(\frac{A-B}{2}\right)\tan\frac{C}{2}$, then show that $x + y + z = -xyz$.

Answer : (2) $\frac{1}{3}$

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5. HALF ANGLE FORMULAE :

$s = \frac{a+b+c}{2}$ = semi-perimeter of triangle.

(a) (i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$ (iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(b) (i) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$ (iii) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(c) (i) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (ii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$ (iii) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

$= \frac{\Delta}{s(s-a)}$ $= \frac{\Delta}{s(s-b)}$ $= \frac{\Delta}{s(s-c)}$

(d) Area of Triangle

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3,$$

where p_1, p_2, p_3 are altitudes from vertices A,B,C respectively.

SOLVED EXAMPLE

Example 15 : If Δ is the area and 2s the sum of the sides of a triangle, then show $\Delta \le \frac{s^2}{3\sqrt{3}}$.

Solution : We have, $2s = a + b + c$, $\Delta^2 = s(s-a)(s-b)(s-c)$

Now, A.M. $\ge$ G.M.

$$\frac{(s-a)+(s-b)+(s-c)}{3} \ge \{(s-a)(s-b)(s-c)\}^{1/3}$$

or $\frac{3s-2s}{3} \ge \left(\frac{\Delta^2}{s}\right)^{1/3}$ or $\frac{s}{3} \ge \left(\frac{\Delta^2}{s}\right)^{1/3}$ or $\frac{\Delta^2}{s} \le \frac{s^3}{27}$ $\Rightarrow$ $\Delta \le \frac{s^2}{3\sqrt{3}}$

Example 16 : In a ΔABC if a, b, c are in A.P., then find the value of $\tan\frac{A}{2}.\tan\frac{C}{2}$

Solution : Since $\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}$ and $\tan\frac{C}{2} = \frac{\Delta}{s(s-c)}$

$\therefore$ $\tan\frac{A}{2}.\tan\frac{C}{2} = \frac{\Delta^2}{s^2(s-a)(s-c)}$ $\because$ $\Delta^2 = s(s-a)(s-b)(s-c)$

$\therefore$ $\tan\frac{A}{2}.\tan\frac{C}{2} = \frac{s-b}{s} = 1 - \frac{b}{s}$ ....(i)

$\because$ it is given that a, b, c are in A.P. $\Rightarrow$ $2b = a + c$

$\because$ $s = \frac{a+b+c}{2} = \frac{3b}{2}$

$\therefore$ $\frac{b}{s} = \frac{2}{3}$ put in equation (i), we get

$\therefore$ $\tan\frac{A}{2}.\tan\frac{C}{2} = 1 - \frac{2}{3}$ $\Rightarrow$ $\tan\frac{A}{2}.\tan\frac{C}{2} = \frac{1}{3}$

Example 17 : In a ΔABC if $b \sin C(b \cos C + c \cos B) = 42$, then find the area of the ΔABC.

Solution : $\because$ $b \sin C\,(b \cos C + c \cos B) = 42$ ....(i) given

$\because$ From **projection rule**, we know that

$a = b \cos C + c \cos B$ put in (i), we get

$ab \sin C = 42$ ....(ii)

$\because$ $\Delta = \frac{1}{2}\, ab \sin C$ $\therefore$ from equation (ii), we get

$\therefore$ $\Delta = 21$ sq. unit

Example 18 : In any ΔABC, prove that $(a + b + c)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = 2c \cot\frac{C}{2}$.

Solution : $\because$ L.H.S. $= (a + b + c)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right)$

$\because$ $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ and $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

$\therefore$ L.H.S. $= (a + b + c)\left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\right]$

$= 2s\sqrt{\frac{s-c}{s}}\left[\sqrt{\frac{s-b}{s-a}} + \sqrt{\frac{s-a}{s-b}}\right]$

$= 2\sqrt{s(s-c)}\left[\frac{s-b+s-a}{\sqrt{(s-a)(s-b)}}\right]$ $\because$ $2s = a + b + c$

$\therefore$ $2s - b - a = c$

$= 2\sqrt{s(s-c)}\left[\frac{c}{\sqrt{(s-a)(s-b)}}\right] = 2c\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$\because$ $\cot\frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$ $= 2c\cot\frac{C}{2}$ = R.H.S.

Hence L.H.S. = R.H.S.

Problems for Self Practice-5 :

(1) Given a = 6, b = 8, c = 10. Find

(i) sinA (ii) tanA (iii) $\sin\frac{A}{2}$

(iv) $\cos\frac{A}{2}$ (v) $\tan\frac{A}{2}$ (vi) Δ

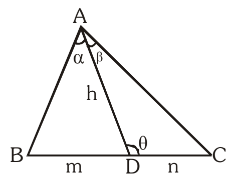
(2) Prove that in any ΔABC, (abcs) $\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2} = \Delta^2$.

Answers : (1) (i) $\frac{3}{5}$ (ii) $\frac{3}{4}$ (iii) $\frac{1}{\sqrt{10}}$ (iv) $\frac{3}{\sqrt{10}}$ (v) $\frac{1}{3}$ (vi) 24

6. m-n THEOREM :

$(m + n) \cot\theta = m \cot\alpha - n \cot\beta$

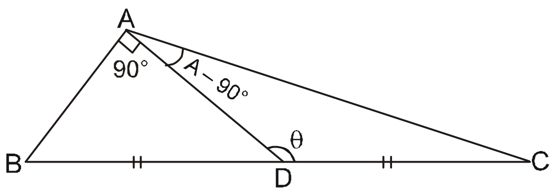
$(m + n) \cot\theta = n \cot B - m \cot C.$



SOLVED EXAMPLE

Example 19 : If the median AD of a triangle ABC is perpendicular to AB, prove that tan A + 2tan B = 0.

Solution : From the figure, we see that $\theta = 90° + B$ (as θ is external angle of ΔABD)



Now if we apply **m-n rule** in ΔABC, we get

$(1 + 1) \cot (90° + B) = 1. \cot 90° - 1.\cot (A - 90°)$

$\Rightarrow$ $-2 \tan B = \cot (90° - A)$

$\Rightarrow$ $-2 \tan B = \tan A$

$\Rightarrow$ $\tan A + 2 \tan B = 0$ **Hence proved.**

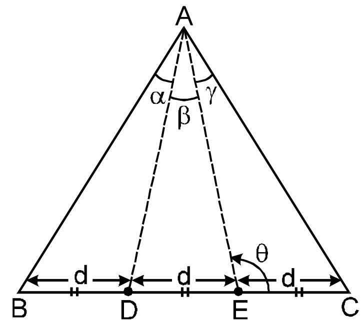
Example 20 : The base of a triangle is divided into three equal parts. If t_1, t_2, t_3 be the tangents of the angles subtended by these parts at the opposite vertex, prove that

$$4\left(1+\frac{1}{t_2^2}\right) = \left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right).$$

Solution : Let point D and E divides the base BC into three equal parts i.e. BD = DE = EC = d (Let) and let α, β and γ be the angles subtended by BD, DE and EC respectively at their opposite vertex.

$\Rightarrow$ $t_1 = \tan\alpha$, $t_2 = \tan\beta$ and $t_3 = \tan\gamma$

Now in ΔABC



$\because$ BE : EC = 2d : d = 2 : 1

$\therefore$ from **m-n rule**, we get

$(2 + 1)\cot\theta = 2\cot(\alpha + \beta) - \cot\gamma$

$\Rightarrow$ $3\cot\theta = 2\cot(\alpha + \beta) - \cot\gamma$ ....(i)

again

$\because$ in ΔADC

$\because$ DE : EC = d : d = 1 : 1

$\therefore$ if we apply **m-n rule** in ΔADC, we get

$(1 + 1)\cot\theta = 1.\cot\beta - 1\cot\gamma$

$2\cot\theta = \cot\beta - \cot\gamma$ ....(ii)

from (i) and (ii), we get $\dfrac{3\cot\theta}{2\cot\theta} = \dfrac{2\cot(\alpha+\beta)-\cot\gamma}{\cot\beta-\cot\gamma}$

$\Rightarrow$ $3\cot\beta - 3\cot\gamma = 4\cot(\alpha + \beta) - 2\cot\gamma$

$\Rightarrow$ $3\cot\beta - \cot\gamma = 4\cot(\alpha + \beta)$

$\Rightarrow$ $3\cot\beta - \cot\gamma = 4\left\{\dfrac{\cot\alpha.\cot\beta - 1}{\cot\beta + \cot\alpha}\right\}$

$\Rightarrow$ $3\cot^2\beta + 3\cot\alpha\cot\beta - \cot\beta\cot\gamma - \cot\alpha\cot\gamma = 4\cot\alpha\cot\beta - 4$

$\Rightarrow$ $4 + 3\cot^2\beta = \cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\alpha\cot\gamma$

$\Rightarrow$ $4 + 4\cot^2\beta = \cot\alpha\cot\beta + \cot\alpha\cot\gamma + \cot\beta\cot\gamma + \cot^2\beta$

$\Rightarrow$ $4(1 + \cot^2\beta) = (\cot\alpha + \cot\beta)(\cot\beta + \cot\gamma)$

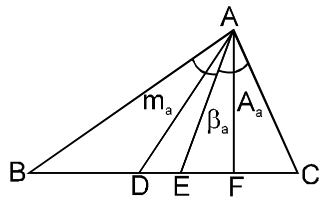
$\Rightarrow$ $4\left(1+\dfrac{1}{\tan^2\beta}\right) = \left(\dfrac{1}{\tan\alpha}+\dfrac{1}{\tan\beta}\right)\left(\dfrac{1}{\tan\beta}+\dfrac{1}{\tan\gamma}\right)$

$\Rightarrow$ $4\left(1+\dfrac{1}{t_2^2}\right) = \left(\dfrac{1}{t_1}+\dfrac{1}{t_2}\right)\left(\dfrac{1}{t_2}+\dfrac{1}{t_3}\right)$ **Hence proved.**

Problems for Self Practice-6 :

(1) In a ΔABC, the median to the side BC is of length $\dfrac{1}{\sqrt{11-6\sqrt{3}}}$ unit and it divides angle A into the angles of 30° and 45°. Prove that the side BC is of length 2 unit.

7. LENGTH OF ANGLE BISECTORS AND MEDIANS AND ALTITUDES :



(i) Length of an angle bisector from the angle A = $\beta_a = \dfrac{2bc\cos\frac{A}{2}}{b+c}$;

(ii) Length of median from the angle A = $m_a = \dfrac{1}{2}\sqrt{2b^2+2c^2-a^2}$

(iii) Length of altitude from the angle A = $A_a = \dfrac{2\Delta}{a}$

NOTE : $m_a^2 + m_b^2 + m_c^2 = \dfrac{3}{4}(a^2 + b^2 + c^2)$

SOLVED EXAMPLE

Example 21 : AD is a median of the ΔABC. If AE and AF are medians of the triangles ABD and ADC respectively, and AD = m_1, AE = m_2, AF = m_3 , then prove that $m_2^2 + m_3^2 - 2m_1^2 = \dfrac{a^2}{8}$.

