CHAPTER

8 APPLICATIONS OF THE INTEGRALS

Syllabus

Applications in finding the area under simple curves, especially lines, arcs of circles/parabolas/ellipses (in standard form only). Area between any of the two above said curves (the region should be clearly identifiable).

Chapter Analysis

The analysis given here gives you an analytical picture of this chapter and will help you to identify the concepts of the chapter that are to be focused more from exam point of view.

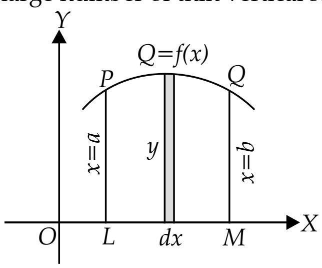
Number of Questions asked in last 3 years

Topic	**2016**		**2017**		**2018**	
	Delhi	**OD**	**Delhi**	**OD**	**Delhi/OD**	
Area Under the curves	2 Q. (6 marks)	2 Q. (6 marks)	2 Q. (6 marks)	2 Q. (6 marks)	2 Q. (6 marks)	–

Revision Notes

➢ **Area Under Simple Curves :**

(i) Let us find the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$. Consider the area under the curve as composed by large number of thin vertical strips.



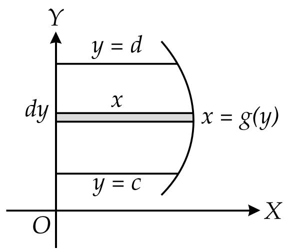
Let there be an arbitrary strip of height y and width dx.

Area of elementary strip $dA = y\,dx$, where $y = f(x)$. Total area A of the region between x-axis ordinates $x = a$, $x = b$ and the curve $y = f(x)$ = sum of areas of elementary thin strips across the region $PQML$.

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

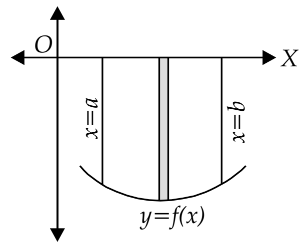
(ii) The area A of the region bounded by the curve $x = g(y)$, y-axis and the lines $y = c$ and $y = d$ is given by

$$A = \int_c^d x\,dy$$



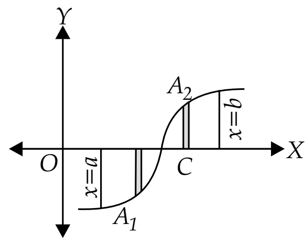
(iii) If the curve under consideration lies below x-axis, then $f(x) < 0$ from $x = a$ to $x = b$, the area bounded by the curve $y = f(x)$ and the ordinates $x = a$, $x = b$ and x-axis is negative. But, if the numerical value of the area is to be taken into consideration, then

$$\text{Area} = \left|\int_a^b f(x)dx\right|$$



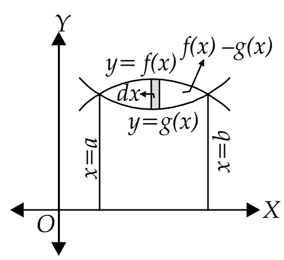
(iv) It may also happen that some portion of the curve is above x-axis and some portion is below x-axis as shown in the figure. Let A_1 be the area below x-axis and A_2 be the area above the x-axis. Therefore, area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is given by

$$A = |A_1| + |A_2|$$



➢ **Area Between Two Curves :**

(i) Let the two curves be $y = f(x)$ and $y = g(x)$, as shown in the figure. Suppose these curves intersect at $f(x)$ with width dx.



Consider the elementary strip of height y,

where $y = f(x)$

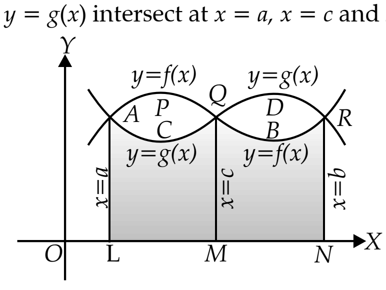
$\therefore$ $dA = y\,dx$

$\Rightarrow$ $A = \int_a^b [f(x) - g(x)]dx$

$= \int_a^b f(x)dx - \int_a^b g(x)dx$

= Area bounded by the curve $\{y = f(x)\}$ – Area bounded by the curve $\{y = g(x)\}$, where $f(x) > g(x)$.

(ii) If the two curves $y = f(x)$ and $y = g(x)$ intersect at $x = a$, $x = c$ and $x = b$, such that $a < c < b$.



If $f(x) > g(x)$ in $[a, c]$ and $g(x) \leq f(x)$ in $[c, b]$, then the area of the regions bounded by the curve = Area of region *PACQP* + Area of region *QDRBQ*.

$$= \int_a^c |f(x) - g(x)|\,dx + \int_c^b |g(x) - f(x)|\,dx.$$

Objective Type Questions **(1 mark each)**

Q. 1. The area of the region bounded by the y-axis, $y = \cos x$ and $y = \sin x$, $0 \leq x \leq \pi/2$ is

(a) $\sqrt{2}$ sq. units **(b) $(\sqrt{2}+1)$ sq. units**

(c) $(\sqrt{2}-1)$ sq. units **(d) $(2\sqrt{2}-1)$ sq. units**

[NCERT Exemp. Ex. 8.3, Q. 24, Page 177, NCERT Exemp. Ex. Q. 19, Page 376]

Ans. Correct option : (c)

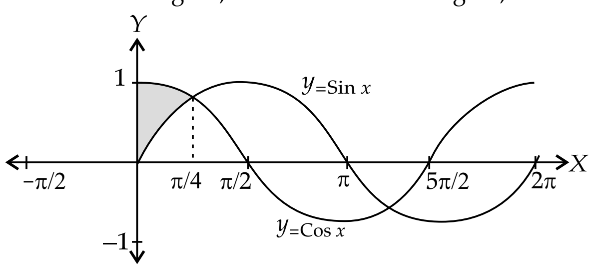
Explanation : We have $y = \cos x$ and $y = \sin x$, where

$$0 \leq x \leq \frac{\pi}{2}$$

We get $\cos x = \sin x$

$$x = \frac{\pi}{4}$$

From the figure, area of the shaded region,



$$A = \int_0^{\frac{\pi}{4}} (\cos x + \sin x)\,dx$$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}}$$

$$= \left[\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin 0 - \cos 0\right]$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 = (\sqrt{2}-1) \text{ sq. units}$$

Q. 2. The area of the region bounded by the curve $x^2 = 4y$ and the straight-line $x = 4y - 2$ is

(a) $\frac{3}{8}$ sq. units **(b) $\frac{5}{8}$ sq. units**

(c) $\frac{7}{8}$ sq. units **(d) $\frac{9}{8}$ sq. units**

[NCERT Exemp. Ex. 8.3, Q. 25, Page 177]

Ans. Correct option : (d)

Explanation :

$x^2 = x + 2$

$x^2 - x - 2 = 0$

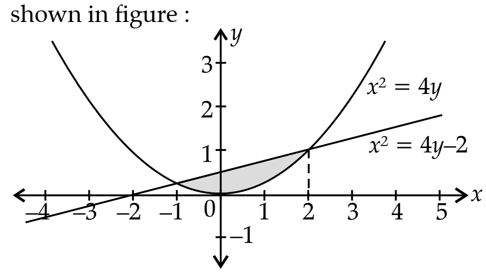
$(x - 2)(x + 1) = 0$

$x = -1, 2$

For $x = -1$, $y = \frac{1}{4}$ and for $x = 2$, $y = 1$

Points of intersection are $(-1, \frac{1}{4})$ and $(2, 1)$.

