CHAPTER – 5

CONTINUITY AND DIFFERENTIABILITY

DEFINITION OF CONTINUITY

(i) The continuity of a real function (f) on a subset of the real numbers is defined when the function exists at point c and is given as- $\lim_{x \to c} f(x) = f(c)$

(ii) A real function (f) is said to be continuous if it is continuous at every point in the domain of f.
Consider a function f(x), and the function is said to be continuous at every point in [a, b] including the endpoints a and b.

Continuity of "f" at a means, $\lim_{x \to a} f(x) = f(a)$

Continuity of "f" at b means, $\lim_{x \to b} f(x) = f(b)$

REMARK: A function f(x) fails to be continuous at x = a for nay of the following reasons.
(i) $\lim_{x \to a} f(x)$ exists but it is not equal to f(a)
(ii) $\lim_{x \to a} f(x)$ does not exist.
(iii) f is not defined at x = a i.e., f(a) does not exist.

Example

Consider the function $\begin{cases} 5-2x & ; x<1 \\ 3; & x=1 \\ x+2 ; & x>1 \end{cases}$ Check whether the function is continuous for all x.

Solution: Let us check the conditions for continuity.
Given f (1) = 3
LHL: $\lim_{x \to 1^-} f(x) = 5 - 2 = 3$
RHL: $\lim_{x \to 1^+} f(x) = 1 + 2 = 3$
LHL = RHL. Also f (1) exists.
So the function is continuous at x = 1

Example

Discuss the continuity of the function f given by f(x) = $x^3 + x^2 - 1$

Solution: Clearly f is defined at every real number c and its value at c is $c^3 + c^2 - 1$

Thus $\lim_{x \to c} f(x) = \lim_{x \to c}(x^3 + x^2 - 1) = c^3 + c^2 - 1$

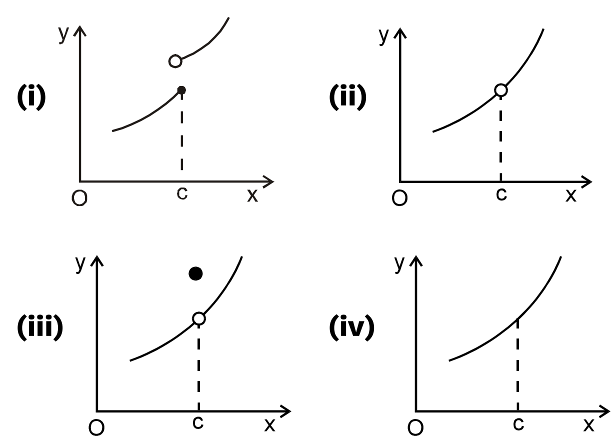
Thus $\lim_{x \to c} f(x) = f(c)$, and hence f is continuous at every real number. This means f is a continuous function.

CONTINUITY AND DERIVABILITY

A function f(x) is said to be continuous at x = c, if $\lim_{x \to c}$ f(x) = f(c) i.e. f is continuous at x =c

if $\lim_{h \to 0^+}$ f(c – h) = $\lim_{h \to 0^+}$ f(c+ h) = f(c).

If a function f(x) is continuous at x = c, the graph of f(x) at the corresponding point (c, f(c)) will not be broken. But if f(x) is discontinuous at x = c, the graph will be broken when x = c



(i), (ii) and (iii) are discontinuous at x = c) (iv) is continuous at x = c)

A function f can be discontinuous due to any of the following three reasons:

(i) $\lim_{x \to c} f(x)$ does not exist i.e. $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$ **[figure (i)]**

(ii) f(x) is not defined at x = c **[figure (ii)]**

(iii) $\lim_{x \to c} f(x) \neq f(c)$ **[figure (iii)]**

Geometrically, the graph of the function will exhibit a break at x= c.

ALGEBRA OF CONTINUOUS FUNCTIONS

THEOREM 1: Let f and g be two real functions , continuous at x = a .Let α be real number .Then,

(i) f + g is continuous at x = a
(ii) f – g is continuous at x = a
(iii) αf is continuous at x = a
(iv) f.g is continuous at x = a
(v) $\frac{1}{f}$ is continuous at x = a , provided that f(a) ≠ 0
(vi) $\frac{f}{g}$ is continuous at x = a , provided that g (a) ≠ 0

THEOREM 2: Let f and g be real functions such that fog is defined .If g is continuous at x =a and f is continuous at g(a) then fog is continuous at x= a

Example

Prove that f(x) = tan x is a continuous function.

Solution: Given,
f(x) = tan x
Since, we know that, by trigonometry formulas,
tan x = sin x/cos x
Thus,
f(x) = sin x/cos x
Now, this function is defined for all real numbers and cos x ≠ 0, x ≠ (2n+1)$\frac{\pi}{2}$.

Sin x and cos x both are continuous functions, therefore, tan x is also continuous, since it is the quotient of sin x/cos x.

CONTINUITY ON AN INTERVAL

CONTINUITY ON AN OPEN INTERVAL : A function f(x) is said to be continuous on an open interval (a ,b) iff it is continuous at every point on the interval (a ,b).

CONTINUITY ON AN CLOSED INTERVAL : A function f(x) is said to be continuous on a closed interval [a ,b] iff (i) f is continuous on (a , b) (ii) $\lim_{x \to a^+} f(x) = f(a)$ and , (iii) $\lim_{x \to b^-} f(x) = f(b)$

CONTINUOUS FUNCTION: A function f(x) is said to be continuous , if it is continuous at each point of its domain.

EVERTWHERE CONTINUOUS FUNCTION: A function f(x) is said to be everywhere continuous if it is continuous on the entire real line (-∞ , ∞).

THEOREM 1: If f and g are two continuous functions on their common domain D , then

(i) f + g is continuous on D
(ii) f – g is continuous on D
(iii) f.g is continuous on D
(iv) αf is continuous on D, where α is any real number.
(v) $\frac{f}{g}$ is continuous on D- { x : g(x) ≠ 0}
(vi) $\frac{1}{f}$ is continuous on D – {x : f(x) ≠ 0 }

THEOREM 2: The composition of two continuous functions is a continuous function.

THEOREM 3: If f is continuous on its domain D , then |f| is also continuous on D.

REMARK: The converse of the above theorem may not be true .

Example

f(x) = $\begin{cases} 1 , if\ x \in Z \\ -1 , if\ x \in R - Z \end{cases}$

Let a be an arbitrary integer , Then $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} -1 = -1$

$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} -1 = -1$and f(1) =1

∴ $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$

So, f is discontinuous at x = a
Now , |f|(x)=|f(x)| = 1 for all x ∈ R . So, |f| is a constant function and hence , it is everywhere continuous.

THEOREM 4: A constant function is everywhere continuous.

THEOREM 5: The identity function is everywhere continuous.

THEOREM 6: A polynomial function is everywhere continuous.

COROLLARY: Every rational function is continuous at every point in its domain.

THEOREM 7: The modulus function is everywhere continuous.

THEOREM 8: The exponential function a^x , a> 0 is everywhere continuous.

COROLLARY: e^x is everywhere continuous.

THEOREM 9: The logarithmic function is continuous in its domain.

THEOREM 10: The sine function is everywhere continuous.

THEOREM 11: The cosine function is everywhere continuous.

THEOREM 12: The tangent function is continuous in its domain.

THEOREM 13: The cosecant , secant , cotangent are continuous in its domain.

REMARK: All inverse trigonometric functions are continuous I their respective domain.

Example

Prove that f(x) = $\sqrt{|x|-x}$ is continuous for all x ≥ **0**

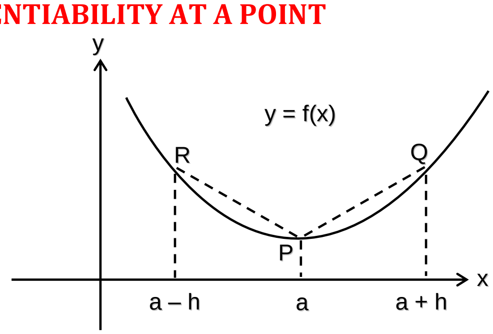
Solution: Let g (x) = |x| -x and h(x) = $\sqrt{x}$ .Clearly , domain(g) = R and domain (h) = [0 , ∞). Also , g(x) and h (x) are continuous in their domains.

We observe that

Domain (hog) = { x∈ $\boldsymbol{Domain\,(g)}: \boldsymbol{g(x)} \in$ $\boldsymbol{Domain\,(h)}$}

$\Rightarrow \boldsymbol{Domain(hog)} = \{\boldsymbol{x \in R : |x| - x \in [0\,,\infty)}\} = \{\boldsymbol{x \in R : x\ \geq 0}\} = [\boldsymbol{0}\,,\infty)$

DIFFRENTIABILITY AT A POINT



(i) The right hand derivative (R.H.D) of f(x) at x = a denoted by f ′(a⁺) is defined by slop of PQ

∴ R.H.D. = f ′(a^+) = $\underset{h\to 0^+}{\text{Limit}}\frac{f(a+h)-f(a)}{h}$, provided the limit exists.

(ii) The left hand derivative (L.H.D) of f(x) at x = a denoted by f′(a^-) is defined by slop of PR

∴ L.H.D. = f ′ (a^-) = $\underset{h\to 0^+}{\text{Limit}}\frac{f(a-h)-f(a)}{-h}$, provided the limit exists.

A function f(x) is said to be differentiable at x = a if $f'(a^+)$ = f ′ (a^-) = finite

By definition f ′(a) = $\underset{h\to 0^+}{\text{Limit}}\frac{f(a+h)-f(a)}{h}$

Example

A function f (x) is defined f(x) = $\begin{cases} -x^2\,; x \leq 0 \\ 5x-4\,; 0 < x \leq 1 \\ 4x^2-3x\,; 1 < x \end{cases}$

Discuss the differentiability off(x).