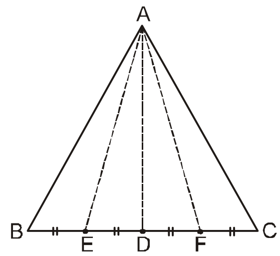
Solution : $\because$ In ΔABC

$AD^2 = \dfrac{1}{4}(2b^2 + 2c^2 - a^2) = m_1^2$ ....(i)

$\because$ In ΔABD, $AE^2 = m_2^2 = \dfrac{1}{4}\left(2c^2 + 2AD^2 - \dfrac{a^2}{4}\right)$ ....(ii)

Similarly in ΔADC, $AF^2 = m_3^2 = \dfrac{1}{4}\left(2AD^2 + 2b^2 - \dfrac{a^2}{4}\right)$ ....(iii)

by adding equations (ii) and (iii), we get



$\because$ $m_2^2 + m_3^2 = \dfrac{1}{4}\left(4AD^2 + 2b^2 + 2c^2 - \dfrac{a^2}{2}\right)$

$= AD^2 + \dfrac{1}{4}\left(2b^2 + 2c^2 - \dfrac{a^2}{2}\right) = AD^2 + \dfrac{1}{4}\left(2b^2 + 2c^2 - a^2 + \dfrac{a^2}{2}\right)$

$= AD^2 + \dfrac{1}{4}(2b^2 + 2c^2 - a^2) + \dfrac{a^2}{8} = AD^2 + AD^2 + \dfrac{a^2}{8}$

$= 2AD^2 + \dfrac{a^2}{8} = 2m_1^2 + \dfrac{a^2}{8}$ $\because$ $AD^2 = m_1^2$

$\therefore$ $m_2^2 + m_3^2 - 2m_1^2 = \dfrac{a^2}{8}$ **Hence Proved**

Example 22 : If in a triangle ABC, CD is the angle bisector of the angle ACB, then prove that CD is equal to-

(i) $\frac{2ab}{a+b}\cos\frac{C}{2}$ (ii) $\frac{b\sin\angle DAC}{\sin(B+C/2)}$

Solution : $\Delta CAB = \Delta CAD + \Delta CDB$

$\Rightarrow \frac{1}{2}absinC = \frac{1}{2}b.CD.\sin\left(\frac{C}{2}\right) + \frac{1}{2}a.CD\sin\left(\frac{C}{2}\right)$

$\Rightarrow CD(a + b)\sin\left(\frac{C}{2}\right) = ab\left(2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)\right)$

So $CD = \frac{2ab\cos(C/2)}{(a+b)}$

and in ΔCAD, $\frac{CD}{\sin\angle DAC} = \frac{b}{\sin\angle CDA}$ (by sine rule)

$\Rightarrow CD = \frac{b\sin\angle DAC}{\sin(B+C/2)}$

Problems for Self Practice-7 :

(1) In a ΔABC if a = 5 cm, b = 4 cm, c = 3 cm. 'G' is the centroid of triangle, then find circumradius of ΔGAB.

(2) In a ΔABC, the lengths of the bisectors of the angle A, B and C are x, y, z respectively. Show that

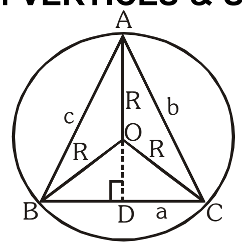
$$\frac{1}{x}\cos\frac{A}{2} + \frac{1}{y}\cos\frac{B}{2} + \frac{1}{z}\cos\frac{C}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

(3) If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of a triangle to the opposite sides, prove that

(i) $p_1p_2p_3 = \frac{a^2b^2c^2}{8R^3}$ (ii) $\Delta = \sqrt{\frac{1}{2}Rp_1p_2p_3}$

Answer : (1) $\frac{5}{12}\sqrt{13}$

8. CIRCUMCIRCLE AND DISTANCE OF CIRCUMCENTRE FROM VERTICES & SIDES OF TRIANGLE:



Circumcentre is the point of intersection of perpendicular bisectors of the sides and distance between circumcentre & vertex of triangle is called circumradius 'R'.

OA = OB = OC = $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$. (Where O is circumcentre)

Distance of O from side BC = R cosA = OD

Distance of O from side AB = R cosC

Distance of O from side AC = R cosB

SOLVED EXAMPLE

Example 23 : In a ΔABC, prove that sinA + sinB + sinC = $\frac{s}{R}$

Solution : In a ΔABC, we know that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$\therefore$ $\sin A = \frac{a}{2R}$, $\sin B = \frac{b}{2R}$ and $\sin C = \frac{c}{2R}$.

$\therefore$ $\sin A + \sin B + \sin C = \frac{a+b+c}{2R} = \frac{2s}{2R}$ $\quad \because \quad a + b + c = 2s$

$\Rightarrow$ $\sin A + \sin B + \sin C = \frac{s}{R}$.

Example 24 : In a ΔABC if a = 13 cm, b = 14 cm and c = 15 cm, then find its circumradius.

Solution : $\because$ $R = \frac{abc}{4\Delta}$ ....(i)

$\because$ $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\because$ $s = \frac{a+b+c}{2} = 21$ cm

$\therefore$ $\Delta = \sqrt{21 \times 8 \times 7 \times 6} = \sqrt{7^2 \times 4^2 \times 3^2}$ $\Rightarrow$ $\Delta = 84$ cm^2

$\therefore$ $R = \frac{13 \times 14 \times 15}{4 \times 84} = \frac{65}{8}$ cm $\quad \therefore \quad R = \frac{65}{8}$ cm. Ans.

Example 25 : In a ΔABC, prove that $s = 4R\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$.

Solution : In a ΔABC,

$\because$ $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$ and $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$ and $R = \frac{abc}{4\Delta}$

$\because$ R.H.S. $= 4R\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$.

$= \frac{abc}{\Delta}.s\sqrt{\frac{s(s-a)(s-b)(s-c)}{(abc)^2}} = s \quad \because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

= L.H.S.

Hence R.H.S = L.H.S. proved.

Example 26 : In a ΔABC, prove that $\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{4R}{\Delta}$.

Solution : $\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{4R}{\Delta}$

$\because$ L.H.S. $= \left(\frac{1}{s-a}+\frac{1}{s-b}\right)+\left(\frac{1}{s-c}-\frac{1}{s}\right)$

$= \frac{2s-a-b}{(s-a)(s-b)}+\frac{(s-s+c)}{s(s-c)} \quad \because \quad 2s = a+b+c$

$= \frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)}$

$= c\left[\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right] = c\left[\frac{2s^2-s(a+b+c)+ab}{\Delta^2}\right]$

$\therefore$ L.H.S. $= c\left[\frac{2s^2-s(2s)+ab}{\Delta^2}\right] = \frac{abc}{\Delta^2} = \frac{4R\Delta}{\Delta^2} = \frac{4R}{\Delta} \quad \because \quad R = \frac{abc}{4\Delta}$

$\Rightarrow$ $abc = 4R\Delta$

$\therefore$ L.H.S. $= \frac{4R}{\Delta}$ = R.H.S.

Problems for Self Practice-8 :

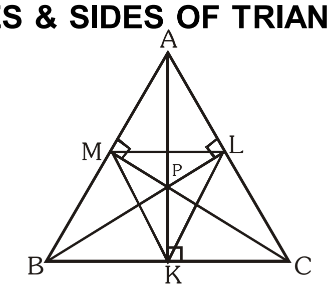
(1) In a ΔABC, prove the following :

(i) $a \cot A + b \cot B + c \cot C = 2(R + r)$. (ii) $4\left(\frac{s}{a}-1\right)\left(\frac{s}{b}-1\right)\left(\frac{s}{c}-1\right) = \frac{r}{R}$.

(2) If α, β, γ are the distances of the vertices of a triangle from the corresponding points of contact with the incircle, then prove that $\frac{\alpha\beta\gamma}{\alpha+\beta+\gamma} = r^2$

(3) If x, y, z are respectively be the perpendiculars from the circumcentre to the sides of ΔABC, then prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$.

9. ORTHOCENTRE AND ITS DISTANCE FROM VERTICES & SIDES OF TRIANGLE:



(a) Point of intersection of altitudes is orthocentre & the triangle KLM which is formed by joining the feet of the altitudes is called the pedal triangle.

(b) The distances of the orthocentre from the angular points of the ΔABC are 2R cosA, 2R cosB, & 2R cosC.

(c) The distance of P from sides are 2R cosB cosC, 2R cosC cosA and 2R cosA cosB.

SOLVED EXAMPLE

Example 27 : If x, y and z are respectively the distances of the vertices of the ΔABC from its orthocentre, then prove that x + y + z = 2(R + r)

Solution :

$\because$ $x = 2R\cos A,\ y = 2R\cos B,\ z = 2R\cos C$

$\because$ $x + y + z = 2R(\cos A + \cos B + \cos C)$

$\because$ in a ΔABC $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\therefore$ $x + y + z = 2R\left(1+4\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}\right) = 2\left(R+4R\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}\right)$

$\because$ $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ $\therefore$ $x + y + z = 2(R + r)$

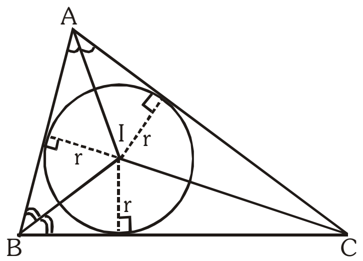
Problems for Self Practice-9:

(1) If x, y, z are the distance of the vertices of ΔABC respectively from the orthocentre, then prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$

(2) In a ΔABC, AD is altitude and H is the orthocentre prove that AH : DH = (tanB + tanC) : tanA

10. INCIRCLE AND DISTANCE OF INCENTRE FROM VERTICES & SIDES OF TRIANGLE:

Point of intersection of internal angle bisectors is incentre and perpendicular distance of incentre from any side is called inradius 'r'.



(i) $r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$

$$= a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = b\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = c\frac{\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$$

(ii) $IA = r\ \text{cosec}\ \frac{A}{2}$; $IB = r\ \text{cosec}\ \frac{B}{2}$; $IC = r\ \text{cosec}\ \frac{C}{2}$

SOLVED EXAMPLE

Example 28 : In a triangle ABC, if a : b : c = 4 : 5 : 6, then find the ratio between its circumradius and inradius.

Solution : $\frac{R}{r} = \frac{abc}{4\Delta}\Big/\frac{\Delta}{s} = \frac{(abc)s}{4\Delta^2} \Rightarrow \frac{R}{r} = \frac{abc}{4(s-a)(s-b)(s-c)}$ ....(i)

$\because$ a : b : c = 4 : 5 : 6 $\Rightarrow \frac{a}{4} = \frac{b}{5} = \frac{c}{6}$ = k (say)

$\Rightarrow$ a = 4k, b = 5k, c = 6k

$\therefore$ $s = \frac{a+b+c}{2} = \frac{15k}{2}$, $s - a = \frac{7k}{2}$, $s - b = \frac{5k}{2}$, $s - c = \frac{3k}{2}$

using (i) in these values $\frac{R}{r} = \frac{(4k)(5k)(6k)}{4\left(\frac{7k}{2}\right)\left(\frac{5k}{2}\right)\left(\frac{3k}{2}\right)} = \frac{16}{7}$

Example 29 : If A, B, C are the angles of a triangle, prove that : cosA + cosB + cosC = $1+\frac{r}{R}$.

Solution : cosA + cosB + cosC = $2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)+\cos C$

$=2\sin\frac{C}{2}.\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2}$ = $1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\sin\left(\frac{C}{2}\right)\right]$

= $1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right]$ $\left\{\because \frac{C}{2}=90°-\left(\frac{A+B}{2}\right)\right\}$

= $1+2\sin\frac{C}{2}.2\sin\frac{A}{2}.\sin\frac{B}{2}=1+4\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$

= $1+\frac{r}{R}$ {as, r = 4R sin A/2 . sinB/2 . sinC/2}

$\Rightarrow$ cosA + cosB + cosC = $1+\frac{r}{R}$. Hence proved.