Graphs of parabola $x^2 = 4y$ and $x = 4y - 2$ are shown in figure :



$$A = \int_{-1}^{2} \left[\frac{x+2}{4} - \frac{x^2}{4}\right] dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[8 - \frac{1}{2} - 3\right]$$

$$= \frac{9}{8} \text{ sq. units}$$

Q. 3. The area of the region bounded by the curve $y = \sqrt{16 - x^2}$ and x-axis is

(a) 8π sq. units **(b) 20π sq. units**

(c) 16π sq. units **(d) 256π sq. units**

[NCERT Exemp. Ex. 8.3, Q. 26, Page 177]

Ans. Correct option : (a)

Explanation : We have $y = \sqrt{16 - x^2}$

$y^2 = 16 - x^2$, $y \geq 0$

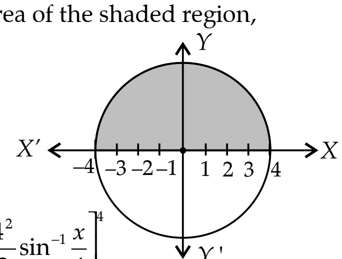
$y^2 + x^2 = 16$, $y \geq 0$

Graph of above function is semi-circle lying above the graph as shown in the adjacent figure.

From the figure, area of the shaded region,

$$A = \int_{-4}^{4} (\sqrt{16 - x^2})\,dx$$

$$= \int_{-4}^{4} (\sqrt{4^2 - x^2})\,dx$$



$$= \left[\frac{x}{2}\sqrt{4^2 - x^2} + \frac{4^2}{2}\sin^{-1}\frac{x}{4}\right]_{-4}^{4}$$

$$= \left[\frac{4}{2}\sqrt{4^2 - 4^2} + 8\sin^{-1}\frac{4}{4}\right] - \left[-\frac{4}{2}\sqrt{4^2 - (-4)^2} + 8\sin^{-1}\left(\frac{-4}{4}\right)\right]$$

$$= 0 + 8\sin^{-1}1 - 0 - 8\sin^{-1}(-1)$$

$$= 4\pi + 4\pi$$

$$= 8\pi \text{ sq. units}$$

Q. 4. Area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$ is

(a) 16π sq. units **(b) 4π sq. units**

(c) 32π sq. units **(d) 24π sq. units**

[NCERT Exemp. Ex. 8.3, Q. 27, Page 178]

Ans. Correct option : (b)

Explanation : We have $y = 0$, $y = x$ and the circle $x^2 + y^2 = 32$ in the first quadrant.

Solving $y = x$ with the circle

$x^2 + x^2 = 32$

$x^2 = 16$

$x = 4$ (In the first quadrant)

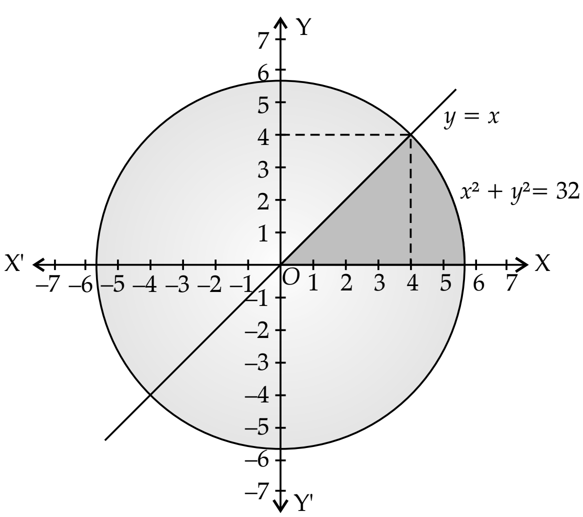
When $x = 4$, $y = 4$ for the point of intersection of the circle with the x-axis.

Put $y = 0$

$x^2 + 0 = 32$

$x = \pm 4\sqrt{2}$

So, the circle intersects the x-axis at $(\pm 4\sqrt{2}, 0)$.



From the above figure, area of the shaded region,

$$A = \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2}\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x}{2}\sqrt{(4\sqrt{2})^2 - x^2} + \frac{(4\sqrt{2})^2}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= \left[\frac{16}{2}\right] + \left[0 + 16\sin^{-1}1 - \frac{4}{2}\sqrt{(4\sqrt{2})^2 - 16^2} - 16\sin^{-1}\frac{4}{4\sqrt{2}}\right]$$

$$= 8 + \left[16\ \pi/2 - 2\sqrt{16} - 16\ \frac{\pi}{4}\right]$$

$$= 8 + [8\pi - 8 - 4\pi]$$

$$= 4\pi \text{ sq. units}$$

Q. 5. Area of the region bounded by the curve y = cos x between $x = 0$ and $x = \pi$ is

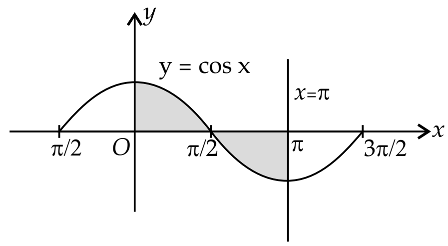
(a) 2 sq. units **(b) 4 sq. units**

(c) 3 sq. units **(d) 1 sq. unit**

[NCERT Exemp. Ex. 8.3, Q. 28, Page 178]

Ans. Correct option : (a)

Explanation : We have $y = \cos x$, $x = 0$, $x = \pi$



From the figure, area of the shaded region,

$$A = \int_0^{\pi} |\cos x|\,dx + \int_0^{\pi/2} \cos x\,dx$$

$$= 2\left[\sin x\right]_0^{\frac{\pi}{2}} = 2 \text{ sq. units}$$

Q. 6. The area of the region bounded by parabola $y^2 = x$ and the straight line $2y = x$ is

(a) $\frac{4}{3}$ sq. units **(b) 1 sq. units**

(c) $\frac{2}{3}$ sq. units **(d) $\frac{1}{3}$ sq. units**

[NCERT Exemp. Ex. 8.3, Q. 29, Page 178]

Ans. Correct option : (a)

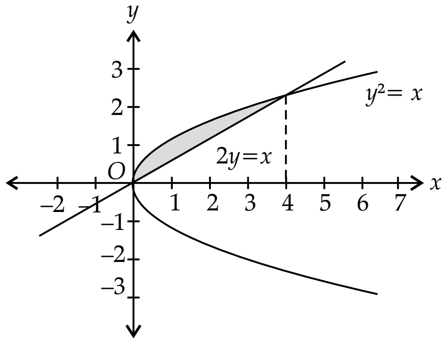
Explanation : When $y^2 = x$ and $2y = x$

Solving we get $y^2 = 2y$

$\Rightarrow y = 0, 2$ and when $y = 2$, $x = 4$

So, points of intersection are (0, 0) and (4, 2).

Graphs of parabola $y^2 = x$ and $2y = x$ are as shown in the adjoining figure :



From the figure, area of the shaded region,

$$A = \int_0^4 \left[\sqrt{x} - \frac{x}{2}\right] dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{2}.\frac{x^2}{2}\right]_0^4$$

$$= \frac{2}{3}.(4)^{3/2} - \frac{16}{4} - 0$$

$$= \frac{16}{3} - 4$$

$$= \frac{4}{3} \text{ sq. units}$$

Q. 7. The area of the region bounded by the curve $y = \sin x$ between the ordinates $x = 0$, $x = \frac{\pi}{2}$ and the axis is

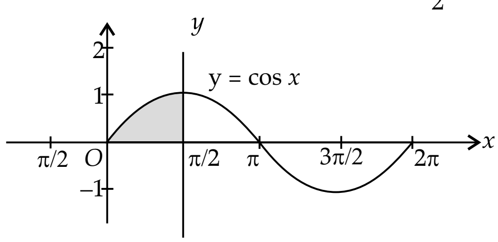
(a) 2 sq. units **(b) 4 sq. units**

(c) 3 sq. units **(d) 1 sq. units**

[NCERT Exemp. Ex. 8.3, Q. 30, Page 178]

Ans. Correct option : (d)

Explanation : We have $y = \sin x$, $0 \le x \le \frac{\pi}{2}$.