Solution: The critical points for this function are x = 0, 1 At x =0

L.H.D =$\lim\limits_{h\to 0}\frac{-(-h)^2-0}{-h} = 0$

R.H.D = $\lim\limits_{h\to 0}\frac{5h-4}{h} = -\infty$

Therefore, this function is non-differentiable at x = 0. At x = 1

L.H.D =$\lim\limits_{h\to 0}\frac{f(1-h)-f(1)}{h}$

=$\lim\limits_{h\to 0}\frac{5(1-h)-4-1}{-h} = 5$

R.H.D =$\lim\limits_{h\to 0}\frac{f(1+h)-f(1)}{h}$

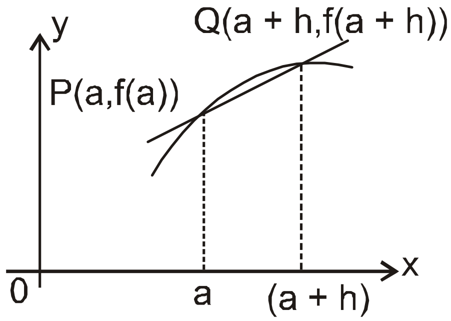
= $\lim\limits_{h\to 0}\frac{4(1+h)^2-3(1+h)-1}{h} = 5$

Since LHL = RHL and LHD = RHD, f (x) is differentiable at x = 1

CONCEPT OF TANGENT AND ITS ASSOCIATION WITH DERIVABILITY

Tangent :- The tangent is defined as the limiting case of a chord or a secant.

Slope of the line joining (a,f(a)) and (a + h, f(a + h)) = $\frac{f(a+h)-f(a)}{h}$



Slope of tangent at P = $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

The tangent to the graph of a continuous function f at the point P(a, f(a)) is

(i) the line through P with slope $f'(a)$ if $f'(a)$ exists ;

(ii) the line x = a if L.H.D. and R.H.D. both are either ∞ or $-\infty$.

If neither (i) nor (ii) holds then the graph of f does not have a tangent at the point P.

In case (i) the equation of tangent is $y - f(a) = f'(a)(x - a)$.

In case (ii) it is x = a

Note: (i) Tangent is also defined as the line joining two infinitesimally close points on a curve.

(ii) A function is said to be derivable at x = a if there exist a tangent of finite slope at that point.$f'(a^+) = f'(a^-)$ = finite value

(iii) $y = x^3$ has x-axis as tangent at origin.

(iv) y = |x| does not have tangent at x = 0 as L.H.D

RELATION BETWEEN DIFFERENTIABILITY AND CONTINUITY

(i) If $f'(a)$ exists, then f(x) is continuous at x = a.

(ii) If f(x) is differentiable at every point of its domain of definition, then it is continuous in that domain.

Note: The converse of the above result is not true i.e. "If 'f' is continuous at x = a, then 'f' is differentiable at x = a is not true.

e.g. the functions f(x) =| x – 2 | is continuous at x = 2 but not differentiable at x = 2.

If f(x) is a function such that R.H.D = $f'(a^+) = \lambda$ and L.H.D. = $f'(a^-) = m$.

Case - I

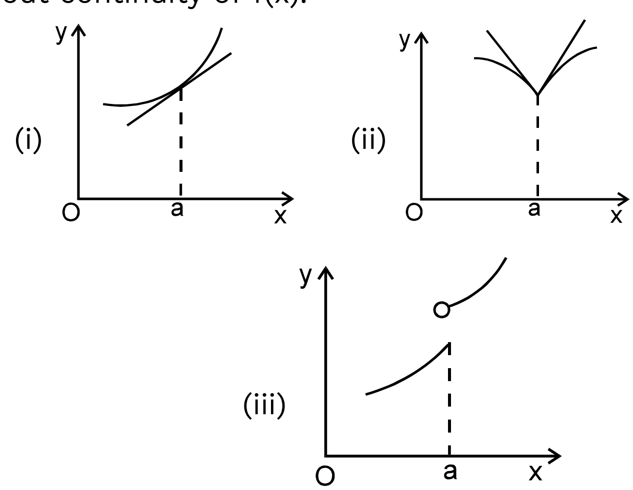
If λ = m = some finite value, then the function f(x) is differentiable as well as continuous.

Case - II

if$\lambda \neq$ m = but both have some finite value, then the function f(x) is non differentiable but it is continuous.

Case - III

If at least one of the λ or m is infinite, then the function is non differentiable but we can not say about continuity of f(x).



continuous and differentiable |continuous but not differentiable| neither continuous nor differentiable

THEOREM : If a function f is differentiable at a point c, then it is also continuous at that point but converse need not to be true.

Example

Take the function: f(x)=x^2+8x, the domain is all real numbers.

Solution:

Take the derivative:

f′(x)=2x+8

The derivative function exists at all points on the domain, so it is safe to say that f(x)=x^2+8x is differentiable and continuous on all real numbers.

Example

consider a function f(x)= |x| is continuous on all real numbers but not differentiable at x= 0

DIFFERNTIABILITY IN A SET

A function f(x) defined on an open interval (a , b) is said to be differentiable or derivable in open interval (a ,b) if it is differentiable at each point of (a ,b)

SOME USEFUL RESULTS ON DIFFERNTIABILITY

(i) Every polynomial function is differentiable at each $x \in R$.

(ii) The exponential function a^x , a > 0 is differentiable at each $x \in R$.

(iii) Every constant function is differentiable at each $x \in R$.

(iv) The logarithmic function is differentiable at each point in its domain.

(v) Trigonometric and inverse – trigonometric functions are differentiable in their respective domains.

(vi) The sum , difference , product and quotient of two differentiable functions is differentiable.

(vii) The composition of differentiable function is a differentiable function.

(viii) If a function f(x) is differentiable at every point in its domain , then $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ or $\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$ is called the derivative or differentiation of f at x and is denoted by $f'(x)$ or $\frac{d\,f(x)}{dx}$

THEOREM (Chain Rule): Let f be a real-valued function which is a composite of two functions f(x) and g(x) ; i.e., f(g(x))

$\frac{d}{dx}$ (f(g(x)) = f' (g(x)) · g' (x)

Example

To find the derivative of $\frac{d}{dx}$(sin 2x), express sin 2x = f(g(x)), where f(x) = sin x and g(x) = 2x.

Then by the chain rule formula,

$\frac{d}{dx}$ (sin2x) = cos 2x · 2 = 2 cos 2x

ALGEBRA OF DERIVATIVE:

Consider two functions f(x) and g(x) whose derivatives are in the same domain, then

Derivative of sum of two functions is same as the sum of derivatives of the functions

d(f(x) +g(x))/dx = d(f(x))/dx + d(g(x))/dx

Derivative of difference of two functions is difference of derivatives of the 2 functions

d(f(x)-g(x)) = d(f(x))/dx – d(g(x))/dx

Leibnitz Rule

While we perform differentiation of two functions either in multiplication and /or division we will use the rules mentioned below

(i) Derivative of product of 2 functions is given by the product rule.
Let ‘u’ and ‘v’ be two functions then (u v)’ = u’ v+ u v’
or d(f(x).g(x))= $\frac{d}{dx}\big(f(x)\big).g(x)+\frac{d}{dx}\big(g(x)\big).f(x)$

(ii) Derivative of quotient of two functions is given by the quotient rule (if the denominator is non zero)
$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{u.d(v)-v.d(u)}{v^2}$

IMPICIT FUNCTIONS

Implicit functions are functions where a specific variable cannot be expressed as a function of the other variable. A function that depends on more than one variable. Implicit Differentiation helps us compute the derivative of y with respect to x without solving the given equation for y, this can be achieved by using the chain rule which helps us express y as a function of x.

Implicit Differentiation can also be used to calculate the slope of a curve, as we cannot follow the direct procedure of differentiating the function y = f(x) and putting the value of the x-coordinate of the point in dy/dx to get the slope. Instead, we will have to follow the process of implicit differentiation and solve for dy/dx.

The method of implicit differentiation used here is a general technique to find the derivatives of unknown quantities.

DERIVATIVE OF IMPLICIT FUNCTIONS

Since the functions cannot be expressed in terms of one specific variable, we have to follow a different method to find the derivative of the implicit function : While computing the derivative of the Implicit function, our aim is to solve for **dy/dx** or any higher-order derivatives depending on the function. To solve dy/dx in terms of x and y, we have to follow certain steps:

Steps to compute the derivative of an implicit function

- Given an implicit function with the dependent variable y and the independent variable x (or the other way around).
- Differentiate the entire equation with respect to the independent variable (it could be x or y).
  After differentiating, we need to apply the chain rule of differentiation.
- Solve the resultant equation for dy/dx (or dx/dy likewise) or differentiate again if the higher-order derivatives are needed.

Example

Find $\frac{dy}{dx}$, if y +sin y = cos x

Solution: We differentiate the relationship directly e.r.t. to x , i.e.,

$\frac{dy}{dx} + \frac{d}{dx}(siny) = \frac{d}{dx}(cosx)$

Which implies using chain rule

$\frac{dy}{dx} + cosy.\frac{dy}{dx} = -sinx$

$\frac{dy}{dx} = \frac{-sinx}{1+cosy}$; y≠ (2n+1)π.

DERIVATIVE OF INVERSE TRIGONOMETRIC FUNCTIONS

Inverse trigonometric functions are continuous functions.

Example

Find the derivative of f given by f(x)= $\sin^{-1} x$ assuming it exists.

Solution: Let y = $\sin^{-1} x$ Then x = sin y

Differentiating both sides w. r. t. x, we get

1 = cos y $\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$

Observe that this is defined only for cos y ≠ 0 i.e., $\sin^{-1} x \neq \frac{-\pi}{2}, \frac{\pi}{2}$ i.e., x ≠ -1,1i.e., x belongs to (-1, 1)

To make this result a bit more attractive, we carry out the following manipulation.

Recall that for x belongs to (-1, 1) ,$\sin(\sin^{-1} x) = x$ and hence $cos^2 y = 1 - sin^2 y = 1 - (\sin(\sin^{-1} x))^2$ = 1 - x^2

Also , since y belongs to $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, cos y is positive and hence cos y = $\sqrt{1-x^2}$

Thus, for x belongs to (-1,1)

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

SECOND ORDER DERIVATIVE

Let y = f(x) be a given function , then $\frac{dy}{dx} = f'(x)$is called the first derivative of y or f(x) and $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the second order derivative of y w.r.t , x and it is denoted by $\frac{d^2y}{dx^2}$ or y''.

