6. CENTRE OF MASS AND THE LAW OF CONSERVATION OF MOMENTUM

1. INTRODUCTION

In this chapter we will study the motion of system of particles or bodies. The individual particles or bodies comprising the system in the general case move with different velocities and accelerations and exert forces on each other and are influenced by external or surrounding bodies as well. We will learn the techniques to simplify the analysis of complicated motion of such a system. We will also learn about the dynamics of extended bodies whose shape and/or mass changes during their motion. We define the linear momentum of a system of particles and introduce the concept of center of mass of a system. The dynamics of center of mass and the law of conservation of linear momentum are important tools in the study of system of particles.

2 CENTER OF MASS

When we study the dynamics of the motion of a system of particles as a whole, then we need not bother about the dynamics of individual particles of the system, but only focus on the dynamics of a unique point corresponding to that system. The motion of this unique point is identical to the motion of a single particle whose mass is equal to the sum of the masses of all the individual particles of the system and the resultant of all the forces exerted on all the particles of the system, by the surrounding bodies, or due to action of a field of force, is exerted directly to that particle. This point is called the center of mass (COM) of the system of particles. The COM behaves as if the entire mass of the system is concentrated there. The concept of COM is very useful in analyzing complicated motion of system of objects, in particular, when two or more objects collide or an object explodes into fragments.

2.1 Center of Mass of a System of Particles

For a system of n particles, having masses $m_1, m_2, m_3 ..... m_n$ and position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n$ respectively with respect to the origin in a certain reference frame, the position vector of center of mass, $\vec{r}_{cm}$ with respect to the origin is given by

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n}{m_1 + m_2 + ... + m_n}$$

If the total mass of the system is M, then $M\vec{r}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n$

Co-ordinates of center of mass are

$$x_{cm} = \frac{x_1m_1 + x_2m_2 + ...x_mm_n}{m_1 + m_2 + ...m_n}; \qquad y_{cm} = \frac{y_1m_1 + y_2m_2 + ...y_mm_n}{m_1 + m_2 + ...m_n}; \qquad z_{cm} = \frac{z_1m_1 + z_2m_2 + ...z_mm_n}{m_1 + m_2 + ...m_n}$$

For a system comprising of two particles of masses m_1, m_2, positioned at co-ordinates (x_1, y_1, z_1) and (x_2, y_2, z_1), respectively, we have

$$X_{com} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}; \qquad Y_{com} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}; \qquad Z_{com} = \frac{m_1z_1 + m_2z_2}{m_1 + m_2}$$

PLANCESS CONCEPTS

For a two-particle system, COM lies closer to the particle having more mass, which is rather obvious. If COM's co-ordinates are made zero, we would clearly observe that distances of individual particles are inversely proportional to their masses.

Vaibhav Gupta (JEE 2009, AIR 54)

Illustration 1: Two particles of masses 1 kg and 2 kg are located at x = 0 and x = 3 m respectively. Find the position of their center of mass. **(JEE MAIN)**

Sol: For the system of particle of masses m_1 and m_2, if the distance of particle from the center of mass are r_1 and r_2 respectively then it is seen that $m_1r_1 = m_2r_2$.

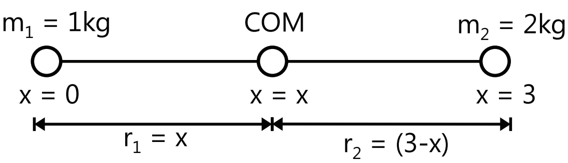


Figure 6.1

Since, both the particles lie on x-axis, the COM will also lie on the x-axis. Let the COM be located at x = x, then r_1 = distance of COM from the particle of mass 1 kg = x

and r_2 = distance of COM from the particle of mass 2 kg = (3 – x)

Using $\frac{r_1}{r_2} = \frac{m_2}{m_1}$ or $\frac{x}{3-x} = \frac{2}{1}$ or x = 2 m

Thus, the COM of the two particles is located at x = 2 m.

Illustration 2: Four particles A, B, C and D having masses m, 2m, 3m and 4m respectively are placed in order at the corners of a square of side a. Locate the center of mass. **(JEE ADVANCED)**

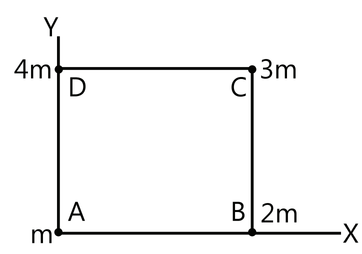


Figure 6.2

Sol: The co-ordinate of center of mass of n particle system are given as

$$X = \frac{\sum_i m_ix_i}{\sum_i m_i}, \quad Y = \frac{\sum_i m_iy_i}{\sum_i m_i}$$

Take the x and y axes as shown in Fig. 6.2. The coordinates of the four particles are as follows:

Particle	Mass	x-coordinate	y-coordinate
A	m	0	0 (taking A as origin)
B	2m	a	0
C	3m	a	a
D	4m	0	a

Hence, the coordinates of the center of mass of the four-particle system are

$$X = \frac{m.0 + 2ma + 3ma + 4m.0}{m + 2m + 3m + 4m} = \frac{a}{2}; \quad Y = \frac{m.0 + 2m.0 + 3ma + 4ma}{m + 2m + 3m + 4m} = \frac{7a}{10}$$

The center of mass is at $\left(\frac{a}{2}, \frac{7a}{10}\right)$.

2.2. Center of Mass of a Continuous Body

For continuous mass distributions, the co-ordinates of center of mass are determined by following formulae,

$$x_{cm} = \frac{\int x\,dm}{\int dm}; \qquad y_{cm} = \frac{\int y\,dm}{\int dm}; \qquad z_{cm} = \frac{\int z\,dm}{\int dm}$$

where x, y and z are the co-ordinates of an infinitesimal elementary mass dm taken on the continuous mass distribution. The integration should be performed under proper limits, such that the elementary mass covers the entire body.

PLANCESS CONCEPTS

Many people have misconception that the center of mass of a continuous body must lie inside the body. Center of mass of a continuous body may lie outside that body also. e.g. Ring.

Vaibhav Krishnan (JEE 2009, AIR 22)

(a) Center of Mass of a Uniform Straight Rod

Center of mass of a rod of length L is at (L/2, 0, 0).

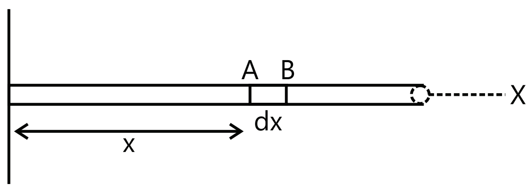


Figure 6.3

(b) Center of Mass of a Uniform Semicircular Wire

Center of mass of a Uniform Semicircular Wire of radius R is (0, 2R/π).

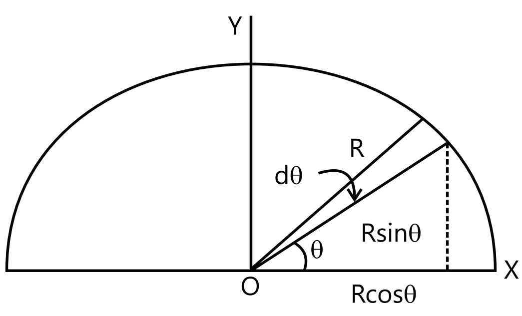


Figure 6.4

(c) Center of Mass of a Uniform Semicircular Plate

Center of mass of a uniform semicircular plate of radius R is (0, 4R/3π)

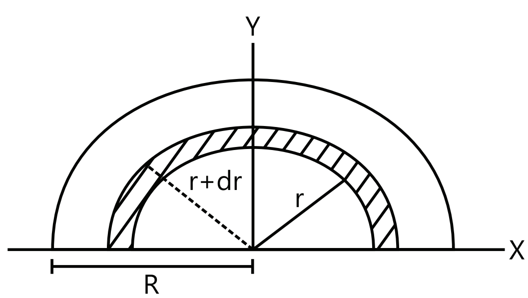


Figure 6.5

(d) Center of mass of a uniform hollow cone

Center of mass of a uniform hollow cone of height H lies on the axis at a distance of H/3 from the center of the bottom.

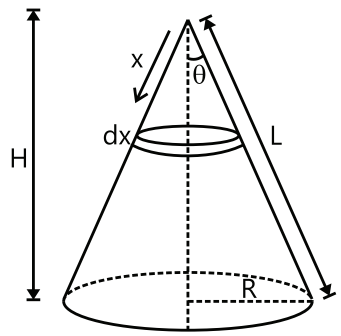


Figure 6.6

PLANCESS CONCEPTS

Student must solve the above integrations to get a better view of how we take infinitesimal segment of a body and the corresponding limits to integrate over whole body.

Please solve the integrations for hollow cone and solid cone to note the difference.

Nivvedan (JEE 2009, AIR 113)

Illustration 3: A rod of length L is placed along the x-axis between x = 0 and x = L. The linear density (mass/length) ρ of the rod varies with the distance x from the origin as ρ = a + bx. Here, a and b are constants. Find the position of center of mass of this rod. **(JEE MAIN)**

Sol: To find C.O.M of continuous mass distributions consider a small element of distribution of mass dm. Then the co-ordinate of C.O.M. is given as

$\therefore x_{COM} = \dfrac{\int x\,dm}{M}$ the limits of integration should be chosen such that the small elements covers entire mass distribution.

Choose an infinitesimal element of the rod of length dx situated at co-ordinates (x, 0, 0) (see Fig.6.7) The linear mass density can be assumed to be constant along the infinitesimal length dx.

Thus the mass of the element dm = rdx = (a + bx) dx

As x varies from 0 to L the element covers the entire rod.

Therefore, x-coordinate of COM of the rod will be

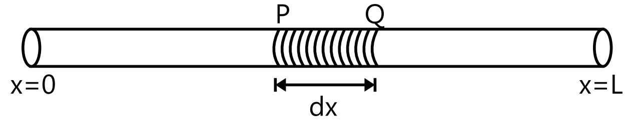


Figure 6.7

$$x_{COM} = \frac{\int_0^L x\,dm}{\int_0^L dm} = \frac{\int_0^L (x)\,(a+bx)dx}{\int_0^L (a+bx)dx} = \frac{\left[\frac{ax^2}{2}+\frac{bx^3}{3}\right]_0^L}{\left[ax+\frac{bx^2}{2}\right]_0^L} = \frac{3aL^2+2bL^3}{6}\times\frac{2}{2aL+bL^2}$$

$$x_{COM} = \frac{3aL+2bL^2}{6a+3bL}\ \text{m}$$

Illustration 4: Determine the center of mass of a uniform solid cone of height h and semi angle α, as shown in Fig. 6.8 **(JEE MAIN)**

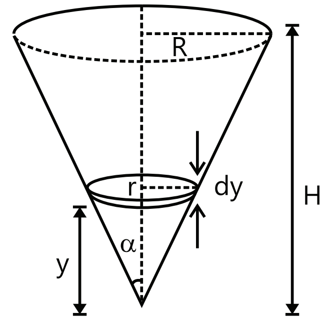


Figure 6.8

Sol: To find C.O.M of continuous mass distributions consider a small element of distribution of mass dm. Then the co-ordinate of C.O.M. is given as

$\therefore Y_{COM} = \frac{\int y\,dm}{M}$ the limits of integration should be chosen such that the small elements covers entire mass distribution.

We place the apex of the cone at the origin and its axis along the y-axis. As the cone is a right circular cone then by symmetry it is clear that the center of mass will lie on its axis i.e. on the y-axis. We consider an elementary disk of radius r and infinitesimal thickness dy whose center is on the y-axis at distance y from the origin as shown in Fig. 6.8. The volume of such a disk is

$dV = \pi r^2\,dy = \pi\,(y\tan\alpha)^2\,dy$

The mass of this elementary disk is dm = rdV. As y varies from 0 to H, the total height of the cone, the elementary disc covers the entire cone. The total mass M of the cone is given by,

$$M = \int dm = \pi\rho\tan^2\alpha\int_0^H y^2 dy = \pi\rho\tan^2\alpha\,\frac{H^3}{3} \qquad \text{.........(i)}$$

The position of the center of mass is given by

$$y_{com} = \frac{1}{M}\int_0^H y\,dm = \frac{1}{M}\pi\rho\tan^2\alpha\int_0^H y^3 dy$$

$$= \frac{1}{M}\pi\rho\tan^2\alpha\,\frac{H^4}{4} \qquad \text{.........(ii)}$$

From equations (i) and (ii), we have $y_{com} = \frac{3H}{4}$

3. CENTER OF GRAVITY

Definition: Center of gravity is a point, near or within a body, at which its entire weight can be assumed to act when considering the motion of the body under the influence of gravity. This point coincides with the center of mass when the gravitational field is uniform.

Note: The center of mass and center of gravity for a continuous body or a system of particles will be different when there is non- uniform gravitational field.

PLANCESS CONCEPTS

You can find the center of gravity and center of mass for a very thin cylinder extending from the surface of earth to the height equal to radius of earth to get the difference. Just sum up all the individual weights of infinitesimal size disks and find the position where gravity will make the same weight of body. This will give center of gravity.

Chinmay S Purandare (JEE 2012, AIR 698)

4. CENTER OF MASS OF THE BODY WHEN A PORTION OF THE BODY IS TAKEN OUT

Suppose there is a body of total mass m and a mass m_1 is taken out from this body. The remaining body will have mass $(m - m_1)$ and its center of mass will be at coordinates,

$$x_{cm} = \frac{mx - m_1x_1}{(m - m_1)}; \quad y_{cm} = \frac{my - m_1y_1}{(m - m_1)}; \quad z_{cm} = \frac{mz - m_1z_1}{(m - m_1)}$$

where (x, y, z) are coordinates of center of mass of original (whole) body and (x_1, y_1, z_1) are coordinates of center of mass of the portion taken out.

Illustration 5: A circular plate of uniform thickness has a diameter of 56 cm. A circular portion of diameter 42 cm is removed from one edge of the plate as shown in Fig. 6.9. Find the center of mass of the remaining position.

(JEE MAIN)

Sol: Let O be the center of circular plate and, O_1, the center of circular portion removed from the plate. The COM of the whole plate will lie at O and the COM of the circular cavity will lie at O_1. Let O be the origin. So OO_1 = 28cm – 21cm = 7cm.

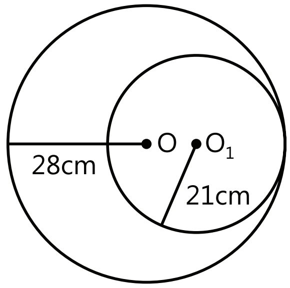


Figure 6.9

The center of mass of the remaining portion will be given as

$$x_{cm} = \frac{mx - m_1x_1}{(m - m_1)} = \frac{\sigma(Ax - A_1x_1)}{\sigma(A - A_1)} = \frac{\pi((28)^2 0 - (21)^2 7)}{\pi((28)^2 - (21)^2)}$$

$$x_{cm} = -9 \text{ cm} = -0.09 \text{ m}.$$

This means that center of mass of the remaining plate is at a distance 9 cm from the center of given circular plate opposite to the removed portion i.e. in this questioon, the new Centre of Mass will shift 9 cm left.

5. MOTION OF THE CENTER OF MASS

For a n-particle system of total mass M and individual particles having mass m_1, m_2, ... m_n, from the definition of center of mass we can write,

$$M\vec{r}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n$$

where $\vec{r}_{cm}$ is the position vector of the center of mass, and $\vec{r}_1$, $\vec{r}_2$, ... $\vec{r}_n$ are the position vectors of the individual particles relative to the same origin in a particular reference frame.

If the mass of each particle of the system remains constant with time, then, for our system of particles with fixed mass, differentiating the above equation with respect to time, we obtain.

$$M\frac{d\vec{r}_{cm}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + ... + m_n\frac{d\vec{r}_n}{dt} \quad ....(i)$$

$$\text{or } M\vec{V}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n$$

Where $\vec{v}_1$, $\vec{v}_2$, ... $\vec{v}_n$ are the velocities of the individual particles, and $\vec{V}_{cm}$ is the velocity of the center of mass. Again differentiating with respect to time, we obtain

$$M\frac{d\vec{V}_{cm}}{dt} = m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + ... + m_n\frac{d\vec{v}_n}{dt}$$

$$M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_n\vec{a}_n \quad ....(ii)$$

Where $\vec{a}_1$, $\vec{a}_2$, ... $\vec{a}_n$ are the accelerations of the individual particles, and $\vec{a}_{cm}$ is the acceleration of the center of mass. Now, from Newton's second law, the force F_i acting on the i^{th} particle is given by $F_i = m_i a_i$. Then, above equation can be written as

$$M\vec{a}_{cm} = \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n = \vec{F}_{internal} + \vec{F}_{external} \quad ....(iii)$$

Internal forces are the forces exerted by the particles of the system on each other. However, from Newton's third law, these internal forces occur in pairs of equal and opposite forces, so their net sum is zero. $\therefore M\vec{a}_{cm} = \vec{F}_{ext}$

This equation states that the center of mass (C.O.M) of a system of particles behaves as if all the mass of the system were concentrated there and the resultant of all the external forces acting on all the particles of the system was applied to it (at C.O.M).

Concept: Whatever may be the rearrangement of the bodies in a system, due to **internal forces** (such as different parts of the system moving away or towards each other or an internal explosion taking place, breaking a body into fragments) provided net F_{ext}=0, we have two possibilities:

(a) If the system as a whole was originally at rest, i.e. the C.O.M was at rest, then the C.O.M. will continue to be at rest.

(b) If before the change, the system as a whole had been moving with a constant velocity (C.O.M was moving with a consant velocity), it will continue to move with a constant velocity.

In presence of a net external force if the C.O.M had been moving with certain acceleration at the instant of an explosion, in a particular trajectory, the C.O.M. will continue to move in the same trajectory, with the same acceleration, as if the system had never exploded at all.

Briefly saying, any internal changes of the body do not effect the motion of C.O.M.

Illustration 6: A man of mass m is standing on a platform of mass M kept on smooth ice (see Fig. 6.10). If the man starts moving on the platform with a speed v relative to the platform, with what velocity relative to the ice does the platform recoil ? **(JEE MAIN)**

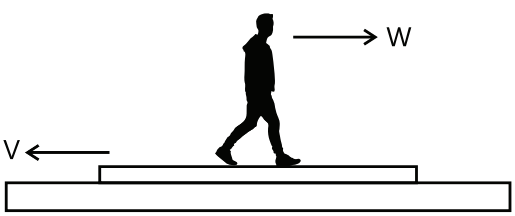


Figure 6.10

Sol: When net external force on system is zero, the C.O.M. will either remain at rest or continue the state of motion. i.e. V_{cm} = constant

Let velocity of platform be $\vec{V}$. If velocity of man relative to platform is $\vec{v}$ then the velocity of man in reference frame of ice is $\vec{w} = \vec{v} + \vec{V}$.

Center of mass of the system comprising of "man and the platform" is initially at rest and as no horizontal external force acts on this system (ice is smooth), the center of mass will continue to remain at rest.

$$\vec{V}_{cm} = 0 = \frac{M\vec{V} + m(\vec{v} + \vec{V})}{M + m}$$

or $\quad (M + m)\vec{V} + m\vec{v} = 0$

or $\quad \vec{V} = -\frac{m\vec{v}}{M + m}\ ms^{-1}$

Negative sign shows that the platform will move in the opposite direction of relative velocity of man.

Illustration 7: Two block of masses m_1 and m_2 connected by a weightless spring of stiffness k rest on a smooth horizontal plane (see Fig 6.11). Block 2 is shifted by a small distance x to the left and released. Find the velocity of the center of mass of the system after block 1 breaks off the wall. **(JEE ADVANCED)**

Sol: Elastic potential energy stored in spring will get converted in kinetic energy of the blocks. If we consider the FBD of mass m_1 at the instant when it breaks off the wall, the normal reaction from the wall is zero, but normal reaction from the wall is equal to is equal to force exerted by spring on mass m_1 so at this instant, force by spring is also zero.

The initial potential energy of compression is $=\frac{1}{2}kx^2$

When the block m_1 breaks off from the wall, the normal reaction from the wall is zero, which in turn means that the tension in the spring is zero. Thus the spring has its natural length at this instant and the kinetic energy of the block m_2 is given by

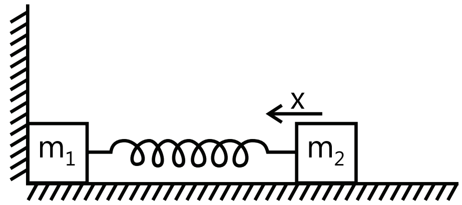


Figure 6.11

$$\frac{1}{2}m_2v_2^2 = \frac{1}{2}kx^2$$

$$v_2^2 = \frac{kx^2}{m_2}$$

$$v_2 = x\sqrt{\frac{k}{m_2}}$$

Velocity of center of mass is

$$V_{cm} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$$

At start $v_1 = 0$

$$\therefore V_{cm} = \frac{m_2}{m_1 + m_2}v_2 = \frac{m_2x}{m_1 + m_2}\sqrt{\frac{k}{m_2}}$$

$\therefore$ Velocity of center of mass of system $V_{cm} = \frac{x\sqrt{km_2}}{m_1 + m_2}$ ms^{-1}

6. LINEAR MOMENTUM

The quantity momentum (denoted as $\vec{P}$) is a vector defined as the product of the mass of a particle and its velocity $\vec{v}$, i.e. $\vec{P} = m\vec{v}$ ....(i)

From Newton's second law of motion, if mass of a particle is constant

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\vec{v}) = \frac{d\vec{P}}{dt}$$

Thus, for constant m, the rate of change of momentum of a body is equal to the resultant force acting on the body and is in the direction of that force.

For a system of n particles with masses m_1, m_2 ..... etc., and velocities v_1, v_2 ..... etc. respectively, the total momentum $\vec{P}$ in a particular reference frame is,

$$\vec{P} = \vec{P}_1 + \vec{P}_2 + ... + \vec{P}_n = m_1\vec{v}_1 + m_1\vec{v}_2 + ... + m_n\vec{v}_n;$$

or $\vec{P} = M\vec{V}_{cm}$

Also, $$\frac{d\vec{P}}{dt} = M\frac{d\vec{V}_{cm}}{dt} = M\vec{a}_{cm} = \vec{F}_{ext}$$

$$\therefore \quad \frac{d\vec{P}}{dt} = \vec{F}_{ext}$$

The magnitude of linear momentum may be expressed in terms of the kinetic energy as well.

$p = mv$

or $$p^2 = m^2v^2 = 2m\left(\frac{1}{2}mv^2\right) = 2mK$$

Thus, $$p = \sqrt{2Km} \text{ or } K = \frac{p^2}{2m}$$

6.1. Law of Conservation of Linear Momentum

"The law of conservation of linear momentum states that if no external forces act on a system of particles, then the vector sum of the linear momenta of the particles of the system remains constant and is not affected by their mutual interaction. In other words the total linear momentum of a closed system remains constant in an inertial reference frame."

Proof: For a system of fixed-mass particles having total mass m we have

$$\vec{F}_{ext} = m\vec{a}_{cm} = m\frac{d\vec{v}_{cm}}{dt} = \frac{d(m\vec{v}_{cm})}{dt} = \frac{d\vec{P}}{dt}$$, where $\vec{P}$ is the total momentum of the system.

Thus, $$\vec{F}_{ext} = \frac{d\vec{P}}{dt}$$

In case the net external force applied to the system is zero, we have

$$\vec{F}_{ext} = \frac{d\vec{P}}{dt} = 0 \quad \text{or} \quad \vec{P} = \text{ constant}$$

Thus for a closed system, the total linear momentum of the system remains constant in an inertial frame of reference.

PLANCESS CONCEPTS

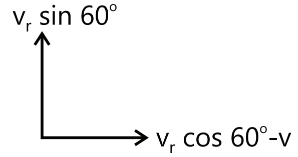
Both linear momentum and kinetic energy are dependent on the reference frame since velocity is inclusively dependent on the frame of reference.

Nitin Chandrol (JEE 2012, AIR 134)

Illustration 8: A gun (mass = M) fires a bullet (mass = m) with speed v_r relative to barrel of the gun which is inclined at an angle of 60° with horizontal. The gun is placed over a smooth horizontal surface. Find the recoil speed of gun. **(JEE MAIN)**

Sol: When a bullet is fired, gun recoils in backward direction. Using law of conservation of linear momentum we can find the recoil velocity of gun.

Let the recoil velocity of gun be $\vec{v}$. The relative velocity of the bullet is $\vec{v}_r$ at an angle of 60° with the horizontal. Taking gun + bullet as the system the net external force on the system in horizontal direction is zero. Let x-axis be along the horizontal and bullet be fired towards the positive direction of x-axis. Initially the system was at rest. Therefore, applying the principle of conservation of linear momentum along x-axis, we get



Components of velocity
of bullet relative to ground

Figure 6. 12

$$Mv_x + m(v_{rx} + v_x) = 0$$

$$-Mv + m(v_r \cos 60° - v) = 0$$

$$v = \frac{mv_r \cos 60°}{M+m}$$

or $$v = \frac{mv_r}{2(M+m)} \text{ ms}^{-1}$$

Illustration 9: The block of mass $m_1 = 2kg$ and $m_2 = 5kg$ are moving in the same direction along a frictionless surface with speeds 10 ms^{-1} and 3 ms^{-1}, respectively m_2 being ahead of m_1 as shown in Fig. 6.13. An ideal spring with spring constant K = 1120 N/m is attached to the back side of m_2. Find the maximum compression of the spring when the blocks move together after the collision. **(JEE ADVANCED)**

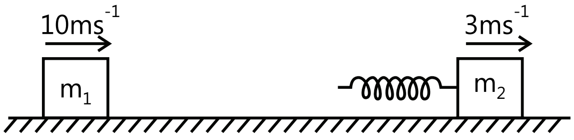


Figure 6.13

Sol: As frictional force on the blocks is zero the total momentum of blocks can be conserved during collision. At the instant of maximum compression some part of initial total K.E. of blocks is stored as elastic P.E. in the spring.

Let v be the final velocity of the system after collision when the blocks move together.

Applying the law of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$

Substituting the values,

$$(2 \times 10) + (5 \times 3) = (2+5)v$$

$$v = \frac{35}{7} = 5 \text{ m/s}$$

Applying the law of conservation of energy we get

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}(m_1 + m_2)v_2^2 + \frac{1}{2}Kx^2$$

$$m_1u_1^2 + m_2u_2^2 = (m_1 + m_2)v_2 + Kx^2$$

$$2 \times (10)^2 + 5 \times (3)^2 = [(2+5) \times (5)^2] + 1120x^2$$

$$x^2 = \frac{70}{1120} = \frac{1}{16} \Rightarrow x = 0.25m$$

PLANCESS CONCEPTS

In the above questions, note that the compression would be maximum when the relative velocity between the blocks is zero.

B Rajiv Reddy (JEE 2012, AIR 11)

7. VARIABLE MASS

From Newton's second law, $\vec{F}_{ext} = m\vec{a}$ is applicable to a system whose total mass m is constant. If total mass of the system is not constant, then this form of Newton's second law is not applicable. If at a certain moment of time the total mass of a system is m and a mass dm is added (or separated) to the system, then we apply $\vec{F}_{ext}=\frac{d\vec{p}}{dt}$ to the system comprising "m+dm" to get

$$\vec{F}_{ext}.dt = d\vec{p} = \vec{p}_{final} - \vec{p}_{initial} = (m+dm)(\vec{v}+d\vec{v}) - [m\vec{v} + dm(\vec{v}+\vec{u})]$$

$$\text{or } \vec{F}_{ext}.dt = md\vec{v} - dm\vec{u}; \qquad (dm.d\vec{v} \simeq 0)$$

$$\text{or } \vec{F}_{ext} = m\frac{d\vec{v}}{dt} - \frac{dm}{dt}\vec{u}$$

$$\text{or } m\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \frac{dm}{dt}\vec{u}$$

where u is velocity of adding or separating mass dm relative to the system having instantaneous mass m and instantaneous velocity v with respect to an inertial reference frame. The term $\frac{dm}{dt}$ can be positive or negative depending upon whether mass is added to the system or mass is separating from the system.

Problems related to variable mass can be solved in following three steps.

(a) Make a list of all the external forces acting on the main mass and draw its FBD.

(b) Apply an additional thrust force or reaction force $\vec{F}_t$ on the main mass, due to the action of the added(separated) mass on the main mass, the magnitude of which is $\left|\vec{u}\left(\pm\frac{dm}{dt}\right)\right|$ and direction is given by the direction of $\vec{u}$ in case the mass is being added or the direction of $-\vec{u}$ if mass is being separated.