From the figure, area of the shaded region,

$$A = \int_0^{\pi/2} \sin x dx$$

$$= \left[-\cos x\right]_0^{\pi/2}$$

$$= \left[-\cos\frac{\pi}{2} + \cos 0\right]$$

$= 0 + 1$

$= 1$ sq. unit

Q. 8. The area of the region bounded by the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is

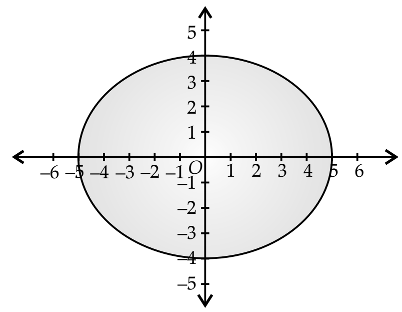
(a) 20π sq. units **(b) $20\pi^2$ sq. units**

(c) $16\pi^2$ sq. units **(d) 25π sq. units**

[NCERT Exemp. Ex. 8.3, Q. 31, Page 178]

Ans. Correct option : (a)

Explanation : We have $\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$, which is ellipse with its axes as coordinate axes.



$$\frac{y^2}{4^2} = 1 - \frac{x^2}{5^2}$$

$$y^2 = 16\left(1 - \frac{x^2}{25}\right)$$

$$y = \frac{4}{5}\sqrt{5^2 - x^2}$$

From the figure, area of the shaded region,

$$A = 4\int_0^5 \frac{4}{5}\sqrt{5^2 - x^2} dx$$

$$= \frac{16}{5}\left[\frac{x}{2}\sqrt{5^2 - x^2} - \frac{5^2}{2}\sin^{-1}\frac{x}{5}\right]_0^5$$

$$= \frac{16}{5}\left[0 + \frac{5^2}{2}\sin^{-1}1 - 0 - 0\right]$$

$$= \frac{16}{5}.\frac{25}{2}.\frac{\pi}{2}$$

$= 20\pi$ sq. units

Long Answer Type Questions-I

(4 marks each)

Q. 1. Find the area of the region bounded by the y-axis, $y = \cos x$ and $y = \sin x$, $0 \le x \le \frac{\pi}{2}$.

R&U **[S.Q.P. Dec. 2016-17]**

Sol. The rough sketch of the bounded region is shown below. **1**

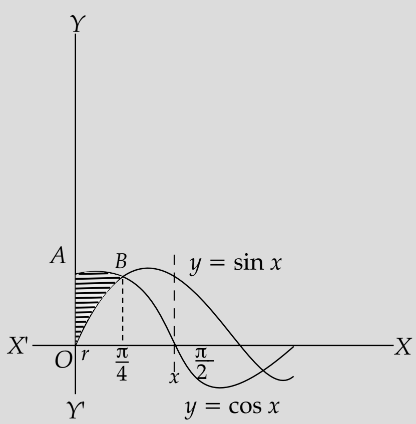
$\therefore$ Required Area = Area $ABCO$ – Area BCO

$$\int_0^{\frac{\pi}{4}} \cos x\, dx - \int_0^{\frac{\pi}{4}} \sin x\, dx \quad \mathbf{1}$$

$$= (\sin x + \cos x)_0^{\frac{\pi}{4}} \quad \mathbf{1}$$

$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin 0 - \cos 0$$

$$= \frac{2}{\sqrt{2}} - 1,\ i.e.,\ \left(\sqrt{2} - 1\right) \text{sq. units} \quad \mathbf{1}$$



[CBSE Marking Scheme 2016]

Commonly Made Error

- Many candidates identify the symmetry of the curve about X-axis incorrectly and either do not apply correct limits.

Answering Tips

- Interpret the graph of the standard functions in detail.

Q. 2. Given that $f[g(x)] = x$, for $x = 0$ to $x = 20$. Find $f(x)$ and $g(x)$ such that the area between $f(x)$ and $y = x^2$ from $x = 0$ to $x = 5$ be A_1 and area between $g(x)$ and $y = \sqrt{x}$ from $y = 0$ to $y = 5$ be A_2. Is $A_1 = A_2$? R&U

Sol.

$$A_1 = \int_{x=0}^{x=5} y\, dx = \int_0^5 x^2 dx \quad \mathbf{1}$$

$$= \frac{1}{3} \left| x^3 \right|_0^5 = \frac{125}{3} \text{ units}^2 \quad \mathbf{1}$$

$$A_2 = \int_{y=0}^{y=5} x\, dy = \int_0^5 y^2 dy$$

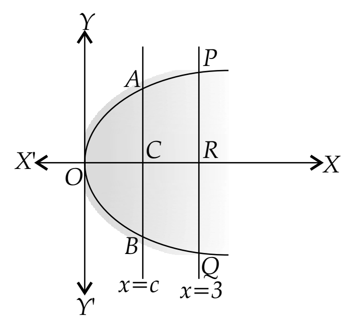
$$\left[\begin{array}{ll} \because & y = \sqrt{x} \\ \text{or} & x = y^2 \end{array}\right]$$

$$A_2 = \left[\frac{y^3}{3}\right]_0^5 = \frac{125}{3} \text{units}^2 \quad \mathbf{1}$$

Thus, $A_1 = A_2$. **½**

Q. 3. A farmer has a field of shape bounded by $x = y^2$ and $x = 3$, he wants to divide this into his two sons equally by a straight line $x = c$. Can you find c ? R&U

Sol.



Given,

Area $OACBO$ = Area of $APRQBCA$

or Area $OACBO$ $= 2\int_0^c y\, dx = 2\int_0^c \sqrt{x}\, dx$

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^c = \frac{4}{3} c^{3/2} \quad \mathbf{1}$$

Area of $APRQBCA$ $= 2\int_c^3 y\, dx = 2\int_c^3 x^{1/2} dx$ **½**

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_c^3 = \frac{4}{3}[3^{3/2} - c^{3/2}]$$

$$\therefore \quad \frac{4}{3} c^{3/2} = \frac{4}{3}[3^{3/2} - c^{3/2}] \quad \mathbf{1}$$

or $c^{3/2} = 3^{3/2} - c^{3/2}$ **½**

or $2c^{3/2} = 3^{3/2}$ or $4c^3 = 3^3$

or $c = \sqrt[3]{27/4} = \frac{3}{2}\sqrt[3]{2}.$ **1**

Long Answer Type Questions-II

(6 marks each)

Q. 1. Using integration, find the area of the region $\{(x, y) : x^2 + y^2 \leq 2ax , y^2 > ax, x, y, \geq 0\}$

R&U [NCERT Exemplar] [Delhi Set I, II, III 2016]

Sol. $\{(x, y) : x^2 + y^2 \leq 2ax, y^2 \geq ax, x, y \geq 0\}$

Considering the inequations as equations

$$x^2 + y^2 = 2ax \quad ...(i)$$

or $x^2 - 2ax + y^2 = 0$

or $x^2 - 2ax + a^2 + y^2 = a^2$

or $(x - a)^2 + y^2 = a^2 \quad ...(ii)$

It represents a circle whose **1**

Centre is $(a, 0)$ and radius $r = a$

$$y^2 = ax \quad ...(iii)$$

Vertex (0, 0)

Axis along x-axis.