INTERMEDIATE VALUE THEOREM

A function f which is continuous in $[a, b]$ possesses the following properties:

(i) If f(a) & f(b) possess opposite signs, then there exists at least one solution of the equation f(x) = 0 in the open interval (a, b).

(ii) If K is any real number between f(a) & f(b), then there exists at least one solution of the equation f(x) = K in the open interval (a, b).

ROLLE'S THEOREM:

Let f : [a, b] → R be continuous on [a, b] and differentiable on (a, b), such that f(a) = f(b), where a and b are some real numbers. Then there exists some c in (a, b) such that f'(c) = 0.

Example

Verify Rolle's theorem for the function $y = x^2 + 2$, a = – 2 and b = 2.

Solution: From the definition of Rolle's theorem, the function $y = x^2 + 2$ is continuous in [– 2, 2] and differentiable in (– 2, 2).

From the given,

$f(x) = x^2 + 2$

$f(-2) = (-2)^2 + 2 = 4 + 2 = 6$

$f(2) = (2)^2 + 2 = 4 + 2= 6$

Thus, f(– 2) = f(2) = 6

Hence, the function f(x) is continuous in [-2, 2].

Now, f'(x) = 2x

Rolle's theorem states that there is a point c ∈ (– 2, 2) such that f′(c) = 0.

At c = 0, f′(c) = 2(0) = 0, where c = 0 ∈ (– 2, 2)

Hence Rolle's theorem is verified.

MEAN VALUE THEOREM:

Let f : [a, b] → R be continuous function on [a, b]and differentiable on (a, b). Then, there exists at least one number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Example

Verify if the function $f(x) = x^2 + 1$ satisfies mean value theorem in the interval [1, 4]. If so, find the value of 'c'.

Solution:

The given function is $f(x) = x^2 + 1$. To verify the mean value theorem, the function $f(x) = x^2 + 1$ must be continuous in [1, 4] and differentiable in (1, 4).

Since f(x) is a polynomial function, both of the above conditions hold true.

The derivative f'(x) = 2x (power rule) is defined in the interval (1, 4)

$f(1) = 1^2 + 1 = 1 + 1 = 2$

$f(4) = 4^2 + 1 = 16 + 1 = 17$

f'(c) = [f(4) - f(1)] / (4 - 1)

= (17 - 2) / (4 - 1) = 15/3 = 5

f'(c) = 5

PARAMETRIC EQUATION

When a group of quantities of one or more independent variables formed as functions, then they are called parametric equations. These are used to represent the coordinates of a point for any geometrical object like curve, surfaces, etc., where the equations of these objects are said to be a parametric representation of that particular object.

The general form of parametric equations are:
x = cos t
y = sin t

Here, (x, y) = (cos t, sin t) form a parametric form of the unit circle such that t is the parameter and (x, y) is the points on the unit circle.

Example

Find the value of $\frac{dy}{dx}$ for the following which are expressed in the parametric form
x = sin t , y = t^2

Solution: Since this function is represented in parametric function format beforehand therefore we need to find out the value $\frac{dy}{dt}$ *and* $\frac{dx}{dt}$

$\frac{dy}{dt} = 2t$ *and* $\frac{dx}{dt} = cost$

Now with the help of chain rule we can write, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Now substituting the value of $\frac{dy}{dt}$ *and* $\frac{dx}{dt}$ into the above equation we can find the derivative ofw.r.t x $\frac{dy}{dx} = \frac{2t}{\cos t}$

This is the required solution of the differentiation of the parametric equation.

STANDARD RESULTS

(i) $\frac{d}{dx}(x^n)$ = nx^{n-1} ∀ x ∈ R

(ii) $\frac{d}{dx}(ax+b)^n$ = n.$(ax+b)^{n-1}$. a ∀ x ∈ R

(iii) (a)The derivative of e^x with respect to x is e^x; i.e., d/dx(e^x) = e^x.

(iv) (b) The derivative of log x with respect to x is 1/x.

(v) i.e., d/dx(log x) =1/x

MCQ QUESTIONS

Q1. The function $f(x) = |x - 5|$ is:

(a) Continuous everywhere but not at $x = 5$.

(b) Continuous everywhere

(c) Continuous everywhere but not at $x = 5$ and $x = 0$

(d) None

Q2. The function $f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$ is continuous at

(a) 0,2 (b) 1,4

(c) 5,1 (d) 0,1

Q3. Points of discontinuity of f where f is defined by:

$f(x) = \begin{cases} 2x + 3, x \le 2 \\ 2x - 3, x > 2 \end{cases}$

(a) $-\frac{3}{2}, \frac{3}{2}$ (b) 2

(c) 0 (d) 3, -3

Q4. All points of discontinuity of f where f is defined by:

$f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$

(a) $1, 0, -1$

(b) $(-\infty, -1) \cup (1, \infty)$

(c) 0

(d) 1, -1

Q5. The following function is continuous in which interval

$f(x) = \begin{cases} 3, & \text{if } x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \end{cases}$

(a) $(-\infty, 1) \cup (1,3) \cup (3, \infty)$

(b) $(-\infty, -1) \cup (-1,3) \cup (3, \infty)$

(c) $(-\infty, -3) \cup (-3,1) \cup (1, \infty)$

(d) None

Q6. Find the relationship between a and b if the following function is continuous at $x = 3$.

$f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$

(a) $a + b = \frac{2}{3}$

(b) $a - b = \frac{-2}{3}$

(c) $a - b = \frac{2}{3}$

(d) $a - b = \frac{3}{2}$

Q7. For what value of λ is the function defined by

$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$ continuous at x = 0 ?

(a) 0

(b) 1

(c) 2

(d) Not possible for any value

Q8. The function defined by $g(x) = x - [x]$ is discontinuous at

(a) All integral points (b) All decimal points

(c) 0 (d) 1

Q9. The function $f(x) = \sin x + \cos x$ is continuous at:

(a) All integral points

(b) Discontinuous at integral points only

(c) Continuous everywhere

(d) Discontinuous at all the multiples of π .

Q10. The secant function is continuous at:

(a) All integral points

(b) Discontinuous at integral points only

(c) Continuous everywhere

(d) Discontinuous at all the multiples of π .

Q11. The value of c in Rolle's theorem for the function, f(x) = sin 2x in [0, π/2] is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$

Q12. The function f(x) = [ln(1+ax)-ln(1-bx)]/x, not defined at x=0. The value should be assigned to f at x=0, so that it is continuous at x =0, is

(a) a (b) a - b

(c) a + b (d) None of these

Q13. Find the values of k so that the function f is continuous at the indicated point

$f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$ at $x = 2$

(a) $\frac{3}{4}$ (b) $\frac{2}{5}$

(c) $\frac{1}{2}$ (d) $\frac{4}{3}$

Q14. Find the values of k so that the function f is continuous at the indicated point"

$$f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$$

(a) $\frac{2}{\pi}$

(b) $\frac{-2}{\pi}$

(c) $\frac{-1}{\pi}$

(d) $\frac{1}{\pi}$

Q15. What are the conditions to satisfy Lagrange's mean value theorem

(a) f is continuous on [a , b]

(b) f is differentiable on (a, b)

(c) f is differentiable and continuous on (a , b)

(d) None of these

Q16. For the function f(x) = $e^x, a = 0, b = 1$, then find the value of c in the mean value theorem.

(a) $\log(e-1)$

(b) $\log(e+1)$

(c) $\log(e)$

(d) None of these

Q17. Differentiate $sin^2x\ w.r.t.\ e^{cosx}$

(a) $\frac{sinx}{e^{cosx}}$

(b) $\frac{-2cosx}{e^{cosx}}$

(c) $\frac{cosx}{e^{cosx}}$

(d) $\frac{sin2x}{e^{cosx}}$

Q18. Find $\frac{dy}{dx}$ in$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

(a) $\frac{2}{\sqrt{1-x^2}}$

(b) $\frac{-2}{\sqrt{1+x^2}}$

(c) $\frac{2}{\sqrt{1+x^2}}$

(d) $\frac{-2}{\sqrt{1-x^2}}$

Q19. Iff and g differentiable function in (0,1) satisfying f(0)= 2= g(1), g(0)= 0 and f(1) = 6 then for some c belongs to (0,1)

(a) $2f'(c) = g'(c)$

(b) $2f'(c) = 3g'(c)$

(c) $f'(c) = g'(c)$

(d) $f'(c) = 2g'(c)$

Q20. Let f(x) = |x| then

(a) Not continuous at x= 0

(b) Differentiable at x = 0

(c) Continuous at x = 0 but not differentiable at x = 0

(d) None of these

Q21. Which one of the following is true

(a) Every continuous function is differentiable

(b) Every differentiable function is continuous

(c) F(x)= cos (x^2) is discontinuous function

(d) None of these

Q22. Find the values of a and b such that the function defined by f(x) = $\begin{cases} 5 if\ x \le 2 \\ ax + b\ if\ 2 < x < 10 \\ 21\ if\ x \ge 10 \end{cases}$ is a continuous function.