<u>Problems for Self Practice</u>-10:

(1) If in ΔABC, a = 3, b = 4 and c = 5, find

(a) Δ (b) R (c) r

(2) In a ΔABC, show that :

(a) $\frac{a^2-b^2}{c}=2R\sin(A-B)$ (b) $r\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{\Delta}{4R}$ (c) $a+b+c=\frac{abc}{2Rr}$

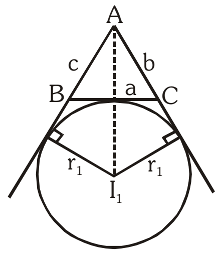
(3) Let Δ & Δ' denote the areas of a Δ and that of its incircle. Prove that

Δ : Δ' = $\left(\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}\right):\pi$

(4) If I be the incentre of ΔABC, then prove that IA . IB . IC = abc $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.

Answers : (1) (a) 6 **(b)** $\frac{5}{2}$ **(c)** 1

11. EX-CIRCLES AND DISTANCE OF EXCENTRES FROM VERTICES & SIDES OF TRIANGLE:



Point of intersection of two external angles and one internal angle bisectors is excentre and perpendicular distance of excentre from any side is called exradius. If r_1 is the radius of escribed circle opposite to $\angle A$ of ΔABC and so on, then -

(i) $$r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

(ii) $$r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} = \frac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}$$

(iii) $$r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$$

I_1, I_2 and I_3 are taken as ex-centre opposite to vertex A, B, C respectively.

(iv) $I_1A = r_1 \text{ cosec } \frac{A}{2}$; $I_1B = r_1 \sec \frac{B}{2}$; $I_1C = r_1 \sec \frac{C}{2}$

$I_2A = r_2 \sec \frac{A}{2}$; $I_2B = r_2 \text{ cosec } \frac{B}{2}$; $I_2C = r_2 \sec \frac{C}{2}$

$I_3A = r_3 \sec \frac{A}{2}$; $I_3B = r_3 \sec \frac{B}{2}$; $I_3C = r_3 \text{ cosec } \frac{C}{2}$

SOLVED EXAMPLE

Example 30 : Find the value of the expression $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}$

Solution : $\frac{(b-c)}{r_1}+\frac{(c-a)}{r_2}+\frac{(a-b)}{r_3} \Rightarrow (b-c)\left(\frac{s-a}{\Delta}\right)+(c-a)\left(\frac{s-b}{\Delta}\right)+(a-b).\left(\frac{s-c}{\Delta}\right)$

$$\Rightarrow \frac{(s-a)(b-c)+(s-b)(c-a)+(s-c)(a-b)}{\Delta}$$

$$=\frac{s(b-c+c-a+a-b)-[ab-ac+bc-ba+ac-bc]}{\Delta}=\frac{0}{\Delta}=0$$

Thus, $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$

Example 31 : If $r_1 = r_2 + r_3 + r$, prove that the triangle is right angled.

Solution : We have, $r_1 - r = r_2 + r_3$

$\Rightarrow \frac{\Delta}{s-a}-\frac{\Delta}{s}=\frac{\Delta}{s-b}+\frac{\Delta}{s-c}$ $\Rightarrow \frac{s-s+a}{s(s-a)}=\frac{s-c+s-b}{(s-b)(s-c)}$

$\Rightarrow \frac{a}{s(s-a)}=\frac{2s-(b+c)}{(s-b)(s-c)}$ {as, $2s = a + b + c$}

$\Rightarrow \frac{a}{s(s-a)}=\frac{a}{(s-b)(s-c)}$ $\Rightarrow s^2 - (b + c)s + bc = s^2 - as$

$\Rightarrow s(-a + b + c) = bc$ $\Rightarrow \frac{(b+c-a)(a+b+c)}{2}=bc$

$\Rightarrow (b + c)^2 - (a)^2 = 2bc$ $\Rightarrow b^2 + c^2 + 2bc - a^2 = 2bc$

$\Rightarrow b^2 + c^2 = a^2$

$\therefore$ $\angle A = 90°$.

Example 32 : In a ΔABC, prove that $r_1 + r_2 + r_3 - r = 4R = 2a \operatorname{cosec} A$

Solution : $\because$ L.H.S $= r_1 + r_2 + r_3 - r = \frac{\Delta}{s-a}+\frac{\Delta}{s-b}+\frac{\Delta}{s-c}-\frac{\Delta}{s}$

$$= \Delta\left(\frac{1}{s-a}+\frac{1}{s-b}\right)+\Delta\left(\frac{1}{s-c}-\frac{1}{s}\right) = \Delta\left[\left(\frac{s-b+s-a}{(s-a)(s-b)}\right)+\left(\frac{s-s+c}{s(s-c)}\right)\right]$$

$$= \Delta\left[\frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right] = c\Delta\left[\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right]$$

$= c\Delta \left[\dfrac{2s^2 - s(a+b+c) + ab}{\Delta^2}\right] = \dfrac{abc}{\Delta}$ $\quad \because \quad a + b + c = 2s$

$\because \quad R = \dfrac{abc}{4\Delta}$

$= 4R = 2a \text{ cosec } A$ $\quad \because \quad \dfrac{a}{\sin A} = 2R = a \text{ cosec } A$

= R.H.S.

Hence L.H.S. = R.H.S.

Example 33 : If the area of a ΔABC is 96 sq. unit and the radius of the escribed circles are respectively 8, 12 and 24. Find the perimeter of ΔABC.

Solution : $\because \quad \Delta = 96$ sq. unit

$r_1 = 8$, $r_2 = 12$ and $r_3 = 24$

$\because \quad r_1 = \dfrac{\Delta}{s-a} \quad \Rightarrow \quad s - a = 12 \quad$ ....(i)

$\because \quad r_2 = \dfrac{\Delta}{s-b} \quad \Rightarrow \quad s - b = 8 \quad$ ....(ii)

$\because \quad r_3 = \dfrac{\Delta}{s-c} \quad \Rightarrow \quad s - c = 4 \quad$ ....(iii)

$\therefore$ adding equations (i), (ii) & (iii), we get

$3s - (a + b + c) = 24$

$s = 24$

$\therefore$ perimeter of $\Delta ABC = 2s = 48$ unit. Ans.

<u>Problems for Self Practice</u>-11 :

(1) In a ΔABC, prove that

(i) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$ (ii) $rr_1 + rr_2 + rr_3 = ab + bc + ca - s^2$

(iii) $\dfrac{r_1 - r}{a} + \dfrac{r_2 - r}{b} = \dfrac{c}{r_3}$. (iv) $\dfrac{1}{4}r^2s^2\left(\dfrac{1}{r} - \dfrac{1}{r_1}\right)\left(\dfrac{1}{r} - \dfrac{1}{r_2}\right)\left(\dfrac{1}{r} - \dfrac{1}{r_3}\right) = R$

(2) If A, A_1, A_2 and A_3 are the areas of the inscribed and escribed circles respectively of a ΔABC, then prove that $\dfrac{1}{\sqrt{A}} = \dfrac{1}{\sqrt{A_1}} + \dfrac{1}{\sqrt{A_2}} + \dfrac{1}{\sqrt{A_3}}$.

(3) In an equilateral ΔABC, R = 2, find

(a) r (b) r_1 (c) a

Answers : (3) (a) 1 (b) 3 (c) $2\sqrt{3}$

12. THE DISTANCES BETWEEN THE SPECIAL POINTS :

(a) The distance between circumcentre and orthocentre is = $R\sqrt{1-8\cos A\cos B\cos C}$

(b) The distance between circumcentre and incentre is $=\sqrt{R^2-2Rr}$

(c) The distance between incentre and orthocentre is = $\sqrt{2r^2-4R^2\cos A\cos B\cos C}$

(d) The distances between circumcentre & excentres are

$$OI_1=R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\sqrt{R^2+2Rr_1}$$ & so on.

(e) $II_1 = 4R\sin\frac{A}{2}$; $II_2 = 4R\sin\frac{B}{2}$; $II_3 = 4R\sin\frac{C}{2}$.

SOLVED EXAMPLE

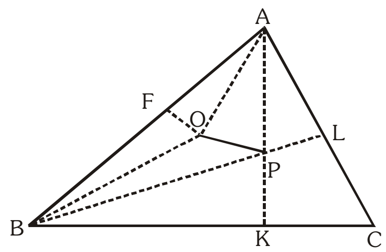
Example 34 : Prove that the distance between the circumcentre and the orthocentre of a triangle ABC is $R\sqrt{1-8\cos A\cos B\cos C}$.

Solution : Let O and P be the circumcentre and the orthocentre respectively. If OF is the perpendicular to AB, we have $\angle OAF = 90° - \angle AOF = 90° - C$. Also $\angle PAL = 90° - C$.

Hence, $\angle OAP = A - \angle OAF - \angle PAL = A - 2(90° - C) = A + 2C - 180°$

$= A + 2C - (A + B + C) = C - B$.

Also OA = R and PA = 2RcosA.



Now in ΔAOP,

$OP^2 = OA^2 + PA^2 - 2OA.\ PA\cos OAP$

$= R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C - B)$

$= R^2 + 4R^2\cos A[\cos A - \cos(C - B)]$

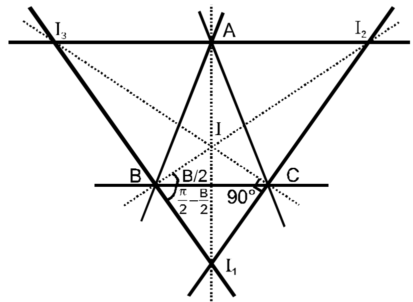
$= R^2 - 4R^2\cos A[\cos(B + C) + \cos(C - B)] = R^2 - 8R^2\cos A\cos B\cos C$.

Hence $OP = R\sqrt{1-8\cos A\cos B\cos C}$.

Example 35 : If I is the incentre and I_1, I_2, I_3 are the centres of escribed circles of the ΔABC, prove that $II_1.\ II_2.\ II_3 = 16R^2r$

Solution : $II_1 .\ II_2 .\ II_3 = abc\sec\frac{A}{2}.\sec\frac{B}{2}.\sec\frac{C}{2}$ ......(i)

$\because$ a = 2R sin A, b = 2R sinB and c = 2R sinC



$\therefore$ equation (i) becomes

$\because$ $II_1 . II_2 . II_3 = (2R \sin A)(2R \sin B)(2R \sin C) \sec\frac{A}{2} \sec\frac{B}{2} \sec\frac{C}{2}$

$$= 8R^3 . \frac{\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)}{\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}$$

$= 64R^3 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ $\qquad \because \quad r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\therefore$ $II_1 . II_2 . II_3 = 16R^2 r$

Problems for Self Practice-12 :

(1) In a ΔABC, if b = 2 cm, c = $\sqrt{3}$ cm and $\angle A = \frac{\pi}{6}$, then find distance between its circumcentre and incentre.

Answer : (1) $\sqrt{2-\sqrt{3}}$ cm

13. SOLUTION OF TRIANGLES :

The three sides a,b,c and the three angles A,B,C are called the elements of the triangle ABC. When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known completely; that is the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the solution of triangles.

* If the three sides a,b,c are given, angle A is obtained from $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

or $\cos A = \frac{b^2+c^2-a^2}{2bc}$ .B and C can be obtained in the similar way.