Point of intersection, from (i) and (iii)

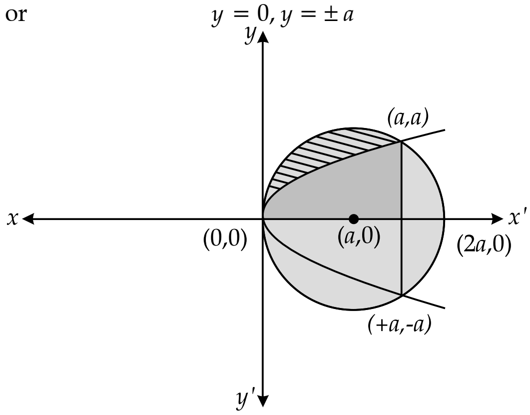
$x^2 + ax = 2ax$

or $x^2 - ax = 0$

or $x(x - a) = 0$

or $x = 0, x = a$

or $y = 0, y = \pm a$ **1**



Point of intersection are (0, 0) (a, a), $(a, -a)$. **1**

$x^2 + y^2 \leq 2ax$

(Area of shaded part)

Required Area $= \int_0^a y \text{ of circle } dx - \int_0^a y \text{ of parabola } dx$

$$= \int_0^a \sqrt{a^2 - (x-a)^2}\, dx - \int_0^a \sqrt{a}\sqrt{x}\, dx$$

$$A = \left[\frac{x-a}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^a - \sqrt{a}\left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^a$$

$$A = \left[0 - \left\{\frac{-a}{2}\sqrt{0} + \frac{a^2}{2}\sin^{-1}(-1)\right\}\right] - \frac{2}{3}\sqrt{a}\left[a^{\frac{3}{2}} - 0\right] \quad \textbf{1½}$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2}\right) - \frac{2}{3}\sqrt{a} \times a\sqrt{a}$$

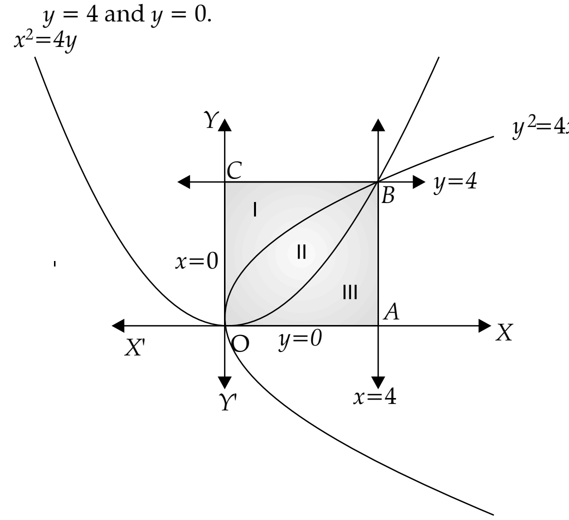
$$= \frac{\pi a^2}{4} - \frac{2a^2}{3} \text{ sq. units} \quad \textbf{1½}$$

Q. 2. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of square bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three equal parts.

R&U [NCERT]

[O.D. Set I, II, III 2016, OD 2015, Delhi 2009]

Sol. Let $OABC$ be the square bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$.



1

$$A(OABC) = 4 \times 4$$

$$= 16 \text{ sq. units} \quad \textbf{½}$$

From, $y^2 = 4x$ and $x^2 = 4y$

$$\left(\frac{x^2}{4}\right)^2 = 4x$$

or $\dfrac{x^4}{16} = 4x$

or $x^4 - 64x = 0$

or $x(x^3 - 64) = 0$

or $x = 0$ or $x = 4$

When $x = 0, y = 0$

$x = 4, y = 4$ **½**

$\therefore$ Point of intersection of the two parabolas is (0, 0) and (4, 4).

Area of part III $= \int_0^4 y\, dx$ (parabola $x^2 = 4y$)

$$= \int_0^4 \frac{x^2}{4}\, dx = \left[\frac{1}{4}\frac{x^3}{3}\right]_0^4$$

$$= \frac{1}{12}(64 - 0) = \frac{64}{12}$$

$$= \frac{16}{3} \text{ sq. units} \quad \textbf{1½}$$

Area of I = Area of square – Area of II and III

$$= 16 - \int_0^4 \sqrt{4x}\, dx$$

$$= 16 - \frac{2\times 2}{3}\left[x^{3/2}\right]_0^4$$

$$= 16 - \frac{32}{3} \text{ sq. units}$$

$$= \frac{16}{3} \text{ sq. units} \qquad \mathbf{1\tfrac{1}{2}}$$

Area of II = Area of square – Area of I – Area of III

$$= 16 - \frac{16}{3} - \frac{16}{3} \text{ sq. units}$$

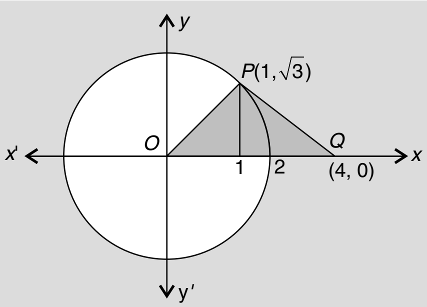
$$= \frac{16}{3} \text{ sq. units} \qquad \mathbf{1}$$

∴ The two curves divide the square into three equal parts.

Q. 3. Using integration, find the area of the triangle formed by positive x-axis and tangent and normal to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.

R&U **[Delhi, 2015]**

Sol.



1

Equation of normal (OP) or $y = \sqrt{3}x$

Equation of tangent (PQ) is **½**

$$y - \sqrt{3} = -\frac{1}{\sqrt{3}}(x-1)$$

or $$y = \frac{1}{\sqrt{3}}(4-x) \qquad \mathbf{1}$$

Co-ordinates of point Q is (4, 0). **½**

$$\therefore \text{ Required Area} = \int_0^1 \sqrt{3}\,x\,dx + \int_1^4 \frac{1}{\sqrt{3}}(4-x)dx \qquad \mathbf{1\tfrac{1}{2}}$$

$$= \sqrt{3}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{\sqrt{3}}\left[4x - \frac{x^2}{2}\right]_1^4 \qquad \mathbf{1}$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}}\left[16 - 8 - 4 + \frac{1}{2}\right]$$

$$= 2\sqrt{3} \text{ sq. units} \qquad \mathbf{\tfrac{1}{2}}$$

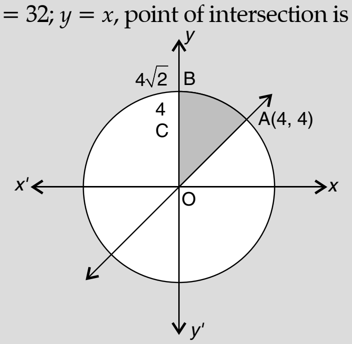
[CBSE Marking Scheme 2015]

AI **Q. 4. Find the area of the region in the first quadrant enclosed by the y-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$, using integration.**

R&U **[Delhi Set I, II, III Comptt. 2015]**

[Delhi Set I, II 2014]

Sol. $x^2 + y^2 = 32$; $y = x$, point of intersection is $y = 4$. **½**



1

$$\text{Required Area} = \int_0^4 y\,dy + \int_4^{4\sqrt{2}} \sqrt{32-y^2}\,dy \qquad \mathbf{1\tfrac{1}{2}}$$

$$= \left[\frac{y^2}{2}\right]_0^4 + \left[\frac{y}{2}\sqrt{32-y^2} + 16\sin^{-1}\frac{y}{4\sqrt{2}}\right]_4^{4\sqrt{2}} \qquad \mathbf{1\tfrac{1}{2}}$$

$$= 8 + \left(0 + 16\cdot\frac{\pi}{2}\right) - \left(8 + 16\cdot\frac{\pi}{4}\right)$$

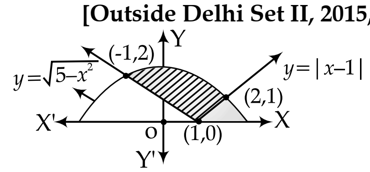
$$= 4\pi \text{ sq. units.} \qquad \mathbf{1\tfrac{1}{2}}$$

[CBSE Marking Scheme 2015]

AI **Q. 5. Sketch the region bounded by the curves $y = \sqrt{5 - x^2}$ and $y = |x - 1|$ and find its area using integration.** R&U **[SQP 2016-17]**

[Outside Delhi Set II, 2015, Delhi 2010]

Sol.