(a) a = 1 ,b = 3

(b) a =2 , b =1

(c) a =1 , b = 2

(d) a = 3 , b = 1

Q23. The function g(x) = [x] , where [x] denotes the greatest integer function , is continuous at

(a) 4

(b) -2

(c) 1

(d) 1.5

Q24. Differentiation of $x^{\sin x} + (\sin\ x)^{\cos x}$

(a) $x^{\sin x}\left(\frac{\sin x}{x} + \cos\ x\log\ x\right) + (\sin\ x)^{\cos x}(\cot\ x \cdot \cos\ x - \sin\ x\log\ \sin\ x)$

(b) -$x^{\sin x}\left(\frac{\sin x}{x} + \cos\ x\log\ x\right) + (\sin\ x)^{\cos x}(\cot\ x \cdot \cos\ x - \sin\ x\log\ \sin\ x)$

(c) $-x^{\sin x}\left(\frac{\sin x}{x} + \cos x\log x\right) - (\sin\ x)^{\cos x}(\cot\ x \cdot \cos\ x - \sin\ x\log\ \sin\ x)$

(d) $x^{\sin x}\left(\frac{\sin x}{x} + \cos\ x\log\ x\right) - (\sin\ x)^{\cos x}(\cot\ x \cdot \cos\ x - \sin\ x\log\ \sin\ x)$

Q25. If x and y are connected parametrically by the equations given by $x = 2at^2, y = at^4$ find dy/dx.
(a) t^2 (b) $-t^2$
(c) t^3 (d) $-t^3$

Q26. If x and y are connected parametrically by the equations given by $x = \sin t, y = \cos 2t$ find dy/dx.
(a) $-4\cos t$ (b) $4\cos t$
(c) $-4\sin t$ (d) $4\sin t$

Q27. Which of the following is differentiable at x = 0
(a) Cos (|x|) +|x|
(b) Cos (|x|)- |x|
(c) Sin(|x|) +|x|
(d) Sin (|x|)-|x|

Q28. Let f(x) = $e^{|x|}$ .Which of the following is false.
(a) Differentiable at x = 0
(b) Continuous at x= 0
(c) Continuous at x = 0 and not differentiable at x = 0
(d) None of these

Q29. Find the second order derivatives of the function $\log x$
(a) $\frac{-1}{x^2}$
(b) $\frac{1}{x^2}$
(c) $\frac{-1}{x}$
(d) $\frac{-2}{x^2}$

Q30. If $y = e^{a\cos^{-1}x}, -1 \le x \le 1$then
(a) $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + a^2y = 0$
(b) $(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - a^2y = 0$
(c) $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$
(d) $(1+x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

SUBJECTIVE QUESTIONS

Q1. Evaluate : $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Q2. If f(x) =$\frac{x+3}{x-3}$and g(x) =$\frac{1}{x-7}$, then discuss the continuity off(x), g(x) and fog (x).

Q3. Show that the function f(x) = (x – a)^2 (x – b)^2 + x take the value $\frac{a+b}{2}$ for some value of x ∈ [a, b].

Q4. Let f(x) =$\lim_{n\to\infty}\frac{1}{1+n\sin^2 x}$, then find f $\left(\frac{\pi}{4}\right)$ and also comment on the continuity at x = 0

Q5. Comment on the differentiability of f(x) =$\begin{cases} x & x<1 \\ x^2 & x\ge 1\end{cases}$at x = 1.

NUMERICAL TYPE QUESTIONS

Q1. $\lim_{x\to 0}\frac{e^x+e^{-x}-2\cos x}{x\sin x}$_____.

Q2. If f(x) = $\begin{cases}\frac{x^2-(a+2)x+a}{x-2} & x\ne 2 \\ 2, & x=2\end{cases}$ is continuous at x = 2, then a is equal to _____.

Q3. If the function f(x) = $\begin{cases}\frac{k\cos x}{\pi-2x}, when\ x\ne\frac{\pi}{2} \\ 3, when\ x=\frac{\pi}{2}\end{cases}$ be continuous at x = $\frac{\pi}{2}$, then k =_____.

Q4. If f(x) = $\begin{cases} e^x ; x<1 \\ a-bx ; x\ge 1\end{cases}$ is differentiable for $x \in R$ then find value of $\left(a-\frac{b}{e}\right)$ _____.

Q5. If f(x) = $\begin{cases}\frac{1}{|x|} ; |x|\ge 1 \\ ax^2+b ; |x|<1\end{cases}$ is differentiable at every point of the domain, then the values of a + b is _______.

TRUE AND FALSE

Q1. Every continuous function is differentiable function.

Q2. $f: R \to R$ s. t. f(x) = $\frac{1}{x}$ is continuous at x = 0.

Q3. Let f : [a, b] → R be continuous on (a, b) and differentiable on (a, b), such that f(a) = f(b), where a and b are some real numbers. Then there exists some c in (a, b) such that f'(c) = 0.

Q4. If x = f(t) and y = g(t) , then $\frac{dy}{dx}$ is equal to $\frac{g'(t)}{f'(t)}$

Q5. Converse of the given theorem ' If f is continuous on its domain D , then |f| is also continuous on D' is true

ASSERTION AND REASON

Directions (Q1 – Q5) Each of these questions contains two statements: Assertion (A) and Reason (R) .Each of these questions also has four alternative choices , any one of which is the correct answer .You have to select one of the codes (a) , (b) , (c) and (d) given below

(a) A is true , R is true ; R is a correct explanation for A.

(b) A is true , R is true ; R is not a correct explanation for A.

(c) A is true , R is False.

(d) A is False , R is true

Q1. Assertion(A) : The function f(x) = $\sqrt[3]{x}$ is continous at all x except at x = 0.

Reason (R): The function f(x) = [x] is continuous at x = 299 , where [] is the greatest integer function.

Q2. Assertion(A) : f(x) is continuous at x = a , iff $\lim_{x\to a} f(x)$ exists and equals to f(a).

Reason (R): If f(x) is continuous at a point , then $\frac{1}{f(x)}$ is also continuous at the point.

Q3. Assertion(A) : Assertion (A) : The function defined by f(x) = cos (x^2) is a continuous functiom.

Reason (R): The sine function is continuous in its domain i.e., $x \in R$.

Q4. f (x) = [x -1] + [x – 2] , where [ ] denotes the greatest integer function.

Assertion(A) : f(x) is discontinuous at x = 2

Reason (R): f(x) is non derivable at x =2.

Q5. Assertion(A) : $\frac{d}{dx}e^{\sin x} = e^{\sin x}(\cos x)$

Reason (R): $\frac{d}{dx}e^{x} = e^{x}$

HOMEWORK (MCQ)

Q1. Evaluate : $\lim_{x\to 0} \frac{\tan x}{\tan 5x}$

(a) 5

(b) -5

(c) $\frac{1}{5}$

(d) - $\frac{1}{5}$

Q2. Evaluate $\lim_{x\to 0^+} x^x$

(a) 1

(b) 2

(c) 3

(d) 5

Q3. f $\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \forall$ x, y ∈ R and f(0) = 1 and f′(0) = – 1 and function is differentiable for all x, then find f(x).

(a) f(x) = 1-x

(b) f(x) = 1+ x

(c) f(x) = 1+ 2x

(d) f(x) = 1- 2x

Q4. If f(x) = $\begin{cases} 4x, & x < 0 \\ 1, & x = 0 \\ 3x^2, & x > 0 \end{cases}$, then function f is

(a) f is not continuous at x = 0

(b) f is continuous at x = 0

(c) f is not continuous at x=3

(d) None of these

Q5. If f(x) = $\begin{cases} 0, & x < 0 \\ x^2, & x \geq 0 \end{cases}$ then L.H.D. of f(x) at x = 0 is

(a) 1

(b) – 1

(c) 0

(d) 2

Q6. If f(x) = $\begin{cases} x^3-1 & x>1 \\ x-1 & ,x\le 1 \end{cases}$ then at x = 1, f(x) is

(a) continuous and differentiable
(b) continuous but not differentiable
(c) discontinuous and differentiable
(d) neither continuous nor differentiable

Q7. Which of the following is false ?

(a)f(x)= |x| at continuous at x = 0
(b)f(x) = $\frac{1}{x-1}$ is not continuous at x = 1
(c) f(x) = |x| is differentiable at x = 0
(d) f(x) = |x| is differentiable at x = 1

Q8. Function f(x) = $\begin{cases} x^2, & x\le 0 \\ 1, & 0<x\le 1 \\ 1/x, & x>1 \end{cases}$ is-

(a) Differentiable at x = 0, 1
(b) Differentiable only at x = 0
(c) Differentiable at only x = 1
(d) Not differentiable at x = 0, 1

Q9. If f(x) = $\begin{cases} x & \text{when } x\in Q \\ -x & \text{when } x\notin Q \end{cases}$, then $\ell\text{im}_{x\to 0}$ f(x) equals-

(a) 0
(b) 1
(c) – 1
(d) Does not exist

Q10. $\ell\text{im}_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$=

(a) 0
(b) 1
(c) ∞
(d) 2

SUBJECTIVE QUESTIONS

Q1. Differentiate cos {sin $(x)^2$} w. r. t. x.

Q2. Is it true that x = $e^{\log x}$ for all real x?

Q3. Use the formula $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$ to find $\lim_{x\to 0}\frac{2^x-1}{(1+x)^{\frac{1}{2}}-1}$

Q4. Let f(x) be a continuous and g(x) be a discontinuous function , prove that f(x) + g(x) is a discontinuous function.

Q5. The domain of the function $\sqrt{\frac{1}{\sqrt{x}}-\sqrt{x+1}}$ is

NUMERICAL TYPE QUESTIONS

Q1. Determine the value of the constant k so that the function f (x) = $\begin{cases} \frac{kx}{|x|}, if\ x<0 \\ 3, if\ x\ge 0 \end{cases}$ is continuous at x = 0_____.

Q2. Evaluate $\ell\text{im}_{x\to-\infty}\frac{\sqrt{3x^2+2}}{x-2}$___________.

Q3. The differentiability of f(x) = x |x| at _______.

Q4. If(x) = $\begin{cases} |x|-3; & |x|\ge 1 \\ -x^2-1; & |x|<1 \end{cases}$, write the sum of doubtful points of differentiability for f(x)_____.

Q5. Find the value of $\lim_{h\to 0}\frac{f(a+2h)-f(a-3h)}{h}$, if f′ (a) = 3_____.

TRUE AND FALSE

Q1. $\ell\text{im}_{x\to\infty}\frac{\ell nx}{x}$ has an indeterminate form of type $\frac{\infty}{\infty}$.

Q2. Expansion of ln (1+x) =$-x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots,$ $for -1<x\le 1$

Q3. A function f is said to be continuous in a closed interval [a, b] if f is continuous in the open interval (a, b).

