Algebraic Identities & Factorisation

Using Identity (x + a)(x + b)

A very important identity that we have to learn is regarding the expression $(x + a)(x + b)$.

We know how to multiply binomials. By finding the product of these binomials, we can find the identity.

Let us see how.

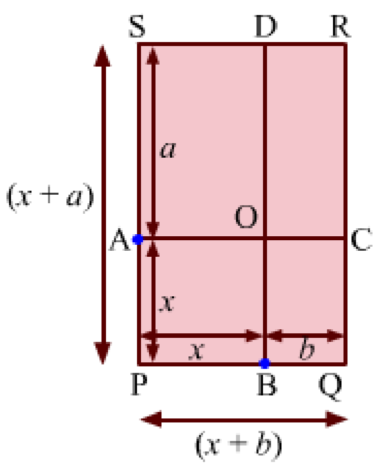
This identity can be derived by geometrical construction as well. Let us learn the same.

Deriving the identity geometrically:

Let us draw a rectangle PQRS of length and breadth $(x + a)$ and $(x + b)$ units respectively.

Also, let us take two points A and B on sides SP and PQ respectively, such that PA = PB = x unit.

Now, let us draw two line segments AC and BD such that AC || PQ and BD || PS. AC intersects QR at point C and BD intersects RS at point D.



Now, we have

l(PS) = $(x + a)$ unit and l(PQ) = $(x + b)$ unit

∴ Area of rectangle PQRS = length × breadth

⇒ Area of rectangle PQRS = *l*(PS) × *l*(PQ)

⇒ Area of rectangle PQRS = (x + a) (x + b) sq. unit ...(i)

Also,

Area of square PBOA = x^2 sq. unit ...(ii)

Area of rectangle BQCO = bx sq. unit ...(iii)

Area of rectangle OCRD = ab sq. unit ...(iv)

Area of rectangle AODS = ax sq. unit ...(v)

From the figure, it can be observed that

Area of rectangle PQRS = Area of square PBOA + Area of rectangle BQCO + Area of rectangle OCRD + Area of rectangle AODS

On substituting the values from (i), (ii), (iii), (iv) and (v), we get

$(x + a)(x + b) = x^2 + bx + ab + ax$

⇒ $(x + a)(x + b) = x^2 + ax + bx + ab$

⇒ $(x + a)(x + b) = x^2 + (a + b)x + ab$

Now, let us solve some examples in which this identity is used.

Example 1:
Find the product of (m + 3) and (m – 5).

Solution:

This expression is of the form (x + a) (x + b).

Using identity, $(x + a)(x + b) = x^2 + (a + b)x + ab$

$$\therefore (m+3)(m-5) = m^2 + (3-5)m + (3)(-5)$$
$$= m^2 - 2m - 15$$

Example 2:

Use the appropriate identity to simplify the following expressions.

(a) $(3p+5q)(3p-7z)$

(b) $\left(z^2-\frac{4}{3}\right)\left(z^2-\frac{3}{2}\right)$

Solution:

(a) The given expression is $(3p+5q)(3p-7z)$.

This expression is of the form (*x* + *a*) (*x* + *b*).

Using identity, (*x* + *a*) (*x* + *b*) = x^2 + (*a* + *b*) *x* + *ab*

$$\begin{aligned}\therefore (3p+5q)(3p-7z) &= (3p)^2 + (5q-7z)(3p) + (5q)(-7z)\\ &= 9p^2 + (5q)(3p) - (7z)(3p) - 35qz\\ &= 9p^2 + 15pq - 21pz - 35qz\end{aligned}$$

(b) The given expression is $\left(z^2-\frac{4}{3}\right)\left(z^2-\frac{3}{2}\right)$.

This expression is of the form (*x* + *a*) (*x* + *b*).

Using identity, (*x* + *a*) (*x* + *b*) = x^2 + (*a* + *b*) *x* + *ab*

$$\begin{aligned}\therefore \left(z^2-\frac{4}{3}\right)\left(z^2-\frac{3}{2}\right) &= \left(z^2\right)^2 + \left\{\left(-\frac{4}{3}\right)+\left(-\frac{3}{2}\right)\right\}z^2 + \left(-\frac{4}{3}\right)\times\left(-\frac{3}{2}\right)\\ &= z^4 + \left(\frac{-8-9}{6}\right)z^2 + 2\\ &= z^4 - \frac{17}{6}z^2 + 2\end{aligned}$$

Example 3:

Find the products of the following pairs of numbers using suitable identities.

(a) 105 × 102

(b) 98 × 103

Solution:

(a) 105 × 102 = (100 + 5) × (100 + 2)

Now, we can use the identity $(x + a)(x + b) = x^2 + (a + b)x + ab$

Here, $x = 100$, $a = 5$, $b = 2$

∴105 × 102 = (100 + 5) × (100 + 2)

$= (100)^2 + (5 + 2) \times 100 + 5 \times 2$

= 10000 + 700 + 10

= 10710

Thus, the product of the numbers 105 and 102 is 10710.

(b) 98 × 103 = (100 − 2) × (100 + 3)

Now, we can use the identity $(x + a)(x + b) = x^2 + (a + b)x + ab$

Here, $x = 100$, $a = -2$, $b = 3$

∴98 × 103 = (100 − 2) × (100 + 3)

$= (100)^2 + (-2 + 3) \times 100 + (-2)(3)$

= 10000 + 1 × 100 − 6

= 10000 + 100 − 6

= 10100 − 6

= 10094

Thus, the product of the numbers 98 and 103 is 10094.

Using Identities for "Square of Sum or Difference of Two Terms"

Let us try to find the square of the number 102. The square of a number, as we know, is the product of the number with itself. One way to do this is by writing the numbers one below the other, and then multiplying them as we normally do. The other way is to break the numbers and then apply distributive property. This will make our work much easier.

Let us see how.

$102^2 = 102 \times 102$

$= (100 + 2)(100 + 2)$

$= 100(100 + 2) + 2(100 + 2)$

$= 100 \times 100 + 100 \times 2 + 2 \times 100 + 2 \times 2$

$= 10000 + 200 + 200 + 4$

$= 10404$

Observing the similar expressions as above, we obtain the following identities.

$$\boldsymbol{(a + b)^2 = a^2 + 2ab + b^2}$$

$$\boldsymbol{(a - b)^2 = a^2 - 2ab + b^2}$$

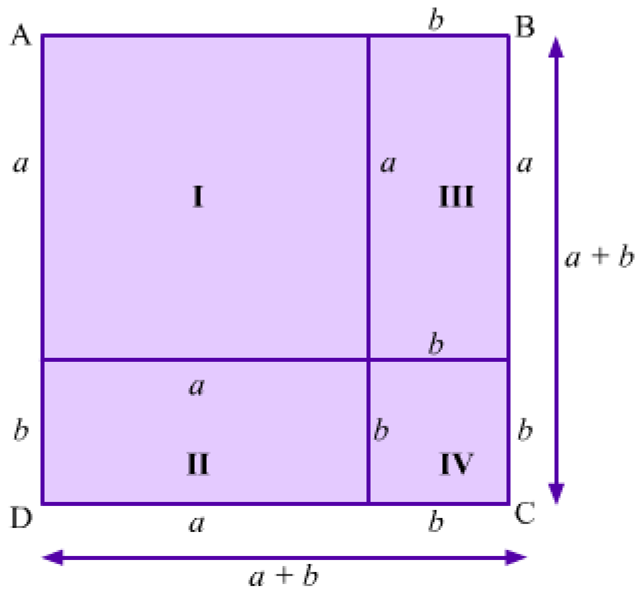
To understand the proof of these identities, look at the following video.

Deriving the identities geometrically:

These identities can be derived by geometrical construction as well. Let us learn the same.

(1) $(a + b)^2 = a^2 + 2ab + b^2$:

Let us consider a square ABCD whose each side measures $(a + b)$ unit.



It can be seen that, we have drawn two line segments at a distance of a unit from A such that one is parallel to AB and other is parallel to AD.

Also, the figure is divided into four regions named as I, II, III and IV.

Now,

∴ Area of square ABCD = $(a + b)^2$ sq. unit ...(i)

Region I is a square of side measuring a unit.

∴ Area of region I = a^2 sq. unit ...(ii)

Each of regions II and III is a rectangle having length and breadth as a unit and b unit respectively.