$y = x - 1$ if $x \geq 1$
$y = -x + 1$ if $x < 1$

intersection point :

$$y = \sqrt{5-x^2}$$
$$y = x - 1$$

so $$\sqrt{5-x^2} = (x-1)$$

$$5 - x^2 = (x-1)^2$$
$$5 - x^2 = x^2 + 1 - 2x$$
$$x^2 - x - 2 = 0$$
$$x^2 - 2x + x - 2 = 0$$
$$x(x-2) + 1(x-2) = 0$$

so $x = 2$ or $x = -1$

$x = -1$ rejected as $x \geq 1$

$$y = \sqrt{5-x^2}$$
$$y = -x + 1$$
$$5 - x^2 = x^2 + 1 - 2x$$

so $x = 2, -1$

$x = 2$ rejected, $x = -1$

Required Area

$$(\text{A}) = \int_{-1}^{2} \sqrt{5-x^2}\,dx - \left[\int_{-1}^{1}(-x+1)dx + \int_{1}^{2}(x-1)dx\right]$$

$$\text{A} = \left|\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin\frac{x}{\sqrt{5}}\right|_{-1}^{2} - \left[\frac{-x^2}{2} + x\right]_{-1}^{1} + \left[\frac{x^2}{2} - x\right]_{1}^{2}$$

$$A = 1 \times 1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} - \left(\frac{-1}{2}\times 2 + \frac{5}{2}\sin^{-1}\left(\frac{-1}{\sqrt{5}}\right)\right) - \left[\frac{-1}{2}[1-1] + 2 + \frac{1}{2}[4-1] - 1\right]$$

$$A = 1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} + 1 + \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \left[0 + 2 + \frac{3}{2} - 1\right]$$

$$A = 2 + \frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right] - \frac{5}{2}$$

$$A = \frac{5}{2}\times\frac{\pi}{2} + 2 - \frac{5}{2} \qquad \left\{\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}} = \frac{\pi}{2}\right\}$$

$$A = \frac{5\pi}{4} - \frac{1}{2} \text{ sq units}$$

Commonly Made Error

- Many candidates do not understand the concept of absolute value function and could not identify the limits of its by using given condition.

Answering Tips

- Learn well about the absolute value functions and to draw the rough sketch of the same by applying the given conditions.

Q. 6. Using integration, find the area bounded by the tangent to the curve $4y = x^2$ at the point (2, 1) and the lines whose equations are $x = 2y$ and $x = 3y - 3$. A **[S.Q.P. 2015-16]**

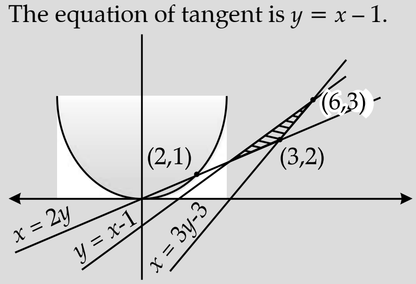
Sol. Given $4y = x^2$

or $4\frac{dy}{dx} = 2x$

or $\frac{dy}{dx} = \frac{x}{2}$

or $\left(\frac{dy}{dx}\right)_{x=2} = 1$ **1**

The equation of tangent is $y = x - 1$.



1

The required Area = Shaded Area of graph

$$= \left[\int_2^3 \{(x-1)\}dx + \int_3^6\left[\frac{(x+3)}{3} - \frac{x}{2}\right]dx\right]$$ **1**

$$= \left[\int_2^3 (x-1)dx + \frac{1}{3}\int_3^6 (x+3)dx - \frac{1}{2}\int_3^6 x\,dx\right]$$

$$= \left[\left[\frac{x^2}{2} - x\right]_2^3 + \frac{1}{3}\left[\frac{x^2}{2} + 3x\right]_3^6 - \frac{1}{4}\left[x^2\right]_3^6\right]$$ **1**

$$= \left[\frac{9}{2} - 3 - 2 + 2\right] + \frac{1}{3}\left[18 + 18 - \frac{9}{2} - 9\right] - \frac{1}{4}[36 - 9]$$ **1**

$$= \frac{3}{2} + \frac{15}{2} - \frac{27}{4}$$

$$= \frac{9}{4} \text{ sq.unit}$$ **1**

[CBSE Marking Scheme 2015]

AI **Q. 7. Using integration, find the area of the region bounded by the curves : $y = |x + 1| + 1$, $x = -3$, $x = 3$, $y = 0$.** A **[Delhi Set I, II, III Comptt. 2014]**

[NCERT Exemplar]

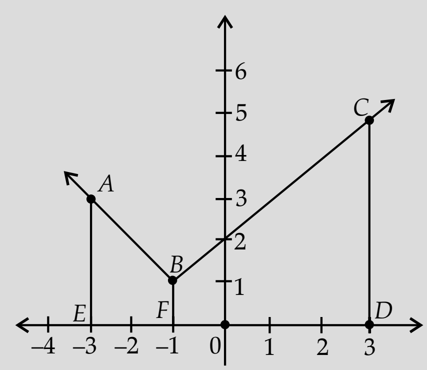
Sol. Area (*EABCDE*) = Area (*ABFE*) + Area (*CBFD*)

$$= \int_{-3}^{-1}(|x+1|+1)dx + \int_{-1}^{3}(|x+1|+1)dx$$ **1**

$$= \int_{-3}^{-1}(-x)dx + \int_{-1}^{3}(x+2)dx$$ **1½**

$$= \left[-\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{(x+2)^2}{2}\right]_{-1}^{3}$$ **1**

$$= \left|\frac{1}{2}(1-9)\right| + \frac{1}{2}(25-1) = 16 \text{ sq. unit.}$$ **1**



1½

[CBSE Marking Scheme 2014]

Q. 8. Find the area bounded by the curves $y = \sqrt{x}$, $2y + 3 = x$ and x – axis. A **[S.Q.P. 2018-19]**

Sol. The given curves are

$$y = \sqrt{x} \qquad ...(1)$$

$$2y + 3 = x \qquad ...(2)$$

Solving equation (1) and (2), we get

$$2y + 3 = y^2$$

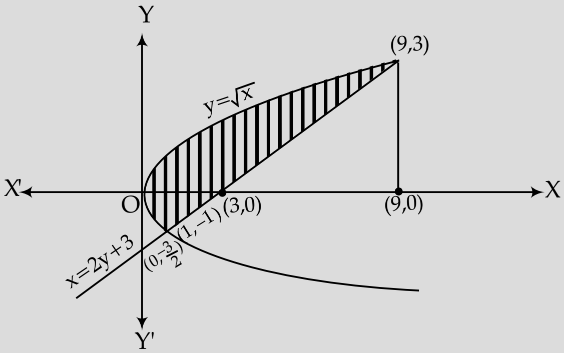
$\Rightarrow$ $y = -1, 3$

$\Rightarrow$ $y = 3$(as y > 0) **½**

substituting value of $y = 3$ in (2) we get

$$x = 2(3) + 3 = 9$$

i.e., (1) and (2) intersects at (9, 3) **1**



Required Area $= \int_0^3 (2y+3)\,dy - \int_0^3 y^2 dy$ **1½**

$= \left[y^2 + 3y - \frac{y^3}{3}\right]_0^3$ **1**

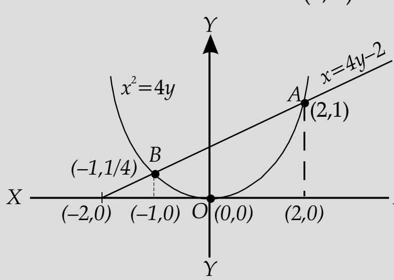
$= 9 + 9 - 9$

$= 9$ sq units. **1**

[CBSE Marking Scheme 2018]

AI **Q. 9. Using integration, find the area of the region bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.** R&U **[Delhi Set III Comptt. 2013, 2014] [NCERT] [NCERT Exemplar]**

Sol. $\therefore$ Points of intersection are (2, 1) and (– 1, 1/4).