* If two sides b and c and the included angle A are given, then $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$ gives $\frac{B-C}{2}$.

Also $\frac{B+C}{2}=90°-\frac{A}{2}$, so that B and C can be evaluated. The third side is given by

a = b $\frac{\sin A}{\sin B}$ or $a^2 = b^2 + c^2 - 2bc \cos A$.

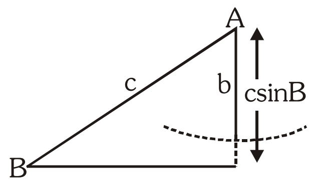
* If two sides b and c and an angle opposite the one of them (say B) are given then

$\sin C=\frac{c}{b}\sin B,\ A=180°-(B+C)$ and $a=\frac{b\sin A}{\sin B}$ given the remaining elements.

Case I :

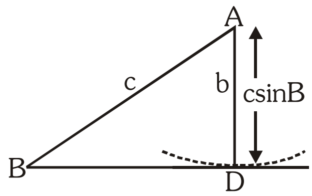
b < c sin B.

We draw the side c and angle B. Now it is obvious from the figure that there is no triangle possible.



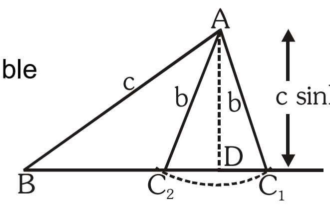
Case II :

b = c sin B and B is an acute angle, there is only one triangle possible. and it is right-angled at C.



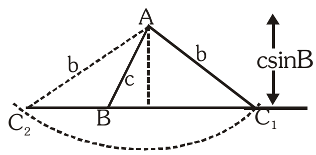
Case III :

b > c sin B, b < c and B is an acute angle, then there are two triangles possible for two values of angle C.



Case IV :

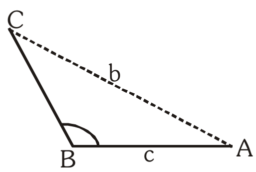
b > c sin B, c < b and B is an acute angle, then there is only one triangle.



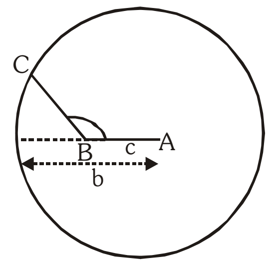
Case V :

b > c sin B, c > b and B is an obtuse angle.

For any choice of point C, b will be greater than c which is a contradication as c > b (given). So there is no triangle possible.



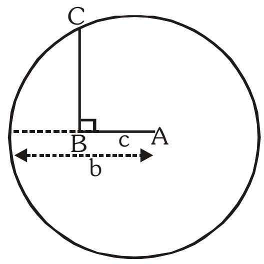
Case VI :



b > c sin B, c < b and B is an obtuse angle.

We can see that the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.

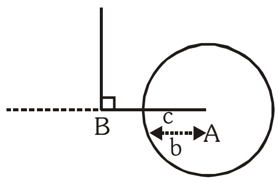
Case VII :



b > c and B = 90°.

Again the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.

Case VIII :



$b \leq c$ and B = 90°.

The circle with A as centre and b as radius will not cut the line in any point. So no triangle is possible.

This is, sometimes, called an ambiguous case.

Alternative Method : By applying cosine rule, we have $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\Rightarrow \quad a^2 - (2c \cos B)a + (c^2 - b^2) = 0 \Rightarrow a = c \cos B \pm \sqrt{(c\cos B)^2 - (c^2 - b^2)}$

$\Rightarrow \quad a = c \cos B \pm \sqrt{b^2 - (c \sin B)^2}$

This equation leads to following cases :

Case-I : If b < csinB, no such triangle is possible.

Case-II: Let b = c sinB. There are further following case :

(a) B is an obtuse angle $\Rightarrow$ cosB is negative. There exists no such triangle.

(b) B is an acute angle $\Rightarrow$ cosB is positive. There exists only one such triangle.

Case-III: Let b > c sin B. There are further following cases :

(a) B is an acute angle $\Rightarrow$ cosB is positive. In this case triangle will exist if and only if

$c \cos B > \sqrt{b^2 - (c \sin B)^2}$ or c > b $\Rightarrow$ Two such triangle is possible. If c < b, only one such triangle is possible.

(b) B is an obtuse angle $\Rightarrow$ cosB is negative. In this case triangle will exist if and only if $\sqrt{b^2-(c\sin B)^2}$ > |c cos B| $\Rightarrow$ b > c. So in this case only one such triangle is possible. If b < c there exists no such triangle. This is called an ambiguous case.

* If one side a and angles B and C are given, then A = 180° – (B + C), and $b=\frac{a\sin B}{\sin A}, c=\frac{a\sin C}{\sin A}$.

* If the three angles A,B,C are given, we can only find the ratios of the sides a,b,c by using sine rule (since there are infinite similar triangles possible).

SOLVED EXAMPLE

Example 36 : In the ambiguous case of the solution of triangles, prove that the circumcircles of the two triangles are of same size.

Solution : Let us say b,c and angle B are given in the ambiguous case. Both the triangles will have b and its opposite angle as B. so $\frac{b}{\sin B}=2R$ will be given for both the triangles. So their circumradii and therefore their sizes will be same.

Example 37 : If a,b and A are given in a triangle and c_1,c_2 are the possible values of the third side, prove that $c_1^2 + c_2^2 - 2c_1c_2 \cos 2A = 4a^2\cos^2 A$.

Solution : $\cos A=\frac{b^2+c^2-a^2}{2bc}$ $\Rightarrow$ $c^2 - 2bc\cos A + b^2 - a^2 = 0$.

$c_1 + c_2 = 2b\cos A$ and $c_1c_2 = b^2 - a^2$.

$\Rightarrow c_1^2 + c_2^2 - 2c_1c_2\cos 2A = (c_1 + c_2)^2 - 2c_1c_2(1 + \cos 2A)$

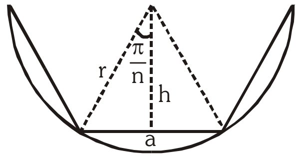
$= 4b^2 \cos^2 A - 2(b^2 - a^2)2 \cos^2 A = 4a^2\cos^2 A$.

Problems for Self Practice-13 :

(1) If b,c,B are given and b<c, prove that $\sin\left(\frac{A_1-A_2}{2}\right)=\frac{a_1-a_2}{2b}$

14. REGULAR POLYGON :

A regular polygon has all its sides equal. It may be inscribed or circumscribed.

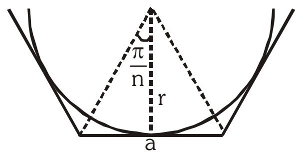


(a) Inscribed in circle of radius r :

(i) a = 2h $\tan\frac{\pi}{n}=2r\sin\frac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides inscribed in a circle of radius r are given by $P=2nr\sin\frac{\pi}{n}$ and $A=\frac{1}{2}nr^2\sin\frac{2\pi}{n}$

(b) Circumscribed about a circle of radius r :



(i) a = $2r\tan\frac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides

circumscribed about a given circle of radius r is given by $P = 2nr\tan\frac{\pi}{n}$ and

$A = nr^2\tan\frac{\pi}{n}$

(c) Area of a cyclic quadrilateral = $\sqrt{(s-a)(s-b)(s-c)(s-d)}$

where a, b, c, d are lengths of the sides of quadrilateral and s = $\frac{a+b+c+d}{2}$.

Problems for Self Practice-14 :

(1) If the perimeter of a circle and a regular polygon of n sides are equal, then

prove that $\frac{\text{area of the circle}}{\text{area of polygon}} = \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}$.

(2) The ratio of the area of n-sided regular polygon, circumscribed about a circle, to the area of the regular polygon of equal number of sides inscribed in the circle is 4 : 3. Find the value of n.

Answers : (2) 6

Exercise # 1

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Sine rule, Cosine rule, Napier's Analogy, Projection rule

A-1. In a ΔABC, prove that :

(i) $a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0$

(ii) $\frac{a^2 \sin(B-C)}{\sin A} + \frac{b^2 \sin(C-A)}{\sin B} + \frac{c^2 \sin(A-B)}{\sin C} = 0$

(iii) $2(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$

(iv) $(a - b)^2 \cos^2\frac{C}{2} + (a + b)^2 \sin^2\frac{C}{2} = c^2$

(v) $b^2 \sin 2C + c^2 \sin 2B = 2bc \sin A$ (vi) $\frac{\sin B}{\sin C} = \frac{c - a\cos B}{b - a\cos C}$

A-2. The angles of a ΔABC are in A.P. (order being A, B, C) and it is being given that $b : c = \sqrt{3} : \sqrt{2}$, then find $\angle$A.

A-3. If K is a point on the side BC of an equilateral triangle ABC and if $\angle$BAK = 15°, then find the ratio of lengths $\frac{AK}{AB}$.

A-4. In a triangle tan A : tan B : tan C = 1 : 2 : 3, then find the ratio $a^2 : b^2 : c^2$

A-5. If the sides of a triangle are $\sin\alpha$, $\cos\alpha$, $\sqrt{1+\sin\alpha\cos\alpha}, 0 < \alpha < \frac{\pi}{2}$, then find the largest angle

A-6. With usual notations, if in a Δ ABC, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$.

A-7. Let a, b and c be the sides of a ΔABC. If a^2, b^2 and c^2 are the roots of the equation $x^3 - Px^2 + Qx - R = 0$, where P, Q & R are constants, then find the value of $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ in terms of P, Q and R.

A-8. In a triangle ABC, prove that for any angle θ, $b \cos (A - \theta) + a \cos (B + \theta) = c \cos \theta$.

A-9. If in a Δ ABC, a = 6, b = 3 and cos(A – B) = 4/5, then find its area.

Section (B) : Half angle formulae, m : n theorem & length of Median/Angle bisector/Altitude

B-1. In a ΔABC, prove that

(i) $2\left[a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}\right]$ = c + a – b.

(ii) $\frac{\cos^2\frac{A}{2}}{a}+\frac{\cos^2\frac{B}{2}}{b}+\frac{\cos^2\frac{C}{2}}{c}=\frac{s^2}{abc}$

(iii) $4\left(bc.\cos^2\frac{A}{2}+ca.\cos^2\frac{B}{2}+ab.\cos^2\frac{C}{2}\right)=(a+b+c)^2$

(iv) $(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}=0$

(v) $4\Delta(\cot A+\cot B+\cot C)=a^2+b^2+c^2$

(vi) $\left(\frac{2abc}{a+b+c}\right).\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}=\Delta$

B-2. If the sides a, b, c of a triangle are in A.P., then find the value of $\tan\frac{A}{2}+\tan\frac{C}{2}$ in terms of cot (B/2).

B-3. If in a triangle ABC, $\angle A = 30^\circ$ and the area of triangle is $\frac{\sqrt{3}a^2}{4}$, then prove that either B = 4 C or C = 4 B.

B-4. If D is the mid point of CA in triangle ABC and Δ is the area of triangle, then show that

$\tan(\angle ADB)=\frac{4\Delta}{a^2-c^2}$.