∴ Area of region II = ab sq. unit ...(iii)

And,

Area of region III = ab sq. unit ...(iv)

Region IV is a square of side measuring b unit.

∴ Area of region IV = b^2 sq. unit ...(v)

From the figure, we have

Area of square ABCD = Area of region I + Area of region II + Area of region III + Area of region IV

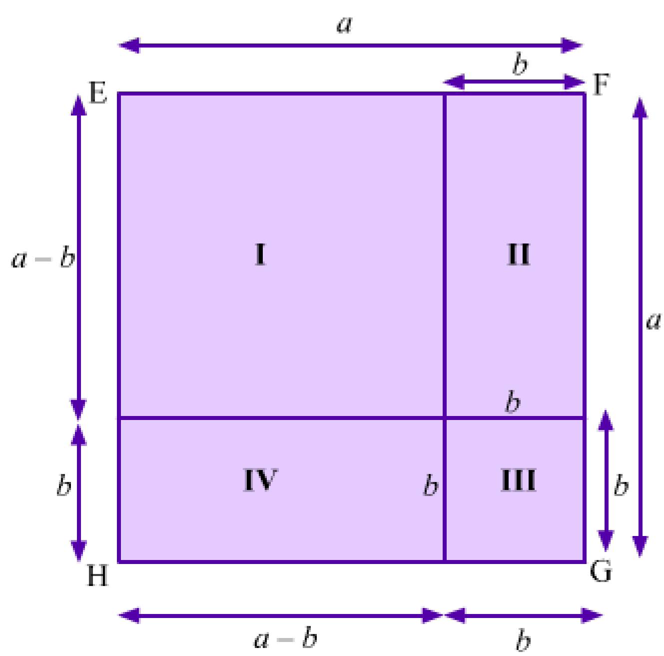
On substituting the values from (i), (ii), (iii), (iv) and (v), we get

$(a + b)^2 = a^2 + ab + ab + b^2$

$\Rightarrow (a + b)^2 = a^2 + 2ab + b^2$

(2) $(a - b)^2 = a^2 - 2ab + b^2$:

Let us consider a square EFGH whose each side measures *a* unit.



It can be seen that, we have drawn two line segments at a distance of *b* unit from G such that one is parallel to GH and other is parallel to FG.

Also, the figure is divided into four regions named as I, II, III and IV.

Now,

∴ Area of square EFGH = a^2 sq. unit ...(i)

Region I is a square of side measuring $(a - b)$ unit.

∴ Area of region I = $(a - b)^2$ sq. unit ...(ii)

Each of regions II and IV is a rectangle having length and breadth as ($a - b$) unit and b unit respectively.

∴ Area of region II = $b(a - b)$ sq. unit ...(iii)

And,

Area of region IV = $b(a - b)$ sq. unit ...(iv)

Region III is a square of side measuring b unit.

∴ Area of region III = b^2 sq. unit ...(v)

From the figure, we have

Area of region I = Area of square ABCD − (Area of region II + Area of region III + Area of region IV)

On substituting the values from (i), (ii), (iii), (iv) and (v), we get

$(a - b)^2 = a^2 - [b(a - b) + b^2 + b(a - b)]$

$(a - b)^2 = a^2 - [ab - b^2 + b^2 + ab - b^2]$

$(a - b)^2 = a^2 - [2ab - b^2]$

⇒ $(a - b)^2 =$
$a^2 - 2ab + b^2$

The identities we have proved above are known as identity because for any value of a and b, the LHS is always equal to the RHS. The difference between an identity and an equation is that for an equation, its LHS and RHS are equal only for some values of the variable.

On the other hand, as we discussed, for an identity, the LHS equals the RHS for any value of the variable.

Many a times, these identities help in shortening our calculations. Let us discuss some examples using the above identities to understand this better.

Example 1:
Simplify the following expressions using suitable identities:
(a) $(2m + 3n)^2$
(b) $(4p – 7q)^2$

Solution:

(a) On comparing the given expression $(2m + 3n)^2$ with $(a + b)^2$, we get
$a = 2m$ and $b = 3n$.

Now,
$(a + b)^2 = a^2 + 2ab + b^2$
Thus,

$$\begin{aligned}(2m+3n)^2 &= (2m)^2 + 2(2m)(3n) + (3n)^2 \\ &= 4m^2 + 12mn + 9n^2\end{aligned}$$

(b) On comparing the given expression $(4p – 7q)^2$ with $(a – b)^2$, we get

$a = 4p$ and $b = 7q$.

Now,
$(a – b)^2 = a^2 – 2ab + b^2$

Thus,
$(4p – 7q)^2 = (4p)^2 – 2(4p)(7q) + (7q)^2$

$= 16p^2 – 56pq + 49q^2$

Example 2:

Simplify the following expressions using suitable identities:

(a) $(3ax + 5by)^2$

(b) $(0.6a^2 − 0.04b^3)^2$

(c) $\left(\frac{3}{7}l + \frac{4}{5}m\right)^2$

Solution:

(a) The given expression is $(3ax + 5by)^2$, which is of the form $(a+ b)^2$.

Thus, we can use the identity $(a + b)^2 = a^2 + 2ab + b^2$.

$\therefore (3ax + 5by)^2 = (3ax)^2 + 2 (3ax) (5by) + (5by)^2$

$= 9a^2x^2 + 30abxy + 25b^2y^2$

(b) The given expression is $(0.6a^2 - 0.04b^3)^2$, which is of the form $(a - b)^2$.

Thus, we can use the identity $(a - b)^2 = a^2 - 2ab + b^2$.

$\therefore (0.6a^2 - 0.04b^3)^2 = (0.6a^2)^2 - 2\,(0.6a^2)\,(0.04b^3) + (0.04b^3)^2$

$= 0.36a^4 - 0.048a^2b^3 + 0.0016b^6$

(c) The given expression is $\left(\frac{3}{7}l + \frac{4}{5}m\right)^2$, which is of the form $(a + b)^2$.

Thus, we can use the identity $(a + b)^2 = a^2 + 2ab + b^2$.

$$\therefore \left(\frac{3}{7}l + \frac{4}{5}m\right)^2 = \left(\frac{3}{7}l\right)^2 + 2\left(\frac{3}{7}l \times \frac{4}{5}m\right) + \left(\frac{4}{5}m\right)^2$$
$$= \frac{9}{49}l^2 + \frac{24}{35}lm + \frac{16}{25}m^2$$

Example 3:

Find the value of $(208)^2$ using a suitable identity.

Solution:

208 = 200 + 8

$\therefore (208)^2 = (200 + 8)^2$

Thus, we can use the identity $(a + b)^2 = a^2 + 2ab + b^2$.

$\therefore (208)^2 = (200 + 8)^2$

$= (200)^2 + 2\,(200)\,(8) + (8)^2$

= 40000 + 3200 + 64

= 43264

Example 4:

Find the value of $(99)^2$ using a suitable identity.

Solution:

99 = 100 − 1

∴ $(99)^2 = (100 - 1)^2$

Thus, we can use the identity $(a - b)^2 = a^2 - 2ab + b^2$.

∴ $(99)^2 = (100 - 1)^2 = (100)^2 - 2\,(100)\,(1) + (1)^2$

= 10000 − 200 + 1

= 9800 + 1

= 9801

Example 5:

(a) If $x - \frac{1}{x} = 3$, then find the value of the expressions $x^2 + \frac{1}{x^2}$ and $x^4 + \frac{1}{x^4}$.

(b) If $2y + \frac{3}{y} = 5$, then find the value of the expression $4y^2 + \frac{9}{y^2}$.

(c) If $3x - 5y = -1$ and $xy = 6$, then find the value of the expression $9x^2 + 25y^2$.

Solution:

(a) It is given that $x - \frac{1}{x} = 3$.