Q4. If f(x) is not differentiable at x = a & g(x) is differentiable at x = a, then the product function F(x) = f(x) .g(x) can still be differentiable at x = a

Q5. If f(x) & g(x) both are non-differentiable at x = a, then the sum function F(x) = f(x)+ g(x) may be a differentiable function.

ASSERTION AND REASON

Directions (Q1 – Q5) Each of these questions contains two statements: Assertion (A) and Reason (R) .Each of these questions also has four alternative choices , any one of which is the correct answer .You have to select one of the codes (a) , (b) , (c) and (d) given below

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(b) A is true , R is true ; R is not a correct explanation for A.

(c) A is true , R is False.

(d) A is False , R is true

Q1 .**Assertion(A) :** |sin x| is continuous for all $x \in R$

Reason (R):sin x and |x| are continuous in R.

Q2. **Assertion(A) :** f(x) = $tan^2 x$ is continous at x = $\frac{\pi}{2}$

Reason (R): g(x) = x^2 is continuous at x = $\frac{\pi}{2}$

Q3. Consider the function f(x) = $\begin{cases} \frac{x^2+3x-10}{x-2}, if\ x \neq 2 \\ k, if\ x = 2 \end{cases}$ which is continuous at x = 2

Assertion(A) : The value of k is 0

Reason (R): f(x) is continuous at x = a , if $\lim_{x \to a} f(x) = f(a)$

Q4. **Assertion(A) :** | sin x | is continuous at x= 0

Reason (R): |sin x| iis differentiable at x = 0

Q5. **Assertion(A) :** If y = $\sin^{-1}\left(6x\sqrt{1-9x^2}\right)$, then $\frac{dy}{dx} = \frac{6}{\sqrt{1-9x^2}}$

Reason (R): $\sin^{-1}\left(6x\sqrt{1-9x^2}\right) = 3\sin^{-1}(2x)$

SOLUTIONS:

S1. **(b)** The modulus function is continuous everywhere.

S2. **(a)** The given function is continuous everywhere except at $x = 1$.

S3. **(b)** Left Hand limit $= \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x+3) = 2\times 2+3 = 7$

Right Hand limit $= \lim_{x\to 2+} f(x) = \lim_{x\to 2^+} (2x-3) = 2\times 2-3 = 1$

Hence, the given function is not continuous at $x = 2$.

S4. **(c)** We check the left-hand limit and right-hand limit of the given function at $x = 0$.

Left Hand limit $= \lim_{x\to 0^-} f(x) = \lim_{z\to 0^-} (-1) = -1$

Right Hand limit $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1) = 1$

As, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

Therefore, $\lim_{x\to 0} f(x)$ does not exist and f(x) is discontinuous at only x = 0.

S5. **(a)** In interval, $0 \le x \le 1, f(x) = 3$

Therefore, f is continuous in this interval.

At $x = 1$,

L.H.L. $= \lim_{x\to 1^-} f(x) = 3$ and R.H.L. $= \lim_{x\to 1^+} f(x) = 4$

As, L.H.L. $\neq$ R.H.L.

Therefore, f(x) is discontinuous at $\mathbf{x} = 1$.

At $x = 3$, L.H.L. $= \lim_{x\to 3^-} f(x) = 4$ and R.H.L. $= \lim_{x\to 3^+} f(x) = 5$

As, L.H.L. $\neq$ R.H.L.

Therefore, $f(x)$ is discontinuous at $x = 3$

Hence, f is discontinuous at $x = 1$ and $x = 3$.

S6. **(c)** Given the function is continuous at $x = 3$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax+1) = \lim_{h\to 0}\{a(3-h)+1\} = \lim_{h\to 0}(3a-ah+1) = 3a+1$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (bx+3) = \lim_{h\to 0}\{b(3+h)+3\} = \lim_{k\to 0}(3b+bh+3) = 3b+3$

Also $f(3) = 3a+1$

Therefore, $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = f(3)$

$\Rightarrow 3b+3 = 3a+1$

$\Rightarrow a-b = \frac{2}{3}$

S7. **(d)** Since f(x) is continuous at $x = 0$.

Therefore,

L.H.L.

$\lim_{x\to 0^-} f(x) = f(0) = \lambda(x^2-2x) = \lambda(0-0) = 0$

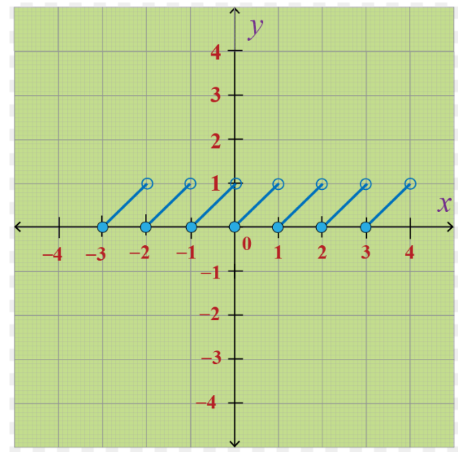
R.H.L.

$\lim_{x\to 0^+} f(x) = f(0) = 4x+1 = 4\times 0+1 = 1$

Here, L.H.L. $\neq$ R.H.L.

This implies $0 = 1$, which is not possible.

S8. (a) See the graph of g(x) = $x - [x]$ is



The function $g(x) = x - [x]$ is discontinuous at all integral points.

S9. (c) Let "a" be an arbitrary real number then

$\lim_{x\to a^-} f(x) \Rightarrow \lim_{h\to 0} f(a+h)$

$\lim_{h\to 0} f(a+h) = \lim_{h\to 0} \sin(a+h) + \cos(a+h)$

$= \lim_{h\to 0}(\sin a \,.\, \cos h + \cos a.\sin h + \cos a.\cos h - \sin a.\sin\ h)$

$= \sin a \,.\, \cos 0 + \cos a.\sin 0 + \cos a.\cos 0 - \sin a.\sin\ 0$

$\{$ As $\cos\ 0 = 1$ and $\sin\ 0 = 0\}$

$= \sin\ a + \cos\ a = f(a)$

Similarly,

$\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h) = \lim_{h\to 0} \sin(a-h) + \cos(a-h)$

$\lim_{h\to 0}(\sin a \,.\, \cos h - \cos a.\sin h + \cos a.\cos h + \sin a.\sin h)$

= sin(a).cos(0) – cos(a).sin(0) + cos(a).cos(0) + sin(a).sin(0)

= sin(a) + cos(a) = f(a)

Therefore, $f(x)$ is continuous at $x = a$.

As, "a" is an arbitrary real number, therefore, $f(x) = \sin\ x + \cos\ x$ is continuous.

S10. (c) Let say "a" be an arbitrary real number then

$f(x) = \sec x = \frac{1}{\cos x}$ and domain $x = \mathrm{R} - (2n+1)\frac{\pi}{2}, n \in \mathrm{I}$

$\Rightarrow \lim_{x\to a^+} \frac{1}{\cos x} = \lim_{h\to 0} \frac{1}{\cos(a+h)}$

$= \lim_{h\to 0} \frac{1}{(\cos a.\cosh - \sin a.\sin h)}$

$= \frac{1}{\cos\ a\cos\ 0 - \sin\ a\sin\ 0}$

$= \frac{1}{\cos\ a(1) - \sin\ a(0)} = \frac{1}{\cos a} = f(a)$

$\Rightarrow \lim_{x\to a^-} \frac{1}{\cos x} = \lim_{h\to 0} \frac{1}{\cos(a-h)}$

$= \lim_{h\to 0} \frac{1}{(\cos a.\cosh + \sin a.\sin h)}$

$= \frac{1}{\cos a \cos 0 + \sin a \sin 0} = \frac{1}{\cos a} = f(a)$

$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

Therefore function f is continuous at x = a

Since, "a" is an arbitrary real number, therefore, $f(x) = \sec\ x$ is continuous.

S11. (a) For the function, f(x) = sin 2x in [0, π/2], f'(c) = 0

Hence, 2 cos 2c =0

2c =π/2 c = π/4

S12. (c) For f(x) to be continuous at x =0,

f(0) = $\lim_{x\to 0}$f(x)

Therefore, f(0) = $\lim_{x\to 0}$[ln(1 + ax) - ln(1 - bx)]/x

$\lim_{x\to 0} \frac{(\ln(1+ax) - \ln(1-bx))}{x} = a + b$

Therefore, f (0) = a + b

S13. (a) Given function is

$$f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases}$$

$$\lim_{x \to 2^-} f(x) = \lim_{k \to 0} f(2+h) = 3$$

$$\lim_{x \to 2^-} f(x) = 3 \text{ and } f(2) = 3$$

$$k \times 2^2 = 3$$

$$\Rightarrow k = \frac{3}{4}$$

S14. (b) Given function is:

$$f(x) = \begin{cases} kx + 1, & \text{if } x \leq \pi \\ \cos\ x, & \text{if } x > \pi \end{cases}$$

$$\lim_{x \to \pi^+} f(x) = \lim_{h \to 0} f(\pi + h) = \lim_{h \to 0} \cos\ (\pi + h) = -\cos\ h = -\cos\ 0 = -1$$

$$\text{and } \lim_{x \to \pi^-} f(x) = \lim_{h \to 0} f(\pi - h) = \lim_{h \to 0}\ (\text{k}\pi + 1) = k\pi + 1$$

Again, As given function is continuous at $x = \pi$, we have

$$\lim_{x \to \pi^-} f(x) = \lim_{z \to \pi^+} f(x) = \lim_{x \to \pi} f(x)$$

$$\Rightarrow k\pi + 1 = -1$$

$$\Rightarrow k\pi = -2$$

$$\Rightarrow k = \frac{-2}{\pi}$$

S15. (c) Mean value theorem: Suppose f(x) is a function that satisfies below conditions:

1. f(x) is Continuous in [a, b]
2. f(x) is Differentiable in (a, b)

Then, there exists a number c, s. t. a < c < b and $f'(c) = \frac{f(b)-f(a)}{b-a}$

S16. (a)