B-5. If α, β, γ are the respective altitudes of a triangle ABC, prove that

(i) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{\cot A+\cot B+\cot C}{\Delta}$ (ii) $\frac{1}{\alpha}+\frac{1}{\beta}-\frac{1}{\gamma}=\frac{2ab}{(a+b+c)\Delta}\cos^2\frac{C}{2}$

Section (C) : Circumcentre, Incentre, Excentres, Orthocentre, Centroid & their distances from vertices

C-1. In any ΔABC, prove that

(i) $Rr(\sin A+\sin B+\sin C)=\Delta$ (ii) $a\cos B\cos C+b\cos C\cos A+c\cos A\cos B=\frac{\Delta}{R}$

(iii) $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr}$. (iv) $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+\frac{r}{2R}$

(v) $a\cot A+b\cot B+c\cot C=2(R+r)$

C-2. In any ΔABC, prove that

(i) $r.\, r_1 .r_2 .r_3 = \Delta^2$ (ii) $r_1 + r_2 - r_3 + r = 4R \cos C$.

(iii) $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{\Delta^2}$ (iv) $\left(\frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)^2 = \frac{4}{r}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)$

(v) $\frac{bc - r_2 r_3}{r_1} = \frac{ca - r_3 r_1}{r_2} = \frac{ab - r_1 r_2}{r_3} = r$

C-3. Show that the radii of the three escribed circles of a triangle are roots of the equation

$x^3 - x^2(4R + r) + x s^2 - r s^2 = 0$.

C-4. The radii r_1, r_2, r_3 of escribed circles of a triangle ABC are in harmonic progression. If its area is 24 sq. cm and its perimeter is 24 cm, find the lengths of its sides.

C-5. If the area of a triangle is 100 sq.cm, $r_1 = 10$ cm and $r_2 = 50$ cm, then find the value of $(b - a)$.

C-6. If in an acute angled ΔABC, line joining the circumcentre and orthocentre is parallel to side AC, then find the value of tan A.tan C.

Section (D) : Distances between special points, solution of triangle, regular polygon

D-1. If the circumcentre of the Δ ABC lies on its incircle, then prove that

$\cos A + \cos B + \cos C = \sqrt{2}$

D-2. If orthocentre of triangle lies on circumcircle of triangle then find the ratio of distance between incentre and orthocentre to in radius.

D-3. If b,c,B are given and b < c, prove that $\cos\left(\frac{A_1 - A_2}{2}\right) = \frac{c \sin B}{b}$.

D-4. In a ΔABC, b,c,B (c > b) are gives. If the third side has two values a_1 and a_2 such that

$a_1 = 3a_2$, show that $\sin B = \sqrt{\frac{4b^2 - c^2}{3c^2}}$

D-5. A regular hexagon & a regular dodecagon are inscribed in the same circle. If the side of the dodecagon is $\left(\sqrt{3}-1\right)$, if the side of the hexagon is $\sqrt[4]{k}$, then find value of k.

D-6. If I_n is the area of n sided regular polygon inscribed in a circle of unit radius and O_n be the area of the polygon circumscribing the given circle, prove that $I_n = \left(\frac{2I_n - O_n}{O_n}\right)^2 + \left(\frac{2I_n}{n}\right)^2 = 1$

PART-II : OBJECTIVE QUESTIONS

Section (A) : Sine rule, Cosine rule, Napier's Analogy, Projection rule

A-1. In a ΔABC, A : B : C = 3 : 5 : 4. Then a + b + c$\sqrt{2}$ is equal to

(A) 2b (B) 2c (C) 3b (D) 3a

A-2. If in a Δ ABC, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then the triangle is :

(A) right angled (B) isosceles (C) equilateral (D) obtuse angled

A-3. In a ΔABC $\frac{bc\sin^2 A}{\cos A + \cos B\cos C}$ is equal to

(A) $b^2 + c^2$ (B) bc (C) a^2 (D) $a^2 + bc$

A-4. In a triangle ABC, B = 60° and C = 45°. Let D divides BC internally in the ratio 1 : 3,

then value of $\frac{\sin\angle BAD}{\sin\angle CAD}$ is

(A) $\sqrt{\frac{2}{3}}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{6}}$ (D) $\frac{1}{3}$

A-5. If R denotes circumradius, then in ΔABC, $\frac{b^2 - c^2}{2aR}$ is equal to

(A) cos (B – C) (B) sin (B – C) (C) cos B – cos C (D) sin(B + C)

A-6. In a triangle ABC, if b = $(\sqrt{3}-1)$ a and $\angle$C = 30°, then the value of (A – B) is equal to

(All symbols used have usual meaning in a triangle.)

(A) 30° (B) 45° (C) 60° (D) 75°

A-7. In triangle ABC, if 2b = a + c and A – C = 90°, then sin B equals

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) $\frac{\sqrt{7}}{5}$ (B) $\frac{\sqrt{5}}{8}$ (C) $\frac{\sqrt{7}}{4}$ (D) $\frac{\sqrt{5}}{3}$

A-8. If in a triangle ABC, (a + b + c) (b + c – a) = k. b c, then :

(A) k < 0 (B) k > 6 (C) 0 < k < 4 (D) k > 4

A-9. If in a triangle ABC, the altitude AM be the bisector of $\angle$BAD, where D is the mid point of side BC, then

$\frac{b^2 - c^2}{a^2}$ is equal to

(A) 1 (B) 2 (C) 0.5 (D) 0.25

A-10. In triangle ABC, if $\sin^3 A + \sin^3 B + \sin^3 C = 3\sin A.\sin B.\sin C$, then triangle is

(A) obtuse angled (B) right angled (C) obtuse right angled (D) equilateral

Section (B) : Half angle formulae, m : n theorem & length of Median/Angle bisector/Altitude

B-1. If in a triangle ABC, right angle at B, s – a = 3 and s – c = 2, then

(A) a = 2, c = 3 (B) a = 3, c = 4 (C) a = 4, c = 3 (D) a = 6, c = 8

B-2. If in a triangle ABC, $b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = \frac{3}{2}c$, then a, c, b are :

(A) in A.P. (B) in G.P. (C) in H.P. (D) None

B-3. If in a ΔABC, $\angle A = \frac{\pi}{2}$, then $\tan\frac{C}{2}$ is equal to

(A) $\frac{a-c}{2b}$ (B) $\frac{a-b}{2c}$ (C) $\frac{a-c}{b}$ (D) $\frac{a-b}{c}$

B-4. A triangle is inscribed in a circle. The vertices of the triangle divide the circle into three arcs of length 3, 4 and 5 units. Then area of the triangle is equal to:

(A) $\frac{9\sqrt{3}(1+\sqrt{3})}{\pi^2}$ (B) $\frac{9\sqrt{3}(\sqrt{3}-1)}{\pi^2}$ (C) $\frac{9\sqrt{3}(1+\sqrt{3})}{2\pi^2}$ (D) $\frac{9\sqrt{3}(\sqrt{3}-1)}{2\pi^2}$

B-5. In a Δ ABC if b + c = 3a, then $\cot\frac{B}{2}\cdot\cot\frac{C}{2}$ has the value equal to:

(A) 4 (B) 3 (C) 2 (D) 1

B-6. If in a ΔABC, $\Delta = a^2 - (b-c)^2$, then tan A is equal to

(A) 15/16 (B) 8/15 (C) 8/17 (D) 1/2

B-7. In a ΔABC, if AB = 5 cm, BC = 13 cm and CA = 12 cm, then the distance of vertex 'A' from the side BC is (in cm)

(A) $\frac{25}{13}$ (B) $\frac{60}{13}$ (C) $\frac{65}{12}$ (D) $\frac{144}{13}$

B-8. Let ABC is a right angled, right angle at B. If D and E be the points on CB such that $\angle ADB = 2\angle ACB$ and $\angle AEB = 3\angle ACB$, then range of $\frac{DE}{CD}$ is

(A) $\left(\frac{1}{3},\frac{1}{2}\right)$ (B) $\left(\frac{1}{6},\frac{1}{3}\right)$ (C) $\left(\frac{1}{5},\frac{1}{3}\right)$ (D) $\left(\frac{1}{6},\frac{1}{4}\right)$

B-9. If 'ℓ' is the length of median from the vertex A to the side BC of a ΔABC, then

(A) $4\ell^2 = b^2 + 4ac\cos B$ (B) $4\ell^2 = a^2 + 4bc\cos A$

(C) $4\ell^2 = c^2 + 4ab\cos C$ (D) $4\ell^2 = b^2 + 2c^2 - 2a^2$

B-10. In a triangle ABC, with usual notations the length of the bisector of internal angle A is not equal to :

(A) $\frac{2bc\cos\frac{A}{2}}{b+c}$ (B) $\frac{2bc\sin\frac{A}{2}}{b+c}$ (C) $\frac{abc\ \text{cosec}\frac{A}{2}}{2R(b+c)}$ (D) $\frac{2\Delta}{b+c}\cdot\text{cosec}\frac{A}{2}$

Section (C) : Circumcentre, Incentre, Excentres, Orthocentre, Centroid & their distances from vertices

C-1. If H is the orthocentre of a triangle ABC, then the radii of the circle circumscribing the triangles BHC, CHA and AHB are respectively equal to :

(A) R, R, R (B) $\sqrt{2}R, \sqrt{2}R, \sqrt{2}R$ (C) 2R, 2R, 2R (D) $\frac{R}{2}, \frac{R}{2}, \frac{R}{2}$

C-2. In a Δ ABC, the value of $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is equal to:

(A) $\frac{r}{R}$ (B) $\frac{R}{2r}$ (C) $\frac{R}{r}$ (D) $\frac{2r}{R}$

C-3. In a triangle ABC, if a : b : c = 3 : 7 : 8, then R : r is equal to

(A) 2 : 7 (B) 7 : 2 (C) 3 : 7 (D) 7 : 3

C-4. In any ΔABC, $\frac{(r_1+r_2)(r_2+r_3)(r_3+r_1)}{Rs^2}$ is always equal to

(A) 8 (B) 27 (C) 16 (D) 4

C-5. Let f, g, h be the lengths of the perpendiculars from the circumcentre of the Δ ABC on the sides BC, CA and AB respectively. If $\frac{a}{f}+\frac{b}{g}+\frac{c}{h} = \lambda \frac{abc}{fgh}$, then the value of '$\lambda$' is:

(A) 1/4 (B) 1/2 (C) 1 (D) 2

C-6. The product of the distances of the incentre from the angular points of a Δ ABC is:

(A) $4R^2r$ (B) $4Rr^2$ (C) $\frac{(abc)R}{s}$ (D) $\frac{4(a\,b\,c)\,r}{s}$

Section (D) : Distances between special points, solution of triangle, regular polygon

D-1. If orthocentre of triangle lies on circumcircle of triangle then $\frac{(a+b)^2-c^2}{R^2\cos A\cos B}$ is equal to (c is largest side)

(A) 2 (B) 4 (C) 6 (D) 8

D-2. In an acute triangle ABC, $\angle$ABC = 45°, AB = 3 and AC = $\sqrt{6}$. The angle $\angle$BAC, is

(A) 60° (B) 65° (C) 75° (D) 15° or 75°

D-3. Two sides of a triangle are of lengths $\sqrt{6}$ and 4 and the angle opposite to smaller side is 30^0. How many such triangles are possible ?

(A) 0 (B) 1 (C) 2 (D) infinite

D-4. The sum of the radii of inscribed and circumscribed circles for an n sided regular polygon of side 'a', is :

(A) $a\cot\left(\frac{\pi}{n}\right)$ (B) $\frac{a}{2}\cot\left(\frac{\pi}{2n}\right)$ (C) $a\cot\left(\frac{\pi}{2n}\right)$ (D) $\frac{a}{4}\cot\left(\frac{\pi}{2n}\right)$

D-5. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A **false** statement among the following is

(A) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$.
(B) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$.
(C) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$.
(D) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$.