On squaring both sides, we get

$$\left(x - \frac{1}{x}\right)^2 = 3^2$$

$$\Rightarrow (x)^2 - 2(x)\left(\frac{1}{x}\right) + \left(\frac{1}{x}\right)^2 = 9 \qquad \left[\text{Using the identity } (a-b)^2 = a^2 - 2ab + b^2\right]$$

$$\Rightarrow x^2 - 2 + \frac{1}{x^2} = 9$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 9 + 2$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 11$$

Now, on squaring both sides again, we get

$$\left(x^2+\frac{1}{x^2}\right)^2=11^2$$

$$\Rightarrow \left(x^2\right)^2+2\left(x^2\right)\left(\frac{1}{x^2}\right)+\left(\frac{1}{x^2}\right)^2=121 \qquad \left[\text{Using the identity } (a+b)^2=a^2+2ab+b^2\right]$$

$$\Rightarrow x^4+2+\frac{1}{x^4}=121$$

$$\Rightarrow x^4+\frac{1}{x^4}=119$$

Thus, the value of the expression $\left(x^2+\frac{1}{x^2}\right)$ is 11 and the value of the expression $\left(x^4+\frac{1}{x^4}\right)$ is 119.

(b) It is given that $2y+\frac{3}{y}=5$.

On squaring both sides, we get

$$\left(2y+\frac{3}{y}\right)^2=5^2$$

$$\Rightarrow (2y)^2+2(2y)\left(\frac{3}{y}\right)+\left(\frac{3}{y}\right)^2=25$$

$$\Rightarrow 4y^2+12+\frac{9}{y^2}=25$$

$$\Rightarrow 4y^2+\frac{9}{y^2}=13$$

Thus, the value of the expression $\left(4y^2+\frac{9}{y^2}\right)$ is 13.

(c) It is given that $3x-5y=-1$.

On squaring both sides, we get

$$(3x-5y)^2=(-1)^2$$
$$\Rightarrow (3x)^2-2(3x)(5y)+(5y)^2=1$$
$$\Rightarrow 9x^2-30xy+25y^2=1$$
$$\Rightarrow 9x^2-30\times 6+25y^2=1 \qquad (xy=6)$$
$$\Rightarrow 9x^2+25y^2=1+180$$
$$\Rightarrow 9x^2+25y^2=181$$

Thus, the value of the expression ($9x^2 + 25y^2$) is 181.

Example 6:

Prove that

(a) $(a+b)^2+(a-b)^2=2(a^2+b^2)$

(b) $\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2=ab$

Solution:

(a) We know that $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$. Thus,

LHS = $(a + b)^2 + (a - b)^2$

= $a^2 + 2ab + b^2 + a^2 - 2ab + b^2$

= $(a^2 + a^2) + (b^2 + b^2) + (2ab - 2ab)$

= $2a^2 + 2b^2$

= $2(a^2 + b^2)$ = RHS

Hence, proved.

(b) We know that $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$. Thus,

$$\text{LHS}=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2$$

$$= \left(\frac{a^2+2ab+b^2}{4}\right)-\left(\frac{a^2-2ab+b^2}{4}\right)$$
$$= \frac{a^2+2ab+b^2-a^2+2ab-b^2}{4}$$
$$= \frac{4ab}{4}$$
$$= ab = \text{RHS}$$

Hence, proved.

Using Identity for Square of Sum of Three Terms

Algebraic Identity:

$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

When we solve an **algebraic equation**, we get the values of the variables present in it. When an algebraic equation is valid for all values of its variables, it is called an **algebraic identity**.

So, an algebraic identity is a relation that holds true for all possible values of its variables. We can use algebraic identities to expand, factorise and evaluate various algebraic expressions.

Many algebraic identities are used in mathematics. One such identity is $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$. In this lesson, we will study this identity and solve some examples based on it.

Proof of the Identity

Let us prove the identity $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

We can write $(x + y + z)^2$ as :

$(a + z)^2$, where $a = x + y$

$= a^2 + 2az + z^2$ [Using the identity $(x + y)^2 = x^2 + 2xy + y^2$]

$= (x + y)^2 + 2(x + y) z + z^2$ (Substituting the value of a)

$= x^2 + 2xy + y^2 + 2xz + 2yz + z^2$ (Using the identity $(x + y)^2 = x^2 + 2xy + y^2$)

$\therefore (x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

The above identity holds true for all values of the variables present in it. Let us verify this by substituting random values for the variables *x*, *y* and *z*.

If $x = 2$, $y = 3$ and $z = 4$, then:

$(2 + 3 + 4)^2 = 2^2 + 3^2 + 4^2 + 2 \times 2 \times 3 + 2 \times 3 \times 4 + 2 \times 4 \times 2$

$\Rightarrow 9^2 = 4 + 9 + 16 + 12 + 24 + 16$

$\Rightarrow 81 = 81$

$\Rightarrow$ LHS = RHS

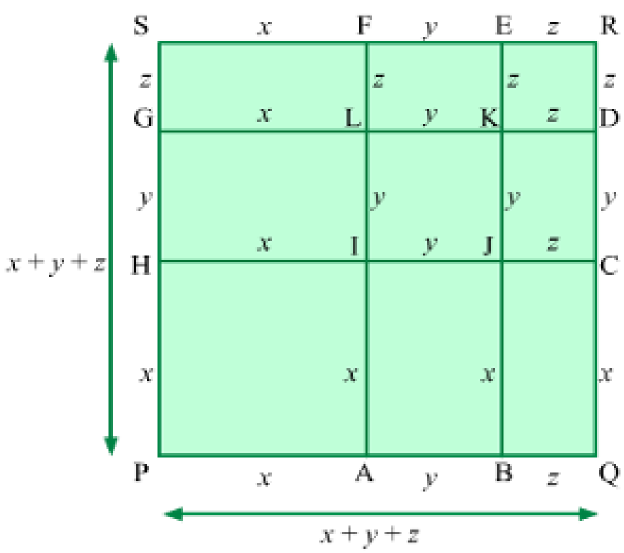
Thus, we see that the identity holds true for random values of the variables present in it.

Let us now use this identity to expand, factorise and evaluate various algebraic expressions.

Deriving Identity Geometrically

The identity $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$ can also be derived with the help of geometrical construction.

The steps of construction are as follows:



(1) Draw a square PQRS of side measuring $(x + y + z)$ taking any convenient values of *x*, *y* and *z*.

(2) Mark two points A and B on side PQ such that *l*(PA) = x and *l*(AB) = y. Thus, *l*(BQ) = z. Also, mark two points H and G on side PS such that *l*(PH) = x and *l*(HG) = y. Thus, *l*(GS) = z.

(3) From points A and B, draw segments AF and BE parallel to side PS and intersecting RS at F and E respectively.

(4) From points H and G, draw segments HC and GD parallel to side PQ and intersecting QR at C and D respectively.

From the figure, it can be observed that

Area of square PQRS = Sum of areas of squares PAIH, IJKL and KDRE + Sum of areas of rectangles ABJI, BQCJ, JCDK, HILG, LKEF and GLFS

$\Rightarrow (x + y + z)^2 = (x^2 + y^2 + z^2) + (xy + zx + yz + xy + yz + zx)$

$\Rightarrow \mathbf{(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx}$

Solved Examples

Easy

Example 1:

Expand the following expressions.

i) $(ab - bc + ca)^2$

ii) $\left(\frac{1}{2}x - \frac{2}{3}y - \frac{3}{4}\right)^2$

Solution:

i) $(ab - bc + ca)^2$

On comparing the expression $(ab - bc + ca)^2$ with $(x + y + z)^2$, we get:

$x = ab$, $y = -bc$ and $z = ca$

On using the identity $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$, we get:

$(ab)^2 + (-bc)^2 + (ca)^2 + 2\,(ab)\,(-bc) + 2\,(-bc)\,(ca) + 2\,(ca)\,(ab)$

$= a^2b^2 + b^2c^2 + c^2a^2 - 2ab^2c - 2abc^2 + 2a^2bc$

$\therefore (ab - bc + ca)^2 = a^2b^2 + b^2c^2 + c^2a^2 - 2ab^2c - 2abc^2 + 2a^2bc$

ii) $\left(\frac{1}{2}x - \frac{2}{3}y - \frac{3}{4}\right)^2$

On comparing the expression $\left(\frac{1}{2}x - \frac{2}{3}y - \frac{3}{4}\right)^2$ with $(a + b + c)^2$, we get:

$a = \frac{1}{2}x$, $b = -\frac{2}{3}y$ and $c = -\frac{3}{4}$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$, we get:

$$\left(\frac{1}{2}x\right)^2 + \left(-\frac{2}{3}y\right)^2 + \left(-\frac{3}{4}\right)^2 + 2\left(\frac{1}{2}x\right)\left(-\frac{2}{3}y\right) + 2\left(-\frac{2}{3}y\right)\left(-\frac{3}{4}\right) + 2\left(-\frac{3}{4}\right)\left(\frac{1}{2}x\right)$$
$$\Rightarrow \frac{1}{4}x^2 + \frac{4}{9}y^2 + \frac{9}{16} - \frac{2}{3}xy + y - \frac{3}{4}x$$

$$\therefore \left(\frac{1}{2}x - \frac{2}{3}y - \frac{3}{4}\right)^2 = \frac{1}{4}x^2 + \frac{4}{9}y^2 + \frac{9}{16} - \frac{2}{3}xy + y - \frac{3}{4}x$$

Example 2:

Expand the expression $(xy + yz + zx)^2$.