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$e^c = \frac{e^b - e^a}{b-a}$$

$$e^c = \frac{e^1 - e^0}{1-0} = e - 1$$

$$\text{c} = \log(e-1)$$

S17. (b) Let u(x) = sin^2x and v(x)= e^{cosx}

Find $\frac{du}{dv} = \frac{du/dx}{dv/dx}$

Clearly $\frac{du}{dx} = 2sinx\ cosx$ and $\frac{dv}{dx} = e^{cosx}(-sinx) = -(sinx)e^{cosx}$

Thus, $\frac{du}{dv} = \frac{-2cosx}{e^{cosx}}$

S18. (d) Given, $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Put $x = \cos\ \theta$

$$y = \sec^{-1}\left(\frac{1}{2\cos^2\ \theta - 1}\right)$$

$$y = \sec^{-1}\left(\frac{1}{\cos\ 2\theta}\right)$$

$$y\& = \sec^{-1}(\sec\ 2\theta)$$

$$y = 2\theta = 2\cos^{-1}\ x$$

$$\Rightarrow \frac{dy}{dx} = 2 \cdot \frac{-1}{\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

S19. (d) Let h(x) = f(x) – 2g(x)…..(i)

h(0) = f(0) – 2g(0)

h(0) = 2 – 0 = 2

Now,

h(1) = f(1) -2g(1)

h(1) = 6 – 4 = 2

h(0) = h(1)

Hence, using Rolle's theorem

$h'(c) = 0, \textit{such that c belongs to } (0,1)$

Differentiating (i) w.r.t x at point c

$h'(x) = f'(x) - 2g'(x)$

$h'(c) = f'(c) - 2g'(c)$

$h'(0) = f'(c) - 2g'(c)$

$0 = f'(c) - 2g'(c)$

$f'(c) = 2g'(c)$

S20. (c) For continuity At x = 0

L.H.L $\lim_{x\to 0^-}(-x) = 0$ and R.H.L $\lim_{x\to 0^+} x = 0$

L.H.L = R. H.L = f (0)

Therefore, f(x) = |x| is continuous at x = 0

For differentiability

L.H.D $\lim_{x\to 0^-} \frac{-x-0}{x-0} = -1$ and R. H. D $\lim_{x\to 0^+} \frac{x-0}{x-0} = 1$

L.H.D ≠ R.H.D

Therefore, f(x) = |x| is not differentiable at x = 0

S21. (b) As we know that every differentiable function is continuous.

S22. (b) L.H.L $\lim_{x\to 2^-} 5 = 5$,R. H.L $\lim_{x\to 2^+} ax + b = 2a + b$and f(2) = 5

Since function is continuous at x = 5 then 2a+b = 5 …. (i)

L.H.L $\lim_{x\to 10^-}(ax + b) = 10a + b$, R.H.L $\lim_{x\to 10^+} 21 = 21$ and f(10)= 21

Since function is continuous at x = 10 then 10a + b = 21… (ii)

From (i) and (ii) then a = 2 and b = 1

S23. (d) F(x) = [x] is discontinuous at every integral points.

S24. (a) Let $y = x^{\sin x} + (\sin x)^{cos\, x}$

Put $u = x^{\sin x}$ and $v = (\sin\ x)^{\text{cosx}}$, we get $y = u + v$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ …(1)

Now $u = x^{\sin x}$

$\log\ u = \log\ x^{\sin x} = \sin\ x \log\ x$

$\frac{d}{dx}\log\ u = \frac{d}{dx}(\sin\ x\log\ x)$

$\frac{1}{u}\frac{du}{dx} = \sin\ x\frac{d}{dx}\log\ x + \log\ x\frac{d}{dx}\sin\ x$

$\frac{1}{u}\frac{du}{dx} = \sin\ x\left(\frac{1}{x}\right) + \log\ x(\cos\ x)$

$\frac{du}{dx} = u\left(\frac{\sin x}{x} + \cos\ x\log\ x\right)$

$\frac{du}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x\log x\right) \dots (2)$

Again $v = (\sin\ x)^{\cos x}$

$\log\ v = \log\,(\sin\ x)^{\cos x} = \cos\ x\log\ \sin\ x$

$\frac{d}{dx}\log\ v = \frac{d}{dx}[\cos\ x\log\,(\sin\ x)]$

$\frac{1}{v}\frac{dv}{dx} = \cos\ x\frac{d}{dx}\log\ \sin\ x + \log\ \sin\ x\frac{d}{dx}\cos\ x$

$\frac{1}{v}\frac{dv}{dx} = \cos\ x\frac{1}{\sin x}\frac{d}{dx}\sin\ x + \log\ \sin\ x(-\sin\ x)$

$\frac{1}{v}\frac{dv}{dx} = \cot\ x \cdot \cos\ x - \sin\ x\log\ \sin\ x$

$\frac{dv}{dx} = v(\cot\ x\cos\ x - \sin\ x\log\ \sin\ x)$

$\frac{dv}{dx} = (\sin x)^{\cos x}(\cot x \cdot \cos x - \sin x \log \sin x) \dots (3)$

Putting values from (2) and (3) in (1)

$\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos\ x\log\ x\right) + (\sin\ x)^{\cos x}(\cot\ x \cdot \cos\ x - \sin\ x\log\ \sin\ x)$

S25. (a) Given functions are $x = 2at^2$ and $y = at^4$

$\frac{dx}{dt} = \frac{d}{dt}(2at^2)$

$\frac{dx}{dt} = 2a\frac{d}{dt}(t^2)$

$= 2a \cdot 2t = 4at$ and

$\frac{dy}{dt} = \frac{d}{dt}(at^4)$

$\frac{dy}{dt} = a\frac{d}{dt}(t^4) = a \cdot 4t^3 = 4at^3$

Now

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4at^3}{4at} = t^2$

S26. (c) Given functions are $x = \sin\ t$ and $y = \cos\ 2t$

$\frac{dx}{dt} = \cos\ t$ and

$\frac{dy}{dt} = -\sin\ 2t\frac{d}{dt}(2t) = -2\sin\ 2t$

Now,

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin\ 2t}{\cos t} = \frac{-2\times 2\sin\ t\cos\ t}{\cos t} = -4\sin\ t$

S27. (d) Let f(x) = sin |x| - |x|

For x$<$ 0

f(x) = - sin x +x

Derivating and putting x = 0 , we have

$f'(x) = -\cos x + 1$

$f'(0) = -1 + 1 = 0$

For x $>$ 0

F(x) = sin x – x

Derivating and putting x = 0

f’(x) = cos x -1

f’ (0)= 1-1 = 0

Derivatives from both the sides are equal . So the function is differentiable .

S28. (a) For continuity at x = 0

f(x) = $e^{|x|}$

$\lim_{x\to 0^-} e^{-x} = 1$ and $\lim_{x\to 0^+} e^{-x} = 1$ and f(0) =1

Therefore function is continuous at x = 0

For differentiability at x = 0

$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h-0}$

L.H.D $\lim_{h\to 0^-} \frac{e^{-h}-1}{h} = -1$

R. H.D $\lim_{h\to 0^+} \frac{e^{h}-1}{h} = 1$

L.H.D ≠ R.H.D

Therefore function is not differentiable at x = 0

S29. (a) Let $y = \log x$

$\frac{dy}{dx} = \frac{1}{x}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1}$

$\frac{d^2y}{dx^2} = (-1)x^{-2} = \frac{-1}{x^2}$

S30. (c) Given expression is $y = e^{a\cos^{-1}x}$

$\frac{dy}{dx} = e^{a\cos^{-1}x}(\frac{d}{dx}a\cos^{-1}x)$

$\frac{dy}{dx} = e^{a\cos^{-1}x}a\left(\frac{-1}{\sqrt{1-x^2}}\right)$

$\frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}}$

This implies

$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$

$(1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$

Differentiating both sides with respect to x, we have

$(1-x^2)2\cdot\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(-2x) = 2a^2y\frac{dy}{dx}$

$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$

$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

SUBJECTIVE QUESTIONS

S1. $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

The form of the given limit is $\frac{0}{0}$ when $x \to 0$.

Rationalizing the numerator, we get

$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\times\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

$= \lim_{x\to 0}\left[\frac{(1+x)-(1-x)}{x\ (\sqrt{1+x}+\sqrt{1-x})}\right]$

$$= \lim_{x\to 0}\left[\frac{2x}{x\ (\sqrt{1+x}+\sqrt{1-x})}\right]$$

$$= \lim_{x\to 0}\left[\frac{2}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

$$= \frac{2}{2}$$

$$= 1$$

S2. $f(x) = \frac{x+3}{x-3}$

f(x) is a rational function it must be continuous in its domain and f is not defined at x = 3

$\therefore$ f is discontinuous at x = 3

$g(x) = \frac{1}{x-7}$

g(x) is also a rational function. It must be continuous in its domain and g is not defined at x = 7

$\therefore$ g is discontinuous at x = 7

Now fog (x) will be discontinuous at

(i) x = 7 (point of discontinuity of g(x))

(ii) g(x) = 3 (when g(x) = point of discontinuity of f(x))

If g(x) = 3

$\Rightarrow \frac{1}{x-7} = 3$

$\Rightarrow$ x = 22/3

$\therefore$ discontinuity of fog(x) should be checked at x = 7 and x = 22/3

$$fog\ (x) = \frac{\frac{1}{x-7}+3}{\frac{1}{x-7}-3}$$

fog (7) is not defined

$$\lim_{x\to 7} fog\ (x) = \frac{\frac{1}{x-7}+3}{\frac{1}{x-7}-3} = \lim_{x\to 7}\frac{1+3x-21}{1-3x+21} = 1$$

$\therefore$ fog (x) is discontinuous at x = 7 and it is removable discontinuity at x = 7

fog (22/3) = not defined

$$\lim_{x\to \frac{22}{3}^{+}} fog\ (x) = \lim_{x\to \frac{22}{3}^{+}}\frac{\frac{1}{x-7}+3}{\frac{1}{x-7}-3} = \infty$$

$$\lim_{x\to \frac{22}{3}^{-}} fog\ (x) = \lim_{x\to \frac{22}{3}^{-}}\frac{\frac{1}{x-7}+3}{\frac{1}{x-7}-3} = -\infty$$

$\therefore$ fog (x) is discontinuous at x = 22/3 and it is non removable discontinuity of II^{nd} kind.