(c) Apply the equation

$$m\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \frac{dm}{dt}\vec{u} \quad (m = \text{instantaneous mass})$$

Illustration 10: A flat cart of mass m_0 at t=0 starts moving to the left due to a constant horizontal force F. The sand spills on the flat cart from a stationary hopper. The rate of loading is constant and equal to μ kg/s. Find the time dependence of the velocity and the acceleration of the flat cart in the process of loading. The friction is negligibly small. **(JEE ADVANCED)**

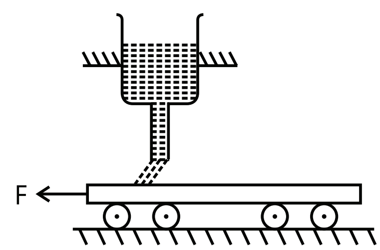


Figure 6.14

Sol: The hopper is at rest in K frame, so in the frame of the cart its initial velocity will be u=-v, where v is velocity of cart in K frame. Here we have used the equation of motion of variable mass $m\frac{dv}{dt} = F_{ext} + \frac{dm}{dt}u$

The rate of increase of mass of the flat car $\frac{dm}{dt} = \mu \text{ kgs}^{-1}$

The hopper is stationary and so its relative velocity is u = 0 - v = - v

The equation of motion is given by

$$m\frac{dv}{dt} = F + \frac{dm}{dt}u = F - \mu v \quad \left[\because \frac{dm}{dt} = \mu \text{ and } u = -v\right]$$

At the instant t, m = $m_0 + \mu t$

$$\therefore \quad \frac{dv}{F-\mu v} = \frac{dt}{m} = \frac{dt}{m_0+\mu t} \quad \Rightarrow \quad \int_0^v \frac{dv}{F-\mu v} = \int_0^t \frac{dt}{m_0+\mu t}$$

$$\text{or,} \quad -\frac{1}{\mu}\log_e\frac{F-\mu v}{F} = \frac{1}{\mu}\log_e\frac{m_0+\mu t}{m_0}$$

or $\log_e \frac{F}{F-\mu v} = \log_e \frac{m_0+\mu t}{m_0} \Rightarrow \quad v = \frac{Ft}{m_0+\mu t}\ ms^{-1}$

The acceleration a is given by

$$a = \frac{dv}{dt} = \frac{Fm_0}{(m_0+\mu t)^2}\ ms^{-2}$$

Alternative:

$$m\frac{dv}{dt} = F + \frac{dm}{dt}u = F - \frac{dm}{dt}v \qquad \text{or } m\frac{dv}{dt} + \frac{dm}{dt}v = F$$

$$\text{or } \frac{d}{dt}(mv) = F \Rightarrow \int_0^{mv} d(mv) = \int_0^t Fdt$$

$$\text{or } mv = Ft \Rightarrow v = \frac{Ft}{(m_0+\mu t)}; \quad (\because m = m_0 + \mu t)$$

8. ROCKET PROPULSION

The propulsion of rocket is an example of a system of variable mass. In the combustion chamber of a rocket, the fuel is burnt in the presence of an oxidizing agent due to which a jet of gases emerges from the tail of the rocket. Thus the mass of the rocket is continuously decreasing. This action due to emission of gases in the backward direction produces a reaction force in the forward direction due to which the rocket moves forward.

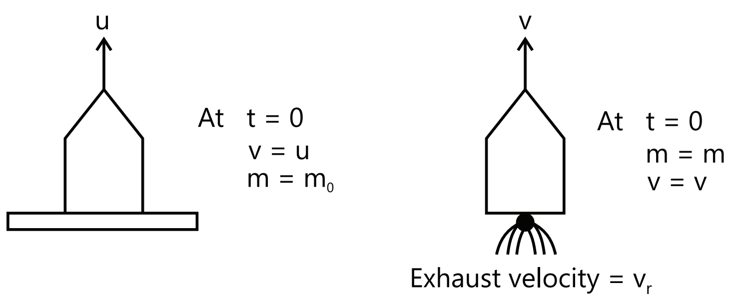


Figure 6.15: Rocket propulsion

Let m_0 be the mass of the rocket and u be its velocity at time t = 0, and m be its mass and v be its velocity at any time t. (see Fig. 6.15)

The mass of the gas ejected per unit time or the rate of change of mass of the rocket is $-\frac{dm}{dt}$ and v_r be the exhaust velocity of the gases relative to the rocket. Usually $-\frac{dm}{dt}$ and v_r are assumed constant throughout the journey of the rocket.

Now using the equation of motion for a system of variable mass derived in the previous article we get,

$$m\frac{d\vec{v}}{dt} = m\vec{g} + \vec{v}_r\frac{dm}{dt}$$

$$\text{or } \frac{d\vec{v}}{dt} = \vec{g} + \frac{\vec{v}_r}{m}\frac{dm}{dt}$$

$$\text{or } d\vec{v} = \vec{v}_r\frac{dm}{m} + \vec{g}.dt$$

This is a vector equation and we do not assume any sign of $\frac{dm}{dt}$. It is taken to be positive. After evaluating the definite integrals, when we substitute the scalar components of the vectors with proper signs we get the correct result.

Integrating on both sides, we get $\int_{\vec{u}}^{\vec{v}} d\vec{v} = \vec{v}_r \int_{m_0}^{m} \frac{dm}{m} + \vec{g}.\int_0^t dt$

or $\vec{v} - \vec{u} = \vec{v}_r \ln\frac{m}{m_0} + \vec{g}.t$

or $\vec{v} = \vec{u} + \vec{g}.t + \vec{v}_r \ln\frac{m}{m_0}$

Now taking upwards direction as positive and downwards as negative (g and v_r are downwards and u is upwards) we get,

$$v = u - g.t + (-v_r)\ln\frac{m}{m_0}$$

Thus, $v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$

Now if $-\frac{dm}{dt} = \mu$ (constant), then m = m_0 - mt

Thus, $v = u - gt + v_r \ln\left(\frac{m_0}{m_0 - \mu t}\right)$

If the initial velocity of the rocket u = 0, and the weight of the rocket is ignored as compared to the reaction force of the escaping gases, the above equation reduces to $v = v_r \ln\left(\frac{m_0}{m}\right)$

PLANCESS CONCEPTS

The concept of variable mass can also be physically visualized by changing the reference frame to the instantaneous velocity of body. In that case mass is either being added by constant speed or being removed by a constant speed. Considering the dm mass and the body as a system, and writing the equations of conservation of momentum one can see the magic!

Anand K (JEE 2011, AIR 47)

Illustration 11: (a) A rocket set for vertical firing weighs 50 kg and contains 450 kg of fuel. It can have a maximum exhaust velocity of 2000 m/s. What should be its minimum rate of fuel consumption?

(i) To just lift it off the launching pad?

(ii) To give it an acceleration of 20 m/s^2?

(b) What will be the speed of the rocket when the rate of consumption of fuel is 10 kg/s after whole of the fuel is consumed? (Take g = 9.8 m/s^2) **(JEE ADVANCED)**

Sol: To lift the rocket upward against gravity, the thrust force in the upward direction due to exiting gases should be greater than or equal to the gravitational force. During motion the mass of rocket decreases till whole of its fuel is consumed. Final velocity of rocket is

$$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right).$$

(a) (i) To just lift the rocket off the launching pad

Initial weight = thrust force

or $\quad m_0 g = v_r\left(-\frac{dm}{dt}\right)$; or $-\frac{dm}{dt} = \frac{m_0 g}{v_r}$

Substituting the values, we get $-\frac{dm}{dt} = \frac{(450+50)(9.8)}{2\times10^3}$ = 2.45 kg/s

(ii) Net acceleration $a = 20\ m/s^2$

$\therefore \quad ma = F_t - mg$

or $\quad m(a+g) = F_t = v_r\left(-\frac{dm}{dt}\right)$

This gives, $\left(-\frac{dm}{dt}\right) = \frac{m(g+a)}{v_r}$

Substituting the values, we get $\left(-\frac{dm}{dt}\right) = \frac{(450+50)(9.8+20)}{2\times10^3}$ = 7.45 kg/s

(b) The rate of fuel consumption is 10 kg/s. So, the time for the consumption of entire fuel is

$$t = \frac{450}{10} = 45\,s$$

The formula for speed of the rocket at time t is, $v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$

Here, u = 0, $v_r = 2\times10^3\ m/s$, $m_0 = 500\,kg$ and m = 50 kg

Substituting the values, we get $v = 0 - (9.8)(45) + (2\times10^3)\ln\left(\frac{500}{50}\right)$

or v = – 441 + 4605.17; or v = 4164.17 m/s; or v = 4.164 km/s

9. COLLISION

An event in which two or more bodies exert forces on each other for a relatively short time is called collision. If net external force acting on the system of bodies is zero, then according to the law of conservation of linear momentum, the total momentum of the system of bodies before and after the collision remains constant.

9.1 Classification of Collisions

Collisions are classified into following types on the basis of the degree of conservation of kinetic energy in a collision:

(a) Elastic Collision: If the total kinetic energy of the colliding particles is conserved before and after the collision, the collision is said to be an elastic collision. If two bodies of masses m_1 and m_2 moving with velocities u_1 and u_2 respectively, collide with each other so that their final velocities after collision are v_1 and v_2 respectively, then the collision will be perfectly elastic if,

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

(b) Inelastic Collision: If the total kinetic energy of the colliding particles is not conserved before and after the collision, the collision is said to be inelastic collision. The kinetic energy is partially converted into other forms of energy like sound, heat, deformation energy etc.

(c) Perfectly Inelastic Collision: The collision is said to be perfectly inelastic if the collision results in "sticking together" of the colliding particles after which they move as a single unit with the same velocity.

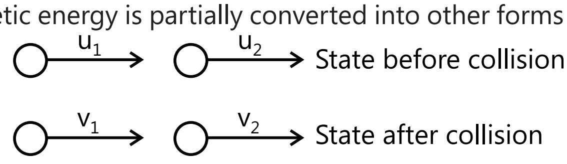


Figure 6.16: Collision in one dimension

Collisions can also be classified on the basis of the line of action of the forces of interaction.

(i) Head- on collisions: A collision is said to be head-on if the direction of the velocities of each of the colliding bodies are along the line of action of the forces of interaction acting on the bodies at the instant of collision.

(ii) Oblique collisions: A collision is said to be oblique if the direction of the velocities of the colliding bodies are not along the line of action of the forces of interaction acting on the bodies at the instant of collision. Just after collision, at least one of the colliding bodies moves in a direction different from the initial direction of motion.

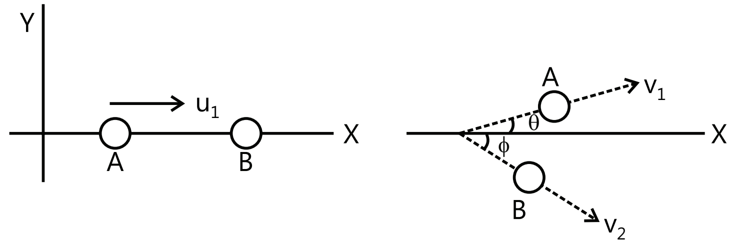


Figure 6.17: Collision in two dimensions

10. COEFFICIENT OF RESTITUTION

Coefficient of restitution is a measure of the elasticity of a collision between two particles. It is defined as the ratio of relative velocity of one of the particles with respect to the other particle after the collision to the relative velocity of the same particle before the collision and the ratio is negative.

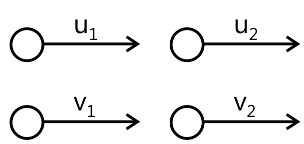


Figure 6.18: Velocities before and after collision

If the velocities of two particles before the collision are u_1 and u_2 respectively and their velocities after the collision are v_1 and v_2 respectively (see Fig. 6.18), then $\frac{v_1 - v_2}{u_1 - u_2} = -e$

The coefficient of restitution is also expressed as the ratio of velocity of separation after collision to the velocity of approach before collision.

$$e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{\text{velocity of separation}}{\text{velocity of approach}}$$

PLANCESS CONCEPTS

- For perfectly inelastic collision e = 0.
- For perfectly elastic collision e = 1.
- For partially inelastic collision 0 < e < 1.

In elastic and inelastic collisioins, momentum is conserved whereas in inelastic collisions, kinetic energy is not conserved.

Yashwanth Sandupatla (JEE 2012, AIR 821)

11. ELASTIC COLLISION

Consider the collision of two small smooth spheres of masses m_1 and m_2 moving with velocities u_1 and u_2 respectively in the same direction along the line joining their centers. Suppose m_1 is following m_2 with $u_1 > u_2$ i.e. m_1 tries to overtake m_2 but as the line of motion is same as the line joining the centers of the spheres, head-on ellastic collision takes place. Let their velocities after the elastic collision are v_1 and v_2 respectively, with $v_2 > v_1$ as shown in the Fig. 6.19

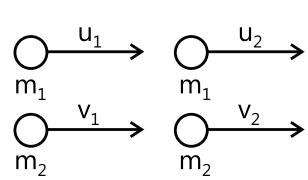


Figure 6.19: Head-on collision between two particles

Conserving momentum before and after collision we get

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$m_1(u_1 - v_1) = m_2(v_2 - u_2) \qquad \text{...(i)}$$

Conserving kinetic energy before and after collision we get

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

$$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \qquad \text{...(ii)}$$

Dividing (ii) by (i)

$$u_1 + v_1 = v_2 + u_2$$

So $v_1 = -u_1 + u_2 + v_2$ ...(iii)

Substitute v_1 in equation (i)

$$m_1(u_1 + u_1 - u_2 - v_2) = m_2v_2 - m_2u_2$$

$$2m_1u_1 + (m_2 - m_1)u_2 = (m_1 + m_2)v_2$$

$$v_2 = \left[\frac{2m_1}{m_1 + m_2}\right]u_1 + \left[\frac{m_2 - m_1}{(m_1 + m_2)}\right]u_2 \qquad \text{...(iv)}$$

Similarly, $v_1 = \left[\frac{m_1 - m_2}{m_1 + m_2}\right]u_1 + \left[\frac{2m_2}{m_1 + m_2}\right]u_2$ ...(v)

Special Cases

(i) When $m_1 = m_2$,

From equation (i)

$u_1 - v_1 = v_2 - u_2$ or $v_1 + v_2 = u_1 + u_2$ ...(vi)

Equation (iii) gives

$v_1 - v_2 = u_2 - u_1$ ...(vii)

Solving (vi) and (vii) we get

$v_1 = u_2$ and $v_2 = u_1$

$\therefore$ In one dimensional elastic collision of two bodies of equal masses, the bodies exchange there velocities after collision.

(ii) When m_2 is at rest i.e. $u_2 = 0$.

$$v_2 = \frac{2m_1u_1}{m_1 + m_2} + \left(\frac{m_2 - m_1}{m_2 + m_1}\right)u_2 \Rightarrow v_2 = \frac{2m_1u_1}{m_1 + m_2}$$

Now there are three possibilities in this case:

(a) If $m_1 = m_2 = m$; $v_2 = \frac{2mu_1}{2m} = u_1, v_1 = 0$.

The first body stops after collision. Both the momentum and the kinetic energy of the first body are completely transferred to the second body.

(b) If $m_2 >> m_1$, $v_1 \simeq -u_1$, $v_2 \simeq 0$

Thus when a light body collides with a much heavier stationary body, the velocity of light body is reversed and heavier body almost remains at rest.

(c) If $m_2 << m_1$, $v_2 \simeq u_1$ and $v_2 \simeq 2u_1$

Thus when a heavy body collides with a much lighter stationary body, the velocity of heavier body remains almost unchanged. The lighter body moves forward with approximately twice the velocity of the heavier body.

12. INELASTIC COLLISION

Consider the situation similar to previous article wherein m_1 is following m_2 with $u_1 > u_2$ i.e. m_1 tries to overtake m_2 but as the line of motion is same as the line joining the centers of the spheres, head-on collision takes place. Let their velocities after the collision are v_1 and v_2 respectively, with $v_2 > v_1$. Now suppose that the collision is inelastic, i.e. kinetic energy is not conserved.

Conserving momentum we get,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Restitution equation gives,

$$v_1 - v_2 = -e(u_1 - u_2)$$

The loss in kinetic energy ΔE in this case, is given by

$$\Delta E = \frac{1}{2}\left(\frac{m_1m_2}{m_1 + m_2}\right)(e^2 - 1)(u_1 - u_2)^2$$

Putting e = 0 in this equation, it is clear that the loss of kinetic energy is maximum in case of pefectly inelastic collision.

Illustration 12: A block of mass m moving at a velcoity v collides head on with another block of mass 2m at rest. If the coefficient of restitution is 0.5, find the velocities of the blocks after the collision. **(JEE MAIN)**

Sol: Solve using law of conservation of momentum, before and after collision and the equation of restitution.

Suppose after the collision the block of mass m moves at a velocity u_1 and the block of mass 2m moves at a velocity u_2. By conservation of momentum,

$$mv = mu_1 + 2mu_2 \quad \text{... (i)}$$

The velocity of sepration is $u_2 - u_1$ and the velocity of approach is v.

So, $$u_2 - u_1 = \frac{v}{2} \quad \text{... (ii)}$$

Solving (i) and (ii) we get, $u_1 = 0\ ms^{-1}$ and $u_2 = \frac{v}{2}\ ms^{-1}$.

Illustration 13: A ball is moving with velocity 2 m/s towards a heavy wall moving towards the ball with speed 1 m/s as shown in Fig. 6.20. Assuming collision to be elastic, find the velocity of ball immediately after the collision. **(JEE MAIN)**

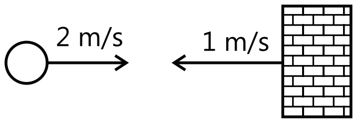


Figure 6.20

Sol: The equation of conservation of momentum will not give us any fruitful result because the mass of the wall is very large and remains at rest before and after the collision. This problem has to be solved by using equation of restitution

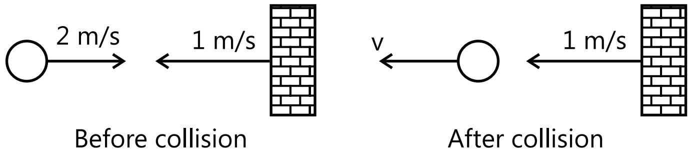


Figure 6.21

The speed of wall will not change after the collision. So, let v be the velocity of the ball after collision in the direction shown in Fig. 6.21. Since, collision is elastic (e = 1).

velocity of Separation=velocity of approach

or $v - 1 = 2 - (-1)$

or $v = 4\ m/s$

Illustration 14: A ball of mass m is projected vertically up from smooth horizontal floor with a speed V_0. Find the total momentum delivered by the ball to the surface, assuming e as the coefficient of restitution of impact.

(JEE MAIN)

Sol: By Newton's third law the impulse delivered by the ball to the surface at each collision will be equal in magnitude to the impulse delivered to the ball by the surface i.e. change in momentum of ball at each collision. The total impulse will be the sum of DP due to all the collisions.

The momentum delivered by the ball at first, second, third impact etc. can be given as the corresponding change in its momentum (ΔP) at each impact.

$$(\Delta\vec{P})_1 = \left|(mV_1)\hat{j} - m(-V_0)\hat{j}\right| \quad \Rightarrow \quad \Delta P_1 = m(V_1 + V_0)$$

Similarly $\Delta P_2 = m(V_1 + V_2)$, $\Delta P_3 = m(V_2 + V_3)$, ... and so on.

$\Rightarrow$ The total momentum transferred $\Delta P = \Delta P_1 + \Delta P_2 + \Delta P_3 + ...$

Putting the values of ΔP_1, ΔP_2 etc., we obtain,

$$\Delta P = m\left[V_0 + 2(V_1 + V_2 + V_3 + ...)\right]$$

Putting $V_1 = eV_0$, $V_2 = e^2V_0$, $V_3 = e^3V_0$

We obtain,

$$\Delta P = mV_0\left[1 + 2(e + e^2 + e^3 + ...)\right] \quad \Rightarrow \quad \Delta P = mV_0\left(1 + 2\frac{e}{1-e}\right) = mV_0\left(\frac{1+e}{1-e}\right)$$

Illustration 15: A stationary body explodes into four identical fragments such that three of them fly off mutually perpendicular to each other, each with same K.E. Find the energy of explosion. **(JEE ADVANCED)**

Sol: As the body is initially at rest, the vector sum of momentum of all fragments will be zero. The energy of explosion will appear as K.E. of fragments.

Let the three fragments move along X, Y and Z axes. Therefore their velocities can be given as

$\vec{V}_1 = V\hat{i}$, $\vec{V}_2 = V\hat{j}$ and $\vec{V}_3 = V\hat{k}$,

where V = speed of each of the three fragments. Let the velocity of the fourth fragment be $\vec{V}_4$ Since, in explosion no net external force is involved, the net momentum of the system remains conserved just before and after explosion. Initially the body is a rest,

$$\Rightarrow \quad m\vec{V}_1 + m\vec{V}_2 + m\vec{V}_3 + m\vec{V}_4 = 0$$

Putting the values of $\vec{V}_1$, $\vec{V}_2$ and $\vec{V}_3$, we obtain, $\vec{V}_4 = -V\left(\hat{i} + \hat{j} + \hat{k}\right)$

Therefore, $V_4 = \sqrt{3}\,V$

The energy of explosion

$$\therefore E = KE_f - KE_i = \left(\frac{1}{2}mV_1^2 + \frac{1}{2}mV_2^2 + \frac{1}{2}mV_3^2 + \frac{1}{2}mV_4^2\right) - (0)$$

Putting $V_1 = V_2 = V_3 = V$ and setting $\frac{1}{2}mV^2 = E_0$, we obtain, $E = 6E_0$.

13. OBLIQUE COLLISION

Let us now consider the case when the velocities of the two colliding spheres are not directed along the line of action of the forces of interaction or the line of impact (line joining the centers). As already discussed this kind of impact is said to be oblique.

Let us consider the collision of two spherical bodies. Since velocities v'_1 and v'_2 of the bodies after impact are unknown in direction and magnitude, their determination will require the use of four independent equations.

We choose the x-axis along the line of impact, i.e. along the common normal to the surfaces in contact, and the y-axis along their common tangent as shown in Fig. 6.22. Assuming the spheres to be perfectly smooth and frictionless, the impulses exerted on the spheres during the collision are along the line of impact i.e., along the x-axis. So,

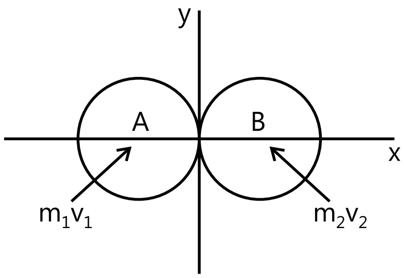


Figure 6.22: Oblique collision of two particles

(i) the component of the momentum of each sphere along the y-axis, considered separately is conserved; hence the y component of the velocity of each sphere remains unchanged. Thus we can write

$$(v_1)_y = (v'_1)_y \quad ....(i)$$

$$(v_2)_y = (v'_2)_y \quad ....(ii)$$

(ii) the component of total momentum of the two spheres along the x-axis is conserved. Thus we can write

$$m_1(v_1)_x + m_2(v_2)_x = m_1(v'_1)_x + m_2(v'_2)_x \quad ....(iii)$$

(iii) The component along the x-axis of the relative velocity of the two spheres after impact i.e. the velocity of separation along x-axis is obtained by multiplying the x-component of their velocity of approach before impact by the coefficient of restitution. Thus we can write

$$(v'_2)_x - (v'_1)_x = e\left[(v_1)_x - (v_2)_x\right] \quad ....(iv)$$

Now the four equations obtained above can be solved to find the velocities of the spheres after collision.

PLANCESS CONCEPTS

It is not advised to break the components of velocity in any other direction even though they are still valid. The only problem will be in using the coefficient of restitution.

Definition of coefficient of restitution can be applied in the normal direction in the case of oblique collision.

G.V. Abhinav (JEE 2012, AIR 329)

Illustration 16: After perfectly inelastic collision between two identical particles moving with same speed in different directions, the speed of the particles becomes half the initial speed. Find the angle between the two before collision. **(JEE MAIN)**

Sol: In case of an oblique collision, the momentum of individual particles are added vectorially in the equation of conservation of linear momentum.

Let θ be the desired angle. Linear momentum of the system will remain conserved.

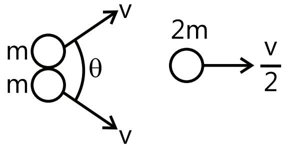


Figure 6.23

Hence $P^2 = P_1^2 + P_2^2 + 2P_1P_2\cos\theta$

or $\left\{2m\left(\frac{v}{2}\right)\right\}^2 = (mv)^2 + (mv)^2 + 2(mv)(mv)\cos\theta$

or $1 = 1 + 1 + 2\cos\theta$ or $\cos\theta = -\frac{1}{2}$

$\therefore$ $\theta = 120°$

Illustration 17: A ball of mass m hits the floor with a speed v making an angle of incidence θ with the normal. The coefficient of restitution is e. Find the speed of the reflected ball and the angle of reflection of the ball.

(JEE MAIN)

Sol: In case of an oblique collision with fixed surface the component of velocity of colliding particle parallel to surface doesn't change. The impulse will act along the normal to the surface so use the equation of restitution along the normal.

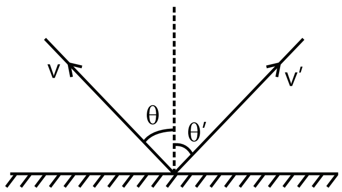


Figure 6.24

See Fig. 6.24. Let the angle of reflection is θ' and the speed after the collision is v'. The impulse on the ball is along the normal to the floor during the collision. There is no impulse parallel to the floor. Thus, the component of the velocity of the ball parallel to the surface remains unchanged before and after the collision. This gives

$v'\sin\theta' = v\sin\theta$ ...(i)

As the floor is stationary before and after the collision, the equation of conservation of momentum in the direction normal to the floor will not give any result. We have to use the formula for coefficient of restitution along the direction normal to the floor.

The velocity of separation along the normal= $v'\cos\theta'$

The velocity of approach along the normal = $v\cos\theta$

Hence, $v'\cos\theta' = ev\cos\theta$ ...(ii)

From (i) and (ii),

$v' = v\sqrt{\sin^2\theta + e^2\cos^2\theta}$ and $\tan\theta' = \frac{\tan\theta}{e}$

For elastic collision, e = 1 so that $\theta' = \theta$ and $v' = v$

PLANCESS CONCEPTS

Inelastic collision doesn't always mean that bodies will stick, which is very clear from the concept of oblique collision. Only the velocities along n-axis become same and may be different in t-direction.

Anurag Saraf JEE 2011, AIR 226

14. CENTER OF MASS FRAME

We can rigidly fix a frame of reference to the center of mass of a system. This frame is called the C-frame of reference and in general is a non-inertial reference frame. Relative to this frame, the center of mass is at rest $(\vec{V}_{com,C} = 0)$ and according to equation $\vec{P} = M\vec{V}_{com}$ the total momentum of a system of particle in the C-frame of reference is always zero.

$\vec{P} = \Sigma\vec{P}_i = 0$ in the C-frame of reference.

Note : When the net external force acting on the system is zero, the C-frame becomes an inertial frame.

14.1 A System of Two Particles

Suppose the masses of the particles are equal to m_1 and m_2 and their velocities in the given reference frame K be $\vec{v}_1$ and $\vec{v}_2$ respectively. Let us find the expressions defining their momentum and total kinetic energy in the C-frame. The velocity of C-frame relative to K-frame is $\vec{v}_c$.

The momentum of the first particle in the C-frame is $\vec{P}_{1/c} = m_1\vec{v}_{1/c} = m_1(\vec{v}_1 - \vec{v}_c)$ where $\vec{v}_c$ is the velocity of the center of mass of the system in the K frame. Substituting the expression for $\vec{v}_c$

$$\vec{v}_c = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

we get $\quad \vec{P}_{1/c} = \mu(\vec{v}_1 - \vec{v}_2)$

where μ is the reduced mass of the system, given by $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

Similarly, the momentum of the second particle in the C-frame is $\vec{P}_{2/c} = \mu(\vec{v}_2 - \vec{v}_1)$

Thus, the momenta of the two particles in the C-frame are equal in magnitude and opposite in direction; the modulus of the momentum of each particle is

$$P_{1/c} = \mu v_{rel}$$

where $v_{rel} = |\vec{v}_1 - \vec{v}_2|$ is the modulus of velocity of one particle relative to another.

Finally, let us consider total kinetic energy. The total kinetic energy of the two particles in the C-frame is

$$K_{sys/c} = K_{1/c} + K_{2/c} = \frac{1}{2}m_1 v_{1/c}^2 + \frac{1}{2}m_2 v_{2/c}^2 = \frac{P_{1/c}{}^2}{2m_1} + \frac{P_{1/c}{}^2}{2m_2}$$

Now, $\quad \mu = \dfrac{m_1 m_2}{m_1 + m_2} \quad$ or $\quad \dfrac{1}{m_2} + \dfrac{1}{m_2} = \dfrac{1}{\mu}$

Then $\quad K_{sys/c} = \dfrac{\vec{P}_{1/c}{}^2}{2\mu} = \dfrac{\mu v_{rel}^2}{2}$

The total kinetic energy of the partices of the system in the K-frame is related to the total kinetic energy in C-frame. The velocity of the ith particle of the system in K-frame can be expressed as:

$$\vec{v}_i = \vec{v}_{i/c} + \vec{v}_c$$

So we can write $K_{sys} = \dfrac{1}{2}\sum m_i v_i^2 = \dfrac{1}{2}\sum m_i(\vec{v}_{i/c} + \vec{v}_c)^2 = \dfrac{1}{2}\sum m_i v_{i/c}^2 + \vec{v}_c \sum m_i \vec{v}_{i/c} + \dfrac{v_c^2}{2}\sum m_i$

In the C-frame, the summation $\sum m_i \vec{v}_{i/c} = M\vec{V}_{com,C} = 0$.

So we get $K_{sys} = \dfrac{1}{2}\sum m_i v_{i/c}^2 + \dfrac{v_c^2}{2}\sum m_i = K_{sys/c} + \dfrac{v_c^2}{2}\sum m_i$

For a two-particle system, we get

$$K_{sys} = \frac{\mu v_{rel}^2}{2} + \frac{M v_c^2}{2} \quad (\text{where } M = m_1 + m_2)$$

Illustration 18: Two blocks of mass m_1 and m_2 connected by an ideal spring of spring constant k are kept on a smooth horizontal surface. Find maximum extension of the spring when the block m_2 is given an initial velocity of v_0 towards right as shown in Fig. 6.25. **(JEE ADVANCED)**

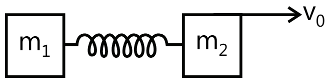


Figure 6.25

Sol: In absence of frictional forces on block, the total mechanical energy of the system comprising the blocks and spring will be conserved. At the time of maximum expansion of spring, the mechanical energy in C frame will be totally stored as elastic P.E. of the spring

This problem can be best solved in the C-frame or the reference frame rigidly fixed to the center of mass of the system of two blocks.