Solution:

On comparing the expression $(xy + yz + zx)^2$ with $(a + b + c)^2$, we get:

$a = xy$, $b = yz$ and $c = zx$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$, we get:

$(xy)^2 + (yz)^2 + (zx)^2 + 2\,(xy)\,(yz) + 2\,(yz)\,(zx) + 2\,(zx)\,(xy)$

$= x^2y^2 + y^2z^2 + z^2x^2 + 2xy^2z + 2xyz^2 + 2x^2yz$

$\therefore (xy + yz + zx)^2 = x^2y^2 + y^2z^2 + z^2x^2 + 2xy^2z + 2xyz^2 + 2x^2yz$

Medium

Example 1:

Factorize the following expressions.

i)$8x^2 + 12y^2 - 8\sqrt{6}xy + 12\sqrt{6}x - 36y + 27$

ii)$8x^4 + 4\sqrt{2}x^3 + 25x^2 + 6\sqrt{2}x + 18$

Solution:

i)$8x^2 + 12y^2 - 8\sqrt{6}xy + 12\sqrt{6}x - 36y + 27$

$= 8x^2 + 12y^2 + 27 - 8\sqrt{6}xy - 36y + 12\sqrt{6}x$

$= (2\sqrt{2}x)^2 + (-2\sqrt{3}y)^2 + (3\sqrt{3})^2 + 2(2\sqrt{2}x)(-2\sqrt{3}y) + 2(-2\sqrt{3}y)(3\sqrt{3}) + 2(3\sqrt{3})(2\sqrt{2}x)$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$, where $a = 2\sqrt{2}x$, $b = -2\sqrt{3}y$ and $c = 3\sqrt{3}$, we are left with $(2\sqrt{2}x - 2\sqrt{3}y + 3\sqrt{3})^2$

$\therefore 8x^2 + 12y^2 - 8\sqrt{6}xy + 12\sqrt{6}x - 36y + 27 = (2\sqrt{2}x - 2\sqrt{3}y + 3\sqrt{3})^2$

ii)$8x^4 + 4\sqrt{2}x^3 + 25x^2 + 6\sqrt{2}x + 18$

$= 8x^4 + 25x^2 + 18 + 4\sqrt{2}x^3 + 6\sqrt{2}x$

$= 8x^4 + x^2 + 18 + 4\sqrt{2}x^3 + 6\sqrt{2}x + 24x^2$ ($\because 25x^2 = x^2 + 24x^2$)

$= (2\sqrt{2}x^2)^2 + (x)^2 + (3\sqrt{2})^2 + 2(2\sqrt{2}x^2)(x) + 2(x)(3\sqrt{2}) + 2(3\sqrt{2})(2\sqrt{2}x^2)$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$, where $a = 2\sqrt{2}x^2$, $b = x$ and $c = 3\sqrt{2}$, we are left with $(2\sqrt{2}x^2 + x + 3\sqrt{2})^2$

$\therefore 8x^4 + 4\sqrt{2}x^3 + 25x^2 + 6\sqrt{2}x + 18 = (2\sqrt{2}x^2 + x + 3\sqrt{2})^2$

Example 2:

Find the value of the expression $4x^2 + 9y^2 + 16z^2 - 12xy - 24yz + 16zx$ for $x = 3$, $y= 4$ and
$z = 5$ without substituting the values of the variables in the expression.

Solution:

$4x^2 + 9y^2 + 16z^2 - 12xy - 24yz + 16zx$

$= (2x)^2 + (-3y)^2 + (4z)^2 + 2\,(2x)\,(-3y) + 2\,(-3y)\,(4z) + 2\,(4z)\,(2x)$

On using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$, where $a = 2x$,
$b = -3y$ and $c = 4z$, we are left with $(2x - 3y + 4z)^2$

It is given that $x = 3$, $y = 4$ and $z = 5$.

On substituting the values of x, y and z, we get:

$(2 \times 3 - 3 \times 4 + 4 \times 5)^2$

$= (6 - 12 + 20)^2$

$= 14^2$

$= 196$

$\therefore 4x^2 + 9y^2 + 16z^2 - 12xy - 24yz + 16zx = 196$

Hard

Example 1:

Find the value of $ab + bc + ca$, where $a + b + c = 1$ and $a^2 + b^2 + c^2 = 29$.

Solution:

We know that $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$\Rightarrow (a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + bc + ca)$

$\Rightarrow (1)^2 = 29 + 2(ab + bc + ca)$

$\Rightarrow 1 - 29 = 2(ab + bc + ca)$

$\Rightarrow -28 = 2(ab + bc + ca)$

$\Rightarrow ab + bc + ca = -\frac{28}{2}$

$\Rightarrow \therefore ab + bc + ca = -14$

Example 2:

If $(x + 2)^2 + (y - 6)^2 + (z - a)^2 - 2x (6 + a) + 2y (2 - a) - 8z + 2 (xy + yz + xz) - 8 (3 - a)$

$= (x + y + z)^2$, then find the value of a.

Solution:

$(x + 2)^2 + (y - 6)^2 + (z - a)^2 - 2x (6 + a) + 2y (2 - a) - 8z + 2 (xy + yz + xz) - 8 (3 - a)$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 - 12x - 2ax + 4y - 2ay - 8z + 2xy + 2yz + 2xz - 24 + 8a$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 - 12x - 2ax + 4y - 2ay - 12z + 4z + 2xy + 2yz + 2xz - 24 + 12a - 4a$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 + 2xy - 12x + 4y - 24 + 2yz - 2ay - 12z + 12a + 2xz - 2ax + 4z - 4a$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 + 2(xy - 6x + 2y - 12) + 2(yz - ay - 6z + 6a) + 2(xz - ax + 2z - 2a)$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 + 2[x(y - 6) + 2(y - 6)] + 2[y(z - a) - 6(z - a)] + 2[x(z - a) + 2(z - a)]$

$= (x + 2)^2 + (y - 6)^2 + (z - a)^2 + 2 (x + 2) (y - 6) + 2 (y - 6) (z - a) + 2 (x + 2) (z - a)$

$= [(x +2) + (y - 6) + (z - a)]^2$

$= (x + y + z - 4 - a)^2$

It is given that
$(x + 2)^2 + (y - 6)^2 + (z - a)^2 - 2x (6 + a) + 2y (2 - a) - 8z + 2 (xy + yz + xz) - 8 (3 - a)$

$= (x + y + z)^2$

$\Rightarrow (x + y + z - 4 - a)^2 = (x + y + z)^2$

$\Rightarrow -4 - a = 0$

$\Rightarrow \therefore a = -4$

Using Identities for Cube of Sum or Difference of Two Terms

Algebraic Identities:

$(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$ and $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$

Consider the number ‘999’. Suppose we have to calculate its cube. One way to find the cube is to multiply 999 by itself three times. However, this method is tedious and, therefore, prone to error.

Here is another way to solve the problem. Let us write 999^3 as $(1000 - 1)^3$. We have thus changed the number into the form $(x - y)^3$. Now, the expansion of $(x - y)^3$ will give the cube of 999. The required calculation will be easy since the values of x and y are simple numbers whose multiplication is also simple.

Thus, we see algebraic identities help make calculations simpler and less tedious. In this lesson, we will study the identities $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$ and $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$. We will also solve some examples based on them.

