Integration by parts

Q.1. Evaluate : $_0\int^{1/2} \sin^{-1} x/(1 - x^2)^{3/2}\, dx$.

Solution : 1

We have , $I = {_0\int^{1/2}} \sin^{-1}x/(1 - x^2)^{3/2}\, dx$,

Put $\sin^{-1} x = t \Rightarrow x = \sin t \Rightarrow dx = \cos t\, dt$

[when $x = 0$, $\sin t = 0 \Rightarrow t = 0$ and when $x = 1/2$, $\sin t = 1/2 \Rightarrow t = \pi/6$]

Therefore, $I = {_0\int^{1/2}} \sin^{-1} x/(1 - x^2)^{3/2}\, dx = {_0\int^{\pi/6}} (t/\cos^3 t).\cos t\, dt$

$= {_0\int^{\pi/6}} t \sec^2 t\, dt$

$= [t\int \sec2\, t\, dt - \int\{d/dt(t).\int \sec^2 t\, dt\}]0^{\pi/6}$ [Integrating by parts]

$= [t.\tan t - \int \tan t dt\,]0\pi/6$

$= [t.\tan t - \log \sec t]0\pi/6$

$= [\pi/6 \tan \pi/6 - \log \sec \pi/6] - [0 - \log \sec 0]$

$= [\pi/6(1/\sqrt{3}) - \log (2/\sqrt{3})] - [0 - 0]$

$= [\pi/(6\sqrt{3}) - \log (2/\sqrt{3})]$

Q.2. Evaluate : $\int e^{-2x} \sin x\, dx$.

Solution : 2

Let $I = \int e^{-2x} \sin x\, dx = \sin x \int e^{-2x} dx - \int \cos x\, (-1/2)\, e^{-2x}\, dx$

$= - 1/2 e^{-2x} \sin x + 1/2 \int e^{-2x} \cos x\, dx$

$\int e^{-2x} \cos x\, dx = \cos x \int e^{-2x} dx - \int(- \sin x)(- 1/2\, e^{-2x})\, dx$

$= - 1/2\, e^{-2x} \cos x - 1/2 \int e -2x \sin x\, dx$

Therefore, $I = - 1/2\, e^{-2x} \sin x + 1/2 \{- 1/2e -2x \cos x - 1/2 \int e^{-2x} \sin x\, dx\}$

$= - 1/2\ e^{-2x} \sin x - 1/4\ e^{-2x} \cos x - 1/4\ I$

Or, $5/4\ I = - 1/2\ e^{-2x} \sin x - 1/4\ e^{-2x} \cos x$

Or, I = – 2/5 e –2x sin x – 1/5 e –2x cos x + c .

Q.3. Evaluate : $\int e^x[(1 + \sin x)/(1 + \cos x)]\ dx$.

Solution : 3

We are given,

$\int e^x\ [(1 + \sin x)/(1 + \cos^x)]dx$

$= \int e^x[\{(1 + \sin x)(1 - \cos^x)\}/\{(1 + \cos^x)(1 - \cos^x)\}]dx$

$= \int e^x[(1 - \cos x + \sin x - \sin x \cos x)/(1 - \cos^2 x)]dx$

$= \int e^x[(1 - \cos x + \sin x - \sin x \cos x)/\sin^2 x]dx$

$= \int e^x\ (\text{cosec}^2 x - \text{cosec}\ x \cot x + \text{cosec}\ x - \cot x)dx$

$= \int e^x\ \text{cosec}^2 x\ dx - \int e^x\ \text{cosec}\ x\ dx + \int ex\ \text{cosec}\ x\ dx - \int e^x \cot x\ dx$

$= e^x \int \text{cosec}^2 x\ dx - \int e^x\ (- \cot x)dx + ex\int(- \text{cosec}\ x \cot x)dx - \int e^x\ \text{cosec}\ x\ dx +$

$e^x\int \text{cosec}\ x\ dx - \int e^x \cot x\ dx$

$= e^x\ (- \cot x) + \int e^x \cot x\ dx + ex\ \text{cosec}\ x - \int ex\text{cosec}\ xdx + \int e^x\ \text{cosec}\ x\ dx - \int e^x \cot x\ dx$

$= e^x\ \text{cosec}\ x - ex \cot x = e^x\ (\text{cosec}\ x - \cot x) + c.$

Q.4. Evaluate : $\int x.\cos x .dx$.

Solution : 4

Let $I = \int x.\cos x.dx = x\int \cos x\ dx - \int\{d/dx(x).\int \cos x\ dx\}.dx$

$= x \sin x - \int 1.\sin x\ dx = x \sin x - \int \sin x\ dx$

$= x \sin x + \cos x + c.$

Q.5. Evaluate : $\int x^2 \sin 2x \, .dx$.