Initially at t=0 when the block m_2 is given velocity v_0, the total kinetic energy of the blocks in C-frame is related to the total kinetic energy in the given frame K by the relation,

$$K_{sys} = \frac{\mu v_{rel}^2}{2} + \frac{(m_1+m_2)v_c^2}{2} = K_{sys/c}(0) + \frac{(m_1+m_2)v_c^2}{2} \qquad \left(\mu = \frac{m_1 m_2}{m_1+m_2};\ v_{rel} = v_0;\ v_c = \frac{m_2 v_0}{m_1+m_2}\right)$$

where the first term on the right hand side of this relation is the total kinetic energy in C-frame at t=0, $K_{sys/c}(0)$, and the second term is the kinetic energy associated with the motion of the system of blocks as a whole in the K-frame. As there are no dissipative external forces acting on the system, the total mechanical energy will remain constant, both in the C-frame and the K-frame. In the C-frame the blocks will oscillate under the action of spring force and the kinetic energy in the C-frame will get converted into the elastic potential energy of the spring and vice-versa, the total mechanical energy remaining constant at each instant, equal to the total kinetic energy in C-frame at t=0, $K_{sys/c}(0)$.

Initially at t=0 when the block m_2 is given velocity v_0, the mechanical energy in C frame will be totally kinetic ($K_{sys/c}(0)$), and at the instant of maximum extension of the spring, the mechanical energy in C-frame will be totally converted into elastic potential energy of the spring. So we have,

$$K_{sys/c}(0) = \frac{1}{2}kx_{max}^2 \Rightarrow \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}v_0^2 = \frac{1}{2}kx_{max}^2$$

Thus, maximum extension is $x_{max} = v_0\sqrt{\frac{m_1 m_2}{k(m_1+m_2)}}$

15. IMPULSE AND MOMENTUM

When two bodies collide during a very short time period, large impulsive forces are exerted between the bodies along the line of impact. Common examples are a hammer striking a nail or a bat striking a ball. The line of impact is a line through the common normal to the surfaces of the colliding bodies at the point of contact.

When two bodies collide, the momentum of each body is changed due to the force on it exerted by the other. On an ordinary scale, the time duration of the collision is very small and yet the change in momentum is sizeable. This means that the magnitude of the force of interaction must be very large on an ordinary scale. Such large forces acting for a very short duration are called impulsive forces. The force may not be uniform during the interaction.

We know that the force is related to momentum as $\vec{F} = \frac{d\vec{P}}{dt} \quad \Rightarrow \quad \vec{F}dt = d\vec{P}$

We can find the change in momentum of the body during a collision (from $\vec{P}_i$ to $\vec{P}_f$) by integrating over the time of collision and assuming that the force during collision has a constant direction, $\vec{P}_f - \vec{P}_i = \int_{P_i}^{P_f} d\vec{P} = \int_{t_i}^{t_f} \vec{F}dt$;

Here the subscripts i (= initial) and f (= final) refer to the times before and after the collision. The integral of a force over the time interval during which the force acts is called impulse.

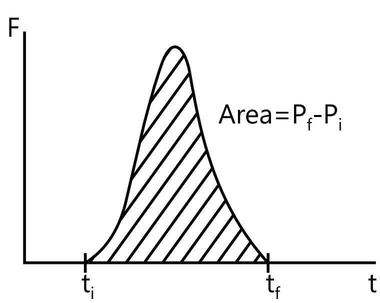


Figure 6.26: Impulse imparted to the particle

Thus the quantity $\int_{t_i}^{t_f} \vec{F} \cdot dt$ is the impulse of the force $\vec{F}$ during the time interval t_i and t_f and is equal to the change in the momentum of the body in which it acts.

The magnitude of impulse $\int_{t_i}^{t_f} \vec{F} dt$ is the area under the force-time curve as shown in Fig. 6.26

Illustration 19: A block of mass m and a pan of equal mass are connected by a string going over a smooth light pulley as shown in Fig. 6.27. Initially the system is at rest when a particle of mass m falls on the pan and sticks to it. If the particle strikes the pan with a speed v find the speed with which the system moves just after the collision. **(JEE MAIN)**

Sol: By Newton's third law, the impulse imparted to the particle in upward direction will be equal in magnitude to the total impulse imparted to the system of block and the pan.

Let N be the contact force between the particle and the pan during the collision.

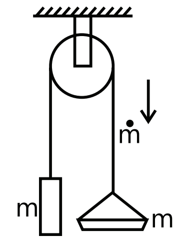


Figure 6.27

Consider the impulse imparted to the particle. The force N will be in upward direction and the impulse imparted to it will be $\int N\,dt$ in the upward direction. This should be equal to the change in momentum imparted to it in the upward direction.

Thus, $\int N\,dt = P_f - P_i = -mV - (-mv) = mv - mV$ ....(i)

Similarly considering the impulse imparted to the pan. The forces acting on it are tension T upwards and contact force N downwards. The impulse imparted to it in the downward direction will be,

$\int (N - T)\,dt = mV - 0 = mV$ ....(ii)

Impulse imparted to the block by the tension T will be upwards,

$\int T\,dt = mV - 0 = mV$ ....(iii)

Adding (ii) and (iii) we get, $\int N dt = 2mV$ ....(iv)

Comparing (i) and (iv) we get, $mv - mV = 2mV$ or $V = \frac{v}{3}\ ms^{-1}$

PROBLEM-SOLVING TACTICS

Applying the principle of Conservation of Linear Momentum

(a) Decide which objects are included in the system.

(b) Relative to the system, identify the internal and external forces.

(c) Verify that the system is isolated.

(d) Set the final momentum of the system equal to its initial momentum. Remember that momentum is a vector.

(e) Always check whether kinetic energy is conserved or not. If it is conserved, it gives you an extra equation. Otherwise use work-energy theorem, carefully.

(f) Try to involve yourself physically in the question, imagine various events. This would help in some problems where some parameters get excluded by conditions. This will also help in checking your answer.

Impulse

(g) Ignore any finite-value forces, while dealing with impulses.

(h) Write impulse equations carefully, because integration which we are unable to calculate will always cancel out.

Collisions

(i) Remembering special cases of collisions would be nice.

FORMULAE SHEET

Position of center of mass of a system: $\vec{r}_{com} = \dfrac{\sum_i m_i \vec{r}_i}{M}$

$$\vec{r}_{COM} = x_{COM}\hat{i} + y_{COM}\hat{j} + z_{COM}\hat{k}$$

$$x_{COM} = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_i m_ix_i}{\sum_i m_i}$$

For continuous bodies $x_{COM} = \dfrac{\int x\,dm}{\int dm} = \dfrac{\int x\,dm}{M}$

For a two-particle system, we have

$$r_1 = \left(\frac{m_2}{m_2 + m_1}\right)d \text{ and } r_2 = \left(\frac{m_1}{m_1 + m_2}\right)d$$

where d is the separation between the particles.

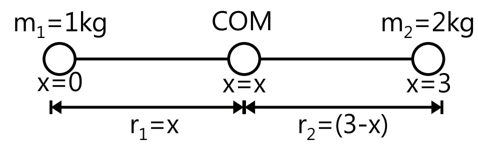


Figure 6.28

Center of Mass of a Uniform Rod $\left(\dfrac{L}{2}, 0, 0\right)$

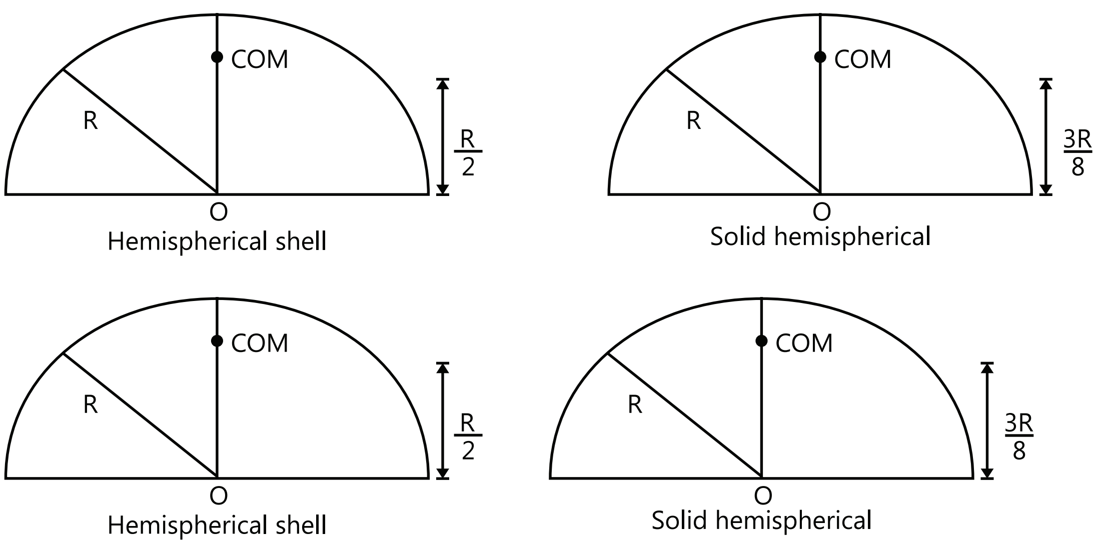


Figure 6.29

If some mass or area is removed from a rigid body, then the position of center of mass of the remaining portion is obtained from the following formula:

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 - m_2\vec{r}_2}{m_1 - m_2}$$

Where m_1 is the mass of the body after filling all cavities with same density and m_2 is the mass filled in the cavity. Cavity mass is assumed negative.

Velocity of COM $\vec{v}_{COM} = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n}{m_1 + m_2 + \ldots + m_n} = \dfrac{\sum_i m_i \vec{v}_i}{\sum_i m_i}$

Total momentum of a n-particle system $\vec{P}_{COM} = \vec{P}_1 + \vec{P}_2 + ... + \vec{P}_n = M\vec{v}_{COM}$

Acceleration of COM $\vec{a}_{COM} = \dfrac{m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_n\vec{a}_n}{m_1 + m_2 + ... + m_n} \dfrac{\sum_i m_i \vec{a}_i}{\sum_i m_i}$

Net force acting on the system $\vec{F}_{COM} = \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n$

Net external force on center of mass is $M\vec{a}_{cm} = \vec{F}_{ext}$

If net force on the system $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ... + \vec{F}_n = 0$ then, $\vec{P}_1 + \vec{P}_2 + \vec{P}_3 + ... + \vec{P}_n$ = constant

Equation of motion of a body with variable mass is:

$$m\left(\frac{d\vec{v}}{dt}\right) = \vec{F} + \left(\frac{dm}{dt}\right)\vec{u}$$

Where $\vec{u}$ is the velocity of the mass being added(separated) relative to the given body of instantaneous mass m and $\vec{F}$ is the external force due to surrounding bodies or due to field of force.

In case of reducing mass of a system $\dfrac{dm}{dt} = \mu \text{ kgs}^{-1}$

For a rocket we have, $m\left(\dfrac{d\vec{v}}{dt}\right) = m\vec{g} + \left(\dfrac{dm}{dt}\right)\vec{v}_r$

Where $\vec{v}_r$ is the velocity of the ejecting gases relative to the rocket.

In scalar form we can write

$$m\left(\frac{dv}{dt}\right) = -mg + v_r\left(-\frac{dm}{dt}\right)$$

Here $-\dfrac{dm}{dt}$ = rate at which mass is ejecting and $v_r\left(-\dfrac{dm}{dt}\right)$ =Thrust force.

Final velocity of rocket $v = u - gt + v_r \ln\left(\dfrac{m_0}{m}\right)$

Impulse of a force: $\vec{J} = \int \vec{F}\,dt = \Delta\vec{p} = \vec{p}_f - \vec{p}_i$

Collision

(a) In the absence of any external force on the system the linear momentum of the system will remain conserved before, during and after collision, i.e.,

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v = m_1v'_1 + m_2v'_2 \qquad ...(i)$$

(b) In the absence of any dissipative forces, the mechanical energy of the system will also remain conserved, i.e.

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}kx_m^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 \qquad ...(ii)$$

Head on Elastic Collision

$$v'_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_1 + \left(\frac{2m_2}{m_1 + m_2}\right)v_2$$

$$v'_2 = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_2 + \left(\frac{2m_1}{m_1 + m_2}\right)v_1$$

$$\frac{\text{separation speed after collision}}{\text{approach speed before collision}} = e$$

$$v'_1 = \left(\frac{m_1 - em_2}{m_1 + m_2}\right)v_1 + \left(\frac{m_2 + em_2}{m_1 + m_2}\right)v_2$$

$$v'_2 = \left(\frac{m_2 - em_1}{m_1 + m_2}\right)v_2 + \left(\frac{m_1 + em_1}{m_1 + m_2}\right)v_1$$

The C-frame: Total kinetic energy of system in K-frame is related to total kinetic energy in C-frame as:

$$K_{sys} = K_{sys/c} + \frac{Mv_c^2}{2};\ M = \sum m_i$$

For a two-particle system: $\vec{P}_{1/c} = -\vec{P}_{2/c} = \frac{m_1 m_2}{m_1 + m_2}\left(\vec{v}_1 - \vec{v}_2\right)$

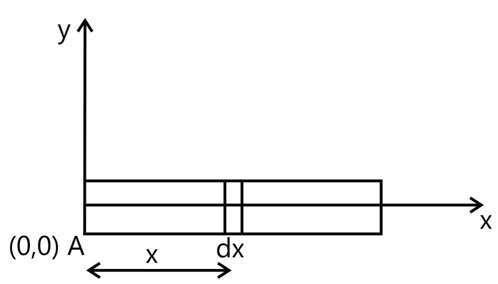
$$\text{Or } P_{1/c} = P_{2/c} = \mu v_{rel} = \frac{m_1 m_2}{m_1 + m_2}\left|\vec{v}_1 - \vec{v}_2\right| \text{ and } K_{sys/c} = \frac{\mu v_{rel}^2}{2} = \frac{P_{1/c}^2}{2}$$

Solved Examples

JEE Main/Boards

Example 1: The linear mass density of rod of a length l=2 m varies from A as (2+x) kg/m. What is the position of center of mass from end A.

Sol: To find C.O.M of continuous mass distributions consider a small element of distribution of mass dm. Then the co-ordinate of C.O.M. is given as



$\therefore x_{COM} = \frac{\int x\, dm}{M}$ the limits of integration should be chosen such that the small elements covers entire mass distribution.

Take an element of the rod of infinitesimal length dx at distance x from point A. The mass of the element will be

$dm = \lambda\ dx = (2 + x)dx$ As x varies from o to l the element covers the entire rod.

Center of mass of rod $X_{cm} = \frac{\int x\,dm}{\int dm}$

$$X_{cm} = \frac{\int_0^l x(2+x)dx}{\int_0^l (2+x)dx} = \frac{\left|(x^2 + \frac{x^3}{3})\right|_0^l}{\left|2x + \frac{x^2}{2}\right|_0^l}$$

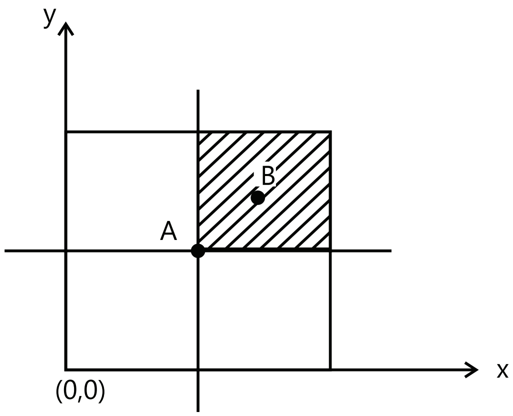
$$X_{cm} = \frac{l^2 + \frac{l^3}{3}}{2l + \frac{l^2}{2}} = \frac{6l + 2l^2}{12 + 3l}$$

For l = 2 m, $X_{cm} = \frac{6\times 2 + 2\times 4}{12 + 3\times 2} = \frac{20}{18} = \frac{10}{9}$m

So center of mass is at a distance $\frac{10}{9}$ m from A.

Example 2: One fourth of the mass of square lamina is cut off (see figure). Where does the center of mass of the remaining part of the square shift.

Sol: To find the C.O.M. of a body having a cavity we first fill the cavity with the same density as body and find the C.O.M. (x,y,z) of the whole body. Then we consider the cavity as second body having negative mass and find the C.O.M (x_1, y_1, z_1) of the cavity. The C.O.M. of the body with cavity is



$$x_{cm} = \frac{mx - m_1x_1}{(m-m_1)};\ y_{cm} = \frac{my - m_1y_1}{(m-m_1)};\ z_{cm} = \frac{mz - m_1z_1}{(m-m_1)}$$

Part of the lamina cut-off is taken as negative mass. Coordinates of center of mass of whole lamina are $\left(\frac{a}{2}, \frac{a}{2}\right)$ and the coordinates of center of mass of cut-off part are $\left(\frac{3a}{4}, \frac{3a}{4}\right)$. So the center of mass of remaining part is given as,

$$X_{cm} = \frac{m\frac{a}{2} - \frac{m}{4}\cdot\frac{3a}{4}}{m - \frac{m}{4}} = \frac{\frac{a}{2} - \frac{3a}{16}}{\frac{3}{4}} = \frac{5a}{12}\text{ m}$$

$$Y_{cm} = \frac{m\frac{a}{2} - \frac{m}{4}\cdot\frac{3a}{4}}{m - \frac{m}{4}} = \frac{5a}{12}\text{ m}$$

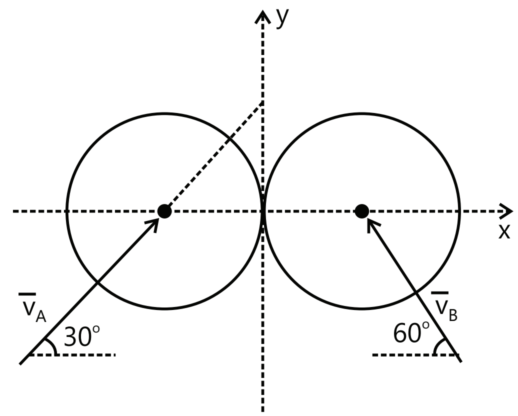
Example 3: The magnitude and direction of the velocities of two identical frictionless balls before they strike each other are as shown in figure. Assuming e=0.90, determine the magnitude and direction of the velocity of each ball after the impact. $v_A = 30\text{ ms}^{-1}$, $v_B = 40\text{ ms}^{-1}$

Sol: In case of an oblique collision, the momentum of individual particles are added vectorially in the equation of conservation of linear momentum. The equation of restitution is used along line of impact

The component of velocity of each ball along the common tangent at the point of impact will remain the same before and after the collisions. Let x and y axes be along the common normal and common tangent respectively.

So $v'_{Ay} = v_{Ay} = v_A \sin 30^o = 15\text{ ms}^{-1}$ ...(i)

$v'_{By} = v_{By} = v_B \sin 60^o = 20\sqrt{3} = 34.6\text{ms}^{-1}$ ...(ii)



Along the x axis we conserve the momentum to get

$$m_A v_{Ax} + m_B v_{Bx} = m_A v'_{Ax} + m_B v'_{Bx}$$

$$m v_A \cos 30 + m(-v_B \cos 60^o) = mv'_{Ax} + mv'_{Bx}$$

$v'_{Ax} + v'_{Bx} = 15\sqrt{3} - 20$ ... (iii)

Velocity of separation = e (velocity of approach)

$$\Rightarrow v'_{Bx} - v'_{Ax} = e(v_{Ax} - v_{Bx})$$

$v'_{Bx} - v'_{Ax} = 0.9\ (15\sqrt{3} + 20)$ ... (iv)

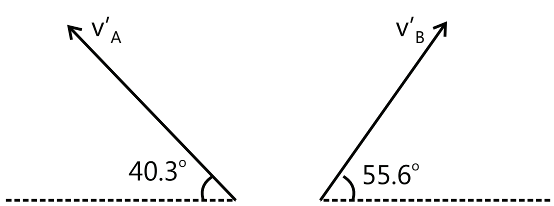
Solving (iii) and (iv) are get

$$v'_{Ax} = -17.7\text{ ms}^{-1}$$

$$v'_{Bx} = 23.68\text{ ms}^{-1}$$

$$\Rightarrow v'_A = \sqrt{v'^2_{Ax} + v'^2_{Ay}} = 23.2\text{ ms}^{-1}$$

and $v'_B = \sqrt{v'^2_{Bx} + v'^2_{By}} = 41.92\text{ ms}^{-1}$



Example 4: The mass of a rocket is 2.8 x 10^6 kg at launch time of this 2x10^6kg is fuel. The exhaust speed is 2500m/s and the fuel is ejected at the rate of 1.4x10^4kg/sec.

(a) Find thrust on the rocket.

(b) What is initial acceleration at launch time?

Ignore air resistance.

Sol: To lift the rocket upward against gravity, the thrust force in the upward direction due to exiting gases should be greater than or equal to the gravitational force. The equation of motion of the rocket can be written in terms of force as $m\frac{dv}{dt} = F_g + v_r\left(\frac{dm}{dt}\right)$

Thrust force

$$F_{th} = v_r\left(\frac{dm}{dt}\right) = 2500\ ms^{-1} \times 1.4 \times 10^4\ kgs^{-1}$$

$$F_{th} = 3.5 \times 10^7\ N$$

Equation of motion of rocket is

$$m\frac{dv}{dt} = -W + v_r\left(\frac{dm}{dt}\right) = -mg + v_r\left(\frac{dm}{dt}\right)$$

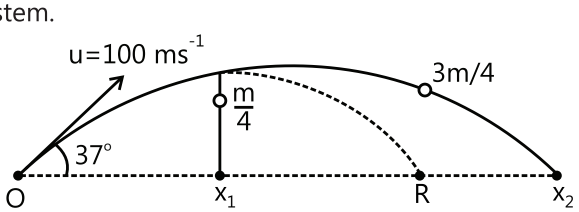
$$\Rightarrow \left.\frac{dv}{dt}\right|_{t=0} = -g + \frac{F_{th}}{m_0} = -9.8 + \frac{3.5 \times 10^7 N}{2.8 \times 10^6 kg}$$

$$\Rightarrow a_0 = \left(\frac{350}{28} - 9.8\right)ms^{-2}$$

$$\Rightarrow a_0 = 12.5 - 9.8 = 2.7\ ms^{-2}$$

Example 5: A projectile is fired at a speed of 100 m/s at an angle of 37° above horizontal (see figure) At the highest point the projectile breaks into two parts of mass ratio 1:3. Find the distance from the launching point to the point where the heavier piece lands. The smaller mass has zero velocity with respect to the earth immediately after explosion.

Sol: The range of center of mass of the system during projectile motion is given by $X_{CM} = R = \frac{u^2 \sin 2\theta}{g}$. This range is not effected by any internal changes in the system.



The C.O.M of the projectile will hit the horizontal plane at the same point where it would have hit without any explosion i.e. the range of COM will not change. Both the smaller and larger mass will reach the ground together because the vertical components of their velocity are equal to zero after the explosion. (Explosion took place at the highest point of the trajectory and the smaller mass comes to rest just after the explosion). At highest point $x_1 = \frac{R}{2}$.

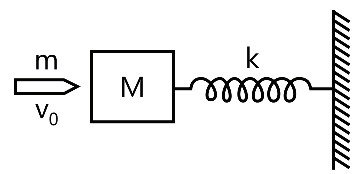
$$X_{cm} = R = \frac{2u^2 \sin\theta\cos\theta}{g} = \frac{2 \times 10^4 \times 0.6 \times 0.8}{10}$$

$$X_{cm} = 960 = \frac{\frac{m}{4}.\frac{960}{2} + \frac{3m}{4}.x_2}{m}$$

$$\Rightarrow 960 = \frac{960}{8} + \frac{3}{4}x_2$$

$$\Rightarrow x_2 = \frac{4}{3} \times 960 \times \frac{7}{8} = 160 \times 7 = 1120m$$

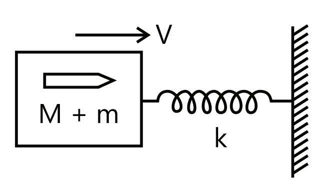
Example 6: A bullet of mass m strikes a block of mass M connected to a light spring of stiffness k, with a speed v_0 and gets embedded into mass M. Find the loss of K.E. of the system just after impact



Sol: During collision as there is no net external force acting on the bullet-block system, hence the momentum of the system can be conserved. As the bullet hits block in-elastically, some of its initial K.E. is lost during the collision.

As the bullet of mass m hits the block of mass M, and gets embedded into it, we can write the equation of conservation of linear momentum at the instant of collision, assuming the force due to spring to be negligible, as at this instant the block M has just started moving and the compression in the spring is negligible (see figure)

$$mV_0 = (m+M)V \Rightarrow V = \frac{mV_0}{m+M}$$



where V is the velocity of block just after collision.

Loss in kinetic energy of the system of bullet and block is,

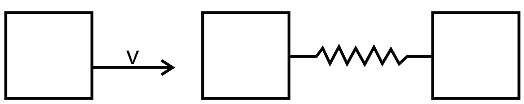
$$\Delta K = \frac{1}{2}mv_0^2 - \frac{1}{2}(M+m)V^2$$

$$= \frac{1}{2}\left[mv_0^2 - (M+m)\frac{m^2v_0^2}{(M+m)^2}\right]$$

$$= \frac{mv_0^2}{2}\left[1 - \frac{m}{M+m}\right]$$

$$\Delta K = \frac{mMv_0^2}{2(M+m)} \text{ J}$$

Example 7: Two blocks B and C of mass m each connected by a spring of natural length l and spring constant k rest on an absolutely smooth horizontal surface as shown in figure A third block A of same mass collides elastically to block B with velocity v. Calculate the velocities of blocks, when the spring is compressed as much as possible and also the maximum compression.



Sol: In absence of frictional forces on block, the total mechanical energy of the system comprising the blocks B and C and spring will be conserved. At the time of maximum compression of spring, the mechanical energy of this system in C-frame will be totally stored as elastic potential energy of the spring.

Block A collides with block B elastically. So conserving momentum between A and B we get, (spring force is negligible at the instant of collision)

$$mv = mv_A = mv_A + mv_B$$

or $v = v_A + v_B$ ...(i)

$v = v_B - v_A$ ...(ii) (restitution equation)

Solving (i) and (ii), we get

$v_A = 0$ and $v_B = v$ ...(iii)

For system comprising blocks B and C, the velocity of center of mass after collision is,

$$v_{cm} = \frac{mv_B}{m+m} = \frac{v}{2} \text{ ms}^{-1} \quad ...(iv)$$

As there are no dissipative forces in the horizontal direction the velocity of COM will remain constant. Let us consider the motion of B and C in the C-frame. At the instant of maximum compression the blocks B and C will come to rest in the C-frame. So there velocity in K- frame will become equal to the velocity of COM.

$$\therefore\ v_B = v_C = \frac{v}{2} \text{ ms}^{-1} \quad ...(v)$$

Kinetic energy of system ''B + C" just after collision in K frame is,

$$K_{sys} = \frac{1}{2} m\, v_B{}^2 + 0 = \frac{1}{2} mv^2$$

Now $K_{sys} = K_{sys/c} + \frac{2mv_{cm}^2}{2}$

$$\Rightarrow \frac{1}{2} mv^2 = K_{sys/c} + m\frac{v^2}{4}$$

$$\Rightarrow K_{sys/c} = \frac{1}{4} mv^2 \quad \left[K_{sys/c} = \frac{\mu v_{rel}^2}{2} = \frac{m}{2}.\frac{v^2}{2}\right]$$

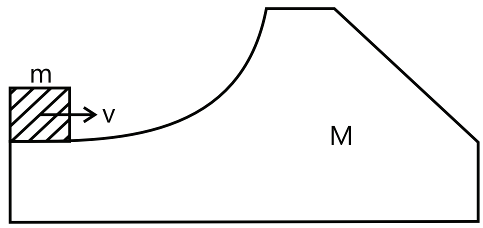
Initially the potential energy of spring is zero and when the compression is maximum the energy in C-frame will be entirely converted into potential energy of the spring, thus we can write

$$\frac{mv^2}{4} = \frac{1}{2} k\, x_{max}^2$$

$$\Rightarrow x_{max} = v\sqrt{\frac{m}{2k}}$$

JEE Advanced/Boards

Example 1: A body of mass M with a small disc of mass m placed on it rests on a smooth horizontal plane as shown in figure. The disc is set in motion in the horizontal direction with velocity v. To what height relative to the initial level will the disc rise after breaking off the body M? The friction is assumed to be absent.



Sol: As there are no external forces acting on the system comprising m and M in the horizontal direction, the momentum is conservative in the horizontal direction. At the instant the disk m breaks – off from block M, it has a component of velocity u in vertical direction and the disk m and block M have a common velocity V in the horizontal direction.

In horizontal direction we can write,

$$mv = (m+M)V \Rightarrow V = \frac{mv}{m+M} \text{ ms}^{-1} \quad ...\text{(i)}$$

Once the disc breaks–off the block, then its horizontal velocity will not change and at the highest point of its trajectory, the vertical component of its velocity becomes zero.