S3. f(a) = a ; f(b) = b. Also f is continuous in [a, b] and $\frac{a+b}{2} \in [a,b]$. Hence using intermediate value theorem, there exist at least one c $\in$ [a, b] such that $f(c) = \frac{a+b}{2}$.

S4. Let $f(x) = \lim_{n\to\infty}\frac{1}{1+n\sin^2 x}$

$\Rightarrow f\left(\frac{\pi}{4}\right) = \lim_{n\to\infty} \frac{1}{1+n.\sin^2\frac{\pi}{4}} = \lim_{n\to\infty} = \frac{1}{1+n\left(\frac{1}{2}\right)} = 0$

Now $\quad f(0) = \lim_{n\to\infty} \frac{1}{n.sin^2(0)+1} = \frac{1}{1+0} = 1$

$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\lim_{n\to\infty} \frac{1}{1+n \rightleftharpoons sin^2 x}\right] = 0$

{Here $\sin^2 x$ is very small quantity but not zero and very small quantity when multiplied with ∞ becomes 0}

$\therefore$ f(x) is not continuous at x = 0

S5. R.H.D. = $f'(1^+)$ $= \text{Limit}_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$

$= \text{Limit}_{h\to 0^+} \frac{(1+h)^2-1}{h} = \text{Limit}_{h\to 0^+} \frac{1+h^2+2h-1}{h} = \text{Limit}_{h\to 0^+} (h+2) = 2$

L.H.D. = $f'(1^-)$ $= \text{Limit}_{h\to 0^+} \frac{f(1-h)-f(1)}{-h} = \text{Limit}_{h\to 0^+} \frac{1-h-1}{-h} = 1$

As L.H.D. ≠ R.H.D. Hence f(x) is not differentiable at x = 1.

NUMERICAL TYPE QUESTIONS

S1. (2) $\ell\text{im}_{x\to 0} \frac{e^x+e^{-x}-2\cos x}{x \sin x} =$

$\frac{0}{0}$ form, then by L – Hôpital rule

$\Rightarrow \lim_{x\to 0} \frac{e^x-e^{-x}+2\sin x}{x\cos x+\sin x}$

$\frac{0}{0}$ form, then by L – Hôpital rule

$\Rightarrow \lim_{x\to 0} \frac{e^x+e^{-x}+2\cos x}{-x\sin x+2\cos x} = \frac{4}{2} = 2$

S2. (0) If $f(x) = \begin{cases} \frac{x^2-(a+2)x+a}{x-2} & x \neq 2 \\ 2, & x = 2 \end{cases}$ is continuous at x = 2, then a is equal to –

$\Rightarrow \lim_{x\to 2} \frac{x^2-(a+2)x+a}{x-2} = \lim_{x\to 2} \frac{2x-(a+2)}{1} = 4-(a+2) = f(2)$

$\Rightarrow 2-a = 2$

$\Rightarrow a = 0$

S3. (6) If the function $f(x) = \begin{cases} \frac{k\cos x}{\pi-2x}, & when\ x \neq \frac{\pi}{2} \\ 3, & when\ x = \frac{\pi}{2} \end{cases}$ be continuous at $x = \frac{\pi}{2}$, then k

$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{k\cos x}{\pi-2x} = \lim_{x\to\frac{\pi}{2}} \frac{-k\sin x}{-2} = f\left(\frac{\pi}{2}\right)$

$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{-k\sin x}{-2} = 3$

$\Rightarrow \frac{k}{2} = 3$

$\Rightarrow k = 6$

S4. (1) For given function to be differentiable, it has to be continuous.

$\Rightarrow L.H.L\,;\ \lim\limits_{x\to 1^-} e^x = e$

$\Rightarrow R.H.L\,;\ \lim\limits_{x\to 1^+} (a-bx) = a-b$

Since f is continuous ,

∴ L.H.L = R.H.L = f(1)

$\Rightarrow a-b=e$ ......(i)

Also , $f'(1-)=e$, $f'(1+)=-b$,

$\Rightarrow\ f'(1-)=f'(1+)$

$\Rightarrow e=\ -b$ ....(ii)

$\Rightarrow b=\ -e$

⇒ Then the value of $\left(a-\frac{b}{e}\right)=0+1=1$

S5. (1) f(x) = $\begin{cases}\frac{1}{|x|}\,,|x|\ge 1\\ ax^2+b\,,|x|<1\end{cases}$

at X = 1 function must be continuous

So, a + b = 1........(i)

Differentiability at x = 1

$\left(\frac{-1}{x^2}\right)_{x=1} = (2ax)_{x=1}$

$\Rightarrow\ -1=2a$

$\Rightarrow a=\ \frac{-1}{2}$

From (i)

$\Rightarrow b=1+\frac{1}{2}=\frac{3}{2}$

$\Rightarrow a+b=\ \frac{-1}{2}+\frac{3}{2}=1$

TRUE AND FALSE

S1. (False) Every continuous function is may or may not be differentiable function.

E.g . $f\colon R\to R$s. t. f(x) = |x| , f is not differentiable but f is continuous function.

S2. (False)

$f\colon R\to R$ s. t. f(x) = $\frac{1}{x}$ is not continuous at x = 0. Since limit of function at x = 0 does not exist.

S3. (False) By Rolle's theorem

Let f : [a, b] → R be continuous on [a, b] and differentiable on (a, b), such that f(a) = f(b), where a and b are some real numbers. Then there exists some c in (a, b) such that f'(c) = 0.

S4. (True) If x = f(t) and y = g(t) , then $\frac{dy}{dx}=\frac{\frac{d}{dt}g(t)}{\frac{d}{dt}f(t)}=\frac{g'(t)}{f'(t)}$

S5. (False) Converse may or may not be true.

Example: f(x) = $\begin{cases}1\,, if\ x\in Z\\ -1\,, if\ x\in R-Z\end{cases}$

Let a be an arbitrary integer , Then $\lim\limits_{x\to a^-} f(x)=\lim\limits_{h\to 0} f(a-h)=\lim\limits_{h\to 0} -1=-1$

$\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h) = \lim_{h\to 0} -1 = -1$and f(1) =1

$\therefore \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \neq f(a)$

So, f is discontinuous at x = a

Now , |f|(x)=|f(x)| = 1 for all x ∈ R . So, |f| is a constant function and hence , it is everywhere continuous.

ASSERTION AND REASON

S1. (d) **Assertion:** Given f(x) = $\sqrt[3]{x}$ or f(x) = $(x)^{\frac{1}{3}}$

Now , we check the continuity of the function at x= 0

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0}(0-h)^{\frac{1}{3}} = 0$

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0}(0+h)^{\frac{1}{3}} = 0$

And f (0) = 0

LHL = RHL = f(0)

So, function is continuous at x = 0

Reason: Given f(x) = [x] , which is greatest integer function.

We know that , the greatest integer function is continuous for all x except integer value of x .So, f(x) = [x] is continuous at x= 299

Hence , A is False , R is true

S2. (c) **Assertion :** We know that

If f(a) = $\lim_{x\to a} f(x)$, then f(x) is continuous at x = a , while both hand must exist.

Reason : If f(x) is continuous at a point , then it is not necessary that $\frac{1}{f(x)}$ is also continuous at that point

Example: f(x) = x is continuous at x = 0 but f(x) = $\frac{1}{x}$ is not continuous at x = 0.

Hence ,A is true , R is False.

S3. (b) **Assertion:** We have , f(x) = cos (x^2)

AT x = c,

LHL = $\lim_{h\to 0} \cos(c-h)^2 = \cos(c^2)$

RHL = $\lim_{h\to 0} \cos(c+h)^2 = \cos(c^2)$

And f(c) = cos c^2

∴ LHL = RHL = f(c)

So, f(x) is continuous at x = c.

Hence , f(x) is continuous for every value of x.

Hence both A and R are true but R is not the correct explanation of A.

S4. (b) Assertion

LHL = $\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h)$

$= \lim_{h\to 0}[2-h-1] + [2-h-2]$

$= \lim_{h\to 0}[1-h] + [-h] = 0$

And f(2) = [2 – 1] + [2-2] = 1

∴ LHL $\neq f(2)$

⇒ f(x) is discontinuous at x = 2.

Reason

$Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h}$

$= \lim_{h\to 0} \frac{[2-h-1]+|2-h-2|-[2-1]-|2-2|}{-h}$

$= \lim_{h\to 0} \frac{0+h-1-0}{-h} = -\infty$

∴ f(x) is not differentiable at x = 2.Hence , both A and R are true and reason is not a correct explanation of assertion.

S5. (a) **Assertion :** Let y = $e^{\sin x}$

Using chain rule , we have

$\frac{dy}{dx} = e^{\sin x}(\cos x)$

Reason : $\frac{d}{dx}(e^x) = e^x$

Hence , both A and R are true and Reason is correct explanation of Assertion.