Solution : 5

$\int x^{2} \sin 2x \, dx = x^{2}\int \sin 2x dx - \int\{d/dx(x^{2}).\int \sin 2x \, dx\}.dx$

$= x^{2} \{(-1/2) \cos 2x\} - \int 2x .\{(-1/2) \cos 2x\}dx$

$= (-1/2)x^{2} \cos 2x + \int x \cos 2x.dx$

$= (-1/2)x^{2} \cos 2x + x.\{(1/2) \sin 2x\} - \int 1.\{(1/2) \sin 2x\}dx$

$= (-1/2) x^{2} \cos 2x + (1/2) x \sin 2x -1/2\{(-1/2) \cos 2x\} + c$

$= (-1/2) x^{2} \cos 2x + (1/2) x \sin 2x + (1/4) \cos 2x + c$.

Q.6. Evaluate : $\int x^2 e^{3x}.dx$.

Solution : 6

$\int x^{2}e^{3x}dx = x^{2}\int e^{3x}.dx - \int[d/dx(x^{2})\int e^{3x}.dx].dx$

$= x^{2}(1/3)e^{3x} - \int 2x(1/3)e^{3x}.dx$

$= (1/3)x^{2}e^{3x} - (2/3)\int xe^{3x}.dx$

$= (1/3)x^{2}e^{3x} - (2/3)[x(1/3)e^{3x} - \int 1.(1/3)e^{3x}.dx]$

$= (1/3)x^{2}e^{3x} - (2/9)xe^{3x} + (2/9)(1/3)e^{3x} + c$

$= [(1/3)x^{2} - (2/9)x + 2/27]e^{3x} + c$.

Q.7. Evaluate : $\int[x^2/(a + bx)^2].dx$.

Solution : 7

$\int[x^{2}/(a + bx)^{2}].dx = \int[x^{2}(a + bx)^{-2}].dx$

$= x^{2}\int(a + bx)^{-2}.dx - \int[d/dx(x^{2})\int(a + bx)^{-2}.dx].dx$

$= x^{2}[(a + bx)^{-1}/(-b)] - \int 2x[(a + bx)^{-1}/(-b)].dx$

$= -[x^{2}/\{b(a + bx)\}] + (2/b^{2})\int[bx/(a + bx)].dx$

$= -[x^{2}/\{b(a + bx)\}] + (2/b^{2})\int[\{(a + bx) - a\}/(a + bx)].dx$

$= -[x^{2}/\{b(a + bx)\}] + (2/b^{2})[\int dx - \int\{a/(a + bx)\}.dx$

$= -[x^{2}/\{b(a + bx)\}] + (2/b^{2})[x - (a/b)\log(a + bx)] + c$

$= -[x^{2}/\{b(a + bx)\}] + 2x/b^{2} - (2a/b^{3})\log(a + bx) + c.$

Q.8. Evaluate : $\int[\log x/x].dx$.

Solution : 8

Let $I = \int[\log x/x].dx = \int(\log x)(1/x).dx$

$= \log x\int(1/x).dx - \int[d/dx(\log x)\int(1/x).dx].dx$

$= (\log x).(\log x) - \int(1/x).(\log x).dx = (\log x)^{2} - I$

Or, $2I = (\log x)^{2}$

Or, $I = 1/2\,(\log x)^{2} + c.$

Q.9. Evaluate : $\int(\log x)^{2}\sqrt{x}.dx$.

Solution : 9

Let I

$= \int(\log x)^{2}\sqrt{x}.dx = (\log x)^{2}\int\sqrt{x}.dx - \int[d/dx\{(\log x)^{2}\}\int(\sqrt{x}).dx].dx$

$= (\log x)^{2}\{(x^{3/2})/(3/2)\} - \int\{(2\log x)(1/x)\}\{(x^{3/2})/(3/2)\}.dx$

$= (2/3)(x^{3/2})(\log x)^{2} - 4/3\int(\log x)x^{1/2}.dx$

$= (2/3)(x^{3/2})(\log x)^{2} - 4/3[\log x\int\sqrt{x}.dx - \int\{d/dx(\log x)\int\sqrt{x}.dx\}.dx]$

$= (2/3)(x^{3/2})(\log x)^{2} - 4/3[\log x\{(x^{3/2})/(3/2)\} - \int(1/x)\{(x^{3/2})/(3/2)\}]$

$= (2/3)(x^{3/2})(\log x)^{2} - 4/3[(2/3)(x^{3/2})\log x - (2/3)\int(x^{1/2})] + c$

$= (x^{3/2})[(2/3)(\log x)^{2} - (8/9)(\log x) + 16/27] + c.$

Q.10. Evaluate : $\int x \tan^{-1} x.dx$

Solution : 10

Let I = $\int x \tan^{-1} x.dx = (\tan^{-1} x)\int x.dx - \int[d/dx(\tan^{-1} x)\int x.dx].dx$

$= (\tan^{-1} x)(x^{2}/2) - \int[\{1/(1 + x^{2})\}(x^{2}/2)].dx$

$= (x^{2}/2)(\tan^{-1} x) - (1/2)\int\{x^{2}/(1 + x^{2})\}dx$

$= (x^{2}/2)(\tan^{-1} x) - (1/2)\int[1^{-1}/(1 + x^{2})].dx$

$= (x^{2}/2)(\tan^{-1} x) - (1/2)[x - \tan -1 x] + c$

$= (1/2)[(x^{2} + 1) \tan^{-1} x - x] + c.$

Q.11. Evaluate : $\int\log \{x + \sqrt{(a^2 + x^2)}\}.dx.$

Solution : 11

Let I = $\int\log \{x + \sqrt{(a^{2} + x^{2})}\}.dx = \int[\log \{x + \sqrt{(a^{2} + x^{2})}\}].1.dx$