PART-III : MATCH THE COLUMN

1. Match the column

Column– I		**Column–II**	
(A)	In a ΔABC, $2B = A + C$ and $b^2 = ac$. Then the value of $\frac{a^2(a+b+c)}{3abc}$ is equal to	(p)	8
(B)	In any right angled triangle ABC, the value of $\frac{a^2+b^2+c^2}{R^2}$ is always equal to (where R is the circumradius of ΔABC)	(q)	1
(C)	In a ΔABC if $a = 2$, $bc = 9$, then the value of $2R\Delta$ is equal to	(r)	5
(D)	In a ΔABC, $a = 5$, $b = 3$ and $c = 7$, then the value of $3 \cos C + 7 \cos B$ is equal to	(s)	9

2. **Match the column**

Column – I		**Column – II**	
(A)	In a ΔABC, $a = 4$, $b = 3$ and the medians AA_1 and BB_1 are mutually perpendicular, then square of area of the ΔABC is equal to	(p)	27
(B)	In any ΔABC, minimum value of $\frac{r_1 r_2 r_3}{r^3}$ is equal to	(q)	7
(C)	In a ΔABC, $a = 5$, $b = 4$ and $\tan\frac{C}{2} = \sqrt{\frac{7}{9}}$, then side 'c' is equal to	(r)	6
(D)	In a ΔABC, $2a^2 + 4b^2 + c^2 = 4ab + 2ac$, then value of $(8 \cos B)$ is equal to	(s)	11

Exercise # 2

PART - I : ONLY ONE OPTION CORRECT TYPE

1. In a triangle ABC, a: b: c = 4: 5: 6. Then 3A + B equals to :

(A) 4C (B) 2π (C) $\pi - C$ (D) π

2. The distance between the middle point of BC and the foot of the perpendicular from A is :

(A) $\frac{-a^2+b^2+c^2}{2a}$ (B) $\frac{b^2-c^2}{2a}$ (C) $\frac{b^2+c^2}{\sqrt{bc}}$ (D) $\frac{b^2+c^2}{2a}$

3. In a ΔABC, a = 1 and the perimeter is six times the arithmetic mean of the sines of the angles. Then measure of $\angle A$ is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$

4. In ΔABC, angle A, B and C are in the ratio 1 : 2 : 3, then which of the following is (are) correct?

(All symbol used have usual meaning in a triangle.)

(A) Circumradius of ΔABC = c (B) a : b : c = 1 : 2 : $\sqrt{3}$

(C) Perimeter of $\Delta ABC = 3 + \sqrt{3}$ (D) Area of $\Delta ABC = \frac{\sqrt{3}}{8} c^2$

5. In ΔABC, angle A is 120°, BC + CA = 20 and AB + BC = 21, then

(A) perimeter of ΔABC is 32 (B) AB < AC

(C) ΔABC is isosceles (D) area of $\Delta ABC = 14\sqrt{3}$

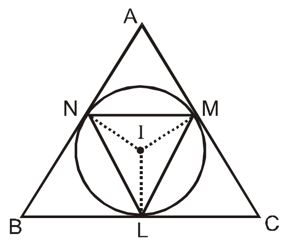
6. AA_1, BB_1 and CC_1 are the medians of triangle ABC whose centroid is G. If points A, C_1, G and B_1 are concyclic, then

(A) $2b^2 = a^2 + c^2$ (B) $2c^2 = a^2 + b^2$ (C) $2a^2 = b^2 + c^2$ (D) $3a^2 = b^2 + c^2$

7. In a triangle ABC, if $\frac{a-b}{b-c} = \frac{s-a}{s-c}$, then r_1, r_2, r_3 are in:

(A) A.P. (B) G.P. (C) H.P. (D) none of these

8. If the incircle of the Δ ABC touches its sides at L, M and N as shown in the figure and if x, y, z be the circumradii of the triangles MIN, NIL and LIM respectively, where I is the incentre, then the product xyz is equal to :



(A) $R r^2$ (B) $r R^2$

(C) $\frac{1}{2} R r^2$ (D) $\frac{1}{2} r R^2$

9. In an acute angled triangle ABC, AP is the altitude. Circle drawn with AP as its diameter cuts the sides AB and AC at D and E respectively, then length DE is equal to

(A) $\frac{\Delta}{2R}$ (B) $\frac{\Delta}{3R}$ (C) $\frac{\Delta}{4R}$ (D) $\frac{\Delta}{R}$

10. A triangle has vertices A, B and C, and the respective opposite sides have lengths a, b and c. This triangle is inscribed in a circle of radius R. If b = c = 1 and the altitude from A to side BC has length $\sqrt{\frac{2}{3}}$, then R equals

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{2}{\sqrt{3}}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{\sqrt{3}}{2\sqrt{2}}$

11. In a triangle ABC, if (a + b + c) (a + b – c) (b + c – a) (c + a – b) = $\frac{8a^2b^2c^2}{a^2+b^2+c^2}$, then the triangle is

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) isosceles (B) right angled (C) equilateral (D) obtuse angled

12. AD and BE are the medians of a triangle ABC. If AD = 4, $\angle DAB = \frac{\pi}{6}, \angle ABE = \frac{\pi}{3}$, then area of triangle ABC equals

(A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) $\frac{32}{3}$ (D) $\frac{32}{9}\sqrt{3}$

PART-II : NUMERICAL QUESTIONS

1. In ΔABC, if a = 2b and A = 3B, then the value of $\left(\frac{c}{b}\right)^2$ is equal to

2. If the angle A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then find the value of expression $E=\left(\frac{a}{c}\sin 2C+\frac{c}{a}\sin 2A\right)^2$.

3. In triangle ABC, if AC = 8, BC = 7 and D lies between A and B such that AD = 2, BD = 4, then the value of $(CD)^2$ equals to

4. In triangle ABC, if cos A + cos B = 4 $\sin^2\left(\frac{C}{2}\right)$, then $\frac{a+b}{c}$ is equal to :

5. If a,b,c are the sides of triangle ABC satisfying $\log\left(1+\frac{c}{a}\right)+\log a-\log b=\log 2$.

Also $a(1 - x^2) + 2bx + c(1 + x^2) = 0$ has two equal roots. Find the value of sinA + sinB + sinC.

6. In a triangle ABC if $a^2 + b^2 = 101c^2$ then find the value of $\frac{\cot C}{\cot A + \cot B}$.

7. The sides of a triangle are consecutive integers *n*, *n* + 1 and *n* + 2 and the largest angle is twice the smallest angle. Find *n*.

8. Given a triangle ABC with AB = 2 and AC = 1. Internal bisector of $\angle$BAC intersects BC at D. If AD = BD and Δ is the area of triangle ABC, then find the value of $12\Delta^2$.

9. Given a triangle ABC with sides a = 7, b = 8 and c = 5. If the value of the expression $\left(\sum \sin A\right)\left(\sum \cot \frac{A}{2}\right)$ can be expressed in the form $\frac{p}{q}$ where p, q $\in$ N and $\frac{p}{q}$ is in its lowest form find the value of (p + q).

10. In a ΔABC, A = $\frac{2\pi}{3}$, b – c = $3\sqrt{3}$ cm and area (ΔABC) = $\frac{9\sqrt{3}}{2}$ cm². Then 'a' is

11. If AD, BE and CF are the medians of a ΔABC, then $\frac{(AD^2 + BE^2 + CF^2)}{(BC^2 + CA^2 + AB^2)}$ is equal to

12. If in a triangle ABC, the line joining the circumcentre and incentre is parallel to BC, then cos B + cos C is equal to :

13. If in a ΔABC, $\frac{r}{r_1} = \frac{1}{2}$, then the value of $\tan\frac{A}{2}\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)$ is equal to :

PART - III : ONE OR MORE THAN ONE CORRECT QUESTION

1. Which of the following holds good for any triangle ABC?

(A) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$

(B) $\frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c} = \frac{3}{2R}$

(C) $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$

(D) $\frac{\sin 2A}{a^2} = \frac{\sin 2B}{b^2} = \frac{\sin 2C}{c^2}$

2. In a triangle ABC, let $2a^2 + 4b^2 + c^2 = 2a(2b + c)$, then which of the following holds good?
[**Note:** All symbols used have usual meaning in a triangle.]

(A) $\cos B = \frac{-7}{8}$

(B) sin (A– C) = 0

(C) $\frac{r}{r_1} = \frac{1}{5}$

(D) sin A : sin B : sin C = 1 : 2 : 1

3. In which of the following situations, it is possible to have a triangle ABC?
(All symbols used have usual meaning in a triangle.)

(A) $(a + c - b)(a - c + b) = 4bc$
(B) $b^2 \sin 2C + c^2 \sin 2B = ab$
(C) $a = 3$, $b = 5$, $c = 7$ and $C = \frac{2\pi}{3}$
(D) $\cos\left(\frac{A-C}{2}\right) = \cos\left(\frac{A+C}{2}\right)$

4. Given an acute triangle ABC such that $\sin C = \frac{4}{5}$, $\tan A = \frac{24}{7}$ and AB = 50. Then-

(A) centroid, orthocentre and incentre of ΔABC are collinear
(B) $\sin B = \frac{4}{5}$
(C) $\sin B = \frac{4}{7}$
(D) area of ΔABC = 1200

5. If in a triangle ABC, cos A cos B + sin A sin B sin C = 1, then the triangle is

(A) isosceles (B) right angled (C) equilateral (D) None of these

6. If in a ΔABC, a = 5, b = 4 and $\cos(A - B) = \frac{31}{32}$, then

(A) c = 6
(B) $\sin A = \left(\frac{5\sqrt{7}}{16}\right)$
(C) area of $\Delta ABC = \frac{15\sqrt{7}}{4}$
(D) c = 8

7. In a triangle ABC, points D and E are taken on side BC such that BD = DE = EC. If angle ADE = angle AED = θ, then:

(A) $\tan\theta = 3 \tan B$
(B) $3 \tan\theta = \tan C$
(C) $\frac{6\tan\theta}{\tan^2\theta - 9} = \tan A$
(D) angle B = angle C

8. In a triangle ABC, if a = 4, b = 8 $\angle C = 60°$, then which of the following relations is (are) correct?
[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) The area of triangle ABC is $8\sqrt{3}$
(B) The value of $\sum \sin^2 A = 2$
(C) Inradius of triangle ABC is $\frac{2\sqrt{3}}{3+\sqrt{3}}$
(D) The length of internal angle bisector of angle C is $\frac{4}{\sqrt{3}}$

9. If $r_1 = 2r_2 = 3r_3$, then

(A) $\frac{a}{b} = \frac{4}{5}$ (B) $\frac{a}{b} = \frac{5}{4}$ (C) $\frac{a}{c} = \frac{3}{5}$ (D) $\frac{a}{c} = \frac{5}{3}$

10. In a ΔABC, following relations hold good. In which case(s) the triangle is a right angled triangle?

(A) $r_2 + r_3 = r_1 - r$ (B) $a^2 + b^2 + c^2 = 8R^2$

(C) $r_1 = s$ (D) $2R = r_1 - r$

11. In a triangle ABC, right angled at B, then

(A) $r = \frac{AB + BC - AC}{2}$ (B) $r = \frac{AB + AC - BC}{2}$ (C) $r = \frac{AB + BC + AC}{2}$ (D) $R = \frac{s - r}{2}$

12. Three equal circles of radius unity touches one another. Radius of the circle touching all the three circles is :

(A) $\frac{2 - \sqrt{3}}{\sqrt{3}}$ (B) $\frac{\sqrt{3} - \sqrt{2}}{\sqrt{2}}$ (C) $\frac{2 + \sqrt{3}}{\sqrt{3}}$ (D) $\frac{\sqrt{3} + \sqrt{2}}{\sqrt{2}}$

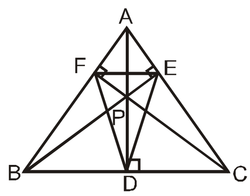
13. With usual notations, in a Δ ABC the value of $\Pi\ (r_1 - r)$ can be simplified as:

(A) $abc\ \Pi \tan\frac{A}{2}$ (B) $4rR^2$ (C) $\frac{(abc)^2}{R(a+b+c)^2}$ (D) $4Rr^2$

PART - IV : COMPREHENSION

Comprehension # 1

The triangle DEF which is formed by joining the feet of the altitudes of triangle ABC is called the Pedal Triangle.