Using law of conservation of energy, we get

$$\frac{1}{2}mv^2 = \frac{1}{2}mV^2 + mgH + \frac{1}{2}MV^2 \quad \text{... (ii)}$$

where H is the height raised by the disc.

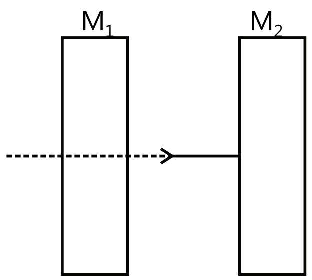
$$\Rightarrow mgH = \frac{1}{2}m(v^2) - \frac{1}{2}(m+M)V^2$$

$$\Rightarrow mgH = \frac{1}{2}mv^2 - \frac{1}{2}\frac{m^2v^2}{(m+M)} \quad \text{(using (i))}$$

$$\Rightarrow H = \frac{v^2}{2g}\left[1 - \frac{m}{m+M}\right]$$

$$H = \frac{v^2}{2g}\cdot\left(\frac{M}{m+M}\right)\ m$$

Example 2: A 20 gm bullet pierces through a plate of mass M_1=1 kg and then comes to rest inside a second plate of mass M_2=2.98 kg as shown in the figure. It is found that the two plates, initially at rest, now move with equal velocities. Find the percentage loss in the initial velocity of the bullet when it is between M_1 and M_2. Neglect any loss of material of the plates, due to action of bullet.



Sol: As the net external force on the system during collision is zero, the momentum of system can be conservative.

Conserve momentum for collision of bullet with fist plate, $mv = mu + M_1V$ ....(i)

Conserve momentum for collision of bullet with second plate, $mu = (m + M_2)V$ ....(ii)

Here the two plates move with equal velocity V after the collision.

Eliminate V from equations (i) and (ii) to get

$$mv = mu + M_1\frac{mu}{(m+M_2)}$$

$$u\left[1 + \frac{M_1}{m+M_2}\right] = v\,;\ \Rightarrow u = \frac{v(m+M_2)}{m+M_1+M_2}$$

$$\Rightarrow \frac{v-u}{v} = 1 - \frac{u}{v} = 1 - \frac{m+M_2}{m+M_1+M_2}$$

$$\Rightarrow \frac{\Delta v}{v}\times 100\% = \frac{M_1}{m+M_1+M_2}\times 100\%$$

Putting the values of m, M_1 and M_2 we get

$$\%\ \text{loss} = \frac{1}{0.020+1+2.98}\times 100\%$$

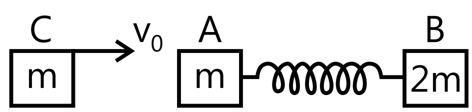
$$\%\ \text{loss} = 25\%$$

Example 3: Two bodies A and B of masses m and 2m respectively are placed on a smooth floor. They are connected by a spring. A third body C of mass m moves with a velocity v_0 along the line joining A and B and collides elastically with A, as shown in figure. At a certain time t_0, it is found that the instantaneous velocities of A and B are the same. Further, at this instant the compression of the spring is found to be x_0.

Find: (a) The common velocity of A and B at the time t_0. (b) The spring constant.

Sol: The collision between the blocks A and C is elastic. In C-frame at time of maximum compression of spring, the total mechanical energy will be stored as elastic potential energy of spring.

Masses of bodies C and A are same and C



collides elastically with body A initially at rest. So after collision C will come to rest and A will take up the velocity of C (spring force during collision is negligible.)

The velocity of center of mass (COM) of the system comprising blocks A and B just after collision is,

$$v_{cm} = \frac{m.v_0}{3m} = \frac{v_0}{3}\ ms^{-1}$$

As there are no external forces acting in horizontal direction, the velocity of COM will be constant.

In the C-frame when the compression in the spring is maximum the blocks will come to rest momentarily. Thus there velocity in K – frame will be equal to the velocity of COM.

$$\Rightarrow v_A = v_B = v_{cm} = \frac{v_0}{3}\ ms^{-1}$$

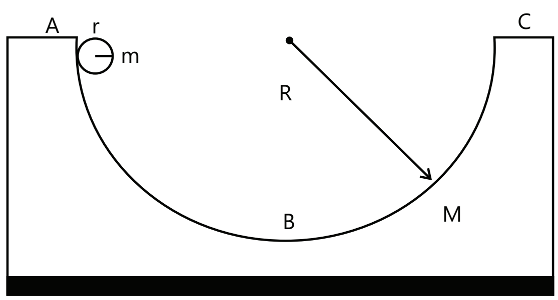
Just after collision the total kinetic energy of blocks A and B in C – frame is,

$$K_{sys/c} = \frac{1}{2}\mu\, v_{rel}^2 = \frac{(m)(2m)}{2\times 3m}.v_0^2 \Rightarrow K_{sys/c} = \frac{mv_0^2}{3}\text{ J}$$

This energy will get converted into the elastic potential energy of the spring at the instant of maximum compression,

$$\frac{mv_0^2}{3} = \frac{1}{2}k\,x_{max}^2 = \frac{1}{2}k\,x_0^2\ ;\ \Rightarrow\ k = \frac{2mv_0^2}{3x_0^2}\text{ J}$$

Example 4: A block of mass M with a semi-circular track of radius R rests on a horizontal frictionless surface. A uniform cylinder of radius r and mass m is released from rest at the point A as shown in the figure. The cylinder slips on the semicircular frictionless track.



(a) How far has the block moved when the cylinder reaches the bottom point B of the track?

(b) How fast is the block moving when the cylinder reaches the bottom of the track?

Sol: As there are no frictional forces acting on the system comprises cylinder and block, the gravitational potential energy of cylinder is converted into the kinetic energy of cylinder and block.

(a) There are no external forces acting on the system comprising cylinder and the block in the horizontal direction. So we can conserve momentum in the horizontal direction, so when cylinder reaches point B on the block, let its velocity in K-frame be v_1 towards right and velocity of block in K-frame be v_2 towards left. So we get

$$0 = mv_1 - Mv_2 \text{ or } mv_1 = Mv_2 \quad \text{....(i)}$$

Also the COM of the system was initially at rest and will continue to remain at rest in absence of horizontal external forces. When m moves towards right a distance of (R – r) relative to block M.

We can write,

$$X_{cm} = \frac{mx_1 + Mx_2}{m+M} \Rightarrow \Delta X_{cm} = \frac{m\,\Delta x_1 + M\Delta x_2}{m+M} = 0$$

Now $\Delta x_1 = (R-r) + \Delta x_2$

$$\Rightarrow m(R - r + \Delta x_2) + M\Delta x_2 = 0$$

$$\Rightarrow \Delta x_2 = -\frac{m}{M+m}(R-r) \quad \text{...(ii)}$$

Now (R – r) is towards right, so Δx_2 will be towards left.

(b) Now as there are no dissipative forces acting on the system, total energy of system is conserved. i.e.

$$mg(R-r) = \frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2 \quad \text{....(iii)}$$

From (i) and (iii) eliminate v_1 to get

$$mg(R-r) = \frac{1}{2}m\frac{M^2v_2^2}{m^2} + \frac{1}{2}Mv_2^2$$

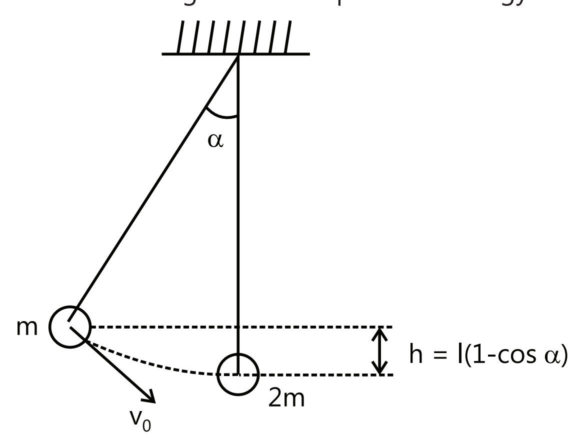
$$\Rightarrow mg(R-r) = \frac{1}{2}Mv_2^2\left[\frac{M}{m}+1\right]$$

$$\Rightarrow mg(R-r) = \frac{M(M+m)v_2^2}{2m}$$

$$\Rightarrow v_2 = m\sqrt{\frac{2g(R-r)}{M(M+m)}}\ \text{ms}^{-1}$$

Example 5: Two balls of masses m and 2m are suspended by two threads of same length l from the same point on the ceiling. The ball m is pulled aside through an angle α and released after imparting to it a tangential velocity v_0 towards the other stationery ball. To what heights will the balls rise after collision, if the collision is perfectly elastic?

Sol: In case of perfectly elastic collision, the kinetic energy of the system is conserved. At the maximum vertical displacement of the ball the total kinetic energy is converted in to gravitational potential energy.



Ball of mass m will collide the ball of mass 2m, which is initially at rest.

The velocity of impact of m be v, then by conserving energy of m we get

$$\frac{1}{2}mv_0^2 + mgl(1-\cos\alpha) = \frac{1}{2}mv^2$$

$$v_0^2 + 2gl(1-\cos\alpha) = v^2 \quad \text{...(i)}$$

Conserve momentum of balls before and after collision to get,

$$mv = mv_1 + 2mv_2 \text{ or } v = v_1 + 2v_2 \quad \text{...(ii)}$$

Equation for coefficient of restitution gives

$$v = v_2 - v_1 \quad \text{...(iii)}$$

Add (ii) and (iii) to get $2v = 3v_2$

$$\text{or } v_2 = \frac{2v}{3}ms^{-1} \text{ and } v_1 = -\frac{v}{3}ms^{-1} \quad \text{...(iv)}$$

Conserve energy for 'm' as it reaches maximum height,

$$\frac{1}{2}m\left(\frac{v}{3}\right)^2 = mg\,h_1$$

$$\text{or } h_1 = \frac{v^2}{18g} = \frac{v_0^2 + 2gl(1-\cos\alpha)}{18g} \text{ [using (i)]}$$

Conserve energy for '2m' as it reaches maximum height,

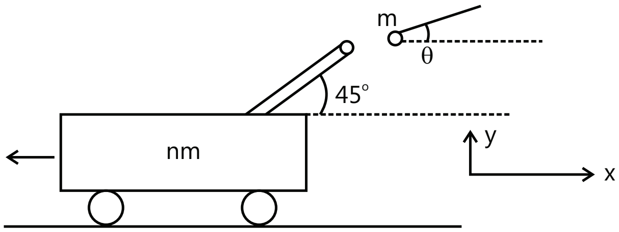
$$\frac{1}{2}2m\left(\frac{2v}{3}\right)^2 = 2mgh_2$$

$$\Rightarrow h_2 = \frac{1}{2g}.\frac{4v^2}{9}$$

$$\Rightarrow h_2 = \frac{2}{9g}\,[v_0^2 + 2gl(1-\cos\alpha)] \text{ [using (i)]}$$

Example 6: A gun is mounted on a gun carriage movable on a smooth horizontal plane and the gun is elevated at an angle 45° to the horizontal. A shot is fired and leaves the gun inclined at an angle θ to the horizontal. If the mass of gun and carriage is n times that of the shot, find the value of θ.

Sol: As the frictional force on the cart is zero, the momentum of cart comprising cart and bullet is conserved in horizontal direction.



Let the mass of the shot be m and the mass of the gun carriage be nm.

Suppose the velocity of the shot relative to the gun be u and its velocity relative to the ground be V. The gun recoils with a speed v. As the system comprising gun and the shot rests on a smooth horizontal plane, the net horizontal external force will be zero, so conserving momentum in the horizontal direction, taken as the x – axis, we get

$$(nm)v_x + mV_x = 0 \Rightarrow nmv_x + m(u_x + v_x) = 0$$

$$\Rightarrow (nm)(-v) + m(u\cos 45 - v) = 0$$

$$\Rightarrow -(n+1)mv + \frac{mu}{\sqrt{2}} = 0 \quad \text{...(i)}$$

$$\text{Again, } V_x = -n\,v_x = -n\,(-v);\Rightarrow V_x = nv \quad \text{...(ii)}$$

Now the component of the velocity of the gun along the vertical i.e. along the y – axis is zero, so the velocity of the shot along the y – axis will be given by

$$V_y = u\ \sin 45\ + 0\ ;\ V_y = \frac{u}{\sqrt{2}} \quad \text{...(iii)}$$

$$\Rightarrow \tan\theta = \frac{V_y}{V_x} = \frac{u/\sqrt{2}}{n\,v} \text{ (using (ii) \& (iii))}$$

$$\Rightarrow \tan\theta = \frac{u}{\sqrt{2}\,n\,v} \quad \text{....(iv)}$$

$$\text{From (i) we get } v = \frac{u}{\sqrt{2}(n+1)}$$

$$\Rightarrow \frac{u}{v} = \sqrt{2}(n+1) \quad \text{....(v)}$$

From (iv) and (v) we get

$$\tan\theta = \frac{n+1}{n} \Rightarrow \theta = \tan^{-1}\left(\frac{n+1}{n}\right)$$

JEE Main/Boards

Exercise 1

Q.1 Show that center of mass of an isolated system moves with a uniform velocity along a straight line path.

Q.2 Locate the center of mass of uniform triangular lamina and a uniform cone.

Q.3 Explain what is meant by center of gravity.

Q.4 Obtain an expression for the position vector of center of mass of a two particle system.

Q.5 Obtain an expression for the position vector of the center of mass of a system of n particle.

Q.6 Prove that center of mass of an isolated system moves with a uniform velocity along a straight line path.

Q.7 Find the center of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 0.10 kg, 0.15 kg and 0.20 kg respectively. Each side of the quilateral triangle is 0.5 m long.

Q.8 Find the center of mass of a triangular lamina.

Q.9 Two bodies of masses 0.5 kg and 1 kg are lying in XY plane at (–1, 2) and (3, 4) respetively. What are the co-ordinates of the center of mass ?

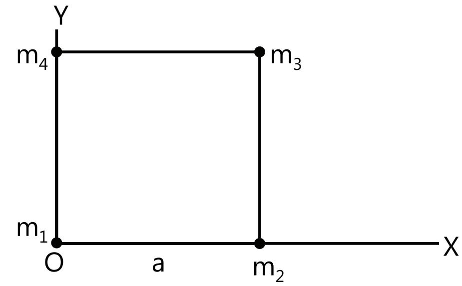
Q.10 Three point masses of 1 kg, 2 kg and 3 kg lie at (1, 2), (0, –1) and (2, –3) respectively. Calculate the co-ordinates of the center of mass of the system.

Q.11 Two particles of mass 2 kg and 1 kg are moving along the same straight line with speeds 2 ms^{-1} and 5 ms^{-1} respectively. What is the speed of the center of mass of the system if both the particles are moving (a) in same direction (b) in opposite direction ?

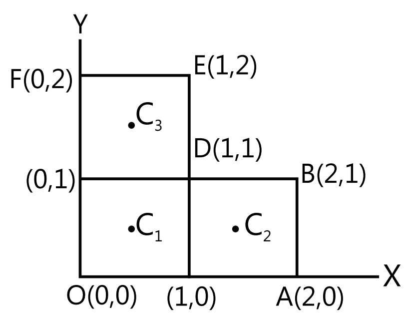
Q.12 Consider a two-particle system with the particles having masses m_1 and m_2. If the first particle is pushed towards the center of mass through a distance d, by what distance should the second particle be moved so as to keep the center of a mass at the same position?

Q.13 Two particles of masses 1 kg and 3 kg are located at $(2\hat{i}+5\hat{j}+13\hat{k})$ and $(-6\hat{i}+4\hat{j}-2\hat{k})$ meter respectively. Find the position of their center of mass.

Q.14 Four particles of masses $m_1 = 1kg$, $m_2 = 2kg$, $m_3 = 3kg$ and $m_4 = 4kg$ are located at the corners of a rectangle as shown in figure. locate the position of center of mass.



Q.15 Find the center of mass of uniform L shaped (a thin flat plate) with dimensions as shown in figure. The mass of the lamina is 3 kg.



Exercise 2

Single Correct Choice Type

Q.1 A bullet of mass m moving with a velocity v strikes a vertically suspended wooden block of mass M and embedded in it. If the block rises to a height h, the initial velocity of the bullet will be

(A) $\sqrt{2hg}$ (B) $\left(\frac{M+m}{m}\right)\sqrt{2hg}$

(C) $\left(\frac{m}{M+m}\right)\sqrt{2hg}$ (D) $\left(\frac{M+m}{m}\right)\sqrt{hg}$

Q.2 A body of mass 1 kg, which was initially at rest, explodes and breaks into three fragments of masses in the ratio of $1:1:3$.

Both the pieces of equal masses fly off perpendicular to each other with a speed of 30 m/s each. The velocity of the heavier fragment is

(A) $\frac{10}{\sqrt{2}}$ ms^{-1} (B) $10\sqrt{2}$ ms^{-1}

(C) 20 ms^{-1} (D) $20\sqrt{2}$ ms^{-1}

Q.3 If the linear momentum of a body is increased by 50%, its kinetic energy will increase by

(A) 50% (B) 100% (C) 125% (D) 150%

Q.4 Two perfectly elastic particles A and B of equal masses travelling along the line joining them with velcity 25 $ms^{-1}$and 20 ms^{-1} respectively collide. Their velocities after the elastic collision will be (in ms^{-1}) respectively.

(A) 0 and 45 (B) 5 and 45

(C) 20 and 25 (D) 25 and 20

Q.5 A body of mass 2.9 kg is suspended from a string of length 2.5 m and is at rest. A bullet of mass 0.1 kg, moving horizontally with a speed of 150 ms^{-1} strikes and sticks to it. What is the maximum angle made by the string with the vertical after the impact?

($g = 10\ ms^{-2}$)

(A) 30° (B) 45° (C) 60° (D) 90°

Q.6 An isolated particle of mass m is moving in a horizontal plane (x-y), along the x-axis, at a certain height above the ground. It suddenly explodes into two fragments of mass $\frac{m}{4}$ and $\frac{3m}{4}$. An instant later, the smaller fragment is at y = +15 cm. The larger fragment at this instant is at

(A) y = -5 cm (B) y = +20 cm

(C) y = +5 cm (D) y = -20cm

Q.7 A ball collides elastically with another ball of the same mass. The collision is oblique and initially one of the ball was at rest. After the collision, the two balls move with same speeds. What will be the angle between the velocity of the balls after the collision?

(A) 30° (B) 45° (C) 60° (D) 90°

Q.8 A body of mass 2kg moving with a velocity of 3 ms^{-1} collides head-on with a body of mass 1 kg moving with a velocity of 4 ms^{-1}. After collision the two bodies stick together and move with a common velocity which in the units m/s is equal to

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$

Q.9 Two particles of masses M and 2M are at a distance D apart. Under the mutual gravitational force they start moving towards each other. The acceleration of their center of mass when they are D/2 apart is:

(A) $2GM/D^2$ (B) $4GM/D^2$

(C) $8GM/D^2$ (D) Zero

Previous Year's Questions

Q.1 Two particles A and B initially at rest, move towards each other by mutual force of attraction. At the instant when the speed of A is v and the speed of B is 2v, the speed of the center of mass of the system is ***(1982)***

(A) 3v (B) v (C) 1.5v (D) zero

Q.2 A shell is fired from a cannon with a velocity v (ms^{-1}) at an angle θ with the horizontal direction. At the highest point in its path it explodes into two pieces of equal mass. One of the pieces retraces its path to the cannon and the speed (ms^{-1}) of the other piece immediately after the explosion is ***(1986)***

(A) $3v\cos\theta$ (B) $2v\cos\theta$

(C) $\frac{3}{2}v\cos\theta$ (D) $\sqrt{\frac{3}{2}}v\cos\theta$

Q. 3 Two particles of masses m_1 and m_2 in projectile motion have velocities $\vec{v}_1$ and $\vec{v}_2$ respectively at time t = 0. They collide at time t_0.Their velocities become $\vec{v}_1'$ and $\vec{v}_2'$ at time $2t_0$ while still moving in air. The value of $|(m_1\vec{v}_1' + m_2\vec{v}_2') - (m_1\vec{v}_1 + m_2\vec{v}_2)|$ is ***(2001)***

(A) Zero (B) $(m_1 + m_2)gt_0$

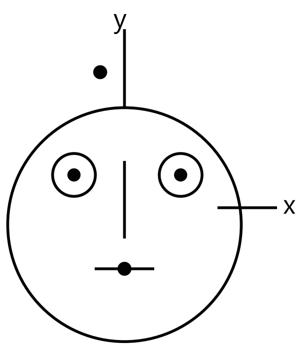
(C) $2(m_1 + m_2)g\ t_0$ (D) $\frac{1}{2}(m_1 + m_2)g\ t_0$

Q.4 Two blocks of masses 10 kg and 4 kg are connected by a spring of negligible mass and placed on a frictionless horizontal surface. An impulse gives a velocity of 14 ms^{-1} to the heavier block in the direction of the lighter block. The velocity of the center of mass is ***(2002)***

(A) 30 ms^{-1} (B) 20 ms^{-1}

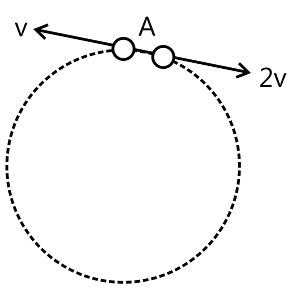
(C) 10 ms^{-1} (D) 5 ms^{-1}

Q.5 Look at the drawing given in the figure, which has been drawn with ink of uniform line-thickness. The mass of ink used to draw each of the two inner circles, and each of the two line segments is m. The mass of the ink used to draw the outer circle is 6m. The coordinates of the center of the different parts are: outer circle (0, 0), left inner circle (-a, a), right inner circle (a, a), vertical line (0, 0) and horizontal line (0, -a). The y-coordinate of the center of mass of the ink in this drawing is ***(2009)***



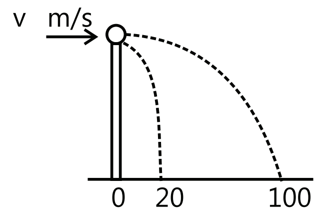
(A) $\frac{a}{10}$ (B) $\frac{a}{8}$ (C) $\frac{a}{12}$ (D) $\frac{a}{3}$

Q.6 Two small particles of equal masses start moving in opposite directions from a point A in a horizontal circular orbit. Their tangential velocities are v and 2v respectively, as shown in the figure. Between collisions, the particles move with constant speed. After making how many elastic collisions, other than that at A, these two particles will again reach the point A? ***(2009)***



(A) 4 (B) 3 (C) 2 (D) 1

Q.7 A ball of mass 0.2 kg rests on a vertical post of height 5 m. a bullet of mass 0.01 kg, travelling with a velocity v ms^{-1} in a horizontal direction, hits the center of the ball. After the collision, the ball and bullet travel independently. The ball hits the ground at a distance of 20 m and the bullet at the distance of 100 m from the foot of the post. The initial velocity v of the bullet is ***(2011)***

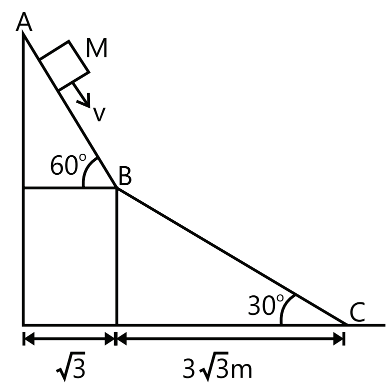


(A) 250 ms^{-1} (B) $250\sqrt{2}$ ms^{-1}

(C) 400 ms^{-1} (D) 500 ms^{-1}

Paragraph: Q.8 - Q.10

A small block of mass M moves on a frictionless surface of an inclined plane, as shown in figure. The angle of the incline suddenly changes from 60° to 30° at point B. the block is initially at rest at A. Assume that collisions between the block and the incline are total inelastic ($g = 10\ ms^{-2}$)



Q.8 The speed of the block at point B immediately after it strikes the second incline is ***(2008)***

(A) $\sqrt{60}$ ms^{-1} (B) $\sqrt{45}$ ms^{-1}

(C) $\sqrt{30}$ ms^{-1} (D) $\sqrt{15}$ ms^{-1}

Q.9 The speed of the block at point C, immediately before it leaves the second incline is ***(2008)***

(A) $\sqrt{120}$ ms^{-1} (B) $\sqrt{105}$ ms^{-1}

(C) $\sqrt{90}$ ms^{-1} (D) $\sqrt{75}$ ms^{-1}

Q.10 If collision between the block and the incline is completely elastic, then the vertical (upward) component of the velocity of the block at point B, immediately after it strikes the second incline is ***(2008)***

(A) $\sqrt{30}$ ms^{-1} (B) $\sqrt{15}$ ms^{-1}

(C) Zero (D) $-\sqrt{15}$ ms^{-1}

Q.11 This question has Statement-I and Statement-II. Of the four choices given after the Statements, choose the one that best describes the two Statements.

Statement-I: A point particle of mass m moving with speed v collides with stationary point particle of mass M. If the maximum energy loss possible is given as $f\left(\frac{1}{2}mv^2\right)$ then $f=\left(\frac{m}{M+m}\right)$.

Statement-II: Maximum energy loss occurs when the particles get stuck together as a result of the collision. ***(2013)***

(A) Statement-I is true, statement-II is true, statement-II is not a correct explanation of statement-I.

(B) Statement-I is true, statement-II is false.

(C) Statement-I is false, statement-II is true

(D) Statement-I is true, statement-II is true, statement-II is a correct explanation of statement-I.

Q.12 Distance of the centre of mass of a solid uniform cone from its vertex is z_0. If the radius of its base is R and its height is h then z_0 is equal to: ***(2015)***

(A) $\frac{3h}{4}$ (B) $\frac{5h}{8}$ (C) $\frac{3h^2}{8R}$ (D) $\frac{h^2}{4R}$

Q.13 A particle of mass m moving in the x direction with speed 2v is hit by another particle of mass 2m moving in the y direction with speed v. If the collision is perfectly inelastic, the percentage loss in the energy during the collision is close to ***(2015)***

(A) 50% (B) 56% (C) 62% (D) 44%

JEE Advanced/Boards

Exercise 1

Q.1 A block of mass 10 kg is suspended from a 3 m long weightless string. A bullet of mass 0.2 kg is fired into the block of horizontally with a speed of 20 ms^{-1} and it gets embedded in the block. Calculate

(a) The speed acquired by the block

(b) The maximum displacement of the block

(c) The energy converted to heat in the collision.

Q.2 A projectile of mass 50 kg shot vertically upwards with an initial velocity of 100 ms^{-1}. After 5 s it explodes into two fragments, one of which having mass 20 kg travels vertically up with a velocity of 150 metres/sec. if g = 9.8 ms^{-2}.

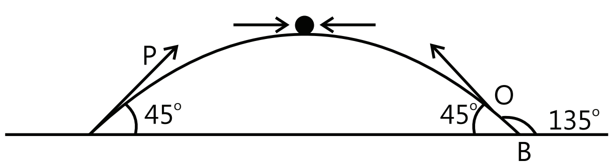
(a) What is the velcoity of the other fragment at that instant ?

(b) Calculate the sum of the momenta of the two fragments 3 s after the explosion. What would have been the momentum of the projectile at this instant if there had been no explosion?

Q.3 Particle P and Q of mass 20 g and 40 g respectively are simultaneously projected from points A and B on the ground. The initial velocities of P and Q make angle 45° and 135° respectively with line AB. Each particle has an initial speed of 49 ms^{-1}. The separation AB is 245 m. Both particles travel in the same vertical plane and undergo a collision. After the collision, P retraces its path. Taking g = 9.8 ms^{-2}, determine

(a) The position of Q when it hits the ground.

(b) How much time, after the collision, does Q take to reach the ground.

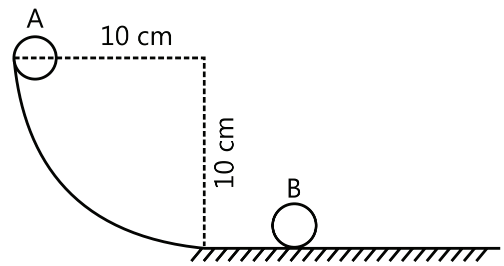


Q.4 A shell of mass 500 kg travelling horizontally at a speed of 100 ms^{-1} explodes into just three parts. The first part of mass 200 kg travels vertically upwards at a speed of 150 ms^{-1} and the second part of mass 150 kg travels horizontal with a speed of 60 ms^{-1}, but in a direction opposite to that of the original shell. What is the velocity fo the third part? What is the path of the center of mass of the fragments after the explosion?

Q.5 A small sphere of mass 10 g is attached to a point of smooth vertical wall by a light string of length 1 m. The sphere is pulled out in vertical plane perpendicular to the wall so that the string makes an angle of 60°

with the wall and is then released. It is found that after the first rebound, the string makes a maximum angle of 30° with the wall. Calculate the coefficient of restitution and the loss of kinetic energy due to impact. If all the energy is converted into heat, find the heat produced by the impact.

Q.6 A small ball A slides down the quadrant of a circle as shown in the figure and hits the ball B of equal mass which is initially at rest. Find the velocities of both the balls after collision. Neglect the effect of friction and assume the collision to be elastic.



Q.7 Two balls A and B of mass 0.10 kg and 0.25 kg respectively are connected by a stretched spring of negligible mass and spring constant 2 Nm^{-1}. Unstretched length of the spring is 0.6 m and placed on a smooth table. When the balls are released simultaneously the initial acceleration of ball B is 50 cm s^{-2} west-ward.

(a) What is the magnitude and direction of the initial acceleration of the ball A?

(b) What is the initial compression of the spring.

(c) What is the maximum distance between balls A and B.

Q.8 Find the center of mass of a uniform disc of radius a from which a circular section of radius b has been removed. The center of the hole is at a distance c from, the center of the disc.