$= \log \{x + \sqrt{(a^{2} + x^{2})}\}\int 1.dx - \int d/dx[\log \{x + \sqrt{(a^{2} + x^{2})}\int 1.dx$

$= x \log \{x + \sqrt{(a^{2} + x^{2})} - \int\{1/\sqrt{(a^{2} + x^{2})}\}.x.dx$

$= x \log \{x + \sqrt{(a^{2} + x^{2})} - (1/2)\int\{(a^{2} + x^{2}) -1/2\}(2x).dx$

$= x \log \{x + \sqrt{(a^{2} + x^{2})} - (1/2)\{(a^{2} + x^{2}) -1/2\}/(1/2) + c$

$= x \log \{x + \sqrt{(a^{2}+ x^{2})} - \sqrt{(a^{2} + x^{2})} + c.$

Q.12. Evaluate : $\int\cos 2x \log \sin x .dx.$

Solution : 12

Let I = $\int \cos 2x \log \sin x.dx = \int (\log \sin x).\cos 2x.dx$

$= (\log \sin x)\int \cos 2x.dx - \int[d/dx(\log \sin x)\int \cos 2x.dx].dx$

$=(\log \sin x)(\sin 2x/2) - \int(\cos x/\sin x)(\sin 2x/2)dx$

$= (\log \sin x)\{(2\sin x.\cos x)/2\} - \int(\cos x/\sin x)\{(2\sin x.\cos x)/2\}.dx$

$= \sin x.\ \cos x.\ \log \sin x - \int \cos2\ x.dx = \sin x.\ \cos x.\ \log \sin x - \int\{(1 + \cos 2x)/2\}.dx$

$= \sin x \cos x \log \sin x - 1/2\ (x + \sin 2x/2) + c$

$= \sin x \cos x \log \sin x - 1/2\ (x + \sin x \cos x) + c.$

Q.13. Evaluate : $\int[(x \sin^{-1} x)/\sqrt{(1 - x^2)}].dx.$

Solution : 13

Let I = $\int[(x \sin^{-1} x)/\sqrt{(1 - x^2)}].dx$ [put $x = \sin\theta$, then $dx = \cos\theta d\theta$]

$I = \int\{\theta.\ \sin\theta)/\sqrt{(1 - \sin^2\theta)}\}.\cos\theta\ d\theta$

$= \int\theta \sin\theta\ d\theta = \theta\ (-\cos\theta) - \int 1\ .(-\cos\theta)d\theta$

$= -\theta\cos\theta + \sin\theta + c = -\theta\sqrt{(1 - \sin^2\theta)} + \sin\theta + c$

$= -\sqrt{(1 - x^2)}\ \sin -1\ x + x + c.$

Q.14. Evaluate : $\int[(x + \sin x)/(1 + \cos x)].dx.$

Solution : 14

Let I = $\int[(x + \sin x)/(1 + \cos x)].dx$

$= \int[(x + 2 \sin 1/2\ x.\cos 1/2\ x)/(1 + 2 \cos2\ 1/2\ x - 1)].dx$

$= \int[(x + 2 \sin 1/2\ x \cos 1/2\ x)/2 \cos2\ 1/2\ x].dx$

$= 1/2 \int x.\ \sec2\ 1/2\ x.dx + \int \tan 1/2\ x.dx$

= 1/2 [x.(2 tan 1/2 x) – ∫1.(2 tan 1/2 x).dx + ∫tan 1/2 x.dx

= x. tan 1/2 x – ∫tan 1/2 x.dx + ∫tan 1/2x.dx

= x. tan 1/2 x + c.

Q.15. Evaluate : $\int[xe^x/(x + 1)^2].dx$.

Solution : 15

Let I = $\int[xe^{\mathbf{x}}/(x + 1)^{\mathbf{2}}].dx = \int(x.e^{\mathbf{x}})\{(x + 1) -2\}.dx$

= $(x.e^{\mathbf{x}})\int[(x + 1)^{\mathbf{-2}}].dx - \int[d/dx(x.e^{\mathbf{x}})\int\{(x + 1)^{\mathbf{-2}}\}d^{\mathbf{x}}].d^{\mathbf{x}}$

= $- xe^{\mathbf{x}}/(x + 1) - \int(xe^{\mathbf{x}} + e^{\mathbf{x}}.1)(-1/(x + 1).dx$

= $- xe^{\mathbf{x}}/(x + 1) + \int e^{\mathbf{x}}(x + 1)/(x + 1).dx$

= $- xe^{\mathbf{x}}/(x + 1) + \int e^{\mathbf{x}}.dx$

= $- xe^{\mathbf{x}}/(x + 1)$ + ex + c

= $e^{\mathbf{x}}/(x + 1) + c$.