Answer The Following Questions :

1. Angle of triangle DEF are

(A) $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$ (B) $\pi + 2A$, $\pi + 2B$ and $\pi + 2C$

(C) $\pi - A$, $\pi - B$ and $\pi - C$ (D) $2\pi - A$, $2\pi - B$ and $2\pi - C$

2. Sides of triangle DEF are

(A) b cosA, a cosB, c cosC (B) a cosA, b cosB, c cosC

(C) R sin 2A, R sin 2B, R sin 2C (D) a cotA, b cotB, c cotC

3. Circumraii of the triangle PBC, PCA and PAB are respectively

(A) R, R, R (B) 2R, 2R, 2R (C) R/2, R/2, R/2 (D) 3R, 3R, 3R

4. Which of the following is/are correct

(A) $\frac{\text{Perimeter of } \Delta DEF}{\text{Perimeter of } \Delta ABC} = \frac{r}{R}$ (B) Area of ΔDEF = $2\Delta \cos A \cos B \cos C$

(C) Area of ΔAEF = $\Delta \cos^2 A$ (D) Circum-radius of ΔDEF = $\frac{R}{2}$

Comprehension # 2

The triangle formed by joining the three excentres I_1, I_2 and I_3 of Δ ABC is called the excentral or excentric triangle and in this case internal angle bisector of triangle ABC are the altitudes of triangles $I_1I_2I_3$

5. Incentre I of Δ ABC is the .... of the excentral $\Delta I_1I_2I_3$.

(A) Circumcentre (B) Orthocentre (C) Centroid (D) None of these

6. Angles of the $\Delta I_1I_2I_3$ are

(A) $\frac{\pi}{2}-\frac{A}{2},\ \frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$ (B) $\frac{\pi}{2}+\frac{A}{2}, \frac{\pi}{2}+\frac{B}{2}$ and $\frac{\pi}{2}+\frac{C}{2}$

(C) $\frac{\pi}{2}-A,\ \frac{\pi}{2}-B$ and $\frac{\pi}{2}-C$ (D) None of these

7. Sides of the $\Delta I_1I_2I_3$ are

(A) $R\cos\frac{A}{2}$, $R\cos\frac{B}{2}$ and $R\cos\frac{C}{2}$ (B) $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ and $4R\cos\frac{C}{2}$

(C) $2R\cos\frac{A}{2}$, $2R\cos\frac{B}{2}$ and $2R\cos\frac{C}{2}$ (D) None of these

8. Value of $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2 =$

(A) $4R^2$ (B) $16R^2$ (C) $32R^2$ (D) $64R^2$

Exercise # 3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

*** Marked Questions may have more than one correct option.**

1. If the angle A, B and C of a triangle are in arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}$ sin 2C + $\frac{c}{a}$ sin 2A is

[IIT-JEE 2010, Paper-1, (3, –1), 84]

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$

2. Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)

[IIT-JEE 2010, Paper-1, (3, 0), 84]

(A) $-\left(2+\sqrt{3}\right)$ (B) $1 + \sqrt{3}$ (C) $2 + \sqrt{3}$ (D) $4\sqrt{3}$

3. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose a = 6, b = 10 and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to

[IIT-JEE 2010, Paper-2, (3, 0), 79]

4. Let PQR be a triangle of area Δ with a = 2, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals

[IIT-JEE 2012, Paper-2, (3, –1), 66]

(A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$

(C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

5.* In a triangle PQR, P is the largest angle and cosP = $\frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are)

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

(A) 16 (B) 18 (C) 24 (D) 22

6. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is

[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

(A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

7. In a triangle XYZ, let x,y,z be the lengths of sides opposite to the angles X,Y,Z, respectively and 2s = x + y + z. If $\frac{s-x}{4}=\frac{s-y}{3}=\frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then-

[JEE(Advanced)-2016, 4(–2)]

(A) area of the triangle XYZ is $6\sqrt{6}$

(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}=\frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right)=\frac{3}{5}$

8. In a triangle PQR, let ∠PQR = 30° and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) ∠QPR = 45°

(B) The area of the triangle PQR is $25\sqrt{3}$ and ∠QRP = 120°

(C) The radius of the incircle of the triangle PQR is $10\sqrt{3}-15$

(D) The area of the circumcircle of the triangle PQR is 100π.

9. In a non-right-angled triangle ΔPQR, let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p=\sqrt{3}$, q = 1, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(A) Area of $\Delta SOE=\frac{\sqrt{3}}{12}$

(B) Radius of incircle of $\Delta PQR=\frac{\sqrt{3}}{2}(2-\sqrt{3})$

(C) Length of $RS=\frac{\sqrt{7}}{2}$

(D) Length of $OE=\frac{1}{6}$

PART - II : JEE(MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A **false** statement among the following is **[AIEEE - 2010 (4, –1), 144]**

(1) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$.
(2) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$.
(3) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$.
(4) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$.

2. ABCD is a trapezium such that AB and CD are parallel and BC $\perp$ CD. If $\angle$ADB = θ, BC = p and CD = q, then AB is equal to - **[JEE - Main 2013]**

(1) $\frac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$
(2) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$
(3) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$
(4) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$

3. In a ΔABC, $\frac{a}{b} = 2+\sqrt{3}$ and $\angle$C = 60º. Then the ordered pair ($\angle$A, $\angle$B) is equal to : **[JEE(Main)-2015]**

(1) (75° , 45°)
(2) (45°, 75°)
(3) (15°, 105°)
(4) (105°, 15°)

4. If 5, 5r, $5r^2$ are the lengths of the sides of a triangle, then r cannot be equal to : **[JEE(Main)-Jan 2019]**

(1) $\frac{3}{2}$ (2) $\frac{3}{4}$ (3) $\frac{5}{4}$ (4) $\frac{7}{4}$

5. With the usual notation, in ΔABC, if $\angle A+\angle B = 120^\circ$, $a=\sqrt{3}+1$ and $b=\sqrt{3}-1$, then the ratio $\angle A : \angle B$, is : **[JEE(Main)-Jan 2019]**

(1) 7 : 1 (2) 5 : 3 (3) 9 : 7 (4) 3 : 1

6. In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y. If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is : **[JEE(Main)-Jan 2019]**

(1) $\frac{y}{\sqrt{3}}$ (2) $\frac{c}{\sqrt{3}}$ (3) $\frac{c}{3}$ (4) $\frac{3}{2}y$

7. Given $\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$ for a ΔABC with usual notation. If $\frac{\cos A}{\alpha}=\frac{\cos B}{\beta}=\frac{\cos C}{\gamma}$, then the ordered triad (α, β, γ) has a value :- **[JEE(Main)-Jan 2019]**

(1) (3, 4, 5) (2) (19, 7, 25) (3) (7, 19, 25) (4) (5, 12, 13)

8. If the lengths of the sides of a triangle are in A.P. and the greatest angle is double the smallest, then a ratio of lengths of the sides of this triangle is : **[JEE(Main)-Apr 2019]**

(1) 5 : 9 : 13 (2) 5 : 6 : 7 (3) 4 : 5 : 6 (4) 3 : 4 : 5

9. The angles A, B and C of a triangle ABC are in A.P. and a : b = 1 : $\sqrt{3}$. If c = 4 cm, then the area (in sq. cm) of this triangle is : **[JEE(Main)-Apr 2019]**

(1) $4\sqrt{3}$ (2) $\frac{2}{\sqrt{3}}$ (3) $2\sqrt{3}$ (4) $\frac{4}{\sqrt{3}}$

Answers

Exercise # 1

PART-I

SECTION-(A)

A-2. 75° **A-3.** $\frac{\sqrt{2}\left(3-\sqrt{3}\right)}{2}$

A-4. 5 : 8 : 9 **A-5.** 120°

A-7. $\frac{P}{2\sqrt{R}}$ **A-9.** 9 sq. unit

SECTION-(B)

B-2. $\frac{2}{3}\cot\frac{B}{2}$

SECTION-(C)

C-4. 6, 8, 10 cm **C-5.** 8

C-6. 3

SECTION-(D)

D-2. $\sqrt{2}$ **D-5.** $\sqrt{2}$

PART-II

SECTION-(A)

A-1. (C) **A-2.** (C)

A-3. (C) **A-4.** (C)

A-5. (B) **A-6.** (C)

A-7. (C) **A-8.** (C)

A-9. (C) **A-10.** (D)

SECTION-(B)

B-1. (B) **B-2.** (A)

B-3. (D) **B-4.** (A)

B-5. (C) **B-6.** (B)

B-7. (B) **B-8.** (A)

B-9. (B) **B-10.** (B)

SECTION-(C)

C-1. (A) **C-2.** (A)

C-3. (B) **C-4.** (D)

C-5. (A) **C-6.** (B)

SECTION-(D)

D-1. (D) **D-2.** (C)

D-3. (C) **D-4.** (B)

D-5. (B)

PART-III

1. (A) $\rightarrow$ (q), (B) $\rightarrow$ (p), (C) $\rightarrow$ (s), (D) $\rightarrow$ (r)

2. (A) $\rightarrow$ (s), (B) $\rightarrow$ (p), (C) $\rightarrow$ (r), (D) $\rightarrow$ (q)

Exercise # 2

PART - I

1. (D) **2.** (B)

3. (C) **4.** (D)

5. (D) **6.** (C)

7. (A) **8.** (C)

9. (D) **10.** (D)

11. (B) **12.** (D)

PART-II

1.	3	**2.**	3
3.	51	**4.**	2
5.	2.4	**6.**	50
7.	4	**8.**	9
9.	107	**10.**	9
11.	0.75	**12.**	1
13.	0.5		

PART - III

1.	(A, (B)	**2.**	(B), (C)
3.	(B), (C)	**4.**	(A), (B), (D)
5.	(A), (B)	**6.**	(A), (B), (C)
7.	(A), (C), (D)	**8.**	(A), (B)
9.	(B), (D)	**10.**	(A), (B), (C), (D)
11.	(A), (D)	**12.**	(A), (C)
13.	(A), (C) (D)		

PART - IV

1.	(A)	**2.**	(B), (C)
3.	(A)	**4.**	(A),(B),(C),(D)
5.	(B),	**6.**	(A)
7.	(B)	**8.**	(B)

Exercise # 3

PART - I

1.	(D)	**2.**	(B)
3.	3	**4.**	(C)
5.	(B, D)	**6.**	(B)
7.	(B, C, D)	**8.**	(B, C, D)
9.	(B, C, D)		

PART - II

1.	(2)	**2.**	(3)
3.	(4)	**4.**	(4)
5.	(1)	**6.**	(2)
7.	(3)	**8.**	(3)
9.	(3)		

SUBJECTIVE QUESTIONS

1. In a triangle ABC, if a tan A + b tan B = (a + b) tan$\left(\frac{A+B}{2}\right)$, prove that triangle is isosceles.

2. In a ΔABC, if a, b and c are in A.P., prove that cos A.cot$\frac{A}{2}$, cos B.cot$\frac{B}{2}$, and cos C.cot$\frac{C}{2}$ are in A.P.