HOMEWORK (MCQ)

S1. (c) $\lim_{x\to 0} \frac{\tan x}{\tan 5x} = \lim_{x\to 0}\left[\frac{\tan x}{x} \cdot \frac{x}{5x} \cdot \frac{5x}{\tan 5x}\right] = \left[\lim_{x\to 0} \frac{\tan x}{x}\right] \cdot \frac{1}{5} \cdot \left[\lim_{5x\to 0} \frac{5x}{\tan 5x}\right] = \frac{1}{5}$

S2. (a) Let y = $\lim_{x\to 0^+} x^x$

$\Rightarrow \ln y = \lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} -\frac{\ln \frac{1}{x}}{\frac{1}{x}} = 0,$

as $\frac{1}{x} \to \infty$

$\Rightarrow$ y = 1

S3. (a) $f'(x) = \lim_{h\to 0} \frac{f\left(\frac{2x+2h}{2}\right) - f\left(\frac{2x+0}{2}\right)}{h} = \lim_{h\to 0} \frac{\frac{f(2x)+f(2h)}{2} - \frac{f(2x)+f(0)}{2}}{h}$

$= \lim_{h\to 0} \frac{f(2h)-f(0)}{2h} = f'(0) = -1$

f′(x) = – 1

integrating both sides, we get

f(x) = – x + c

Since c = + 1 (as f(0) = 1)

∴ f(x) = – x + 1 = 1 – x

S4. (a) If $f(x) = \begin{cases} 4x, & x < 0 \\ 1, & x = 0 \\ 3x^2, & x > 0 \end{cases}$, then f is

LHL $\lim_{x\to 0^-} 4x = 0$

RHL $\lim_{x\to 0^+} 3x^2 = 0$ and f(0) =1

Hence , f is not continuous at x = 0

S5. **(c)** If f(x) = $\begin{cases} 0, & x < 0 \\ x^2, & x \geq 0 \end{cases}$ then L.H.D. of f(x) at x = 0 is

$\lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0} = 0$

S6. **(b)** If f(x) = $\begin{cases} x^3 - 1 & x > 1 \\ x - 1 & , x \leq 1 \end{cases}$ then at x = 1, f(x) is

For continuity at x = 1

LHL $\lim_{x \to 1^-} (x - 1) = 0$

RHL $\lim_{x \to 1^+} (x^3 - 1) = 0$ and f(1) = 0

∴ f is continuous at x = 1

For differentiability at x = 1

LHD $\lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1^-} \frac{x-1}{x-1} = 1$

RHD $\lim_{x \to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1^+} \frac{x^3-1}{x-1} = \lim_{x \to 1^+} x^2 + x + 1 = 3$

∴ LHD ≠ RHD

Hence, f is not differentiable at x = 1

Hence, f is continuous at x = 1 and not differentiable at x = 1

S7. **(c)** As we know that

f(x) = |x| is not differentiable at x = 0

S8. **(d)** Function f(x) = $\begin{cases} x^2, & x \leq 0 \\ 1, & 0 < x \leq 1 \\ 1/x, & x > 1 \end{cases}$ is-

LHD at x = 0 $\lim_{x \to 0^-} \frac{x^2}{x} = 0$

RHD at x = 0 $\lim_{x \to 0^+} \frac{1-0}{x} = \infty$

f is not differentiable at x = 0

LHD at x = 1 $\lim_{x \to 1^-} \frac{1-1}{x-1} = 0$

RHD at x= 1 $\lim_{x \to 1^+} \frac{\frac{1}{x}-1}{x-1} = -\infty$

f is not differentiable at x = 1

S9 . **(a)** If f(x) = $\begin{cases} x & \text{when } x \in Q \\ -x & \text{when } x \notin Q \end{cases}$, then $\lim_{x \to 0}$ f(x) equals-

If limit exist , then x = -x

⇒ 2x = 0

⇒ x = 0

Then $\lim_{x \to 0}$ f(x) = 0

S10. **(a)** $\lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x + \sqrt{x}}}} = \lim_{x \to \infty} \frac{1}{\sqrt{1 + \sqrt{x + \sqrt{x}}}} = 0$

SUBJECTIVE QUESTIONS

S1. Let y = cos {sin $(x)^2$}.

∴ $\frac{dy}{dx}$ = – sin {sin $(x)^2$}. $\frac{d}{dx}${sin $(x)^2$}

= – sin {sin $(x)^2$}. $\cos(x)^2$ $\frac{d}{dx}(x^2)$

= – sin {sin $(x)^2$}. $\cos(x)^2 \cdot 2x$

= -2x $\cos(x)^2$ sin {sin$(x)^2$}.

S2. The given equation is x = e $^{\log x}$

This is not true for non-positive real numbers.

[∵ Domain of log function is R+]

Now, let y = e $^{\log x}$

If y > 0, taking logs.,

log y = log (e $^{\log x}$) = log x.log e

= log x . 1 = log x

⇒ y = x.

Hence, x = e $^{\log x}$ is true only for positive values of x.

S3. $\lim\limits_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1} = \lim\limits_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}\times\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$

$= \lim\limits_{x\to 0}\frac{(2^x-1)(\sqrt{1+x}+1)}{1+x-1}$

$= \lim\limits_{x\to 0}\frac{2^x-1}{x}.\lim\limits_{x\to 0}\left(\sqrt{1+x}+1\right)$

$= \ln 2.(1+1) = 2\ln 2$

S4. Let h(x) = f(x) + g(x) be continuous.

Then , g(x) = h(x) − f(x)

Now , h(x) and f(x) both are continuous functions.

∴h(x) − f(x) must also be continuous .But it is a contradiction as given that g(x) is discontinuous .

Therefore our assumption that f(x) + g(x) a continuous function is wrong and hence f(x) + g(x) is discontinuous.

S5. $x > 0$ and $\frac{1}{\sqrt{x}} \geq \sqrt{x+1}$

$\therefore\ \frac{1}{x} \geq (x+1)$ or $x^2 + x \leq 1$ or $\left(x+\frac{1}{2}\right)^2 \leq \frac{5}{4}$

$\therefore x\ \leq \frac{\sqrt{5}-1}{2}$

NUMERICAL TYPE QUESTIONS

S1. (-3) Let f(x) = $\begin{cases}\frac{kx}{|x|}, if\ x<0\\ 3, if\ x\geq 0\end{cases}$ is continuous at x = 0

Then , $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x) = f(0)$

$\Rightarrow \lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} f(0-h) = f(0)$

$\Rightarrow 3 = \lim\limits_{h\to 0}\frac{k(-h)}{|-h|} = 3$

$\Rightarrow \lim\limits_{h\to 0}\frac{-kh}{h} = 3 \Rightarrow \lim\limits_{h\to 0}(-k) = 3$

$\therefore k = -3$

S2. $(-\sqrt{3})$ $\lim\limits_{x\to -\infty}\frac{\sqrt{3x^2+2}}{x-2}$

(Put x = $-\frac{1}{t}$, x → − ∞ ⇒ t → 0$^+$)

$= \lim\limits_{t\to 0^+}\frac{\sqrt{3+2t^2}\,.\,\frac{1}{\sqrt{t^2}}}{\frac{(-1-2t)}{t}} = \lim\limits_{t\to 0^+}\frac{\sqrt{3+2t^2}}{-(1+2t)}\frac{t}{|t|} = \frac{\sqrt{3}}{-1} = -\sqrt{3}.$

S3. (0) f(x) = $\begin{cases} x^2 & , \quad x \geq 0 \\ -x^2 & , \quad x < 0 \end{cases}$

R.H.D = $\frac{h|h|-0}{h}$ = 0

L.H.D = $\frac{-h|-h|-0}{-h}$ = 0

Differentiable at x = 0

S4. (0) x = 1 and x = – 1 are doubtful points as the definition of the function changes at these points.

Also. x = 0 is a doubtful points because of |x|.

∴ The sum of doubtful points = 1 -1 + 0 = 0

S5. (15) $\lim_{h\to 0}\frac{f(a+2h)-f(a-3h)}{h} = \lim_{h\to 0}\frac{f(a+2h)-f(a-3h)}{5h} . 5$

= f′(a) × 5 = 3 × 5 = 15

TRUE AND FALSE

S1. (True) $\ell\text{im}_{x\to\infty}\frac{\ell nx}{x} = \lim_{x\to\infty}\frac{\ln(\infty)}{\infty} = \frac{\infty}{\infty}$ form

Now ,apply L Hôpital rule and find limit.

S2. (False) Expansion of ln (1+x) = $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots,\quad \text{for } -1 < x \leq 1$

S3. (False) A function f is said to be continuous in a closed interval [a, b] if:

(a) f is continuous in the open interval (a, b),

(b) f is right continuous at 'a' i.e. $\underset{x\to a^+}{Lim}$ f(x) = f(a) = a finite quantity and

(c) f is left continuous at 'b' i.e. $\underset{x\to b^-}{Lim}$ f(x) = f(b) = a finite quantity.

S4. (True) If f(x) is not differentiable at x = a & g(x) is differentiable at x = a, then the product function F(x) = f(x) .g(x) can still be differentiable at x = a e.g. f(x) = $|x|$ and g(x) =x^2.

S5. (True) If f(x) & g(x) both are non-differentiable at x = a, then the sum function F(x) = f(x)+ g(x) may be a differentiable function. e.g. f(x) = $|x|$ & g(x) = $-|x|$.

ASSERTION AND REASON

S1. (a) Sin x and | x| are continuous in R and hence R is true.

Consider the function f(x) = sin x and g(x) = |x| both of which are continuous in R.

gof (x) = g(f(x)) = g(sin x) = | sin x|.

Since f(x) and g(x) are continuous in R , gof(x) is also continuous in R. Hence A is true.

R is the correct explanation of A.

S2. (d) g(x) = x^2 is a polynomial function. It is continuous for all $x \in R$. Hence R is true.

f(x) = $tan^2 x$ is not defined when x = $\frac{\pi}{2}$. Therefore f$\left(\frac{\pi}{2}\right)$ does not exist and hence f(x) is not continuous at x = $\frac{\pi}{2}$. A is false

S3. (d) f(x) is continuous at x = a , if $\lim_{x\to a} f(x) = f(a)$

∴ R is true.

$\lim_{x\to 2} f(x) = f(2) = k$

$\lim_{x\to 2}\frac{(x+5)(x-2)}{x-2} = k$ ∴ k = 7

Hence A is false

S4. (c) Since sin x and | x| are continuous function in R, |sin x| is continuous at x = 0. Hence A is true.

|sin x | = $\begin{cases} -\sin x\,; x < 0 \\ \sin x\,; x \geq 0 \end{cases}$

f(0) = 0

LHD $f'(0-) = \lim_{x \to 0} \frac{-\sin x}{x} = -1$

RHD $f'(0+) = \lim_{x \to 0} \frac{\sin x}{x} = 1$

At x = 0 , LHD ≠ RHD

So, f(x) is not differentiable at x = 0. Hence R is false.

S5. (c) Put $3x = \sin\theta$ or $\theta = \sin^{-1} 3x$

y = $\sin^{-1}\left(6x\sqrt{1-9x^2}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1} 3x$

$\therefore \frac{dy}{dx} = \frac{6}{\sqrt{1-9x^2}}$

A is true , R is false.