Q.9 A man of mass m climbs a rope of length L suspended below a balloon of mass M. The ballon is stationary with respect to ground,

(a) If the man begins to climb up the rope at a speed v_{rel} (relative to rope) in what direction and with what speed (relative to ground) will the balloon move?

(b) How much has the balloon by climbing the rope.

(c) What is the state of motion after the man stops climbing?

Q.10 Prove that in case of oblique elastic collision of two particles of equal mass out of which one is at rest, the recoiling particles always move off at right angles to each other.

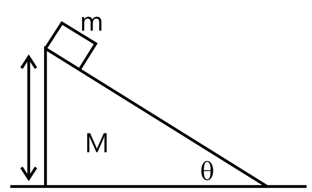
Q.11 A uniform thin rod of mass M and length L is standing vertically along the y-axis on a smooth hroizontal surface, with its lower end at the origin (0,0). A slight disturbance at t = 0 causes the lower end to slip on the smooth surface along the positive x-axis, and the rod starts falling.

(a) What is the path followed by the center of mass of the rod during its fall?

(b) Find the equation of trajectory of a point on the rod located at a distance r from the lower end.

Q.12 Two blocks of masses m_1 and m_2 are connected by a light inextensible string passing over a smooth fixed pulley of negligible mass. Find the acceleration of the center of mass of the system when blocks move under gravity.

Q.13 A block of mass m is resting on the top of a smooth prism of mass M which is resting on a smooth table. Calculate the distance moved by the prism when the block reaches the bottom.



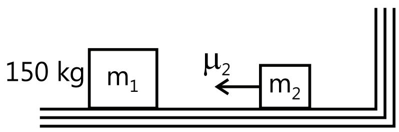
Q.14 A shell is fired from a cannon with a velocity v m/s at an angle θ with the horizontal direction. At the highest point of its path is explodes into two pieces of equal masses. What is the speed of other piece immediately after explosion, if one of the piece retraces its path to the cannon?

Q.15 A particle of mass 4m which is at rest explodes into three fragments. Two of the fragments each of mass m are found to move with a speed v each is mutually perpendicular directions. Calculate the energy released in the process of explosion.

Q.16 A moving particle of mass m makes a head on elastic collision with a particle of mass 2m which is initially at rest. Show that the colliding particle losses (8/9)th of its energy after collision.

Q.17 A ball is dropped on the ground from a height h. If the coefficient of restitution is e, then find the total distance travelled by the ball before coming to rest and the total time elapsed.

Q.18 A block of mass $m_1 = 150$ kg is at rest on a very long frictionless table, one end of which is terminated in a wall. Another block of mass m_2 is placed between the first block and the wall, and set in motion towards m_1 with constant speed u_2.

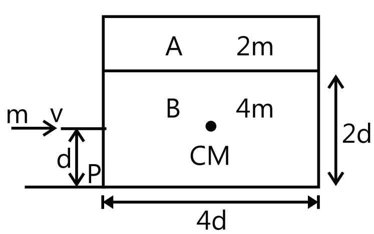


Assuming that all collisions are completely elastic, find the value of m_2 for which both blocks move with the same velocity after m_2 has collided once with m_1 and once with the wall. The wall has effectively infinite mass.

Q.19 A simple pendulum is suspended from a peg on a vertical wall. The pendulum is pulled away from the wall to a horizontal position and released. The ball hits the wall, the coefficient of restitution being $\left(\frac{2}{\sqrt{5}}\right)$.

What is the minimum number of collisions after which the amplitude of oscillation becomes less than 60°?

Q.20 A block A of mass 2m is placed on another block b of mass 4 m which in turn in placed on a fixed table. The two blocks have the same length 4d and they are placed as shown in the figure.



The coefficient of friction (both static and kinetic) between the block B and the table is μ. There is no friction between the two blocks. A small object of mass m moving horizontally along a line passing through the center of mass of the block B and perpendicular to its face with a speed v collides elastically with the block B at a height d above the table.

(a) What is the minimum value of v (call it v_0), required to make the block A topple?

(b) If $v = 2v_0$ find the distance (from the point P) at which the mass m falls on the table after collision.

Q.21 A 60 kg man and a 50 kg woman are standing on opposite ends of a platform of mass 20 kg. The platform is placed on a smooth horizontal ground. The man and the woman begin to approach each other. Find the displacement of the platform when the two meet in terms of the displacement x_0 of the man relative to the platform. The length of the platform is 6m.

Q.22 A rope thrown over a pulley has a ladder with a man A on one of its ends and a counter balancing mass M on it other end. The man whose mass is m, climb upwards by $\Delta\vec{r}$ relative to the ladder and the stops. Ignoring the masses of the pulley and the rope, as well as the friction in the pulley axis, find the displacement of the center of mass of this system.

Q.23 A drinking straw of length $\frac{3a}{2}$ and mass 2m is placed on a square table of side 'a' parallel to one of its sides such that one third of its length extends beyond the table. An insect of mass $\frac{m}{2}$ lands on the inner end of the straw (i.e., the end which lies on the table) and walks along the straw until it reaches the outer end. It does not topple even when another insect lands on top of the first one. Find the largest mass of the second insect that can have without toppling the straw. Neglect friction.

Q.24 A boy throws a ball with initial speed $2\sqrt{ag}$ at an angle θ to the horizontal. It strikes a smooth vertical wall and returns to his hand. Show that if the boy is standing at a distance 'a' from the wall, the coefficient of restitution between the ball and the wall equals $\frac{1}{(4\sin 2\theta - 1)}$. Also show that θ cannot be less than 15°.

Q.25 A ball is projected from a point A on a smooth inclined plane which makes an angle α to the horizontal. The velocity of projection makes an angle θ with the plane upwards. If on the second bounce the ball is moving perpendicular to the plane, find e in terms of α and θ. Here e is the coefficient of restitution between the ball and the plane.

Q.26 Two identical smooth balls are projected toward each ther from points A and B on the horizontal ground with same speed of projection. The angle of projection.

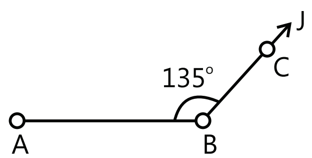
The angle of projection in each case is 30°. The distance between A and B is 100 m. The balls collide in air and return to their respective points of projection. If coefficient of restitution is e = 0.7, find

(a) The speed of projection of either ball.

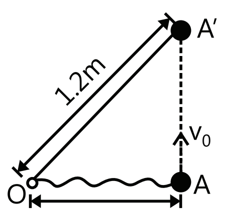
(b) Coordinates of point with respect to A where the balls collide.(Take g = 10 ms^{-2})

Q.27 Three identical particles A, B and C lie on a smooth horizontal table. (see figure) Light inextensible strings which are just taut connect AB and BC and $\angle ABC$ is 135°. An impulse J is applied to the particle C in the direction BC.

Find the initial speed of each particle. The mass of each particle in m.



Q.28 A 2 kg sphere A is connected to a fixed point O by an inextensible cord of length 1.2 m (see figure). The sphere is resting on a frictionless horizontal surface at a distance of 0.5 m from O when it is given a velocity v_0 in a direction perpendicular to the line OA. It moves freely until it reaches position A' when the cord becomes taut.

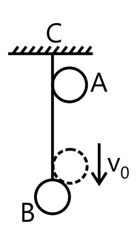


Determine

(a) The maximum allowable velocity v_0 if the impulse of the force exerted on the cord is not to exceed 3 Ns.

(b) The loss of energy as the cord becomes taut, if the sphere is given the maximum allowable velocity v_0.

Q.29 An open car of mass 1000 kg is running at 25 m/s holds three men each of mass 75 kg. Each man runs with a speed of 5 ms^{-1} relative to the car and jumps off from the back end. Find the speed of the car if the three men jump off.



(a) In succession

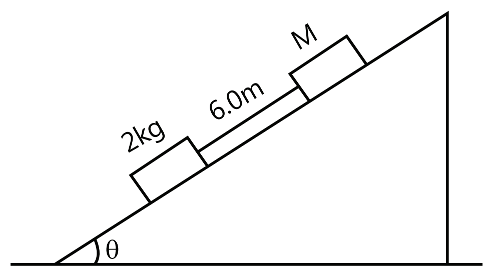
(b) All together.

Neglect friction between the car and the ground.

Q.30 Ball B is hanging from an inextensible cord BC. An identical ball A is released from rest when it is just touching the cord and acquires a velocity v_0 before striking ball B. Assuming perfectly elastic impact (e = 1) and no friction, determine the velocity of each ball immediately after impact.

Q.31 A particle whose initial mass is m_0 is projected vertically upwards at time t = 0 with speed gT, where T is a constant. The particle gradually acquires mass on its way up and at time t the mass of the particle has increased to $m_0 e^{dT}$. If the added mass is at rest relative to the particle when it is acquired, find the time when the particle is at highest point and its mass at that instant.

Q.32 Two blocks of mass 2kg and M are at rest on an inclined plane and are separated by a distance of 6.0 m as shown. The coefficient of friction between each block and the inclined plane is 0.25. the 2 kg block is given a velocity of 10.0 ms^{-1} up the inclined plane.



It collides with M, comes back and has a velocity of 1.0 m/s when it reaches its initial position. The other blocks M after the collision moves 0.5 m up and comes to rest. Calculate the coefficient of restitution between the blocks and the mass of the block M.

(Take $\sin\theta \approx \tan\theta = 0.05$ and g = 10 ms^{-2})

Exercise 2

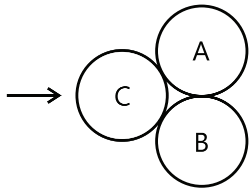
Single Correct Choice Type

Q.1 A bullet of mass m moving with a velocity v strikes a vertically suspended wooden block of mass M and embedded in it. If the block rises to a height h, the initial velocity of the bullet will be

(A) $\sqrt{2hg}$ (B) $\left(\frac{M+m}{m}\right)\sqrt{2hg}$

(C) $\left(\frac{m}{M+m}\right)\sqrt{2hg}$ (D) $\left(\frac{M+m}{m}\right)\sqrt{hg}$

Q.2 Two identical billiard balls A and B of equal mass and radius are in contact on a horizontal table. A similar third ball C strikes these balls symmetrically in the middle and remains at rest after the impact, the coefficient of restitution of the balls is



(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{\sqrt{3}}{2}$

Q.3 A sphere of mass m moving with a constant velocity u hits another stationary sphere of the same mass. If e is the coefficient of the restitution, then the ratio of the velocities of the two spheres after collision will be

(A) $\left(\frac{1-e}{1+e}\right)$ (B) $\left(\frac{1+e}{1-e}\right)$ (C) $\left(\frac{e+1}{e-1}\right)$ (D) $\left(\frac{e-1}{e+1}\right)$

Q.4 A cannon ball is fired with a velocity of 200 ms^{-1} at an angle of 60° with the horizontal. At the highest point it explodes into three equal fragments. One goes vertically upwards with a velocity of 100 ms^{-1}, the second one falls vertically downwards with a velocity of 100 ms^{-1}. The third one moves with a velcoity of

(A) 100 ms^{-1} horizontally

(B) 300 ms^{-1} horizontally

(C) 200 ms^{-1} at 60° with the horizontal

(D) 300 ms^{-1}at 60° with the horizontal

Q.5 A bullet of mass 0.01 kg, travelling at a speed of 500 m/s, strikes a block of mass 2kg, which is suspended by a string of length 5 m, and emerges out. The block rises by a vertical distance of 0.1 m. The speed of the bullet after it emerges from the block is

(A) 55 ms^{-1} (B) 110 ms^{-1}

(C) 220 ms^{-1} (D) 440 ms^{-1}

Q.6 A 1 kg ball, moving at 12 ms^{-1}collides head-on with a 2 kg ball moving in the opposite direction at 24 m/s. If the coefficient of restitution is $\frac{2}{3}$, then the energy lost in the collision is

(A) 60 J (B) 120 J

(C) 240 J (D) 480 J

Q.7 A body of mass m_1 and speed v_1 makes a head-on, elastic collision with a body of mass m_2, initially at rest. The velocity of m_1 after the collision is

(A) $\frac{m_1+m_2}{m_1m_2}v_1$ (B) $\frac{m_1-m_2}{m_1+m_2}v_1$

(C) $\frac{2m_1v_1}{m_1+m_2}$ (D) $\frac{2m_2v_1}{m_1+m_2}$

Q.8 In the above example, the velocity of mass m_2 after the collision is

(A) $\frac{m_1+m_2}{m_1m_2}v_1$ (B) $\frac{m_1-m_2}{m_1+m_2}v_1$

(C) $\frac{2m_1v_1}{m_1+m_2}$ (D) $\frac{2m_2v_1}{m_1+m_2}$

Q.9 A ball of mass m approaches a moving wall of infinite mass with speed v along the normal to the wall. The speed of the wall is u towards the ball. The speed of the ball after an elastic collision with the wall is

(A) u + v away from the wall

(B) 2u + v away from the wall

(C) u – v away from the wall

(D) v – 2u away from the wall.

Q.10 A neutron is moving with velocity u. It collides head on and elastically with an atom of mass number A. If the initial K.E. of the neutron is E, how much K.E. is retained by neutron after collision?

(A) $[A/(A+1)]^2E$ B) $[A/(A+1)^2]E$

(C) $[(1-A)/(A+1)^2]E$ (D) $[(A-1)/(A+1)^2]E$

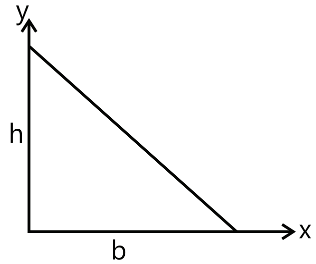
Q.11 A ball is dropped from a height h on the ground. If the coefficient of restitution is e, the height to which the ball goes up after it rebounds for the nth time is

(A) he^2n (B) he^2 (C) $\frac{e^2n}{h}$ (D) $\frac{h}{e^{2n}}$

Q.12 Two equal spheres A and B lie on a smooth horizontal circular groove at opposite ends of diameter. A is projected along the groove and at the end of time t impinges on B. If e is coefficient of restitution, the second impact will occur after a time

(A) $\frac{2t}{e}$ (B) $\frac{t}{e}$ (C) $\frac{\pi t}{e}$ (D) $\frac{2\pi t}{e}$

Q.13 The center of mass of triangle shown in the figure. has co-ordinates.



(A) $x = \frac{h}{2}; y = \frac{b}{2}$ (B) $x = \frac{b}{2}; y = \frac{h}{2}$

(C) $x = \frac{b}{3}; y = \frac{h}{3}$ (D) $x = \frac{h}{3}; y = \frac{b}{3}$

Q.14 A cart of mass M is tied to one end of a massless rope of length 10 m. The other end of the rope is in the hands of a man of mass M, the entire system is on a smooth horizontal surface. The man is at x = 0 and the cart at x = 10 m. if the man pulls the cart by a rope, the man and the cart will meet at the point :

(A) x = 0 (B) x = 5 m

(C) x = 10 m (D) They will nevemeet

Multiple Correct Choice Type

Q.15 Which one of the following statements does not hold god when two balls of masses m_1 and m_2 undergo elastic collision?

(A) when $m_1 < m_2$ and m_2 at rest, there will be maximum transfer of momentum.

(B) when $m_1 > m_2$ and m_2 at rest, after collision the ball of mass m_2 moves with four times the velocity of m_1

(C) when $m_1 = m_2$ and m_2 at rest, there will be maximum transfer of K.E.

(D) when collision is oblique and m_2 at rest with $m_1 = m_2$, after collision the ball moves in opposite directions.

Assertion Reasoning Type

Each of the questions given below consists of two statements, an assertion (A) and reason (R). Select the number corresponding to the appropriate alternative as follows.

(A) If both A and R are true and R is the correct explanation of A.

(B) If both A and R are true but R is not the correct explanation of A.

(C) If A is true but R is false

(D) If A is flase but R is true

Q.16 Assertion: When two bodies of different masses are just released from different position above the ground, then acceleration of their center of mass is zero.

Reason: When bodies move, their center may change position but is not accelerated.

Q.17 Assertion: The center of mass of a proton and an electron, released from their respective positions remains at rest.

Reason: The proton and electron attract and move towards each other. No external force is applied, therefore, their center of mass remains at rest.

Q.18 Assertion: The center of mass of a body may lie where there is no mass.

Reason: Center of mass of a body is a point, where the whole mass of the body is supposed to be concentrated.

Q.19 Assertion: When a body dropped from a height explodes in mid air, the pieces fly in such a way that their center of mass keeps moving vertically downwards.

Reason: Explosion occurs under internal forces only. External force = 0.

Q.20 Assertion: The center of mass of a circular disc lies always at the center of the disc.

Reason: Circular disc is a symmetrical body.

Q.21 Assertion: At the center of earth, a body has center of mass, but no center of gravity.

Reason: This is because g = 0 at the center of earth.

Q.22 Assertion: The center of mass of a body may lie where there is no mass.

Reason: The center of mass has nothing to do with the mass.

Comprehension Type

In physics, we come across many examples of collisions. The molecules of a gas collide with one another and with the container. The collisions of a neutron with an atom is well known. In a nuclear reactor, fast neutrons produced in the fission of uranium atom have to be slowed down. They are, therefore, made to collide with hydrogen atom. The term collision does not necessarily mean that a particle or a body must actually strike another. In fact, two particles may not even touch each other and yet they are said to collide if one particle influences the motion of the other. When two bodies collide, each body exerts an equal and opposite force on the other. The fundamental conservation law of physics are used to determine the velocities of the bodies after the collision. Collision may be elastic or inelastic. Thus a collision may be defined as an event in which two or more bodies exert relatively strong forces on each other for a relatively short time. The forces that the bodies exert on each other are internal to the system.

Almost all the knowledge about the sub-atomic particles such as electrons, protons, neutrons, muons, quarks, etc. is obtained from the experiments involving collisions.

There are certain collisions called nuclear reactions in which new particles are formed. For example, when a slow neutron collides with a U^{235} nucleus, new nuclei barium-141 and Kr^{92} are formed. This collisioin is called nuclear fission. In nuclear fusion, two nuclei deuterium and tritium collide (or fuse) to form a helium nucleus with the emission of a neutron.

Q.23 Which one of the following collisions is not elastic?

(A) A hard steel ball dropped on a hard concrete floor and rebounding to its original height.

(B) Two balls moving in the same direction collide and stick to each other

(C) Collision between molecules of an ideal gas.

(D) Collisions of fast neutrons with hydrogen atoms in a fission reactor.

Q.24 Which one of the following statemnts is true about inelastic collision?

(A)The total kinetic energy of the particles after collision is equal to that before collision.

(B) The total kinetic energy of the particle after collision is less than that before collision.

(C) The total momentum of the particles after collision is less than that before collision.

(D) Kinetic energy and momentum are both conserved in the collision.

Q.25 In elastic collision

(A) Only energy is conserved.

(B) Only momentum is conserved.

(C) Neither energy nor momentum is conserved.

(D) Both energy and momentum are conserved.

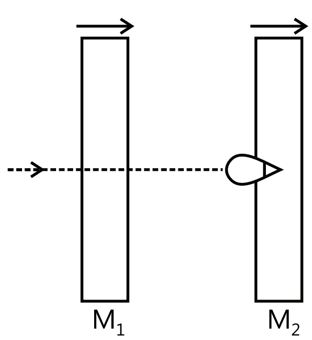
Previous Years' Questions

Q.1 A body of mass m moving with a velocity v in the x-direction collides with another body of mass M moving in the y-direction with a velocity V. They coalesce into one body during collsion. Find

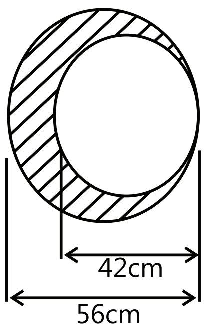
(a) The direction and magnitude of the momentum of the composite body.

(b) The fraction of the initial kinetic energy transformed into heat during the collision. ***(1978)***

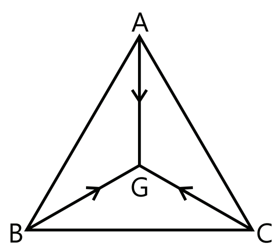
Q.2 A 20 g bullet pierces through a plate of mass $M_1 = 1kg$ and then comes to rest inside a second plate of mass $M_2 = 2.98kg$ as shown in the figure. It is found that the two plates initially at rest, now move with equal velocities. Find the percentage loss in the initial velocity of the bullet when it is between M_1 and M_2. Neglect any loss of material of the plates due to the action of bullet. Both plates are lying on smooth table. ***(1979)***



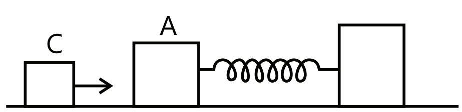
Q.3 A circular plate of uniform thickness has a diameter of 56 cm. A circular portion of diameter 42 cm is removed from one edge of the plate as shown in figure. Find the position of the center of mass of the remaining portion. ***(1980)***



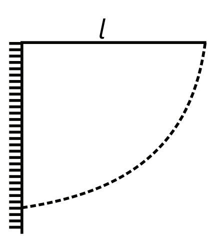
Q.4 Three particles A, B and C of equal mass move with equal speed v along the medians of an equilateral triangle as shown in figure. They collide at the centroid G of the triangle. After the collision, A comes to rest, B retraces its path with speed v. What is the velocity of C? ***(1982)***



Q.5 Two bodies A and B of masses m and 2m respectively are placed on a smooth floor. They are connected by a spring. A third body C of mass m moves with velocity v_0 along the line joining A and B and collides elastically with A as shown in figure. At a certain instant of time t_0 after collision, it is found that the instantancous velocities of A and B are the same. Further at this instant the compression of the spring is found to be x_0. Determine (a) the common velocity of A and B at time t_0 and (b) the spring constant. ***(1984)***



Q.6 A simple pendulum is suspended from a peg on a vertical wall. The pendulum is pulled away from the wall to a horizontal position (see figure) and released. The ball hits the wall, the coefficient of restitution being $\frac{2}{\sqrt{5}}$. What is the minimum number of collisions after which the amplitude of oscillations becomes less than 60 degrees? ***(1987)***

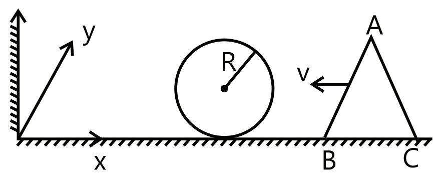


Q.7 A uniform thin rod of mass M and length L is standing vertically along the y-axis on a smoth horizontal surface, with its lower end at the origin (0, 0). A slight disturbance at t = 0 causes the lower end to slip on the smooth surface along the positive x-axis, and the rod starts falling.

(a) What is the path followed by the center of mass of the rod during its fall?

(b) Find the equation of the trajectory of a point on the rod located at a distance r from the lower end. What is the shape of the path of this point? ***(1993)***

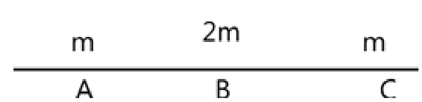
Q.8 A wedge of mass m and triangular cross-section $(AB = BC = CA = 2R)$ is moving with a constant velocity $(-v\hat{i})$ towards a sphere of radius R fixed on a smooth horizontal table as shown in the figure. The wedge makes an elastic collision with the fixed sphere and returns along the same path without any rotation. Neglect all friction and suppose that the wedge remains in contact with the sphere for a very short time Δt during which the sphere exerts a constant force $\vec{F}$ on the wedge. ***(1998)***



(a) Find the force $\vec{F}$ and also the normal force $\vec{F}$ exerted by the table on the wedge during the time Δt.

(b) Let h denote the perpendicular distance between the center of mass of the wedge and the line of action of force. Find the magnitude of the torque due to the normal force $\vec{N}$ about the center of the wedge during the interval Δt.

Q.9 Three objects A, B and C are kept in a straight line on a frictionless horizontal surface (see figure). These have masses m, 2m and m, respectively. The object A moves towards B with a speed $9ms^{-1}$ and makes an elastic collision with it. Thereafter, B makes completely inelastic collision with C. All motions occur on the same straight line. Find the final speed (in ms^{-1}) of the object C. ***(2009)***



Q.10 A particle of mass m is projected from the ground with an initial speed u_0 at an angle α with the horizontal.

At the highest point of its trajectory, it makes a completely inelastic collision with another identical particle, which was thrown vertically upward from the ground with the same initial speed u_0. The angle that the composite system makes with the horizontal immediately after the collision is ***(2013, 14, 15, 16)***

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{4}+\alpha$ (C) $\frac{\pi}{4}-\alpha$ (D) $\frac{\pi}{2}$

Q.11 A bob of mass m, suspended by a string of length l_1, is given a minimum velocity required to complete a full circle in the vertical plane. At the highest point, it collides elastically with another bob of mass m suspended by a string of length l_2, which is initially at rest. Both the strings are mass-less and inextensible. If the second bob, after collision acquires the minimum speed required to complete a full circle in the vertical plane, the ratio l_1 / l_2 is ***(2013)***

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q. 7 Q.9 Q.16

Exercise 2

Q.1 Q.9

Previous Years' Questions

Q.2 Q.3 Q.5

Q.8 Q.9 Q.10

JEE Advanced/Boards

Exercise 1

Q.3 Q.6 Q.7

Q.10 Q.20 Q.21

Q.28 Q.32

Exercise 2

Q.1 Q.2 Q.3

Q.4 Q.9

Previous Years' Questions

Q.2 Q.5 Q.8

Answer Key

JEE Main/Boards

Exercise 1

Q.9 $\frac{5}{3}, \frac{10}{3}$

Q.10 $\frac{7}{6}, -\frac{3}{2}$

Q.11 (a) 3 ms^{-1} (b) $\frac{1}{3}$ ms^{-1} in the direction of motion of 1 kg

Q.12 $\frac{m_1}{m_2}d$

Q.13 $-\hat{i} + \frac{17}{4}\hat{j} + \frac{7}{4}\hat{k}$

Q.14 $0.5a\hat{i} + 0.7b\hat{j}$

Q.15 $\frac{5}{6}m; \frac{5}{6}m$

Exercise 2

Single Correct Choice Type

Q.1 B	**Q.2** B	**Q.3** C	**Q.4** C	**Q.5** C	**Q.6** A
Q.7 D	**Q.8** C	**Q.9** D			

Previous Years Questions

Q.1 D	**Q.2** A	**Q.3** C	**Q.4** C	**Q.5** A	**Q.6** C
Q.7 D	**Q.8** B	**Q.9** B	**Q.10** C	**Q.11** C	**Q.12** A
Q.13 B					

JEE Advanced/Boards

Exercise 1

Q.1 (a) 0.39 m/s (b) 0.220 m (c) 39.32 J

Q.2 15 m/s, 1080 kg ms^{-1}

Q.3 (a) 122.5 m (b) $5\sqrt{2}$ second.

Q.4 441.25 m/s, $-27°$

Q.5 0.518, 0.0359 J, 0.0085 cals

Q.6 $v_A = 0$, $v_B = 1.4$ m/s

Q.7 (a) 1.25 cm/s^2 (eastwards) (b) 6.25 cm (c) 66.25 cm

Q.8 At a distance $\frac{cb^2}{a^2 - b^2}$ from O on the other side of the hole.

Q.9 (a) $-m\vec{v}_{rel}$ / (M + m) (b) L $\frac{m}{M+m}$ (c) system is stationary

Q.11 (a) Straight line (b) $\frac{x^2}{[L/2-r]^2} + \frac{y^2}{r^2} = 1$

Q.12 $\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 g$

Q.13 $\frac{mh\cot\theta}{M+m}$

Q.14 $3v\cos\theta$

Q.15 $\frac{3}{2}mv^2$

Q.17 $\frac{h(1+e^2)}{1-e^2}, \sqrt{\frac{2h}{g}}\left[\frac{1+e}{1-e}\right]$

Q.18 50 kg

Q.19 4

Q.20 (a) $\frac{5}{2}\sqrt{6\mu gd}$ (b) $= -6d\sqrt{3\mu}$

Q.21 $\frac{30-11x_0}{13}$

Q.22 $\frac{m}{2M}\overrightarrow{\Delta r}$

Q. 23 $m' - m = \frac{m}{2}$

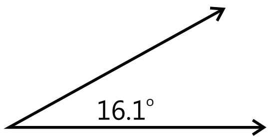
Q.25 $\frac{\cot\theta\cot\alpha}{2} - 1$

Q.27 $\frac{\sqrt{2}}{7m}, \frac{\sqrt{10}}{7m}, \frac{3J}{7m}$

Q.28 (a) 1.65 m/s (b) 2.25 J

Q.29 (a) 25.97 m/s (b) 25.92 m/s

Q.30 $|v'_B| = 0.721v_0, |v'_A| = 0.693v_0$



Q.31 T ln (2), $2m_0$

Q.32 0.84, 15.011 kg

Exercise 2

Single Correct Choice Type

Q.1 B **Q.2** C **Q.3** A **Q.4** B **Q.5** C **Q.6** C

Q.7 B **Q.8** C **Q.9** B **Q.10** C **Q.11** A **Q.12** A

Q.13 C **Q.14** B

Multiple Correct Choice Type

Q.15 C, D

Assertion Reasoning Type

Q.16 D **Q.17** A **Q.18** B **Q.19** A **Q.20** D **Q.21** A

Q.22 B

Comprehension Type

Q.23 B **Q.24** B **Q.25** D

Previous Years' Questions

Q.1 (a) $\theta = \tan^{-1}\frac{MV}{mv}$, $P = \sqrt{m^2v^2 + M^2V^2}$ (b) $\frac{\Delta K}{K_i} = \frac{Mm(v^2+V^2)}{(M+m)(mv^2+MV^2)}$

Q.2 25%

Q.3 9 cm

Q.4 Opposite to velocity of B

Q.5 (a) $v_0/3$ (b) $\frac{2mv_0^2}{3x_0^2}$

Q.6 4

Q.7 (a) a straight line (b) $\frac{x^2}{\left(\frac{L}{2}-r\right)^2} + \frac{y^2}{r^2} = 1$

Q.8 (a) $\overline{F} = \frac{2mv}{\Delta t}\hat{i} - \frac{2mv}{\sqrt{3}\Delta t}\hat{k}$, $\overline{N} = \left(\frac{2mv}{\sqrt{3}\Delta t} + mg\right)\hat{k}$ (b) $\Rightarrow |\overline{\tau}_N| = \frac{4\,mvh}{\sqrt{3}\Delta t}$

Q.9 4 ms^{-1}

Q.10 A

Q.11 5

Solutions

JEE Main/Boards

Exercise 1

Sol 1: Isolated system, so external force = 0

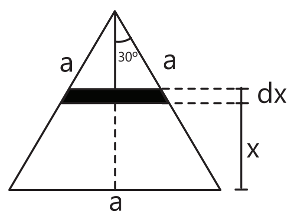
F_{ext} = 0, therefore acceleration of centre of mass = 0.