3. ABCD is a trapezium such that AB, DC are parallel and BC is perpendicular to them. If angle ADB = θ, BC = p and CD = q, show that AB = $\frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$.

4. If in a triangle ABC, $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$ prove that the triangle ABC is either isosceles or right angled.

5. In a Δ ABC, $\angle$ C = 60° and $\angle$ A = 75°. If D is a point on AC such that the area of the Δ BAD is $\sqrt{3}$ times the area of the Δ BCD, find the $\angle$ ABD.

6. In a triangle ABC, prove that the area of the incircle is to the area of triangle itself is,

π : cot $\left(\frac{A}{2}\right)$. cot $\left(\frac{B}{2}\right)$. cot$\left(\frac{C}{2}\right)$.

7. Three circles, whose radii are a, b and c, touch one another externally and the tangents at their points of contact meet in a point, prove that the distance of this point from either of their points of contact is $\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}$.

8. In any ΔABC, prove that

(i) $(r_3+ r_1)(r_3+ r_2)\sin C = 2 r_3 \sqrt{r_2r_3 + r_3r_1 + r_1r_2}$

(ii) $\frac{\tan\frac{A}{2}}{(a-b)(a-c)} + \frac{\tan\frac{B}{2}}{(b-a)(b-c)} + \frac{\tan\frac{C}{2}}{(c-a)(c-b)} = \frac{1}{\Delta}$

(iii) $(r+r_1)\tan\frac{B-C}{2} + (r + r_2)\tan\frac{C-A}{2} + (r + r_3)\tan\frac{A-B}{2} = 0$

(iv) $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - a^2 - b^2 - c^2$.

9. If $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)$ = 2, then prove that the triangle is right angled.

10. In a triangle ABC, AD is the altitude from A. Given b > c, angle C = 23° and AD = $\frac{abc}{b^2-c^2}$, then find angle B.

11. DEF is the triangle formed by joining the points of contact of the incircle with the sides of the triangle ABC; prove that

(i) its sides are $2r\cos\frac{A}{2}$, $2r\cos\frac{B}{2}$ and $2r\cos\frac{C}{2}$,

(ii) its angles are $\frac{\pi}{2}-\frac{A}{2}$, $\frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$

and

(iii) its area is $\frac{2\Delta^3}{(abc)s}$, i.e. $\frac{1}{2}\frac{r}{R}\Delta$.

12. In acute angled triangle ABC, a semicircle with radius r_a is constructed with its base on BC and tangent to the other two sides. r_b and r_c are defined similarly. If r is the radius of the incircle of triangle ABC then prove that,

$\frac{2}{r}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}$.

13. Let ABC and ABC′ be two non-congruent triangles with sides AB = 4, AC = AC′ = $2\sqrt{2}$ and angle B= 30°. Find the absolute value of the difference between the areas of these triangles.

14*. In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2\frac{A}{2}$. If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C respectively, then prove that

(i) b + c = 2a

(ii) locus of points A is an ellipse

15. ABC is a triangle with incentre I. Let P and Q be the feet of perpendiculars from A to BI and CI respectively, then prove that $\frac{AP}{BI}+\frac{AQ}{CI}=\cot\frac{A}{2}$

16. Let ΔABC be equilateral. On side AB produced, we chose a point P such that A lies between P and B. Denote 'a' as the length of sides of ΔABC ; r_1 as the radius of incircle of ΔPAC; and r_2 as the exradius of ΔPBC with respect to side BC. Then find the value of $(r_1 + r_2)$

17. The diagonals of a parallelogram are inclined to each other at an angle of 45°, while its sides a and b (a > b) are inclined to each other at an angle of 30°. Find the value of $\frac{a}{b}$.

18. Let ABC be a triangle of area Δ and A' B' C' be the triangle formed by the altitudes of ΔABC as its sides with are Δ' and A'' B'' C'' be the triangle formed by the altitudes of ΔA'B'C' as its sides with area Δ''. If $\Delta' = 30$ and $\Delta'' = 20$ then find the value of Δ .

19. In a right triangle ABC, right angle at A. The radius of the inscribed circle is 2 cm. Radius of the circle touching the side BC and also sides AB and AC produced is 15 cm. Find the length of the side BC measured in cm.

20. If p_1, p_2, p_3 are the altitudes of a triangle which circumscribes a circle of diameter $\frac{4}{3}$ units, then find the least value of $(p_1 + p_2 + p_3)$

Answers

5.	$\angle$ ABD = 30°	**10.**	$\angle$ B = 113°	**13.**	4	**16.**	$\frac{\sqrt{3}}{2}a$
17.	$\frac{\sqrt{5}+1}{2}$	**18.**	45	**19.**	13	**20.**	6

Self Assessment Paper

JEE ADVANCED

Maximum Marks : 62 **Total Time : 1:00 Hr**

SECTION-1 : ONE OPTION CORRECT (Marks - 12)

1. In ΔABC, if AB = 10, BC = 8, CA = 12 and point D is taken on AB such that AD : DB is 3 : 2, then the length of CD is

(A) $\sqrt{24}$ (B) $\sqrt{36}$ (C) $\sqrt{84}$ (D) $\sqrt{72}$

2. Given a ΔOAB has sides OA = 7, AB = 11 and OB = 14. T is the mid point of OA and a point S is taken on BT dividing it in the ratio 2 : 1 then area of ΔSOT is

(A) $2\sqrt{5}$ (B) $2\sqrt{10}$ (C) $12\sqrt{10}$ (D) $6\sqrt{10}$

3. A point 'O' is situated on a circle of radius R and with centre O, another circle of radius $\frac{3R}{2}$ is described. Inside the crescent shaped are intercepted between these circles, a circle of radius R/8 is placed. If the same circle moves in contact with the original circle of radius R, then find the length of the arc described by its centre in moving from one extreme position to the other.

(A) $\frac{7\pi R}{12}$ (B) $\frac{5\pi R}{12}$ (C) $\frac{7\pi R}{10}$ (D) $\frac{5\pi R}{9}$

4. In ΔABC, if $\frac{s-a}{11}=\frac{s-b}{12}=\frac{s-c}{13}$, then $\tan^2\frac{A}{2}$ is equal to

[All symbols used has unsual meaning in triangle ABC.]

(A) $\frac{143}{342}$ (B) $\frac{13}{33}$ (C) $\frac{11}{39}$ (D) $\frac{12}{37}$

SECTION-2 : ONE OR MORE THAN ONE CORRECT (Marks - 32)

5. In ΔABC, which of the following is/are possible (where notations have usual meaning)

(A) sinA : sinB : sinC = 1 : 2 : 3 (B) $\Delta=\frac{bc}{4}$

(C) (a + b + c)(a + b – c) = 3ab (D) $b^2 - c^2 = aR$

6. In ΔABC with usual notations if $a^2 + 4 + b^2 + 4c^2 = 4a + 4bc$, then

(A) length of internal angle bisector through A is $\frac{b}{3}\cos\frac{A}{2}$

(B) $\frac{\sin B}{\sin C} = 2$

(C) $\frac{\sin B}{\sin C} = \frac{1}{2}$

(D) length of internal angle bisector through A is $\frac{2b}{3}\cos\frac{A}{2}$

7. If the sides of a triangle are in A.P. with common difference 1 and whose circumradius is $\frac{8}{\sqrt{15}}$, then which of the following can be side(s) of a triangle

(A) 2 (B) 3 (C) 4 (D) 5

on solving we get a =3

8. A triangle ABC is such that a = 3, b = 4 and area of triangle is maximum, then -

(A) Distance between orthocentre & circumcentre is 5.

(B) Inradius of ΔABC is 1.

(C) Perpendicular distance of orthocentre from side AB is $\frac{12}{5}$

(D) Sum of ex-radii of ΔABC is 11.

9. Consider an isosceles triangle ABC, whose none of the angle is right angle. If a & tanB are rational and b = c , then which of the following is/are always rational ?

(A) tanA (B) altitude through vertex A

(C) $\sin\frac{A}{2}$ (D) $\cos\frac{A}{2}$

10. In a triangle if the length of two longer sides are 8 and 7 and its angles are in A.P., then smaller side can be-

(A) 3 (B) 4 (C) 5 (D) 6

11. Let P is an interior point of ΔABC for which $\angle A = 45°$, $\angle B = 60°$ & $\angle C = 75°$, then PA : PB : PC is -

(A) 1 : 1 : 1 if P is the circumcentre

(B) $2:\sqrt{2}:(\sqrt{3}-1)$ if P is the orthocentre

(C) $\text{cosec}\,22\frac{1}{2}^{\circ} : \text{cosec}\,30^{\circ} : \text{cosec}\,37\frac{1}{2}^{\circ}$ if P is the incentre

(D) $2:\sqrt{2}:(\sqrt{3}+1)$ if P is the orthocentre

12. Let ABC & ABC' are two non-congruent triangles such that AB = 5, BC = 4 = BC' and $\angle A = \tan^{-1}\left(\frac{3}{4}\right)$, then-

(A) $|AC - AC'| = 2\sqrt{7}$

(B) Non-negative difference between perimeters of two triangles is $2\sqrt{7}$

(C) $\sin C = \sin C' = \frac{3}{4}$

(D) Both triangles are acute angled triangle

SECTION-3 : NUMERICAL VALUE TYPE (Marks - 18)

13. In a triangle ABC, BD is a median. If l (BD) = $\frac{\sqrt{3}}{4}$. l (AB) and $\angle$DBC = $\frac{\pi}{2}$. Determine the $\angle$ABC.

14. ABCD is a rhombus. the circumradii of ΔABD are 12.5 and 25 respectively. Find the area of rhombus.

15. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the area of the quadrilateral is $4\sqrt{3}$, find the sum of remaining two sides.

16. ABC is a triangle. Circles with radii as shown are drawn inside the triangle each touching two sides and the incircle. Find the radius of the incirlce of the ΔABC.

17. In a scalene triangle ABC the altitudes AB and CF are dropped from the vertices A and C to the sides BC and AB. The area of ΔABC is known to be equal to 18, the area of triangle BDF is equal to 2 and length of segment DF is equal to $2\sqrt{2}$. Find the radius of the circle circumscribed.

18. A circle is inscribed in a right triangle ABC, right angled at C. The circle is tangent to the segment AB at D and length of segments AD and DB are 7 and 13 respectively. Area of triangle ABC is equal to

Answers

1.	(D)	**2.**	(B)	**3.**	(A)	**4.**	(B)
5.	(B,C,D)	**6.**	(B,D)	**7.**	(A,B,C)	**8.**	(B,C,D)
9.	(A,B)	**10.**	(A,C)	**11.**	(A,B,C)	**12.**	(A,B,C)
13.	120°	**14.**	400	**15.**	5	**16.**	11
17.	4.5	**18.**	91				