Understanding the Identities

We have the two algebraic identities as follows:

- $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$ OR $(x + y)^3 = x^3 + y^3 + 3x^2y + 3xy^2$
- $(x - y)^3 = x^3 - y^3 - 3xy\,(x - y)$ OR $(x - y)^3 = x^3 - y^3 - 3x^2y + 3xy^2$

The above identities hold true for all values of the variables present in them. Let us verify this by substituting random values for the variables x and y in the first identity.

If $x = 2$ and $y = 3$, then:

$(2 + 3)^3 = 2^3 + 3^3 + 3 \times 2 \times 3 \times (2 + 3)$

$\Rightarrow 5^3 = 8 + 27 + 18 \times 5$

$\Rightarrow 125 = 8 + 27 + 90$

$\Rightarrow 125 = 125$

$\Rightarrow$ LHS = RHS

Thus, we see that the first identity holds true for random values of the variables present in it. We can prove the same for the second identity as well.

Here are some other ways in which the two identities can be represented

- $x^3 + y^3 = (x + y)^3 - 3xy\,(x + y)$ OR $x^3 + y^3 = (x + y)\,(x^2 - xy + y^2)$
- $x^3 - y^3 = (x - y)^3 + 3xy\,(x - y)$ OR $x^3 - y^3 = (x - y)\,(x^2 + xy + y^2)$

Proof of the Identities:
$(x + y)^3 = \ldots$ and $x^3 + y^3 = \ldots$

Let us prove the identity $(x + y)^3 = x^3 + y^3 + 3x^2y + 3xy^2$ OR $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$

We can write $(x + y)^3$ as:

$(x + y)\,(x + y)^2$

$= (x + y)\,(x^2 + 2xy + y^2)$

$= x^3 + 2x^2y + 2xy^2 + x^2y + 2xy^2 + y^3$

$= x^3 + y^3 + 3x^2y + 3xy^2$

$\therefore$ **$(x + y)^3 = x^3 + y^3 + 3x^2y + 3xy^2$**

$\Rightarrow$ **$(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$** … (1)

Let us prove the identity $x^3 + y^3 = (x + y)^3 - 3xy\,(x + y)$ OR $x^3 + y^3 = (x + y)\,(x^2 - xy + y^2)$

We can rewrite equation 1 as:

$x^3 + y^3 = (x + y)^3 - 3xy\,(x + y)$

$\Rightarrow x^3 + y^3 =(x + y)\,[(x + y)^2 - 3xy]$

$\Rightarrow x^3 + y^3 = (x + y)\,(x^2 + 2xy + y^2 - 3xy)$

$\Rightarrow$ **$x^3 + y^3 = (x + y)\,(x^2 - xy + y^2)$**

Proof of the Identities:

$(x - y)^3 = \ldots$ and $x^3 - y^3 = \ldots$

Let us prove the identity $(x - y)^3 = x^3 - y^3 - 3x^2y + 3xy^2$ OR $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$

We can write $(x - y)^3$ as:

$(x - y)(x - y)^2$

$= (x - y)(x^2 - 2xy + y^2)$

$= x^3 - 2x^2y + xy^2 - x^2y + 2xy^2 - y^3$

$= x^3 - y^3 - 3x^2y + 3xy^2$

$\therefore$ **$(x-y)^3 = x^3-y^3- 3x^2y + 3xy^2$**

$\Rightarrow$ **$(x - y)^3 = x^3 - y^3 - 3xy(x - y)$** … (1)

Let us prove the identity $x^3 - y^3 = (x - y)^3 + 3xy(x - y)$ OR $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

We can rewrite equation 1 as:

$x^3 - y^3 = (x-y)^3 + 3xy(x-y)$

$\Rightarrow x^3 - y^3 = (x - y)[(x - y)^2 + 3xy]$

$\Rightarrow x^3 - y^3 = (x - y)(x^2 - 2xy + y^2 + 3xy)$

$\Rightarrow$ **$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$**

Example Based on the Identity $x^3 - y^3 =$ …

Solved Examples

Easy

Example 1:

Factorise the following expressions.

i) $a^3 - 125b^3 - 15a^2b + 75ab^2$

ii) $27p^3 + 125q^3$

Solution:

i)$a^3 - 125b^3 - 15a^2b + 75ab^2$

$= (a)^3 - (5b)^3 - 15ab\ (a - 5b)$

$= (a)^3 - (5b)^3 - 3 \times a \times 5b\ (a - 5b)$

On using the identity $(x - y)^3 = x^3 - y^3 - 3xy\ (x - y)$, where $x = a$ and $y = 5b$, we are left with $(a - 5b)^3$

$\therefore a^3 - 125b^3 - 15a^2b + 75ab^2 = (a - 5b)^3$

ii)$27p^3 + 125q^3$

$= (3p)^3 + (5q)^3$

On using the identity $x^3 + y^3 = (x + y)\ (x^2 - xy + y^2)$, where $x = 3p$ and $y = 5q$, we get:

$(3p + 5q)\ [(3p)^2 - (3p)\ (5q) + (5q)^2]$

$= (3p + 5q)\ (9p^2 - 15pq + 25q^2)$

$\therefore 27p^3 + 125q^3 = (3p + 5q)\ (9p^2 - 15pq + 25q^2)$

Example 2:

Evaluate the following expressions using identities.

i)1003^3

ii)98^3

Solution:

i)We can write 1003^3 as:

$(1000 + 3)^3$

On using the identity $(x + y)^3 = x^3 + y^3 + 3xy\ (x + y)$, where $x = 1000$ and $y = 3$, we get:

$(1000 + 3)^3 = 1000^3 + 3^3 + 3 \times 1000 \times 3 \times (1000 + 3)$

$= 1000000000 + 27 + 9000 \times (1000 + 3)$

$= 1000000000 + 27 + 9000000 + 27000$

= 1009027027

ii)We can write 98^3 as:

$(100 - 2)^3$

On using the identity $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$, where $x = 100$ and $y = 2$, we get:

$(100 - 2)^3 = 100^3 - 2^3 - 3 \times 100 \times 2 \times (100 - 2)$

$= 1000000 - 8 - 600 \times (100 - 2)$

$= 1000000 - 8 - 60000 + 1200$

$= 941192$

Medium

Example 1:

Expand the following expressions.

i) $\left(\frac{x}{a} + \frac{y}{b}\right)^3$

ii) $(2x + 5y)^3 - (2x - 5y)^3$

Solution:

i) $\left(\frac{x}{a} + \frac{y}{b}\right)^3$

On using the identity $(p + q)^3 = p^3 + q^3 + 3pq(p + q)$, where $p = \frac{x}{a}$ and $q = \frac{y}{b}$, we get:

$$\left(\frac{x}{a} + \frac{y}{b}\right)^3 = \left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 + 3 \times \left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right)$$
$$= \frac{x^3}{a^3} + \frac{y^3}{b^3} + \frac{3xy}{ab}\left(\frac{x}{a} + \frac{y}{b}\right)$$
$$= \frac{x^3}{a^3} + \frac{y^3}{b^3} + \frac{3x^2y}{a^2b} + \frac{3xy^2}{ab^2}$$

ii)$(2x + 5y)^3 - (2x - 5y)^3$

We have two terms in the given expression—$(2x + 5y)^3$ and $(2x - 5y)^3$.

On using the identity $(p + q)^3 = p^3 + q^3 + 3pq\,(p + q)$, where $p = 2x$ and $q = 5y$, we get:

$(2x + 5y)^3 = (2x)^3 + (5y)^3 + 3 \times (2x)\,(5y)\,(2x + 5y)$

$= 8x^3 + 125y^3 + 30xy\,(2x + 5y)$

$= 8x^3 + 125y^3 + 60x^2y + 150xy^2$

Similarly, on using the identity $(p - q)^3 = p^3 - q^3 - 3pq\,(p - q)$, we get

$(2x - 5y)^3 = (2x)^3 - (5y)^3 - 3 \times (2x)\,(5y)\,(2x - 5y)$

$= 8x^3 - 125y^3 - 30xy\,(2x - 5y)$

$= 8x^3 - 125y^3 - 60x^2y + 150xy^2$

So,

$(2x + 5y)^3 - (2x - 5y)^3 = 8x^3 + 125y^3 + 60x^2y + 150xy^2 - [8x^3 - 125y^3 - 60x^2y + 150xy^2]$

= $8x^3 + 125y^3 + 60x^2y + 150xy^2 - 8x^3 + 125y^3 + 60x^2y - 150xy^2$

= $250y^3 + 120x^2y$

Alternate method

On using the identity $p^3 - q^3 = (p - q)\,(p^2 + pq + q^2)$, where $p = (2x + 5y)$ and $q = (2x - 5y)$, we get:

$(2x + 5y)^3 - (2x - 5y)^3$

$= [(2x + 5y) - (2x - 5y)]\,[(2x + 5y)^2 + (2x + 5y)\,(2x - 5y) + (2x - 5y)^2]$

$= (2x + 5y - 2x + 5y)\,(4x^2 + 20xy + 25y^2 + 4x^2 + 10xy - 10xy - 25y^2 + 4x^2 - 20xy + 25y^2)$

$= 10y\,(12x^2 + 25y^2)$

$= 120x^2y + 250y^3$

Example 2:

The side of a cube is *a*. If each side of the cube is increased by b/5, then by how much does its volume increase?