So the centre of mass moves with a constant velocity along a straight line path (I[st] law of motion).

Sol 2: (i) Lamina: mass per unit area

$$= \frac{M}{\frac{\sqrt{3}a^2}{4}} = \rho \text{ (say)}$$

then $X_{COM} = \int_0^{\frac{\sqrt{3}}{2}a} \frac{(\rho.dA).x}{M}$



where dA = area of the strip of thickness dx (shaded reg.)

$$= 2\left(\frac{\sqrt{3}}{2}a - x\right). \tan 30^\circ \, dx$$

$$dA = 2\left(\frac{a}{2} - \frac{x}{\sqrt{3}}\right). dx$$

$$\text{so, } X_{COM} = \int_0^{\frac{\sqrt{3}a}{2}} \frac{\rho \times 2 \times \left(\frac{a}{2} - \frac{x}{\sqrt{3}}\right).x.dx}{M}$$

$$= \frac{\rho}{M} \times 2 \times \int_0^{\frac{\sqrt{3}a}{2}} \left(\frac{ax}{2} - \frac{x^2}{\sqrt{3}}\right) dx$$

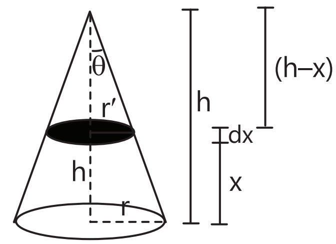
$$= \frac{4}{\sqrt{3}a^2} \times 2 \times \left[\frac{ax^2}{4} - \frac{x^2}{\sqrt{3}}\right]_0^{\frac{\sqrt{3}a}{2}}$$

$$= \frac{4}{\sqrt{3}a^2} \times 2 \times \left[\frac{a}{4} \times \frac{3}{4}a^2 - \frac{1}{3\sqrt{3}} \times \frac{3\sqrt{3}a^3}{8}\right]$$

$$= \frac{4}{\sqrt{3}a^2} \times 2 \times \left[\frac{3a^3}{16} - \frac{a^3}{8}\right] = \frac{4}{\sqrt{3}a^2} \times \frac{3a^3}{16}$$

$$= \frac{a}{2\sqrt{3}} \text{ (from bottom)}$$

(ii)



$$\tan \theta = \frac{r}{h}$$

$$\text{Let } \rho = \frac{3M}{\pi r^2 h} = \frac{\text{mass}}{\text{volume}}$$

then dm = ρdV, where dV = volume of shaded region

dV = π. r². dx

$$= \pi \times [(h - x). \tan\theta]^2 \, dx = \frac{\pi.(h-x)^2.r^2}{h^2}. dx$$

$$\text{so, } x_{COM} = \frac{\int_0^h x\,dm}{M} = \int_0^h \frac{x.\rho.dV}{M} = \rho.\int_0^h \frac{x.dV}{M}$$

$$= \frac{3M}{\pi r^2 h.M}\int_0^h x.dV = \frac{3}{\pi r^2 h}\int_0^h \frac{\pi.x.(h-x)^2.r^2}{h^2}dx$$

$$= \frac{3}{h^3}\int_0^h x.(h-x)^2 dx = \frac{3}{h^3}.\frac{h^4}{12}; \quad x_{COM} = \frac{h}{4}$$

Sol 3: Centre of gravity is the point at which all the force of gravity is assumed to be applied i.e., there is a force of gravity on each point of the body and hence the complications are reduced by finding a point where all the force is assumed to be applied, this point is centre of gravity.

Sol 4: Now we have

$$M_{tot}\vec{a}_{COM} = m_1\vec{a}_1 + m_2\vec{a}_2$$

where m = mass

a = acceleration

$$\text{so } M_{Tot.}\frac{\partial \vec{V}_{COM}}{dt} = m_1\frac{\partial \vec{V}_1}{dt} + m_2\frac{\partial \vec{V}_2}{dt}$$

x dt, on integrating w. r. t dt, we get

$$M_{Tot.}\vec{V}_{COM} = m_1\vec{V}_1 + m_2\vec{V}_2$$

$$\Rightarrow M_{Tot.}\frac{\partial \vec{x}_{COM}}{dt} = m_1\frac{\partial \vec{x}_1}{dt} + m_2\frac{\partial \vec{x}_2}{dt}$$

$\Rightarrow$ On multiplying by dt, and integrating

$$M_{Tot}\vec{x}_{COM} = m_1\vec{x}_1 + m_2\vec{x}_2$$

$$\text{So } \vec{x}_{COM} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{M_{tot.}}$$

Sol 5: $M_{Tot.}\cdot\vec{a}_{COM} = m_1\vec{a}_1 + m_2\vec{a}_2 + \dots m_n\vec{a}_n$

(Now, just like above question, question-4, we can find that

$$\vec{x}_{COM} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2 + \dots m_n\vec{x}_n}{M_{tot.}}$$

Sol 6: We have

$$M_{tot.}\cdot\vec{a}_{COM} = m_1\vec{a}_1 + m_2\vec{a}_2 + \dots m_n\vec{a}_n$$

Now, $\vec{a}_{COM} = 0$, so we have

$$0 = m_1\frac{d\vec{V}_1}{dt} + m_2\frac{d\vec{V}_2}{dt} + \dots m_n\frac{d\vec{V}_n}{dn}$$

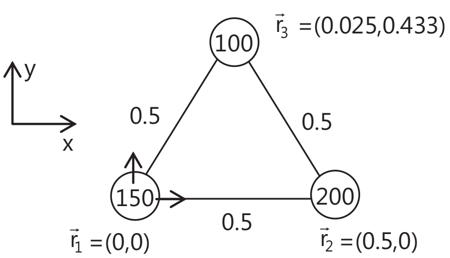
x dt, and integrating, we get

$$C = m_1\vec{V}_1 + m_2\vec{V}_2 + \dots m_n\vec{V}_n = M_{Tot.}V_{COM}$$

$$\text{So } V_{COM} = \frac{C}{M_{Tot.}} = \text{constant}$$

Hence proved.

Sol 7:



$$\text{So } \vec{x}_{COM} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2 + m_3\vec{x}_3}{M_{Tot.}}$$

$$= \frac{150\times(10) + 200\times(0.5) + 100\times(0.25)}{450}$$

$$= \frac{25+180}{450} = \frac{125}{450}$$

$$\vec{x}_{COM} = 0.277\,\hat{i}$$

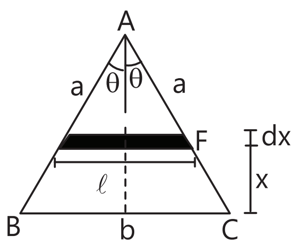
$$\vec{y}_{COM} = \frac{m_1\vec{y}_1 + m_2\vec{y}_2 + m_3\vec{y}_3}{M_{Tot.}}$$

$$= \frac{150\times 0 + 200\times(0) + 100\times 0.433}{450} = \frac{43.3}{450}$$

$$\vec{y}_{COM} = 0.096\,\hat{j}$$

$$\text{So } \vec{r}_{COM} = (0.277\,\hat{i} + 0.096\,\hat{j})$$

Sol 8:



$$\sin\theta = \frac{b}{2a},\ \cos\theta = \frac{\sqrt{4a^2-b^2}}{2a},$$

$$\tan\theta = \frac{b}{\sqrt{4a^2-b}}$$

$$\text{Now, } \rho = \frac{M}{\frac{1}{2}a^2\sin 2\theta} = \frac{M}{a^2\sin\theta\cos\theta}$$

$$\ell = \overbrace{2\underbrace{(a\cos\theta - x)}_{\substack{AE\\(AD-DE)}}.\tan\theta}^{EF}$$

So, $dA = \ell.\,dx$

So, $dm = \rho.\,dA = \rho.\,\ell.\,dx$

$$\text{So } \vec{x}_{COM} = \frac{\int_0^{a\cos\theta} x.dx}{M}$$

$$= \frac{\rho}{M}.\int_0^{a\cos\theta} x\times 2x(a\cos\theta - x).\tan\theta.dx$$

$$= \frac{1\times 2}{a^2\sin\theta.\cos\theta}.\int_0^{a\cos\theta} x(a\sin\theta - x\tan\theta)dx$$

$$= \frac{2}{a^2\sin\theta.\cos\theta}.\left[\frac{x^2}{2}a\sin\theta - \frac{x^3\tan\theta}{3}\right]_0^{a\cos\theta}$$

$$= \frac{2}{a^2\sin\theta.\cos\theta}.\left[\frac{a^3\sin\theta.\cos 2\theta}{2} - \frac{a^3\sin\theta.\cos^2\theta}{3}\right]$$

$$= \frac{a^3 \sin\theta.\cos^2\theta}{3a^2 \sin\theta.\cos\theta} = \frac{a\cos\theta}{3}$$

$$= \frac{\sqrt{4a^2 - b^2}}{6}$$

Sol 9: $\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M_{Tot.}}$

$$= \frac{0.5(-1,2) + 1(3,4)}{0.5 + 1}$$

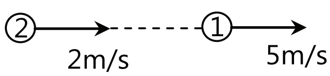
$$= \frac{(-0.5,\ 1) + (3,\ 4)}{1.5} = \frac{(2.5,\ 5)}{1.5}$$

$$\vec{r}_{COM} = \left(\frac{5}{3}, \frac{10}{3}\right)$$

Sol 10: Same as question (9)

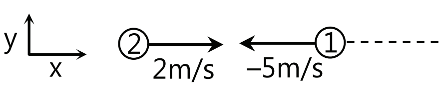
$$\vec{x}_{COM} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2 + m_3\vec{x}_3}{m_1 + m_2 + m_3} \text{ and}$$

$$\vec{y}_{COM} = \frac{m_1\vec{y}_1 + m_2\vec{y}_2 + m_3\vec{y}_3}{m_1 + m_2 + m_3}$$

Sol 11: (a) 

$$\vec{v}_{COM} = \frac{\vec{m}_1\vec{v}_1 + \vec{m}_2\vec{v}_2}{m_1 + m_2}$$

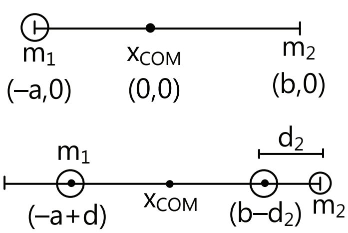
$$= \frac{2\times 2 + 5\times 1}{2 + 1} = \frac{9}{3} = 3 \text{ m/s}$$

(b) 

$$\vec{v}_{COM} = \frac{2\times 2 - 5\times 1}{2 + 1} = -1/3 \text{ m/s}$$

(–ve direction ⇒ velocity is negative direction)

Sol 12:



and $m_2b - m_1a = 0$

$\Rightarrow m_2b = m_1a$ ...(i)

Take origin at x_{COM} for simplicity.

Assuming the x_{COM} at origin, we have

$m_1(-a + d) + m_2(b - d_2) = 0$

$\Rightarrow m_2(b - d_2) = m_1(a - d)$

$\Rightarrow m_2b - m_2d_2 = m_1a - m_1d$...(2)

from (1), $m_2b = m_1a$, putting this in (2)

$m_1a - m_2d_2 = m_1a - m_1d$

$$\Rightarrow \boxed{\frac{m_1 d}{m_2} = d_2}$$

Sol 13: $\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$

$$\vec{x}_{COM} = \frac{2\times 1 + 3(-6)}{1 + 3} = \frac{-18 + 2}{4} = -1$$

$$\vec{y}_{COM} = \frac{5\times 1 + 4\times 3}{1 + 3} = \frac{12 + 5}{4} = \frac{17}{4}$$

$$\vec{z}_{COM} = \frac{13\times 1 + 3\times(-2)}{1 + 3} = \frac{13 - 6}{1 + 3} = \frac{7}{4}$$

So, $\vec{r}_{COM} = \left(-1, \frac{17}{4}, \frac{7}{4}\right)$

Sol 14: $x_{COM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4}$

$$= \frac{1\times 0 + 2\times a + 3\times a + 4\times 0}{1 + 2 + 3 + 4} = \frac{5a}{10} = \frac{a}{2}\hat{i}$$

$$y_{COM} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4}{m_1 + m_2 + m_2 + m_4}$$

$$= \frac{1\times 0 + 2\times 0 + 3\times b + 4\times b}{10} = \frac{7b}{10}\hat{j}$$

So $\vec{r}_{COM} = \frac{a}{2}\hat{i} + \frac{7b}{10}\hat{j}$

Sol 15: Divide the lamina in 3 equal parts with centre of masses as C_1, C_2, C_3 respectively.

So, the centre of mass of the whole plate can be found using the centre of mass of these three plates.

Now, from symmetry, we can say that the centre of mass of square plate lies at its centre.

So, $\vec{r}_{C_1} = (0.5, 0.5)$

$\vec{r}_{C_2} = (1.5, 0.5)$ and $\vec{r}_{C_3} = (0.5, 1.5)$

mass of each plate = 1 kg

So $x_{COM} = \dfrac{0.5\times1+1.5\times1+0.5\times1}{(1+1+1)} = \dfrac{2.5}{3}$

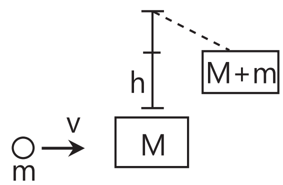
$y_{COM} = \dfrac{0.5\times1+0.5\times1+1.5\times1}{3} = \dfrac{2.5}{3}$

So $\vec{r}_{CM} = \left(\dfrac{2.5}{3}, \dfrac{2.5}{3}\right) = \left(\dfrac{5}{6}, \dfrac{5}{6}\right)$

Exercise 2

Single Correct Choice Type

Sol 1: (B)



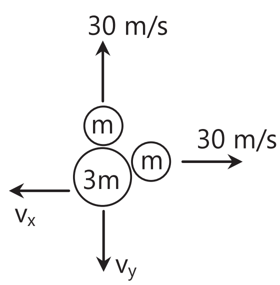
velocity of system after collision = $\sqrt{2gh}$

so using momentum constant

$(M + m)\sqrt{2gh} = mv$

$\Rightarrow v = \dfrac{(M+m)}{m}\sqrt{2gh}$

Sol 2: (B) m_1 = 200g, m_2 = 200 g, m_3 = 600 g



from momentum conservation

$x \Rightarrow 30 \times m = 3m \times v_x = 10$ m/s

$y \Rightarrow 30 \times m = 3m \times v_y = 10$ m/s

$\Rightarrow v = \sqrt{10^2+10^2} = 10\sqrt{2}$ m/s

Sol 3: (C) Momentum = mv (mass = constant)

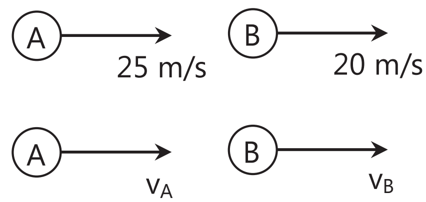
so new $v_n = \dfrac{3v}{2}$

New, K. E. $= \dfrac{1}{2}\times m\times v_n^2 = \dfrac{1}{2}\times mv^2\times\left(\dfrac{9}{4}\right)$

So increase $= \dfrac{9}{4}\times\left(\dfrac{1}{2}mv^2\right) - \dfrac{1}{2}mv^2$

$= \dfrac{5}{4}\times\left(\dfrac{1}{2}mv^2\right) = \dfrac{5}{4}$ of initial K. E.

Sol 4: (C)



$e = \dfrac{25-20}{v_B - v_A} = \dfrac{5}{v_B - v_A} = 1$

$\Rightarrow v_B - v_A = 5$

Momentum conservation $\Rightarrow$

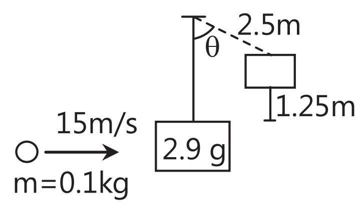
$m.(25) + m.(20) = m.(v_A) + m(v_B)$

$\Rightarrow v_A + v_B = 45$

$\Rightarrow v_B = 25$ m/s

$v_A = 20$ m/s

Sol 5: (C)



Using momentum conservation

$1.50 \times 0.1 = (2.9 + 0.1)\,v$

$\Rightarrow 15 = 3 \times v \quad \Rightarrow v = 5$ m/s

so $\dfrac{1}{2}mv^2 = mgh \qquad \Rightarrow h = \dfrac{v^2}{2g}$

$\Rightarrow h = \dfrac{25}{2\times10} = \dfrac{5}{4} = 1.25$ m

So $L.(1-\cos\theta) = 1.25$ m

$\Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = 60^\circ$

Sol 6: (A) The center of mass will be on x-axis, so y_{COM} = 0.

$\Rightarrow 15 \times \dfrac{m}{4} + y \times \dfrac{3m}{4} = 0$

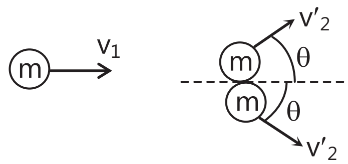
$\Rightarrow y = -5$ cm

Sol 7: (D) Elastic collision $\Rightarrow$ Energy is conserved.

$\frac{1}{2} \times mv_1^2 = \frac{1}{2} \times m \times v_2^2 + \frac{1}{2} m \times v_2^2$

$\Rightarrow v_1^2 = 2v_2^2$

$\Rightarrow v_1 = \sqrt{2}\, v'_2$



Using momentum conservation

$\Rightarrow 2mv \Rightarrow_2 \cos\theta = mv_1$

$\Rightarrow \cos\theta = \frac{v_1}{v'_2} \times \frac{1}{2}$

$\boxed{\cos\theta = \frac{1}{\sqrt{2}}}$ $\Rightarrow \theta = 45^o \Rightarrow 90^o$

Sol 8: (C) Momentum conservation

$\Rightarrow 2 \times 3 - 1 \times 4 = (2 + 1)\, v$

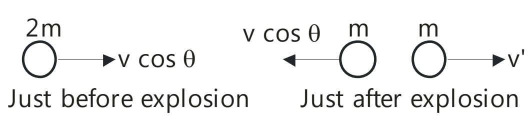
$\Rightarrow v = \frac{2}{3}$ m/s

Sol 9: (D) No external force $\Rightarrow a_{COM} = 0$

Previous Years' Questions

Sol 1: (D) Net force on centre of mass is zero. Therefore, centre of mass always remains at rest.

Sol 2: (A) Let v′ be the velcoity of second fragment. From conservation of linear momentum,



$2m(v\cos\theta) = mv' - m(v\cos\theta)$

$\therefore v' = 3v\cos\theta$

Sol 3: (C) $|(m_1\vec{v}_1 + m_2\vec{v}_2) - (m_1\vec{v}_1 + m_2\vec{v}_2)|$

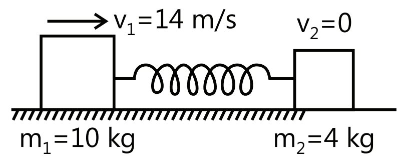
= |Change in momentum of the two particle|

= |External force on the system| × time interval

$= (m_1 + m_2)g\,(2t_0) = 2\,(m_1 + m_2)gt_0$

Sol 4: (C) $v_{CM} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$

$= \frac{10 \times 14 + 4 \times 0}{10 + 4} = \frac{140}{14} = 10$ m/s



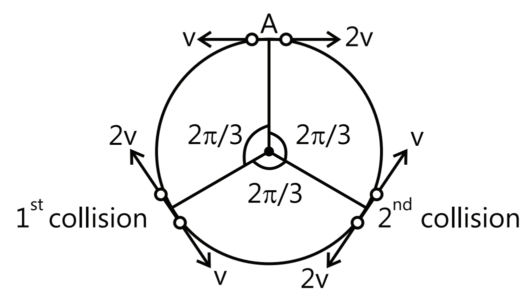
Sol 5: (A) y_{CM}

$= \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4 + m_5y_5}{m_1 + m_2 + m_3 + m_4 + m_5}$

$= \frac{(6m)(0) + (m)(a) + m(a) + m(0) + m(-a)}{6m + m + m + m + m}$

$= \frac{a}{10}$

Sol 6: (C) At first collision one particle having speed 2v will rotate 240° $\left(\text{or } \frac{4\pi}{3}\right)$ while other particle having speed v will rotate 120° $\left(\text{or } \frac{2\pi}{3}\right)$. At first collision they will exchange their velocities. Now as shown in figure, after two collisions they will again reach at point A.



Sol 7: (D) $R = u\sqrt{\frac{2h}{g}}$

$\Rightarrow 20 = v_1\sqrt{\frac{2 \times 5}{10}}$ and $100 = v_2\sqrt{\frac{2 \times 5}{10}}$

$\Rightarrow \quad v_1 = 20$ m/s, $v_2 = 100$ m/s

Applying momentum conservation just before and just after the collision.

(0.01)(v) = (0.2) (20) + (0.01)(100)

v = 500 m/s

Sol 8: (B) Between A and B, height fallen by block

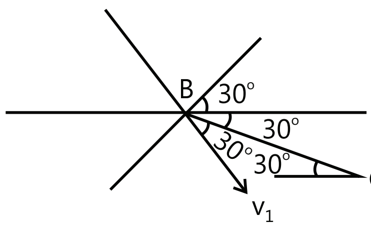
$$h_1 = \sqrt{3}\tan 60^\circ = 3\text{m}.$$

∴ Speed of block just before striking the second incline,

$$v_1 = \sqrt{2gh_1} = \sqrt{2\times 10\times 3} = \sqrt{60}\ \text{ms}^{-1}$$

In perfectly inelastic collision, component of v_1 perpendicular to BC will become zero, while component of v_1 parallel to BC will remain unchanged.

∴ Speed of block B immediately after it strikes the second inline is,



v_2 = component of v_1 along BC

$$= v_1\cos 30^\circ = (\sqrt{60})\left(\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{45}\ \text{ms}^{-1}$$

Sol 9: (B) Height fallen by the block from B to C

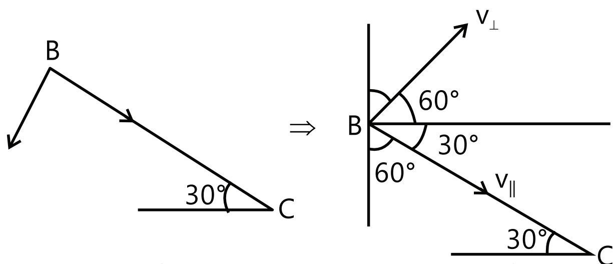
$$h_2 = 3\sqrt{3}\ \tan 30^\circ = 3\text{ m}$$

Let v_3 be the speed of block, at point C, just before it leaves the second incline, then:

$$v_3 = \sqrt{v_2^2 + 2gh_2}$$

$$= \sqrt{45+2\times 10\times 3} = \sqrt{105}\ \text{ms}^{-1}$$

Sol 10: (C) In elastic collision, component of v_1 parallel to BC will remain unchanged, while component perpendicular to BC will remain unchanged in magnitude but its direction will be reversed.



Just before   Just after

$$v_\parallel = v_1\cos 30^\circ = (\sqrt{60})\left(\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{45}\ \text{ms}^{-1}$$

$$v_\perp = v_1\sin 30^\circ = (\sqrt{60})\left(\frac{1}{2}\right)$$

$$= \sqrt{15}\ \text{ms}^{-1}$$

Now vertical component of velocity of block

$$v = v_\perp\cos 30^\circ - v_\parallel\cos 60^\circ$$

$$= (\sqrt{15})\left(\frac{\sqrt{3}}{2}\right) - (\sqrt{45})\left(\frac{1}{2}\right) = 0$$

Sol 11: (C) Loss of energy is maximum when collision is inelastic as in an inelastic collision there will be maximum deformation.

KE in COM frame is $\frac{1}{2}\left(\frac{Mm}{M+n}\right)V_{rel}^2$

$$KE_i = \frac{1}{2}\left(\frac{Mm}{M+m}\right)V^2 \quad KE_f = 0\ (\because V_{rel} = 0)$$

Hence loss in energy is $\frac{1}{2}\left(\frac{Mm}{M+m}\right)V^2$

$$\Rightarrow f = \frac{M}{M+m}$$

Sol 12: (A) $z_0 = h - \frac{h}{4} = \frac{3h}{4}$

Sol 13: (B) $E_{initial} = \frac{1}{2}m(2v)^2 + \frac{1}{2}2m(v)^2 = 3mv^2$

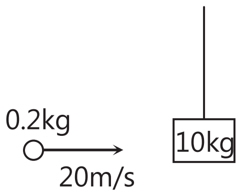
$$E_{final} = \frac{1}{2}3m\left(\frac{4}{9}v^2 + \frac{4}{9}v^2\right) = \frac{4}{3}mv^2$$

$$\therefore \text{ Fractional loss } = \frac{3-\frac{4}{3}}{3} = \frac{5}{9} = 56\%$$

JEE Advanced/Boards

Exercise 1

Sol 1: (a)



From the conservation of momentum

we have,

$m_1v_1 + m_2v_2 = m_{Tot} \times v$

$0.2 \times 20 + 10 \times 0 = (10 + 0.2) \times v$

$\Rightarrow 4 = 10.2 \times v$

$v = \frac{4}{10.2}$ m/s = 0. 392 m/s

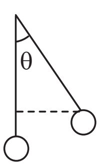
(b) From conservation of energy (Force by string is perpendicular to displacement, hence no work done by string)

$\frac{1}{2} \times m \times v^2 = mgh_{vert.}$

$\Rightarrow h = \frac{v^2}{2g} = \frac{(0.392)^2}{2 \times 9.81} = 0.0078$ m

= 7. 8 mm $\approx$ 0. 008 m

In horizontal direction:



$R(1 - \cos\theta) = 0.008$ m

$\Rightarrow \cos\theta = \frac{1 - 0.08}{R} = 1 - 0.0027 = 0.9973$

So $R \sin\theta = 3 \times \sqrt{(1 - \cos^2\theta)}$

$= 3 \times \sqrt{1 - (0.9937)^2} = 0.220$ m

So total displacement = 0. 220 m

(c) Initial energy:

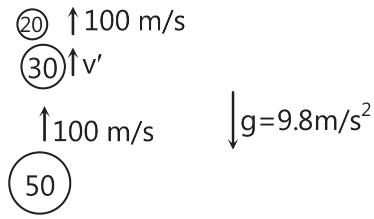
$\frac{1}{2} \times m_B \times v_B^2 = \frac{1}{2} \times (0.2) \times (20)^2 = 40$ J

Final energy:

$\frac{1}{2} \times m_{Tot} \times v_{Tot}^2 = \frac{1}{2} \times 10.2 \times (0.392)^2 = 0.8$ J

So energy lost = 40 J – 0.8 J = 39.2 J

Sol 2:



(a) Now, from equation of motion

$v = u + at = 100 - 9.8 \times (5) = 51$ m/s

so v = 51 m/s

Now at this velocity, particle exploded, Δt is very small and hence momentum can be conserved.