1½

Required Area of region ($\Delta OABO$)

$= \int_{-1}^{2} \frac{x+2}{4}dx - \int_{-1}^{2} \frac{x^2}{4}dx$ **1½**

$= \frac{1}{4}\left[\frac{(x+2)^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$ **1½**

$= \frac{1}{4}\left[\frac{16}{3} - \frac{5}{6}\right]$

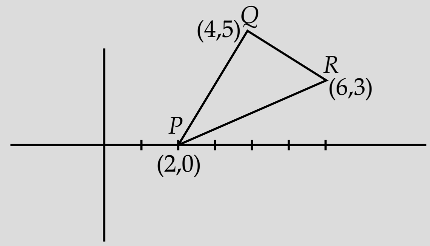
$= \frac{27}{24}$ sq.units

$= \frac{9}{8}$ sq.units **1½**

[CBSE Marking Scheme 2014]

AI **Q. 10. Using integration, find the area of the ΔPQR co-ordinates whose vertices are $P(2, 0)$, $Q(4, 5)$ and $R(6, 3)$.** R&U **[Dehli Set I, II III Comptt. 2016] [NCERT] [O.D. Set I Comptt. 2014]**

Sol.



1

Eqns. of PQ, QR and PR are :

$PQ : y = \frac{5}{2}(x-2)$

$QR : y = 9 - x$ **1½**

$PR : y = \frac{3}{4}(x-2)$ **1**

Req. Area $= \int_2^4 \frac{5}{2}(x-2)dx + \int_4^6 (9-x)dx$

$-\int_2^6 \frac{3}{4}(x-2)dx$ **1**

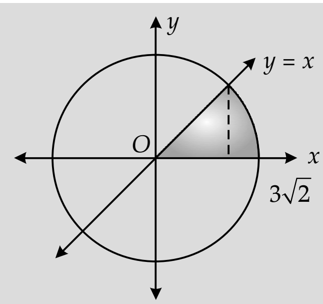
$= \left[\frac{5}{4}(x-2)^2\right]_2^4 - \frac{1}{2}\left[(9-x)^2\right]_4^6 - \frac{3}{8}\left[(x-2)^2\right]_2^6$ **1**

$= 5 + 8 - 6 = 7$ sq. units. **½**

[CBSE Marking Scheme 2014]

Q.11. Using integration, find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 18$. R&U **[O.D. Set II Comptt. 2014]**

Sol.



1

Point of intersection of Circle and Line is (3, 3). **1**

$\therefore$ Area $= \int_0^3 x\,dx + \int_3^{3\sqrt{2}} \sqrt{18-x^2}\,dx$ **½+1**

$= \left[\frac{x^2}{2}\right]_0^3 + \left[\frac{x}{2}\sqrt{18-x^2} + 9\sin^{-1}\frac{x}{3\sqrt{2}}\right]_3^{3\sqrt{2}}$ **½+1**

$= \frac{9}{2} + \frac{9\pi}{2} - \frac{9}{2} - \frac{9\pi}{4}$

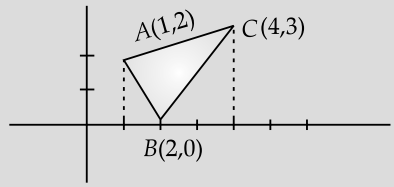
$= \frac{9\pi}{4}$ sq. units. **1**

[CBSE Marking Scheme 2014]

Q. 12. Using integration, find the area of the region bounded by the line $2x + y = 4$, $3x - 2y = 6$ and $x - 3y + 5 = 0$. R&U **[NCERT] [O.D. Set III Comptt. 2014, Foreign 2017]**

Sol. Let the line AB, BC and CA have equations $2x + y = 4$, $3x - 2y = 6$ and $x - 3y + 5 = 0$ respectively.

$B(2, 0)$, $C(4, 3)$ and $A(1, 2)$



2

$$\text{Area} = \int_1^4 \frac{1}{3}(x+5)dx - \int_1^2 (4-2x)dx - \int_2^4 \frac{1}{2}(3x-6)dx \quad \mathbf{1}$$

$$= \frac{1}{3}\left[\frac{(x+5)^2}{2}\right]_1^4 + 2\left[\frac{(2-x)^2}{2}\right]_1^2 - \frac{3}{2}\left[\frac{(x-2)^2}{2}\right]_2^4 \quad \mathbf{1}$$

$$= \left(\frac{81}{6} - \frac{36}{6}\right) + (0-1) - \frac{3}{4}\cdot 4 \quad \mathbf{1}$$

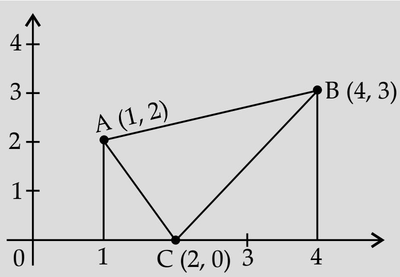
$$= \frac{15}{2} - 1 - 3 = \frac{7}{2} \text{ sq. units.} \quad \mathbf{1}$$

[CBSE Marking Scheme 2014]

Q. 13. Using the method of integration, find the area of the triangle ABC, coordinates of whose vertices are A(1, 2), B(2, 0) and C(4, 3).

R&U **[Foreign 2017]**

Sol.



1

Equation of AB : $y = \dfrac{x+5}{3}$

Equation of BC : $y = \dfrac{3x}{2} - 3$

Equation of AC : $y = 4 - 2x$ **1½**

$$\therefore \text{ Area } (A) = \int_1^4 \frac{x+5}{3}dx - \int_1^2 (4-2x)dx - \int_2^4 \left(\frac{3x}{2} - 3\right)dx \quad \mathbf{1}$$

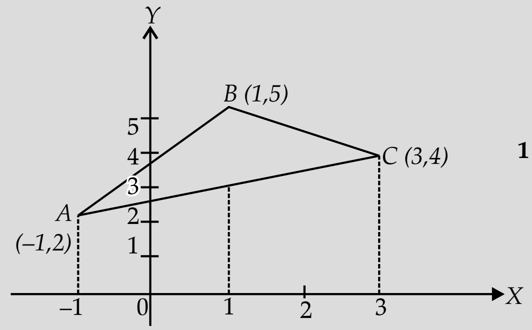
$$= \frac{1}{3}\left[\frac{x^2}{2} + 5x\right]_1^4 - [4x - x^2]_1^2 - \left[\frac{3x^2}{4} - 3x\right]_2^4 \quad \mathbf{1½}$$

$$= \frac{15}{2} - 1 - 3 = \frac{7}{2} \text{ sq. units} \quad \mathbf{1}$$

[CBSE Marking Scheme 2017]

Q. 14. Using integration, find the area of the region bounded by the triangle whose vertices are (–1, 2), (1, 5) and (3, 4). R&U **[O.D. Set I, II, III, 2014]**

Sol.