Solution:

Let the side of the cube be *a*.

Original volume of the cube = $a \times a \times a = a^3$

After the increase, each side becomes $\left(a+\frac{b}{5}\right)$.

New volume = $\left(a+\frac{b}{5}\right)\left(a+\frac{b}{5}\right)\left(a+\frac{b}{5}\right)=\left(a+\frac{b}{5}\right)^3$

On using the identity $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$, where $x = a$ and y = b/5, we get:

$$\begin{aligned}\left(a+\frac{b}{5}\right)^3 &= a^3+\left(\frac{b}{5}\right)^3+3(a)\left(\frac{b}{5}\right)\left(a+\frac{b}{5}\right)\\ &= a^3+\frac{b^3}{125}+\frac{3ab}{5}\times\left(a+\frac{b}{5}\right)\\ &= a^3+\frac{b^3}{125}+\frac{3a^2b}{5}+\frac{3ab^2}{25}\end{aligned}$$

Now,

Increase in volume = New volume − Original volume

$$\begin{aligned}&= a^3+\frac{b^3}{125}+\frac{3a^2b}{5}+\frac{3ab^2}{25}-a^3\\ &= \frac{b^3}{125}+\frac{3a^2b}{5}+\frac{3ab^2}{25}\end{aligned}$$

Thus, the volume of the cube increases by $\frac{b^3}{125}+\frac{3a^2b}{5}+\frac{3ab^2}{25}$.

Hard

Example 1:

Find the values of the following expressions.

i) $x^3 + \frac{1}{x^3}$ when $x + \frac{1}{x} = 5$

ii) $8y^3 - \frac{27}{y^3}$ when $4y^2 + \frac{9}{y^2} = 37$

iii) $125x^3 - 27y^3$ when $5x - 3y = 1$ and $xy = 6$

Solution:

i) $x^3 + \frac{1}{x^3}$ when $x + \frac{1}{x} = 5$

We have $x + \frac{1}{x} = 5$

On cubing both sides, we get:

$$\left(x + \frac{1}{x}\right)^3 = 5^3$$

On using the identity $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$, where $a = x$ and $b = 1/x$, we get:

$$\left(x + \frac{1}{x}\right)^3 = 125$$
$$\Rightarrow x^3 + \frac{1}{x^3} + 3(x)\left(\frac{1}{x}\right)\left(x + \frac{1}{x}\right) = 125$$
$$\Rightarrow x^3 + \frac{1}{x^3} = 125 - 3\left(x + \frac{1}{x}\right)$$
$$\Rightarrow x^3 + \frac{1}{x^3} = 125 - 3 \times 5$$
$$\Rightarrow x^3 + \frac{1}{x^3} = 125 - 15$$
$$\Rightarrow x^3 + \frac{1}{x^3} = 110$$

ii) $8y^3 - \frac{27}{y^3}$ when $4y^2 + \frac{9}{y^2} = 37$

We have $4y^2 + \frac{9}{y^2} = 37$

$$\Rightarrow (2y)^2 + \left(\frac{3}{y}\right)^2 = 37$$

On using the identity $a^2 + b^2 = (a - b)^2 + 2ab$, where $a = 2y$ and $b = 3/y$, we get:

$$\Rightarrow \left(2y - \frac{3}{y}\right)^2 + 2(2y)\left(\frac{3}{y}\right) = 37$$
$$\Rightarrow \left(2y - \frac{3}{y}\right)^2 + 12 = 37$$
$$\Rightarrow \left(2y - \frac{3}{y}\right)^2 = 37 - 12$$
$$\Rightarrow \left(2y - \frac{3}{y}\right)^2 = 25$$
$$\Rightarrow \left(2y - \frac{3}{y}\right) = 5 \qquad ...(1)$$

On cubing both sides, we get:

$$\left(2y - \frac{3}{y}\right)^3 = 5^3$$

On using the identity $(a - b)^3 = a^3 - b^3 - 3ab\,(a - b)$, we get:

$$(2y)^3 - \left(\frac{3}{y}\right)^3 - 3(2y)\left(\frac{3}{y}\right)\left(2y - \frac{3}{y}\right) = 125$$
$$\Rightarrow 8y^3 - \frac{27}{y^3} - 18\left(2y - \frac{3}{y}\right) = 125$$

On using equation 1, we get:

$$8y^3 - \frac{27}{y^3} - 18 \times 5 = 125$$

$$\Rightarrow 8y^3 - \frac{27}{y^3} = 125 + 90$$

$$\Rightarrow 8y^3 - \frac{27}{y^3} = 215$$

iii)$125x^3 - 27y^3$ when $5x - 3y = 1$ and $xy = 6$

We have $5x - 3y = 1$

On cubing both sides, we get:

$(5x - 3y)^3 = 1^3$

On using the identity $(a - b)^3 = a^3 - b^3 - 3ab\,(a - b)$, where $a = 5x$ and $b = 3y$, we get:

$(5x)^3 - (3y)^3 - 3\,(5x)\,(3y)\,(5x - 3y) = 1$

$\Rightarrow 125x^3 - 27y^3 - 45xy\,(5x - 3y) = 1$

On substituting the values of xy and $(5x - 3y)$,we get:

$125x^3 - 27y^3 - 45 \times 6 \times 1 = 1$

$\Rightarrow 125x^3 - 27y^3 = 1 + 270$

$\Rightarrow 125x^3 - 27y^3 = 271$

Example 2:

If $x + y = 8$ and $x^2 + y^2 = 42$, then find the value of $x^3 + y^3$.

Solution:

It is given that $x + y = 8$

$\Rightarrow (x + y)^2 = 8^2$

$\Rightarrow x^2 + y^2 + 2xy = 64$

$\Rightarrow 42 + 2xy = 64\ (\because x^2 + y^2 = 42)$

⇒ $2xy = 64 - 42$

⇒ $2xy = 22$

⇒ $xy = \frac{22}{2}$

⇒ ∴ $xy = 11$

Now, $(x + y)^3 = 8^3$

⇒ $x^3 + y^3 + 3xy(x + y) = 512$

⇒ $x^3 + y^3 + 3 \times 11 \times 8 = 512$

⇒ $x^3 + y^3 + 264 = 512$

⇒ $x^3 + y^3 = 512 - 264$

⇒ ∴ $x^3 + y^3 = 248$

Example 3:

Prove that $\frac{\mathbf{0.77 \times 0.77 \times 0.77 + 0.23 \times 0.23 \times 0.23}}{\mathbf{0.77 \times 0.77 - 0.77 \times 0.23 + 0.23 \times 0.23}} = \mathbf{1}$.

Solution:

$$\frac{0.77 \times 0.77 \times 0.77 + 0.23 \times 0.23 \times 0.23}{0.77 \times 0.77 - 0.77 \times 0.23 + 0.23 \times 0.23}$$
$$= \frac{(0.77)^3 + (0.23)^3}{(0.77)^2 - 0.77 \times 0.23 + (0.23)^2}$$
$$= \frac{x^3 + y^3}{x^2 - xy + y^2}, \text{ where } x = 0.77 \text{ and } y = 0.23$$
$$= \frac{(x + y)(x^2 - xy + y^2)}{(x^2 - xy + y^2)}$$
$$= (0.77 + 0.23)$$
$$= 1$$

Factorisation of Algebraic Expressions Using Method of Common Factors

You know about the prime factorization of numbers. Let us revise the method of prime factorization by taking the example of the number 210.