So $50 \times 51 = 20 \times 150 + 30 \times v'$

$\Rightarrow 2550 = 3000 + 30 \times v'$

$\Rightarrow \boxed{v' = \frac{-450}{30} = -15\,m/s}$

(b) When no explosion:

$v = u + at$

$\Rightarrow v = 100 - 9.8 \times 8 = 21.6$ m/s

so momentum = m × v

= 50 × 21. 6= 1080 kg. m/s

When explosion:

For 20 kg v = u + at

= 150 – 9. 8 × (s) = 120.6 m/s

For 30 kg v = u + at

= – 15 – 9. 8 × 3 = –44.4 m/s

So total momentum

$= 20 \times (120.6) - (44.4) \times (30)$

= 2412 – 1332 = 1080 kg m/s

Sol 3: (a) The particle are meet at the mid-point of the trajectory (i. e. vertical velocity = 0)

u = 49 m/s

So $t = \frac{u\sin\theta}{g}$

Thus, v_x = horizontal velocity = $u\cos\theta$

Now to retrace the path velocity of P must be ucosθ in the (–)ve direction, so now using momentum balance

$m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2$

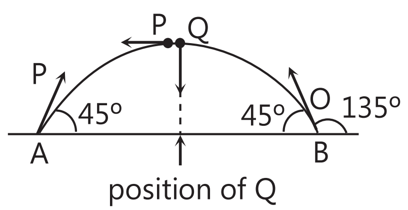
$\Rightarrow 20\times(u\cos\theta) + 40\times(-u\cos\theta)$

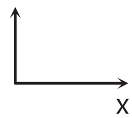
$= 20\times(u\cos\theta) + 40\times v'_2$

$\Rightarrow 40\times v'_2 = 0 \Rightarrow v'_2 = 0$

so the horizontal velocity of particle Q after collision would be 0.

so position of Q would be just below the point of collision





So position of Q = $\dfrac{u\cos45^\circ \times u\sin45^\circ}{g}$

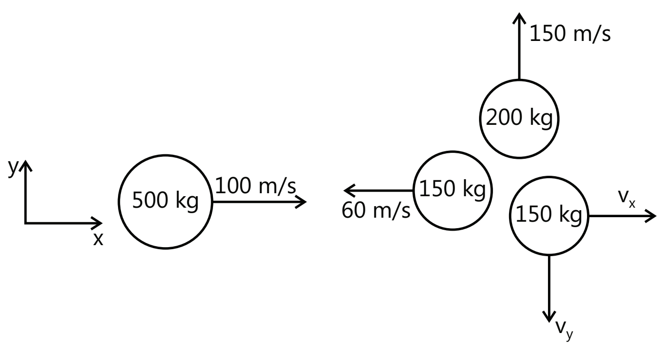
$= \dfrac{u^2\sin90^\circ}{2g} = \dfrac{u^2}{2g} = \dfrac{(49)^2\times10}{2\times9.8} = 122.5$ m

From position A in the (+)ve x-direction

(b) Time take would be same as the vertical component has not changed, so

$t = \dfrac{u\sin\theta}{g} = \dfrac{49\times\sin45^\circ}{9.8} = 3.54$ sec

Sol 4:



Now, since there is no external force

Using momentum conservation in x-direction

$500\times100 = 150\times v_x + 150\times(-60)$

$\Rightarrow 150\times v_x = 5\times10^4 + 9000$

$150\times v_x = 59\times10^3$

$\Rightarrow v_x = 393.33$ m/s

Similarly in y-direction

$0 = 150\times200 + 150\times(-v_y)$

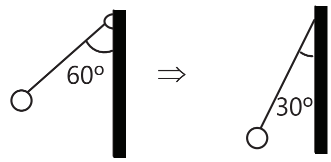
$\Rightarrow v_y = 200$ m/s

So $v_{III} = 393.33\,\hat{i} - 200\,\hat{j}$

$\| v_{III} \| = 441.26$ m/s

and $\theta = \tan^{-1}\dfrac{-200}{393.33} = -27^\circ$

Sol 5:



Velocity of ball just before impact

$= \sqrt{2gL(1-\cos\theta)} = \sqrt{2\times9.8\times1/2}$

$= \sqrt{9.8} = 3.13$ m/s

v of ball after impact $\Rightarrow mg(\Delta h) = \dfrac{1}{2}mv^2$

(energy conservation)

$\Rightarrow \sqrt{2gL.(1-\cos30^\circ)} = v_f$

$\Rightarrow \sqrt{2\times9.8\times\left(1-\dfrac{\sqrt{3}}{2}\right)} = v_f$

$\Rightarrow v_f = \sqrt{2.626}$

$\Rightarrow v_f = 1.62$ m/s

so coefficient of rest. $= \dfrac{1.62}{3.13} = 0.517$

Loss of kinetic energy = heat produced

$= \dfrac{1}{2}\times m\times(v_i^2 - v_f^2)$

$= \dfrac{1}{2}\times(0.01)\times[(3.13)^2 - (1.62)^2]$

$= \dfrac{0.0717}{2} = 0.036$ J

= 0.0085 cal.

Sol 6: No friction ⇒ no torque, so its pure translational motion

Now, conservation of energy

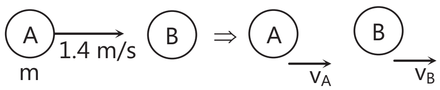
$\Rightarrow \dfrac{1}{2}\times mv^2 = mgh$

$\Rightarrow v = \sqrt{2gh}$

$= \sqrt{2 \times 9.8 \times 0.1}$

v = 1. 4 m/s

Now, as collision is elastic



so $v_B - v_A$ = 1. 4 m/s ...(i)

and conservation of momentum gives:

$m_A v_A + m_B v_B = m_A \times 1.4$ m/s

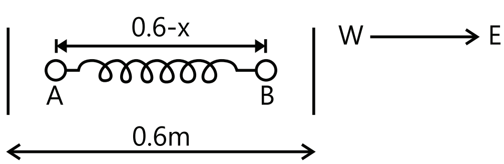
$\Rightarrow v_A + v_B = 1.4$ m/s ...(ii)

on solving (i) and (ii)

v_B = 1. 4 m/s

v_A = 0 m/s

Sol 7:



(a) Initial acceleration of $B = 0.5\ m/s^2$ W

So, kx = (0.5) (0.25)

As the magnitude of force would be the same for A,

Initial acceleration of A = $\frac{kx}{0.1} = \frac{(0.5)(0.25)}{0.1}$

$= 1.25\ m/s^2$ E

$= 1.25\ cm/s^2$ E

(b) $x = \frac{(0.5)(0.25)}{k} = \frac{(0.5)(0.25)}{2}$

= 0.0625 m

= 6.25 cm

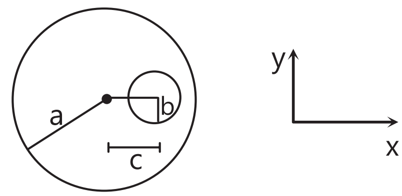
(c) Max distance would be when spring is fully elongated. And, symmetry of conservation of energy implies that expansion would be equal to companion.

So, Maximum distance between

A and B = 60 cm + 6.25 cm

= 66.25 cm

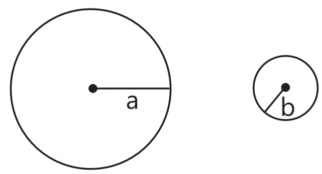
Sol 8:



Now $\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$

Take the disks as these two bodies and treat m_2 as negative

Given body =

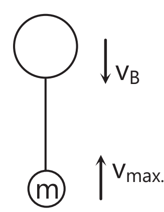


So $x_{COM} = \frac{\rho.\pi a^2.(0) - \rho.\pi b^2 \times c}{\rho.(\pi a^2 - \pi b^2)}$

where $\rho = \frac{m}{area}$

$x_{COM} = \frac{-b^2 c}{(a^2 - b^2)}$

Sol 9: (a) No external force, hence v_{COM} = 0



thus,

$m.\ v_{max.} - M.\ v_B = 0$

$\Rightarrow mv_{max.} = M.\ v_B$ ...(i)

and $v_{rel.} = v_{max.} + v_B$ ...(ii)

$v_{rel.} = \left(\frac{M}{m} + 1\right) v_B$

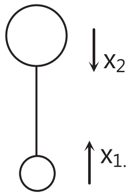
$\Rightarrow v_B = \frac{v_{rel}.m}{(m+M)}$

$\Rightarrow v_B = \frac{m.v_{rel}}{(m+M)}$ in (–)ve direction

(b) $D_{rel.}$ = L

Let x_{COM} = 0

then $m.\ x_1 - M.\ x_2 = 0$



$x_1 \neq x_2$ = distance covered by man and balloon in ground frame, so

$x_1 + x_2 = L$

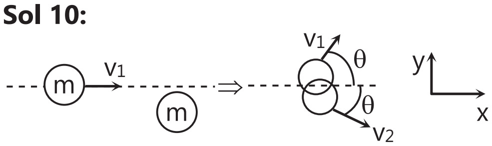
$\Rightarrow \left(\frac{M}{m}+1\right)x_2 = L$

$\Rightarrow x_2 = \left(\frac{mL}{m+M}\right)$

in the downward direction

(c) No external force: initial velocity = final velocity = 0

Sol 10:



No generation of momentum would be there in the y-direction

Energy conservation ⇒

$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$

$\Rightarrow v^2 = v_1^2 + v_2^2$ ...(i)

Also, $mv_1\sin\theta = mv_2\sin\theta$

$\Rightarrow v_1 = v_2$ ... (ii)

Now, using (i) and (ii), we get

$v^2 = 2v_1^2$

$\Rightarrow v = \sqrt{2}\,v_1$

and using momentum conservation in x-direction:

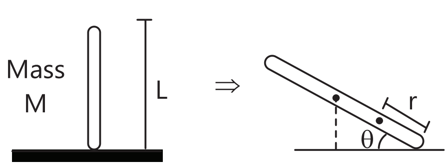
$mv = 2mv_1\cos\theta$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = 45^\circ$ so angle between them = 90°

Hence proved.

Sol 11:



(a) Now as there is no horizontal force on rod, there would be no displacement of COM of the rod. Thus, the path followed will be a straight line.

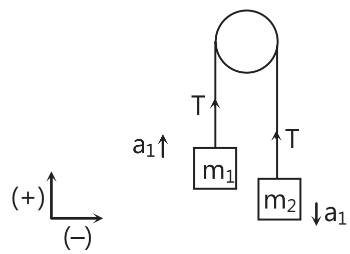
(b) x comp. of r = $\frac{L}{2}\cos\theta - r\cos\theta$

$x = \left(\frac{L}{2}-r\right)\cos\theta$

and y comp. of r = $r\sin\theta = y$

$\Rightarrow \frac{x^2}{(L/2-r)^2} + \frac{y^2}{r^2} = 1$

Sol 12:



So, $T - m_1g = m_1a_1$

$m_2g - T = m_2a_1$

$\Rightarrow (m_2 - m_1)g = (m_1 + m_2)a$

$\Rightarrow a = \frac{(m_2-m_1)g}{m_1+m_2}$

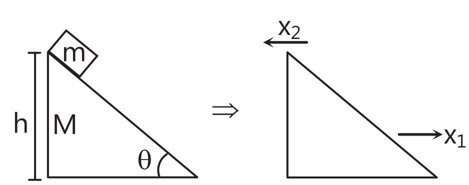
Now $a_{COM} = \frac{m_1a_1+m_2a_2}{m_1+m_2}$

$$= \frac{m_2\cdot\left[\frac{-(m_2-m_1)g}{(m_1+m_2)}\right]+m_1\cdot\left[\frac{m_2-m_1}{m_1+m_2}\right]g}{m_1+m_2}$$

$$= \frac{-m_2^2g+m_1m_2g+m_1.m_2g-m_1^2g}{(m_1+m_2)^2}$$

$$= \frac{-(m_1-m_2)^2}{(m_1+m_2)^2}g \Rightarrow \text{in the (–)ve direction.}$$

Sol 13: There is no external force in horizontal direction, so x_{COM} is same after this relative horizontal distance = h cotθ



Let x_1 and x_2 be the distance travelled by the block and prism, respectively in ground frame.

so, $x_1 + x_2 = h\cos\theta$ ...(i)

and $m_1x_1 - Mx_2 = 0$

[taking x_{COM} as origin]

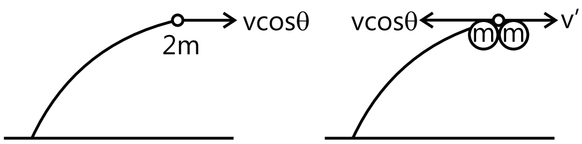
$\Rightarrow m_1x_1 = Mx_2$

$\Rightarrow x_1 = \frac{M}{m}x_2$, putting this in (i)

$\left(\frac{M}{m}+1\right)x_2 = h\cot\theta$

$\Rightarrow \quad x_2 = \frac{m.h\cot\theta}{(m+M)}$

Sol 14:



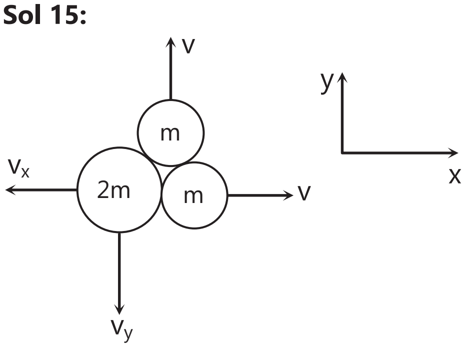
If the particle retraces its path, its velocity must be same as before i.e. vcosθ in the opposite direction (independent of mass), so using the momentum conservation:

$2m.(v\cos\theta) = m.(-v\cos\theta) + m.v'$

$\Rightarrow mv' = 3mv\cos\theta$

$\Rightarrow \quad v' = 3v\cos\theta$

Sol 15:



Momentum conservation in x-direction

$m.v + 2m.(-v_x) = 0$

$\Rightarrow v_x = \frac{v}{2}$

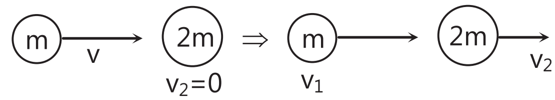
Similarly, $v_y = \frac{v}{2}$

so total energy

$= \frac{1}{2}\times m\times v^2 + \frac{1}{2}\times m\times v^2 + \frac{1}{2}\times(2m)(v_x^2 + v_y^2)$

$= mv^2 + m.\left[\frac{v^2}{4}+\frac{v^2}{4}\right] = \frac{3mv^2}{2}$

Sol 16:



Elastic collision $\Rightarrow \frac{v_2 - v_1}{v - 0} = 1$

$\Rightarrow v_2 - v_1 = v$ ...(i)

and using momentum conservation

$mv = mv_1 + 2mv_2$

$\Rightarrow mv = m(v_2 - v_1) + 2mv_2$

$\Rightarrow 2mv = 3mv_2$

$\Rightarrow v_2 = \frac{2v}{3}$ and thus $v_1 = \frac{-v}{3}$

so kinetic energy before collision:

$= \frac{1}{2}mv^2$

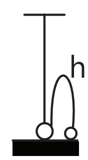
kinetic energy after collision

$= \frac{1}{2}\times m\times\frac{v^2}{9} = \frac{mv^2}{18}$

loss $= \frac{1}{2}mv^2 - \frac{mv^2}{18} = \frac{8mv^2}{18} = \frac{8}{9}\times\frac{1}{2}mv^2$

$= \frac{8}{9}\times$(initial K. E.)

Sol 17:



v at first impact $= \sqrt{2gh}$

and time at first impact $= \sqrt{\frac{2h}{g}}$

(eq. of motion)

Now, u_n = velocity after nth impact

$= e^n.\sqrt{2hg}$

so total distance $= h + \sum_{n=1}^{\infty}\frac{u_n^2}{2g}\times 2$

$[\frac{u_n^2}{2g}\times 2 = \frac{u^2}{g} \Rightarrow$ distance between two impacts]

$= h + 2e^2.h + 2e^4.h$ .... .

$= h.[1 + 2e^2(1 + e^2 + e^4 .... .)]$

$= h.\left[1+\frac{2e^2}{1-e^2}\right] = \frac{h.[1+e^2]}{[1-e^2]}$

Similarly total time

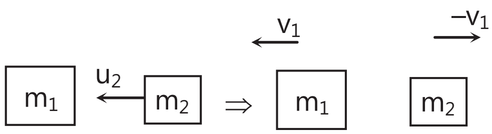
$= \sqrt{\frac{2h}{g}} + \sum_{n=1}^{\infty}\frac{2u_n}{g}\left[\frac{2\times u_n}{8} = \text{time between two impacts}\right]$

$= \sqrt{\frac{2h}{g}} + \sqrt{\frac{2h}{g}}.2e + \sqrt{\frac{2h}{g}}.2e^2 + \ldots$

$= \sqrt{\frac{2h}{g}}\,[1 + 2e(1 + e + e^2 \ldots .)]$

$= \sqrt{\frac{2h}{g}}\left[1+\frac{2e}{1-e}\right] = \sqrt{\frac{2h}{g}}.\left[\frac{1+e}{1-e}\right]$

Sol 18: If the block m_2 is moving with same velocity after wall collision ⇒ the velocity of block m_1 and m_2 has same magnitude.



Now $\frac{u_2}{v_1-(-v_1)} = 1$

$\Rightarrow u_2 = 2v_1$ .... (i)

and using momentum conservation

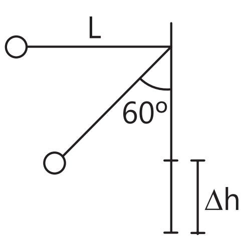
$u_2m_2 = m_1v_1 - m_2v_1$

$\Rightarrow 2m_2v_1 = m_1v_1 - m_2v_1 \Rightarrow 3m_2v_1 = m_1v_1$

$\Rightarrow 3m_2 = m_1$

$\Rightarrow \quad m_2 = 50$ kg

Sol 19:



For the amplitude to be less than 60º.

$\frac{1}{2}mv^2 < mg(\Delta h)$

$\Rightarrow v^2 < 2g\,(\Delta h)$

$v^2 < 2gL\,(1 - \cos 60º)$

$\Rightarrow v^2 < gL$

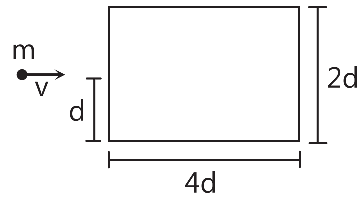
v after n collisions $= e^n.\sqrt{2gL}$

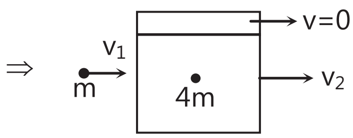
$\Rightarrow e^{2n}.\,(2gL) < gL$

$\Rightarrow e^{2n} < \frac{1}{2} \qquad \Rightarrow\left(\frac{4}{5}\right)^n < \frac{1}{2}$

⇒ n = 4 is the largest value satisfying.

Sol 20: (a)





Now, using momentum conservation $\Rightarrow 4mv_2 + mv_1 = mv$

$\Rightarrow 4v_2 + v_1 = v$ ... (i)

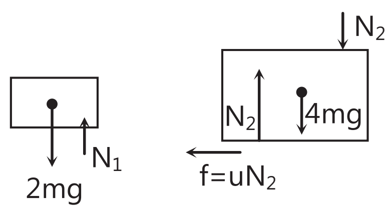
Elasticity $\frac{v_2 - v_1}{v} = 1$

$\Rightarrow v_2 - v_1 = v$ ...(ii)

(i) + (ii) $\Rightarrow 5v_2 = 2v \Rightarrow v_2 = \frac{2v}{5}$ and

$v_1 = \frac{-3v}{5}$

To topple, the distance by 4m block should be 2d. (No horizontal force on 2m block, and hence that block remains stationary)



$N_1 + 4mg = N_2,$

Now $N_1 = 2mg$

$\Rightarrow N_2 = 6$ mg

So f = 6μmg

thus acceleration (or deceleration)

$a = \frac{f}{4m} = \frac{6\mu mg}{4m} = \frac{3\mu g}{2}$

So now the velocity (or v_2) required

$\Rightarrow v^2 = u^2 + 2as$

$\Rightarrow 0 = v_2^{\,2} + 2\left(\frac{-3\mu g}{2}\right)\times(2d) \Rightarrow v_2 = \sqrt{6\mu g d}$

so, $v_2 = \frac{2v}{5}$ from above,

$\Rightarrow \quad v = \frac{5}{2}v_2 = \frac{5}{2}\sqrt{6\mu g d}$

(b) Now $v = 2v_0 = \frac{10}{2}\sqrt{6\mu g d}$

$v = 5\sqrt{6\mu g d}$ m/s

Now

so $v_2 - v_1 = v$ (elasticity)

and $m \times v = m \times v_1 + 4m \times v_2$

[Momentum conservation]

$\Rightarrow v = v_1 + 4v_2$ ...(ii)

$\Rightarrow 2v = 5v_2$

$\Rightarrow v_2 = \frac{2v}{5}$ and $v_1 = \frac{-3v}{5}$

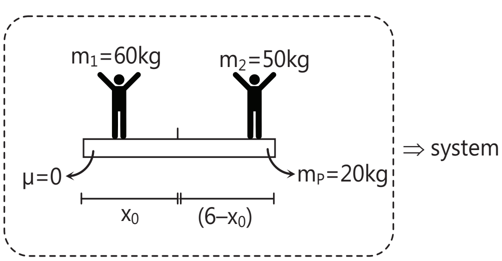
So $v_1 = -3\sqrt{6\mu g d}$

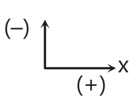
Now time $\Rightarrow \sqrt{\frac{2d}{g}}$

so distance$= v_1 \times t$

$= -3 \times \sqrt{6\mu g d} \times \sqrt{\frac{2d}{g}} = -6d.\ \sqrt{3\mu}$

Sol 21:





Now no external force thus x_{COM}=constant≈ 0.

$x_1, x_2, x_3 \Rightarrow$ displacement of man, woman and platform respectively w.r.t. ground frame then,

$60 \times x_1 + 50 \times x_2 + 20 \times x_3 = 0$

$\Rightarrow 6x_1 + 5x_2 + 2x_3 = 0$ ...(i)

and distance by man w.r.t. platform

$x_0 = x_1 - x_3$ ...(iii)

and displacement of woman w. r. t platform $= x_2 - x_3$

also $x_1 - x_3 - (x_2 - x_3) = 6$

$\Rightarrow x_1 - x_2 = 6$ ...(ii)

$x_2 = x_1 - 6$ and $x_1 = x_0 + x_3$

so thus putting these in (i)

$6.\ (x_0 + x_3) + 5(x_1 - 6) + 2x_3 = 0$

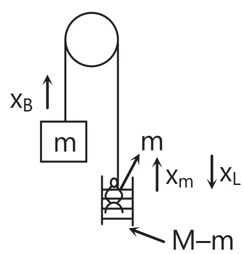
$6x_0 + 6x_3 + 5(x_0 + x_3) - 30 + 2x_3 = 0$

$11x_0 + 13x_3 - 30 = 0$

$\Rightarrow x_3 = \frac{30 - 11x_0}{13}$

so displacement $= \frac{30 - 11x_0}{13}$ in (+)ve direction.

Sol 22:



Now, we have

$x_B = x_L$(constraint)

Now $x_m - (-x_L) = \Delta\vec{r}$

Now displacement of centre of mass

$$\Rightarrow \Delta\vec{S} = \frac{M.x_B + m.x_m - (M-m)x_L}{M + M - m + m}$$

$$= \frac{M.x_B + m.x_m - (M-m)x_L}{2M}$$

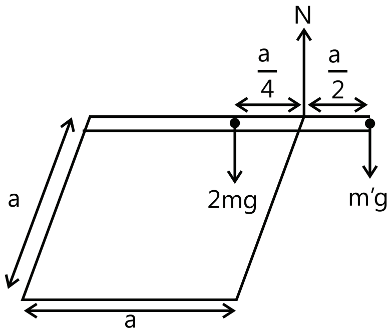
$$= \frac{M(x_B - x_L) + m(x_m + x_L)}{2M}$$

$[(x_B - x_L) = 0]$

$$= \frac{m.\Delta\vec{r}}{2M}$$

Sol 23: Let the total mass of both insects be 'm'

For the limiting case of toppling, normal reaction would pass through the corner.



$N = 2mg + m'g$

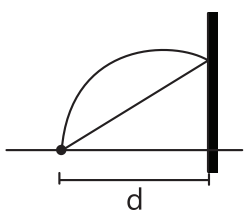
$$m'g\left(\frac{a}{2}\right) = 2mg\left(\frac{a}{4}\right)$$

$\Rightarrow m' = m$

As the mass of first insect is $\frac{m}{2}$, the second insect would also have the same mass.

Hence, mass of the other insect $= m' - m = \frac{m}{2}$

Sol 24:



Horizontal velocity: $2\sqrt{ag}\,.\cos\theta$

after collision: $\dfrac{2\sqrt{ag}\cos\theta}{(4\sin 2\theta - 1)}$

so time (total)

$$= \frac{d}{2\sqrt{ag}\cos\theta} + \frac{d(4\sin 2\theta - 1)}{2\sqrt{ag}\cos\theta} = \frac{d.(4\sin 2\theta)}{2\sqrt{ag}\cos\theta}$$

Now this time must be equal to time of flight

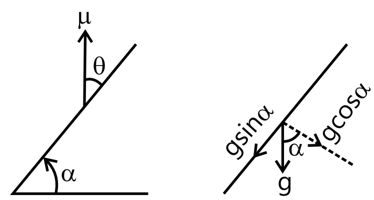
$$= \frac{2u\sin\theta}{g} \Rightarrow \frac{2.2\sqrt{ag}}{g} = \frac{d(4\sin 2\theta)}{2\sqrt{ag}\cos\theta}$$

$$\Rightarrow \frac{a.g.\sin 2\theta}{g} = d.\sin 2\theta \quad \Rightarrow d = a$$

also, $e < 1 \Rightarrow \dfrac{1}{4\sin 2\theta - 1} < 1$

$\Rightarrow \sin 2\theta > \dfrac{1}{2} \Rightarrow 2\theta > 30^\circ \Rightarrow \theta > 15^\circ$

Sol 25:



Time for first collision, $T_1 = \dfrac{2\mu\sin\theta}{g\cos\alpha}$

Time for second collision, $T_2 = e\left(\dfrac{2\mu\sin\theta}{g\cos\alpha}\right)$

As velocity becomes perpendicular to the surface, its horizontal component (along the surface) must go to 'O'.

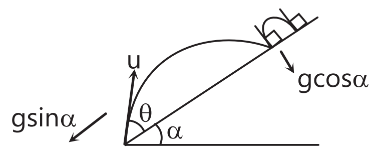
$$O = \mu\cos\theta - g\sin\alpha(T_1 + T_2)$$

$$\Rightarrow \mu\cos\theta = g\sin\alpha\left(\frac{2\mu\sin\theta}{g\cos\alpha}\right)(1+e)$$

$$\Rightarrow 1 + e = \frac{\cot\theta\cot\alpha}{2}$$

$$\Rightarrow e = \frac{\cot\theta\cot\alpha}{2} - 1$$

Sol 26:



here time of flight $= \dfrac{2u\sin\theta}{g\cos\alpha}$

total time for second bounce

$$= \frac{2u\sin\theta}{g\cos\alpha} + \frac{2ue\sin\theta}{g\cos\alpha}$$

Also equation of motion along incline $= \dfrac{u\cos\theta}{g\sin\alpha}$

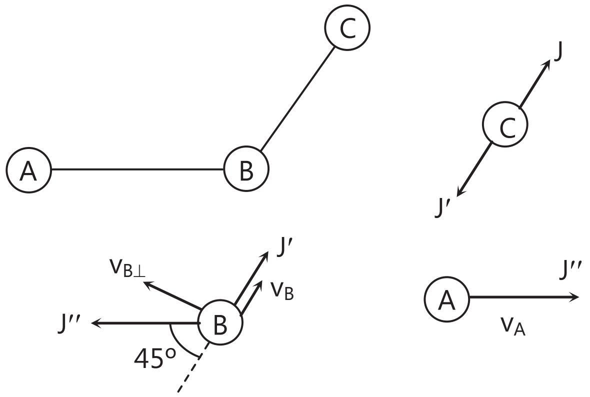
$$\text{so } \frac{u\cos\theta}{g\sin\alpha} = \frac{2u\sin\theta}{g\cos\alpha} + \frac{2ue\sin\theta}{g\cos\alpha}$$

$$\Rightarrow \frac{\cos\theta}{\sin\alpha} - \frac{2\sin\theta}{\cos\alpha} = \frac{2e\sin\theta}{\cos\alpha}$$

$$\Rightarrow \frac{(\cot\theta.\cot\alpha - 2)}{2} = e$$

$$\Rightarrow e = \frac{1}{2}\cot\alpha\,.\cot\theta - 1$$

Sol 27:



Now we have,

$J - J' = mv_c$ ... (i) (moment eq^n)

$v_c = v_{B||}$ ...(ii) (constraint eq^n)

$J' - J'' \cos45° = mv_{B||}$ ...(iii)

(impulse momentum eq^n)

$J'' \sin45° = mv_{B\perp}$ ...(iv)

(impulse momentum eq^n)

$J'' = mv_A$ ...(v)

(impulse momentum eq^n)

$(v_{B||} - v_{B\perp})\dfrac{1}{\sqrt{2}} = v_A$ ...(vi)

(constraint equation)

so we have 6 variables (J', J'', $v_{B||}$, $v_{B\perp}$, v_A and v_c) and six equations,

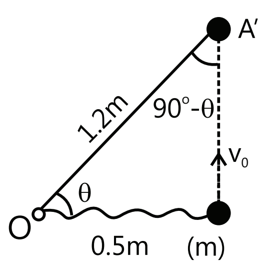
on solving, we get

$v_c = \dfrac{3J}{7m}$,

$v_B = \dfrac{\sqrt{10}}{7m}$,

$v_A = \dfrac{\sqrt{2}}{7m}$

Sol 28:



(a) The comp. ⊥ to the string will only be there and the momentum along the thread will be lost.

So $mv_0 \cos(90° - \theta) < 3\ N_s$

$\Rightarrow 2 \times v_0 \times \sin\theta < 3$

$\Rightarrow v_0 < \dfrac{3}{2\sin\theta}$

$\Rightarrow v_0 < \dfrac{3\times1.2}{2\times\sqrt{(1.2)^2-(0.5)^2}}$

$\Rightarrow v_0 < 1.\ 65$ m/s

(b) The energy remaining

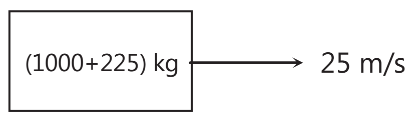
$= \dfrac{1}{2}\times m(v_0\cos\theta)^2 = \dfrac{1}{2}mv_0^2\cos^2\theta$

so loss $= \dfrac{1}{2}mv_0^2 - \dfrac{1}{2}mv_0^2\cos^2\theta$

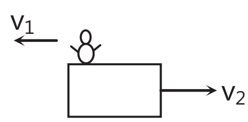
$= \dfrac{1}{2}mv_0^2(1-\cos^2\theta) = \dfrac{1}{2}mv_0^2.\ \sin^2\theta$

$= \dfrac{1}{2}\times2\times(1.\ 65)^2\times\dfrac{(1.2)^2-(0.5)^2}{(1.2)^2} = 2.\ 25$ Joules.