1

Equation of :

AB is : $y = \dfrac{1}{2}(3x+7)$ **½**

BC is : $y = \dfrac{1}{2}(11-x)$ **½**

AC is : $y = \dfrac{1}{2}(x+5)$ **½**

Required Area

$$= \frac{1}{2}\int_{-1}^{1}(3x+7)dx + \frac{1}{2}\int_{1}^{3}(11-x)dx - \frac{1}{2}\int_{-1}^{3}(x+5)dx \quad \mathbf{1}$$

$$= \left[\frac{1}{12}(3x+7)^2\right]_{-1}^{1} - \frac{1}{4}\left[(11-x)^2\right]_1^3 - \frac{1}{4}\left[(x+5)^2\right]_1^3 \quad \mathbf{1½}$$

$= 7 + 9 - 12 = 4$ sq. units **1**

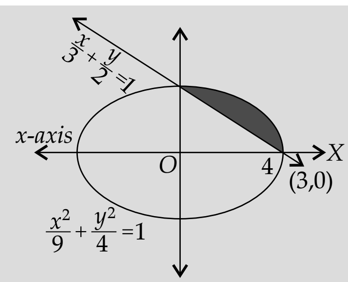
[CBSE Marking Scheme 2014]

Q.15. Find the area of the smaller region bounded by the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ and the line $\dfrac{x}{3} + \dfrac{y}{2} = 1$.

R&U **[NCERT]**

[Foreign Set I, II, III, 2014, O.D Comptt. 2009]

Sol.



1

Area of shaded region

$$= \int_0^3 \left\{\frac{2}{3}\sqrt{9-x^2} - \frac{2}{3}(3-x)\right\}dx \quad \mathbf{2}$$

$$= \frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} + \frac{(3-x)^2}{2}\right]_0^3 \quad \mathbf{2}$$

$$= \frac{2}{3}\left[\left(0 + \frac{9}{2}\cdot\frac{\pi}{2} + 0\right) - \left(0 + 0 + \frac{9}{2}\right)\right]$$

$$= \frac{2}{3}\left(9\frac{\pi}{4}-\frac{9}{2}\right)$$

$$= 3\left(\frac{\pi}{2}-1\right) \text{ sq. unit.} \quad \mathbf{1}$$

[CBSE Marking Scheme 2014]

Q. 16. Using integration, find the area of the region bounded by the curves $y = x^2$ and $y = x$.

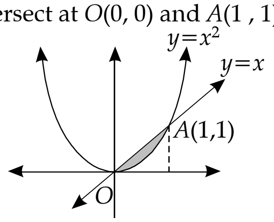
R&U [NCERT] [Delhi Set I Comptt. 2013]

Sol. The given curves are

$y = x^2$ (parabola)

$y = x$ (line)

These intersect at $O(0, 0)$ and $A(1, 1)$. **1**



The area bounded by the curves

$= $ Shaded area **2**

$$= \int_0^1 (y_2 - y_1)dx \quad \mathbf{1}$$

$$= \int_0^1 (x - x^2)dx \quad \mathbf{½}$$

$$= \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 \quad \mathbf{½}$$

$$= \frac{1}{2}-\frac{1}{3}$$

$$= \frac{1}{6} \text{ sq. units.} \quad \mathbf{1}$$

Q. 17. Using integration, find the area of the region enclosed by the curves $y^2 = 4x$ and $y = x$.

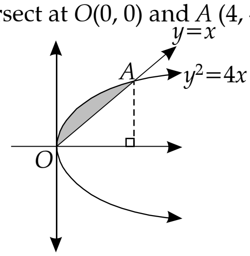
R&U [Delhi Set II Comptt. 2013]

Sol. The given curves are

$y^2 = 4x$ (Parabola) ...(i)

$y = x$ (line) ...(ii) **2**

These intersect at $O(0, 0)$ and A (4, 4). **1**



The required area

$= $ Shaded Area

$$= \int_0^4 (y_2 - y_1)dx \quad \mathbf{1}$$

$$= \int_0^4 (2\sqrt{x} - x)dx$$

$$= \left[2\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{x^2}{2}\right]_0^4 \quad \mathbf{1}$$

$$= \frac{4}{3}(4\sqrt{4})-\frac{4^2}{2}$$

$$= \frac{32}{3}-8$$

$$= \frac{32-24}{3}$$

$$= \frac{8}{3} \text{ sq. units.} \quad \mathbf{1}$$

Q. 18. Find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$.

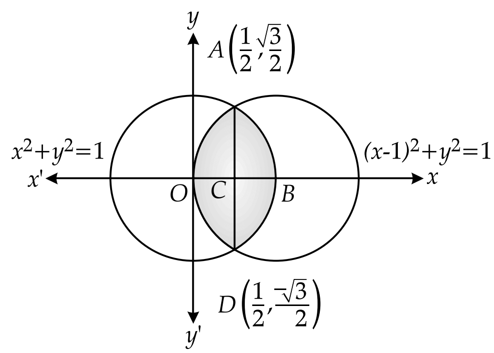
A [O.D. Set I Comptt. 2013] [S.Q.P. 2013]

Sol. We have $x^2 + y^2 = 1$ ...(i)

and $(x - 1)^2 + y^2 = 1$ ...(ii)

From eqns. (i) and (ii), we get points of intersection as :

$$A\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right), D\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right) \quad \mathbf{1}$$



2

Required Area = 2(Area $OABCO$)

$$= 2\left[\int_0^{1/2} y_{(ii)}dx + \int_{1/2}^1 y_{(i)}dx\right]$$

$$= 2\left[\int_0^{1/2}\sqrt{1-(x-1)^2}dx + \int_{1/2}^1\sqrt{1-x^2}dx\right] \quad \mathbf{1}$$

$$= 2\left[\frac{(x-1)}{2}\sqrt{1-(x-1)^2}+\frac{1}{2}\sin^{-1}\left(\frac{x-1}{1}\right)\right]_0^{1/2}$$

$$+2\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_{1/2}^1$$

$$= \left[(x-1)\sqrt{1-(x-1)^2}+\sin^{-1}(x-1)\right]_0^{1/2}$$

$$+\left[x\sqrt{1-x^2}+\sin^{-1}(x)\right]_{1/2}^1 \quad \mathbf{1}$$

$$= \left[-\frac{\sqrt{3}}{4}+\sin^{-1}\left(-\frac{1}{2}\right)-\sin^{-1}(-1)\right]$$

$$+\left[\sin^{-1}(1)-\frac{\sqrt{3}}{4}-\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \left[-\frac{\sqrt{3}}{4}-\frac{\pi}{6}+\frac{\pi}{2}\right]+\left[\frac{\pi}{2}-\frac{\sqrt{3}}{4}-\frac{\pi}{6}\right]$$

$$\therefore \text{ Required Area} = \left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right) \text{ sq. unit.} \quad \mathbf{1}$$

Commonly Made Error

- Some students do not solve the equations properly. The solution of two equation is intersection of two curve which is needed to calculate the bounded Part.

Q. 19. Using Integration, find the area of the region enclosed between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$. A **[O.D. Set II Comptt. 2013] [NCERT] [Delhi Set I 2013]**

OR

Sketch the region enclosed between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$. Using integration, find the area of this enclosed region.