2	210
3	105
5	35
7	7
	1

We can write 210 as a product of 2, 3, 5, and 7.

Hence, 210 = 2 × 3 × 5 × 7

Here, 2, 3, 5, and 7 are the prime factors of 210. In the same way, we can factorize any expression, i.e., we can write any expression as a product of its factors.

For example, $2xyz = 2 \times x \times y \times z$

Here, 2, x, y, and z are the factors of $2xyz$, and we cannot further reduce them. Thus, we say that 2, x, y, and z are the irreducible factors of $2xyz$.

Let us try to find the factors of $3x + 9$ in the given video.

Now let us take another example in the given video to understand the concept of factors of a polynomial.

The process of writing a given algebraic expression as a product of two or more expressions is called factorization. Each of the expressions which form the product is called a factor of the given expression.

Let us discuss some more examples based on the above concept.

Example 1:

Find the common factors of the terms $6pq$, $8p^2$, and $4pq^2$.

Solution:

Write the factors of each term.

$6pq = 2 \times 3 \times p \times q$

$8p^2 = 2 \times 2 \times 2 \times p \times p$

$4pq^2 = 2 \times 2 \times p \times q \times q$

Here, 2 and p are the common factors of the given terms $6pq$, $8p^2$, and $4pq^2$.

Example 2:

Factorize $6x^2 - 18x$.

Solution:

Write the factors of each terms.

$6x^2 = 2 \times 3 \times x \times x$

$18x = 2 \times 3 \times 3 \times x$

HCF of $6x^2$ and $18x = 2 \times 3 \times x = 6x$

$\therefore 6x^2 - 18x = 6x(x - 3)$

Example 3:

Factorize the following expressions:

(i) $7x^2 + 14x$

(ii) $4a^2bcx^2y^3z^4 - 5ab^2cx^3y^4z^2 + 7abc^2x^4y^2z^3$

(iii) $-4a^2 + 3p^3 - 5b$

Solution:

(i) Write the factors of each term.

$7x^2 = 7 \times x \times x$

$14x = 2 \times 7 \times x$

Here, 7 and x are the common factors.

$7x^2 + 14x = 7 \times x \times x + 2 \times 7 \times x$

$= 7 \times x(x + 2)$

$= 7x(x + 2)$

(ii) Write the factors of each term.

$$4a^2bcx^2y^3z^4 = 2 \times 2 \times \underline{a} \times a \times \underline{b} \times \underline{c} \times \underline{x} \times \underline{x} \times \underline{y} \times \underline{y} \times y \times \underline{z} \times \underline{z} \times z \times z$$

$$-5ab^2cx^3y^4z^2 = -5 \times \underline{a} \times \underline{b} \times b \times \underline{c} \times \underline{x} \times \underline{x} \times x \times \underline{y} \times \underline{y} \times y \times y \times \underline{z} \times \underline{z}$$

$$7abc^2x^4y^2z^3 = 7 \times \underline{a} \times \underline{b} \times \underline{c} \times c \times \underline{x} \times \underline{x} \times x \times x \times \underline{y} \times \underline{y} \times \underline{z} \times \underline{z} \times z$$

Here, a, b, c, x, x, y, y, z, and z are the common factors of given terms.

Hence, $4a^2bcx^2y^3z^4 - 5ab^2cx^3y^4z^2 + 7abc^2x^4y^2z^3$

$$= a \times b \times c \times x^2 \times y^2 \times z^2 \begin{Bmatrix} 2 \times 2 \times a \times y \times z \times z - 5 \times b \times x \times y \times y \\ + 7 \times c \times x \times x \times z \end{Bmatrix}$$

$$= abcx^2y^2z^2\left(4ayz^2 - 5bxy^2 + 7cx^2z\right)$$

.

(iii) Write the factors of each term.

$-4a^2 = -2 \times 2 \times a \times a$

$3p^3 = 3 \times p \times p \times p$

$-5b = -5 \times b$

There is no common factor of these terms other than 1.

Factorisation of Algebraic Expressions Using Method of Regrouping Terms

Can you factorize the algebraic expression $4ab + a + 4b + 1$?

All four terms in the expression do not have any common factor except 1. Thus, we cannot factorize the expression by taking the common factors of each term.

Now, let us apply the method of regrouping to the above expression and see if we can factorize it or not.

Let us look at the following example based on the above concept.

Example 1:

Factorize the following expressions:

(i) $2x + ax - 2y - ay$

(ii) $2a + 3b - 2 - 3ab$

Solution:

(i) The given expression is $2x + ax - 2y - ay$.

We can factorize this expression as

$2x + ax - 2y - ay = (2x - 2y) + (ax - ay)$

$= 2\ (x - y) + a\ (x - y)$

$= (x - y)\ (2 + a)$

$= (x - y)\ (2 + a)$

(ii) The given expression is $2a + 3b - 2 - 3ab$.

We can factorize this expression as

$2a + 3b - 2 - 3ab = (2a - 2) + (-\ 3ab + 3b)$

$= 2\ (a - 1) + (-3b)\ (a - 1)$

$= 2\ (a - 1) - 3b\ (a - 1)$

$= (a - 1)\ (2 - 3b)$

Using Identity for "Difference of Two Squares"

Suppose we need to find the product of the numbers 79 and 81. Instead of multiplying these two numbers, we can use the identity $(a + b)\ (a - b)$. This identity is very important and is applicable in various situations.

Let us first understand this identity.

$(a + b)\ (a - b) = a\ (a - b) + b\ (a - b)$ (By distributive property)

$= a^2 - ab + ba - b^2$

$= a^2 - ab + ab - b^2$ $(ab = ba)$

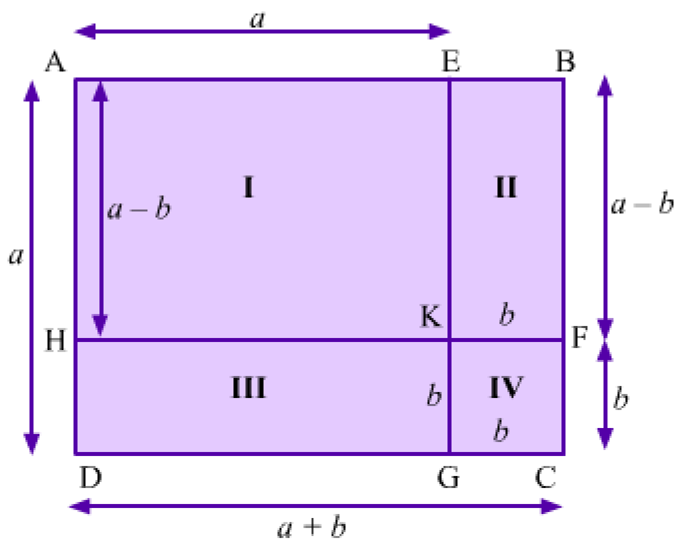
$= a^2 - b^2$

∴ $(a + b)(a - b) = a^2 - b^2$

Deriving the identity geometrically:

This identity can be derived from geometric construction as well.

For this, let us consider a square AEGD whose each side measures *a* units.



It can be seen that, a segment BC is drawn outside the square AEGD such that BC is parallel to EG and at a distance of *b* unit from G.

Another segment HF is drawn inside the square AEGD such that HF is parallel to GD and at a distance of *b* unit from G.

From the figure, it can be observed that

Area of rectangle ABFH = Area of rectangle ABCD – (Area of rectangle HKGD + Area of square KFCG)

⇒ $(a + b)(a - b) = a(a + b) - [(a \times b) + (b \times b)]$

⇒ $(a + b)(a - b) = a^2 + ab - ab - b^2$

⇒ $(a + b)(a - b) = a^2 - b^2$

Now, let us solve some examples in which the above identity can be applied.

Example 1:

Simplify the following expressions.

(a) (*x* + 3) (*x* − 3)

(b) (11 + *y*) (11 − *y*)

Solution:

(a) $(x + 3)(x - 3)$

This expression is of the form $(a + b)(a - b)$.

Hence, we can use the identity $(a + b)(a - b) = a^2 - b^2$.