Sol 29:



(a) (i)



So $v_2 - v_1 = 5 \Rightarrow v_1 = v_2 - 5$

and

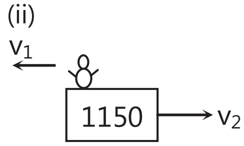
$(1225) \times 25 = 1150 \times v_2 + 75 \times v_1$

$1225\times25 = 1150\times v_2 + 75 \times (v_2 - 5)$

$1240 \times 25 = 1225\ v_2$

$\Rightarrow v_2 = \dfrac{1240\times25}{1225} = 25.\ 306$ m/s

(ii)



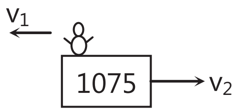
$1150 \times 25.\ 306 = 1075\times v_2 + 75\times v_1$

$\Rightarrow 1150\times25.\ 306 = 1075v_2+75(v_2-5)$

$\Rightarrow \dfrac{1150\times25.306+75\times5}{1075} = v_2$

$\Rightarrow v_2 = 25.\ 63$ m/s

(iii)



$1075 \times 25.63 = 1000 \times v_2 + 75 \times v_1$

$\Rightarrow 1075 \times 25.63 = 1000v_2 + 75(v_2 - 5)$

$\Rightarrow \dfrac{1075 \times 25.63 + 75 \times 5}{1075} = v_2$

$\Rightarrow v_2 = 25.97$ m/s

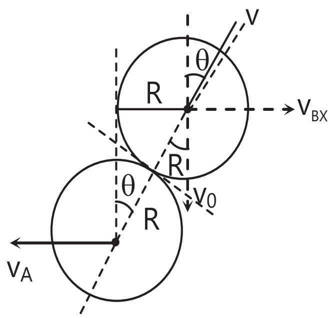
(b)All together

$\Rightarrow 1225 \times 25 = 1000 \times v_2 + 225 \times v_1$

$\Rightarrow 1225 \times 25 = 1000v_2 + 225(v_2 - 5)$

$\Rightarrow \dfrac{1225 \times 25 + 225 \times 5}{1225} = v_2 = 25.92$ m/s

Sol 30:



$\sin\theta = \dfrac{R}{2R} = \dfrac{1}{2} \Rightarrow \theta = 30^\circ$

Now, using elasticity, (along line of impact)

$$\frac{v_A \cos 60^\circ + v_{By}.\frac{\sqrt{3}}{2} + v_{Bx}.\frac{1}{2}}{v_0.\frac{\sqrt{3}}{2}} = 1$$

$\Rightarrow \dfrac{v_A}{2} + \dfrac{\sqrt{3}v_{By}}{2} + \dfrac{v_{Bx}}{2} = \dfrac{v_0\sqrt{3}}{2}$

$\Rightarrow v_A + v_{By}.\sqrt{3} + v_{Bx} = v_0\sqrt{3}$ ...(i)

using energy balance,

$\dfrac{1}{2}mv_0^2 = \dfrac{1}{2}mv_A^2 + \dfrac{1}{2}m(v_{Bx}^2 + v_{By}^2)$

$\Rightarrow v_0^2 = v_A^2 + v_{Bx}^2 + v_{By}^2$ ...(ii)

Momentum along x-direction

$\Rightarrow v_A - v_{Bx} = 0 \Rightarrow v_A = v_{Bx}$ ...(iii)

so on, solving $v_0 - \dfrac{2v_A}{\sqrt{3}} = v_{By}$

Putting this in (ii)

$\Rightarrow v_0^2 = 2v_A^2 + \left(v_0 - \dfrac{2v_A}{\sqrt{3}}\right)^2$

$\Rightarrow v_A = \dfrac{2\sqrt{3}v_0}{\sqrt{3}} \Rightarrow 0.693\ v_0$

So, $v_{Bx} = 0.693\ v_0$ and

$v_{By} = v_0 - \dfrac{2v_A}{\sqrt{3}} = v_0 - \dfrac{4}{5}v_0 = \dfrac{v_0}{5} = 0.2\ v_0$

So, $v_B = \sqrt{(0.693)^2 + (0.2)^2}\ .v_0$

$\Rightarrow v_B = 0.721\ v_0$

Sol 31: $\dfrac{dP}{dt} = F = \dfrac{d(mu)}{dt} = \dfrac{udm}{dt} + \dfrac{m.du}{dt}$

$-mg = \dfrac{u}{T}.m_0e^{t/T} + m.\dfrac{du}{dt}$

$-g = \dfrac{u}{T} + \dfrac{du}{dt}$

$\Rightarrow \displaystyle\int_{u_0,0}^{u,t} d(u.e^{t/T}) = \int_0^t -g.e^{t/T}.dt$

$u.\ e^{t/T} - u_0 = -gT.\ (e^{t/T} - 1)$

$\Rightarrow u(t) = u_0.\ e^{-t/T} - gT.\ (1 - e^{-t/T})$

so here, $u(t) = 0$

$\Rightarrow gT.\ e^{-t/T} - gT(1 - e^{-t/T}) = 0$

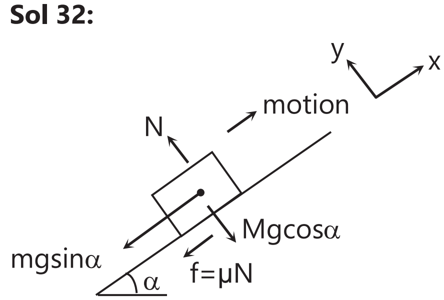
$\Rightarrow e^{-t/T} = \dfrac{1}{2}$

$\Rightarrow e^{t/T} = 2 \Rightarrow t = T \ln 2$

So $m = m_0.\ e^{t/T}$

$m = 2m_0$

Sol 32:



$N = Mg\cos\alpha$

So $f = \mu N = \mu Mg\cos\alpha$

so acceleration of 2 kg block

$= \dfrac{\mu mg\cos\alpha + mg\sin\alpha}{m} = \mu g\cos\alpha + g\sin\alpha$

So for a distance of 6m, v just before impact would be:

$v^2 = u^2 - 2as$

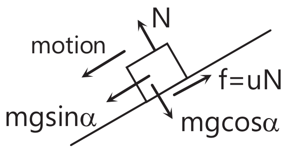
$\Rightarrow v^2 = (10)^2 - 2 \times 6 \times [0.25 \times 10 \times 0.998 + 10 \times 0.05]$

$= 100 - 12 \times [3]$

$v^2 = 64$

$v_{in.} = 8$ m/s

Now, after collision:



so acc. = $\mu g \cos\alpha - g \sin\alpha$

$= 2.5 - 0.5 = 2$ m/s^2

So $v^2 = u^2 - 2as$

$1^2 = u^2 - 2 \times 2 \times 6$

$\Rightarrow u^2 = 25 \Rightarrow u = 5$ m/s

so after collision v of 2 kg block = 5 m/s

For M block:

a_{cc} = same as in 1st part (ind. of mass,)

$= \mu g\cos\alpha + g\sin\alpha = 3$m/s^2 (in (–)ve direction)

So $v^2 = u^2 + 2as$

$0 = u^2 + 2 \times (-3) \times (0.5)$

$\Rightarrow u = 1.732$ m/s

so coefficient of restitution

$= \dfrac{5+1.732}{8} = \dfrac{6.732}{8} = 0.84$

Using momentum conservation, just before impact

$m \times v_i = m \times v_f + M \times v_m$

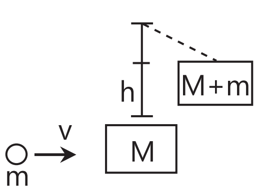
$2 \times 8 = 2 \times (-5) + M \times 1.732$

$\Rightarrow \dfrac{26}{1.732} = M = 15.011$ kg

Exercise 2

Single Correct Choice Type

Sol 1: (B)



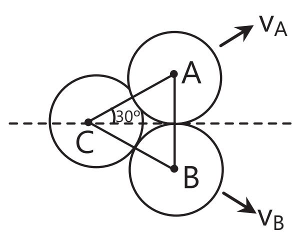
v of system after collision = $\sqrt{2gh}$

so using momentum conservation.

$(M + m)\sqrt{2gh} = mv$

$\Rightarrow v = \dfrac{(M+m)}{m}\sqrt{2gh}$

Sol 2: (C)



Let the initial velocity of

C = v

then momentum conservation in y gives

$v_A = v_B$

and using momentum conservation in x,

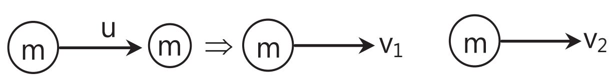
$mv = 2mv_A \cos 30°$

$\Rightarrow v = \sqrt{3}\, v_A \quad \Rightarrow v_A = \dfrac{v}{\sqrt{3}}$

so coefficient of restitution

$= \dfrac{\text{final relative velocity}}{\text{initial relative velocity}} = \dfrac{v/\sqrt{3}}{v\sqrt{3}/2} = \dfrac{2}{3}$

Sol 3: (A)



$e = \dfrac{v_2 - v_1}{u} \Rightarrow v_2 - v_1 = ue$ ...(i)

and momentum conservation ⇒

$mu = mv_1 + mv_2$

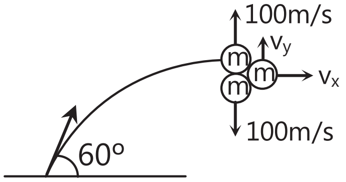
$\Rightarrow v_1 + v_2 = u$ ...(ii)

So $2v_2 = u(e + 1)$

$\Rightarrow v_2 = \dfrac{u(e+1)}{2}$ and thus,

$v_1 = \dfrac{(1-e)u}{2}$

Sol 4: (B)



We have

Momentum cons.: in y-direction

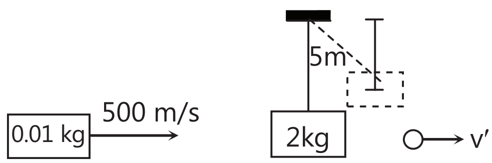
$m \times 100 + m(-100) + mv_y = 0$

$\Rightarrow v_y = 0$

x-direction: $3m.\ u \cos 60^\circ = mv_x$

$\Rightarrow v_x = 3 \times 200 \times \frac{1}{2} = 300$ m/s

Sol 5: (C)



v of block after the bullet emerges can be found using energy conservation

$mgh = \frac{1}{2}mv^2$

$\Rightarrow v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 0.1} = 1.4$ m/s

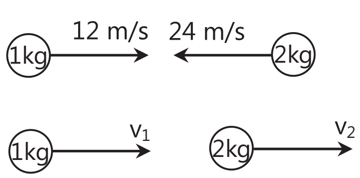
so now using momentum conservation

$500 \times 0.01 = 2 \times 1.4 + 0.01 \times v'$

$\Rightarrow 5 - 2.8 = 0.01 \times v'$ $\Rightarrow 2.2 = 0.01 \times v'$

$v' = 220$ m/s

Sol 6: (C)



So $e = \frac{v_2 - v_1}{36} = \frac{2}{3}$

$\Rightarrow v_2 - v_1 = 24$ ...(i)

Using momentum conservation

$12 \times 1 - 2 \times (24) = 1 \times v_1 + 2 \times v_2$

$\Rightarrow -36 = v_1 + 2v_2$ ...(ii)

$\Rightarrow 3v_2 = -12$

$\Rightarrow v_2 = -4$ m/s and $v_1 = -28$ m/s

$E_{initial} = \frac{1}{2} \times (1) \times 12^2 + \frac{1}{2} \times 2 \times 24^2$

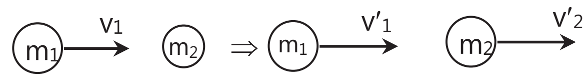
$= 72 + 576 = 648$ J

$E_{final} = \frac{1}{2} \times 1 \times (28)^2 + \frac{1}{2} \times 2 \times 4^2$

$= 16 + 392 = 408$ J

So $\Delta E = 240$ J

Sol 7: (B)



$e = 1 = \frac{v'_2 - v'_1}{v_1} \Rightarrow v'_2 - v'_1 = v_1$ ...(i)

and $m_1v_1 = m_1v'_1 + m_2v'_2$

$\Rightarrow v_1 = v'_1 + \frac{m_2}{m_1}.\ v'_2$ ...(ii)

so (1) + (2) $\Rightarrow \left(1 + \frac{m_2}{m_1}\right)v'_2 = 2v_1$

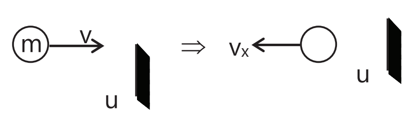
$\Rightarrow v'_2 = \frac{2v_1.m_1}{(m_1 + m_2)}$

and $v'_1 = v'_2 - v_1 = \frac{2v_1.m_1}{(m_1 + m_2)} - v_1$

$= \frac{m_1v_1 - m_2v_1}{m_1 + m_2} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_1$

Sol 8: (C) $v'_2 = \frac{2v_1m_1}{(m_1 + m_2)}$

Sol 9: (B)

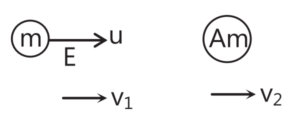


$e = 1 = \frac{v_x - (-u)}{-v - u}$

$v_x = -u - v - u = -v - 2u$

$2u + v \Rightarrow$ away from wall.

Sol 10: (C)



$e = \frac{v_2 - v_1}{u} = 1 \Rightarrow v_2 - v_1 = u$ ...(i)

Momentum conservation

$mu = Amv_2 + mu_1$

$\Rightarrow Av_2 + v_1 = u$

$\Rightarrow (A + 1)\, v_2 = 2u$ or $v_2 = \dfrac{2u}{(A+1)}$

$v_1 = v_2 - u = \dfrac{2u}{(A+1)} - u = \dfrac{(1-A)u}{(1+A)}$

$E = \dfrac{1}{2} \times m \times \dfrac{(1-A)^2 u^2}{(1+A)^2} = \dfrac{E.(1-A)^2}{(1+A)^2}$

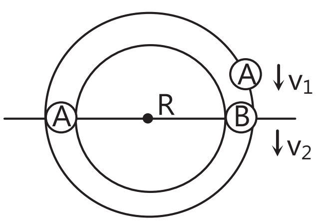
Sol 11: (A) v at first imp. = $\sqrt{2gh}$

v at after 1st imp. = $e\sqrt{2gh}$

v at after nth imp. = $e^n\sqrt{2gh}$

$h = \dfrac{v^2}{2g} = e^{2n}.\dfrac{2gh}{2g} = e^{2n}.\, h$

Sol 12: (A)



$v_A = \dfrac{\pi R}{t}$,

$v_2 - v_1 = ev_A$(elasticity)

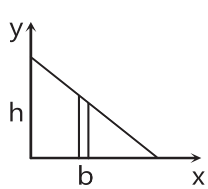
and $v_2 + v_1 = v_A$(mom. cons.)

$\Rightarrow v_2 = \dfrac{(e+1)v_A}{2}$ and $v_1 = \dfrac{(1-e)v_A}{2}$

so total time = $\dfrac{D_{rel}}{v_{rel}} = \dfrac{2\pi r}{\left[\dfrac{e+1}{2} - \left(\dfrac{1-e}{2}\right)\right] v_A}$

$= \dfrac{2\pi r \times t}{e.\pi r} = \dfrac{2t}{e}$

Sol 13: (C)



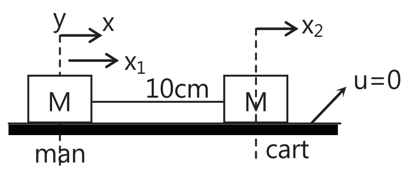
Equation $\Rightarrow \dfrac{y}{h} + \dfrac{x}{b} = 1$,

$\rho = \dfrac{M}{hb/2} = \dfrac{2M}{hb}$ So $x_{COM} = \int \dfrac{x.dm}{M}$

$x_{COM} = \int_0^b \dfrac{x.\rho.h.\left(1 - \dfrac{x}{b}\right)dx}{M} = \dfrac{2}{b}\left[\dfrac{x^2}{2} - \dfrac{x^3}{3b}\right]_0^b = \dfrac{b}{3}$

Similarly, $y_{COM} = \dfrac{h}{3}$

Sol 14: (B)



At x = 5 m

$M_1\vec{x}_1 + M_2\vec{x}_2 = 0$

$\Rightarrow \vec{x}_1 + \vec{x}_2 = 0$ and $\vec{x}_1 - \vec{x}_2 = 10$

$\Rightarrow \vec{x}_1 = 5$ m & $\vec{x}_2 = -5$m

Multiple Correct Choice Type

Sol 15: (C, D)

Elastic collision $\Rightarrow$ 100 % energy transfer

The relative velocity along tangent is zero but in oblique collision the tangent direction is not the one perpendicular to the line joining centres.

Assertion Reasoning Type

Sol 16: (D) $a_{COM} \neq 0$ as $F_{ext} \neq 0$. (in COM frame it is zero)

Sol 17: (A) Proper exp.

Sol 18: (B) It is average and it may be outside body.

Sol.19: (A) In explosion only internal forces are involved.

Sol 20: (D) Disk may be non-uniform.

Sol 21: (A) (A) true, (R) true $\Rightarrow$ correct reason

Sol 22: (B) A $\rightarrow$ true, R $\rightarrow$ true. But R not explanation of A.

Comprehension Type

Sol 23: (B) Obvious (inelastic)

Sol 24: (B) Inelastic collision leads to loss of energy.

Sol 25: (D) Basic concept.

Previous Years' Questions

Sol 1: (a) From conservation of linear momentum, momentum of composite body

$$\vec{p} = (\vec{p}_i)_1 + (\vec{p}_i)_2 = (mv)\hat{i} + (MV)\hat{j}$$

$$\therefore \quad |\vec{p}| = \sqrt{(mv)^2 + (MV)^2}$$

Let it makes an angle α with positive x-axis, then

$$\alpha = \tan^{-1}\left(\frac{p_y}{p_x}\right) = \tan^{-1}\left(\frac{MV}{mv}\right)$$

(b) Fraction of initial kinetic energy transformed into heat during collision

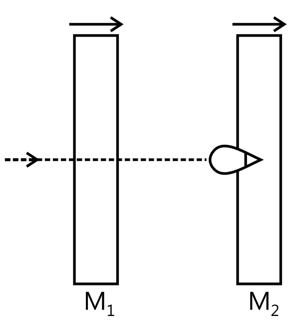
$$= \frac{K_f - K_i}{K_f} = \frac{K_f}{K_i} - 1$$

$$= \frac{p^2/2(M+m)}{\frac{1}{2}mv^2 + \frac{1}{2}MV^2} - 1$$

$$= \frac{(mv)^2 + (MV)^2}{(M+m)(mv^2 + MV^2)} - 1$$

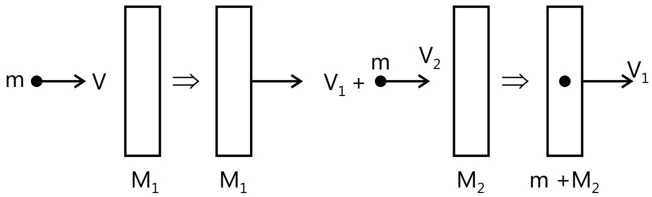
$$= \frac{Mm(v^2 + V^2)}{(M+m)(mv^2 + MV^2)}$$

Sol 2: Applying conservation of linear momentum twice. We have



$$mv = M_1v_1 + mv_2 \quad \text{....(i)}$$

$$mv_2 = (M_2 + m)v_1 \quad \text{....(ii)}$$



Solving Eqs. (i) and (ii), we get

$$\frac{v_2}{v} = \frac{M_2 + m}{M_1 + M_2 + m}$$

Substituting the values of m : M_1 and M_2 we get, percentage of velocity retained by bullet.

$$\frac{v_2}{v} \times 100 = \left(\frac{2.98 + 0.02}{1 + 2.98 + 0.02}\right) \times 100 = 75\%$$

$\therefore$ %loss = 25%

Sol 3: Suppose r_1 be the distance of centre of mass of the remaining portion from centre of the bigger circle, then

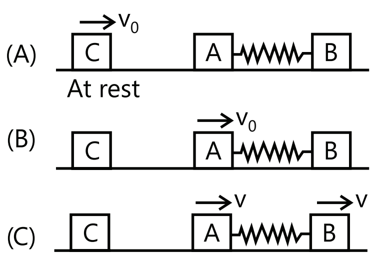
$$A_1r_1 = A_2r_2$$

$$r_1 = \left(\frac{A_2}{A_1}\right)r_2$$

$$r_1 = \frac{\pi(42)^2}{\pi[(56)^2 - (42)^2]} \times 7 = 9 \text{ cm}$$

Sol 4: Before collision net momentum of the system was zero. No external force is acting on the system. Hence, momentum after collision should also be zero. A has come to rest. Therefore, B and C should have equal and opposite momenta or velocity of C should be V in opposite direction of velocity of B.

Sol 5: Collision between A and C is elastic and mass of both the blocks is same. Therefore, they will exchange their velocities i.e., C will come to rest and A will be moving will velocity v_0. Let v be the common velocity of A and B, then from conservation of linear momentum, we have



$$m_Av_0 = (m_A + m_B)v$$

or $mv_0 = (m+2m)v$ or $v = \frac{v_0}{3}$

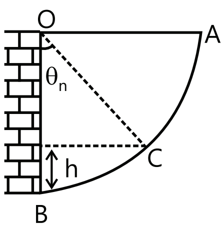
(b) From conservation of energy, we have

$$\frac{1}{2}m_A v_0^2 = \frac{1}{2}(m_A + m_B)v^2 + \frac{1}{2}kx_0^2$$

or $\frac{1}{2}mv_0^2 = \frac{1}{2}(3m)\left(\frac{v_0}{3}\right)^2 + \frac{1}{2}kx_0^2$

or $\frac{1}{2}kx_0^2 = \frac{1}{3}mv_0^2$ or $k = \frac{2mv_0^2}{3x_0^2}$

Sol 6: As shown in figure initially when the bob is at A, its potential energy is mgl. When the bob is released and it strikes the wall at B, its potential energy mgl is converted into its kinetic energy. If v be the velocity with which the bob strikes the wall, then



$$mgl = \frac{1}{2}mv^2 \text{ or } v = \sqrt{(2gl)} \quad .....(i)$$

Speed of the bob after rebounding (first time)

$$v_1 = e\sqrt{(2gl)} \quad .....(ii)$$

The speed after second rebound is $v_2 = e^2\sqrt{(2gl)}$

In general after n rebounds, the speed of the bob is

$$v_n = e^n\sqrt{(2gl)} \quad .....(iii)$$

Let the bob rises to a height h after n rebounds. Applying the law of conservatioin of energy, we have

$$\frac{1}{2}mv_n^2 = mgh$$

$$\therefore \quad h = \frac{v_n^2}{2g} = \frac{e^{2n}.\,2gl}{2g} = e^{2n}.l$$

$$= \left(\frac{2}{\sqrt{5}}\right)^{2n}.\,l = \left(\frac{4}{5}\right)^n l \quad .....(iv)$$

If θ_n be the angle after n collisions, then

$$h = l - l\cos\theta_n = l(1-\cos\theta_n)$$

From Eqs. (iv) and (v), we have

$$\left(\frac{4}{5}\right)^n l = l(1-\cos\theta_n) \text{ or } \left(\frac{4}{5}\right)^n = (1-\cos\theta_n)$$

For θ_n to be less than 60°, i.e., $\cos\theta_n$ is greater than $\frac{1}{2}$, i.e., $(1-\cos\theta_n)$ is less than $\frac{1}{2}$, we have

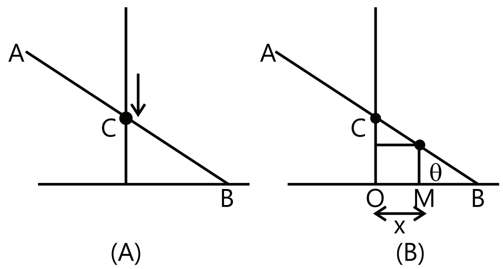
$$\left(\frac{4}{5}\right)^n < \left(\frac{1}{2}\right)$$

The condition is satisfied for n = 4.

∴ Required number of collisions = 4.

Sol 7: (a) Since, only two forces are acting on the rod, its weight Mg (vertically downwards) and a normal reaction N at point of contact B (vertically upwards).

No horizontal force is acting on the rod (surface is smooth).



(A) (B)

Therefore, CM will fall vertically downwards towards negative y-axis i.e., the path of CM is a straight line.

(b) Refer figure (B). We have to find the trajectory of a point P(x, y) at a distance r from end B.

CB = L/2

∴ OB = (L/2) $\cos\theta$

MB = r $\cos\theta$

∴ x = OB – MB = $\cos\theta$ {(L/2 – r)}

$$\text{or} \quad \cos\theta = \frac{x}{\{(L/2)-r\}} \quad ....(i)$$

Similarly, y = r sin θ

$$\text{or} \quad \sin\theta = \frac{y}{r} \quad ....(ii)$$

Squaring and adding Eqs. (i) and (ii), we get

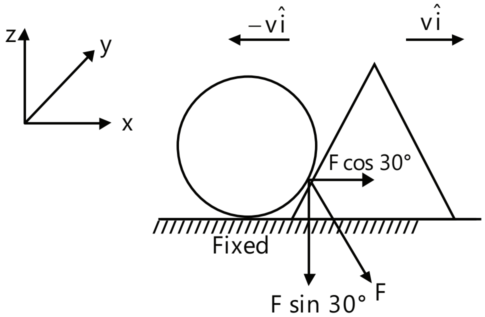
$$\sin^2\theta + \cos^2\theta = \frac{x^2}{\{(L/2)-r\}^2} + \frac{y^2}{r^2}$$

$$\text{or} \quad \frac{x^2}{\{(L/2)-r\}^2} + \frac{y^2}{r^2} = 1 \quad ....(iii)$$

This is an equation of an ellipse. Hence, path of point P is an ellipse whose equation is given by (iii).

Sol 8: (a) Since, the collision is elastic, the wedge will return with velocity $v\hat{i}$

(i)



Now, linear impulse in x-direction

= change in momentum in x-direction.

$\therefore \quad (F\cos 30°)\ \Delta t = mv - (-mv) = 2mv$

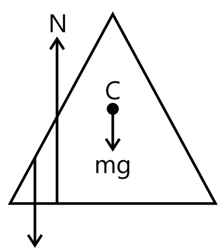
$\therefore \quad F = \dfrac{2mv}{\Delta t \cos 30°} = \dfrac{4mv}{\sqrt{3}\,\Delta t}$

$F = \dfrac{4mv}{\sqrt{3}\,\Delta t}$

$\therefore \quad \vec{F} = (F\cos 30°)\hat{i} - (F\sin 30°)\hat{k}$

or $\quad \vec{F} = \left(\dfrac{2mv}{\Delta t}\right)\hat{i} - \left(\dfrac{2mv}{\sqrt{3}\,\Delta t}\right)\hat{k}$

(ii) Taking the equilibrium of wedge in vertical z-direction during collision.



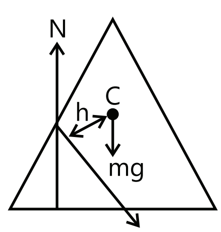
$N = mg + F\sin 30°$

$N = mg + \dfrac{2mv}{\sqrt{3}\,\Delta t}$

or in vector form

$\vec{N} = \left(mg + \dfrac{2mv}{\sqrt{3}\,\Delta t}\right)\hat{k}$

(b) For rotational equilibrium of wedge [about (CM) anticlockwise torque of F = clockwise torque due to N.



$\therefore$ Magnitude of torque of N about CM = magnitude of torque of F about CM

$= F\,.\,h$

$|\vec{\tau}_N| = \left(\dfrac{4mv}{\sqrt{3}\,\Delta t}\right)h$

Sol 9: After elastic collision,

$v'_A = \left(\dfrac{m-2m}{m+2m}\right)(9) + \dfrac{2(2m)}{m+2m}(0) = -3ms^{-1}$

Now from conservation of linear momentum after all collisions are complete,

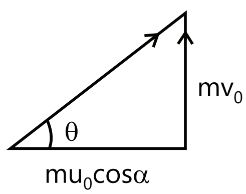
$m(+9ms^{-1}) = m(-3\ ms^{-1}) + 3m\ (v_C)$

or $\quad v_C = 4ms^{-1}$

Sol 10: (A) Velocity of particle performing projectile motion at highest point

$= v_1 = v_0 \cos\alpha$

Velocity of particle thrown vertically upwards at the position of collision



$= v_2^2 = u_0^2 - 2g\dfrac{u^2\sin^2\alpha}{2g} = v_0\ \cos\alpha$

So, from conservation of momentum

$\tan\theta = \dfrac{mv_0\ \cos\alpha}{mu_0\cos\alpha} = 1$

$\Rightarrow\ \theta = \pi/4$

Sol 11: The initial speed of 1st bob (suspended by a string of length l_1) is $\sqrt{5gl_1}$.

The speed of this bob at highest point will be $\sqrt{gl_1}$.

When this bob collides with the other bob there speeds will be interchanged.

$\sqrt{gl_1} = \sqrt{5gl_1} \Rightarrow \dfrac{l_1}{l_2} = 5$.