$(x + 3)(x - 3) = x^2 - 3^2 = x^2 - 9$

(b) $(11 + y)(11 - y)$

This expression is of the form $(a + b)(a - b)$.

Hence, we can use the identity $(a + b)(a - b) = a^2 - b^2$.

$(11 + y)(11 - y) = 11^2 - y^2 = 121 - y^2$

Example 2:

Simplify the following expressions.

(a) $\left(\frac{3}{7}l + \frac{4}{5}m\right)\left(\frac{3}{7}l - \frac{4}{5}m\right)$

(b) $\left(x^2 - y^3\right)\left(x^2 + y^3\right) + \left(y^3 - z^4\right)\left(y^3 + z^4\right) + \left(z^4 - x^2\right)\left(z^4 + x^2\right)$

Solution:

(a) The given expression is $\left(\frac{3}{7}l + \frac{4}{5}m\right)\left(\frac{3}{7}l - \frac{4}{5}m\right)$.

Using the identity $(a + b)(a - b) = a^2 - b^2$, we get

$$\therefore \left(\frac{3}{7}l+\frac{4}{5}m\right)\left(\frac{3}{7}l-\frac{4}{5}m\right)=\left(\frac{3}{7}l\right)^2-\left(\frac{4}{5}m\right)^2$$
$$=\frac{9}{49}l^2-\frac{16}{25}m^2$$

(b) $\left(x^2-y^3\right)\left(x^2+y^3\right)+\left(y^3-z^4\right)\left(y^3+z^4\right)+\left(z^4-x^2\right)\left(z^4+x^2\right)$

Using the identity $(a+b)(a-b)=a^2-b^2$, we get

$$\left\{\left(x^2\right)^2-\left(y^3\right)^2\right\}+\left\{\left(y^3\right)^2-\left(z^4\right)^2\right\}+\left\{\left(z^4\right)^2-\left(x^2\right)^2\right\}$$
$$=x^4-y^6+y^6-z^8+z^8-x^4$$
$$=\left(x^4-x^4\right)+\left(-y^6+y^6\right)+\left(-z^8+z^8\right)$$
$$=0$$

Example 3:

Find the values of the following expressions using suitable identities.

(a) 195 × 205

(b) $(993)^2-(7)^2$

(c) 24.5 × 25.5

Solution:

(a) 195 = 200 − 5

and, 205 = 200 + 5

∴ 195 × 205 = (200 − 5) × (200 + 5)

$=(200)^2-(5)^2$ [∵ $(a+b)(a-b)=a^2-b^2$]

= 40000 − 25

= 39975

(b) $(993)^2-(7)^2$

= (993 + 7) (993 − 7) [∵ $(a+b)(a-b)=a^2-b^2$]

= (1000) (986)

= 986000

(c) 24.5 × 25.5

= (25 − 0.5) (25 + 0.5)

= $(25)^2 - (0.5)^2$ [∵ $(a + b)(a - b) = a^2 - b^2$]

= 625 − 0.25

= 624.75

Factorisation of Quadratic Trinomial

Factorisation of Quadratic Trinomial of the Form $ax^2 + bxy + cy^2$

Consider the following algebraic expression.

$2x^2 - x - 6$

This is a quadratic trinomial.

Factorisation of this expression can be done in the following manner:

$$\begin{aligned}&2x^2 - x - 6\\&= 2x^2 - 4x + 3x - 6 \qquad [\text{Since}, (-6)\times 2 = (-12), (-4)\times 3 = (-12) \text{ and } (-4)+3 = (-1)]\\&= 2x(x-2) + 3(x-2)\\&= (2x+3)(x-2)\end{aligned}$$

This type of factorisation is based on the concept that $x^2 + (a+b)x + ab = (x+a)(x+b)$.

Quadratic trinomial of the form $ax^2 + bxy + cy^2$ can also be factorised by the method similar to the factorisation of $ax^2 + bx + c$.

So, let us factorise the trinomial $2x^2 + 3xy - 5y^2$.

Observe that here, 2 × (−5) = (−10)

Also, 5 × (−2) = (−10) and 5 − 2 = 3

$$\therefore 2x^2+3xy-5y^2 = 2x^2+5xy-2xy-5y^2$$
$$= x(2x+5y)-y(2x+5y)$$
$$= (x-y)(2x+5y)$$

Similarly, we can factorise a given trinomial of form $ax^2+bxy+cy^2$.

Sometimes, we find a few algebraic expressions which are not quadratic. But, they can still be expressed in the form ax^2+bx+c or $ax^2+bxy+cy^2$. Thus, they can be factorised.

Consider the following algebraic expression.

$$x^4+x^2y^2-12y^4$$

Let us assume that $x^2 = a \text{ and } y^2 = b$.

$$\therefore x^4 = a^2,\ y^4 = b^2 \text{ and } x^2y^2 = ab$$

$$\therefore x^4+x^2y^2-12y^4 = a^2+ab-12b^2 \qquad (\text{Form } ax^2+bxy+cy^2)$$
$$= a^2+4ab-3ab-12b^2$$
$$= a(a+4b)-3b(a+4b)$$
$$= (a-3b)(a+4b)$$

On re-substituting the values of *x* and *y*, we get

$$x^4+x^2y^2-12y^4 = (x^2-3y^2)(x^2+4y^2)$$
$$= \left\{(x)^2-\left(\sqrt{3}y\right)^2\right\}(x^2+4y^2)$$
$$= (x-\sqrt{3}y)(x+\sqrt{3}y)(x^2+4y^2)$$

Let us understand this concept better by solving some examples based on this method.

Example 1: Factorise the following algebraic expressions.

(i) $2x^2-xy-6y^2$

(ii) $12a^2-7ab-10b^2$

Solution:

$$\begin{aligned}
&\text{(i)}\ 2x^2 - xy - 6y^2 \\
&= 2x^2 - 4xy + 3xy - 6y^2 \quad [\text{Since}, 2\times(-6) = (-12),\ (-4)\times 3 = (-12) \text{ and } (-4)+3 = (-1)] \\
&= 2x(x-2y) + 3y(x-2y) \\
&= (2x+3y)(x-2y)
\end{aligned}$$

$$\begin{aligned}
&\text{(ii)}\ 12a^2 - 7ab - 10b^2 \\
&= 12a^2 - 15ab + 8ab - 10b^2 \quad [\text{Since}, 12\times(-10) = (-120),\ (-15)\times 8 = (-120) \text{ and } (-15)+8 = (-7)] \\
&= 3a(4a-5b) + 2b(4a-5b) \\
&= (3a+2b)(4a-5b)
\end{aligned}$$

Example 2:

Factorise the following algebraic expressions.

(i) $2(y^2-y)\left(y^2-y-3\right)-8$

(ii) $x^4 - 4x^2y^2 - 21y^2$

Solution:

$$\begin{aligned}
&\text{(i)}\ 2(y^2-y)\left(y^2-y-3\right)-8 \\
&\quad \text{Let } y^2 - y = a \\
&\quad \therefore 2(y^2-y)\left(y^2-y-3\right)-8 = 2a(a-3)-8 \\
&\qquad = 2\left(a^2-3a-4\right) \\
&\qquad = 2(a^2-4a+a-4) \qquad [\text{Since}, 1\times(-4) = -4 \text{ and } 1+(-4) = -3] \\
&\qquad = 2(a-4)(a+1)
\end{aligned}$$

On re-substituting the value of *a*, we get

$$2(y^2-y)\left(y^2-y-3\right)-8 = 2(y^2-y-4)(y^2-y+1)$$

(ii) $x^4 - 4x^2y^2 - 21y^2$

Let $x^2 = a$ and $y^2 = b$

$$\therefore x^4 - 4x^2y^2 - 21y^2 = a^2 - 4ab - 21b^2$$

$$= a^2 - 7ab + 3ab - 21b^2$$

[Since, $1 \times (-21) = (-21), (-7) \times (3) = (-21)$ and $(-7) + 3 = (-4)$]

$$= a(a - 7b) + 3b(a - 7b)$$

$$= (a + 3b)(a - 7b)$$

Re-substituting the values of *a* and *b*, we get

$$x^4 - 4x^2y^2 - 21y^2 = (x^2 + 3y^2)(x^2 - 7y^2)$$

$$= (x^2 + 3y^2)(x + \sqrt{7}y)(x - \sqrt{7}y)$$