[O.D. Set I Comptt. 2012] [S.Q.P. 2011]

Sol. $x^2 + y^2 = 4$ ...(i)

and $(x - 2)^2 + (y - 0)^2 = 4$ ...(ii)

Clearly from eq. (i), $x^2 + y^2 = 4$ or $x^2 + y^2 = 2^2$ represents a circle with centre (0, 0) and radius 2. To find the points of intersection of the given curves, we solve (i) and (ii) simultaneously.

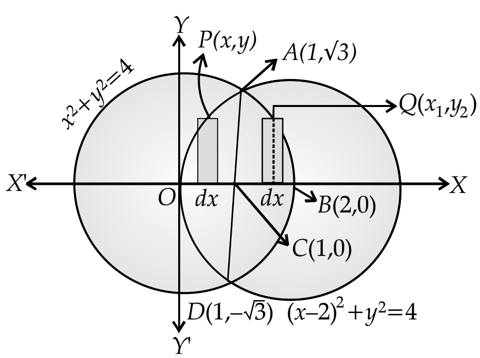
Solving eqns. (i) and (ii), we find that the two curves intersect at $A(1, \sqrt{3})$ and $D(1, -\sqrt{3})$. **1**

Required Area = 2 (Area $OABCO$)

Area $OABCO$ = Area $OACO$ + Area $CABC$

$$\therefore \quad \text{Area } OACO = \int_0^1 y_1 dx = \int_0^1 \sqrt{4-(x-2)^2}\,dx$$

$$\text{Area } CABC = \int_1^2 y_2\, dx = \int_1^2 \sqrt{4-x^2}\,dx \qquad \textbf{1}$$



½

Required Area

$$= 2\left[\int_0^1 \sqrt{4-(x-2)^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx\right] \qquad ½$$

$$= 2\left[\left\{\frac{1}{2}(x-2)\cdot\sqrt{4-(x-2)^2} + \frac{1}{2}\times 4\sin^{-1}\left(\frac{x-2}{2}\right)\right\}_0^1 + \left\{\frac{1}{2}x\cdot\sqrt{4-x^2} + \frac{1}{2}\times 4\sin^{-1}\left(\frac{x}{2}\right)\right\}_1^2\right] \qquad ½$$

$$= \left[\left\{(-1)\cdot\sqrt{4-(-1)^2} + 4\sin^{-1}\left(-\frac{1}{2}\right) - 4\sin^{-1}(-1)\right\} + \left\{4\sin^{-1}(1) - \sqrt{3} - 4\sin^{-1}\left(\frac{1}{2}\right)\right\}\right] \qquad ½$$

$$= \left[-\sqrt{3} - \frac{4\pi}{6} + \frac{4\pi}{2} + \frac{4\pi}{2} - \sqrt{3} - \frac{4\pi}{6}\right] \qquad \textbf{1}$$

$$= \left(\frac{8\pi}{3} - 2\sqrt{3}\right) \text{ sq. units.} \qquad \textbf{1}$$

Q. 20. Using integration, find the area of the region enclosed between the two circles $x^2 + y^2 = 9$ and $(x - 3)^2 + y^2 = 9$.

A **[Delhi 2009, O.D. Set III Comptt. 2013]**

Sol. Try yourself Similar to Q.18 Long Answer Type-II

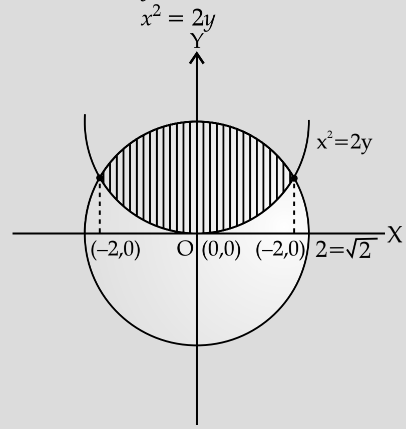
Q. 21. Find the area of the region.

$\{(x, y) : x^2 + y^2 \le 8, x^2 \le 2y\}$ A **[S.Q.P. 2018-19]**

Sol. The given curves are

$$x^2 + y^2 = 8 \qquad ...(1)$$

$$x^2 = 2y \qquad ...(2)$$



Solving (1) and (2)

$8 - y^2 = 2y \Rightarrow y = 2, -4 \Rightarrow y = 2$ (as $y > 0$)

Substituting y = 2 in (2) we get $x^2 = 4 \Rightarrow x = -2$ or 2

$$\text{Required Area} = \int_{-2}^{2} \sqrt{8-x^2}\,dx - \int_{-1}^{2} \frac{x^2}{2}dx$$

$$= 2\left[\int_0^2 \sqrt{\left(2\sqrt{2}\right)^2 - x^2}\,dx - \int_0^2 \frac{x^2}{2}dx\right]$$

$$= 2\left[\frac{x}{2}\sqrt{8-x^2} + \frac{8}{2}\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right)\right]_0^2 - \frac{1}{2}\left[x^3\right]_0^2$$

$$= 2\left[2 + 4\left(\frac{\pi}{4}\right) - 0\right] - \frac{1}{3}[8 - 0]$$

$$= 4 + 2\pi - \frac{8}{3}$$

$$= \left(2\pi + \frac{4}{3}\right) \text{ sq. units}$$

[CBSE Marking Scheme 2018]

Commonly Made Error

- In most of the cases, rough sketch of the curve is not drawn correctly.

Answering Tip

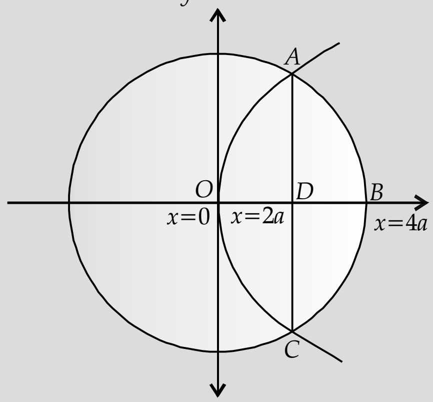
- Give sufficient practice in solving problems based on areas under a given curve and circle, etc.

Q. 22. Find the area of the region $\{(x, y) : y^2 \le 6ax$ and $x^2 + y^2 \le 16a^2\}$ using method of integration.

A [NCERT Exemplar] [O.D. Set II 2013]

Sol. Here the curves are :

$$x^2 + y^2 = 16a^2 \quad \text{...(i)}$$
$$y^2 = 6ax \quad \text{...(ii)}$$



1

From equations (i) and (ii),

$$x^2 + 6ax - 16a^2 = 0 \quad \textbf{1}$$

or $x^2 + 8ax - 2ax - 16a^2 = 0$

or $x(x + 8a) - 2a(x + 8a) = 0$

or $(x - 2a)(x + 8a) = 0$ **1**

or $x = -8a, 2a$

Area bounded by region *OAC* :

$$A_1 = 2\int_0^{2a} \sqrt{6ax}\, dx$$
$$= 2\sqrt{6a}\int_0^{2a} \sqrt{x}\, dx$$
$$= 2\sqrt{6a}\,\frac{2}{3}\left[x^{3/2}\right]_0^{2a} \quad \text{½}$$
$$= \frac{4\sqrt{6a}}{3}\,[(2a)^{3/2} - 0] = \frac{8}{3}\sqrt{12}a^2$$
$$= \frac{16}{3}\sqrt{3}a^2 \text{ sq. units} \quad \text{½}$$

Area bounded by region *ABC* :

$$A_2 = 2\int_{2a}^{4a} \sqrt{16a^2 - x^2}\, dx$$
$$= 2\int_{2a}^{4a} \sqrt{(4a)^2 - (x)^2}\, dx = 2\left[\frac{x\sqrt{16a^2 - x^2}}{2} + \frac{16a^2}{2}\sin^{-1}\frac{x}{4a}\right]_{2a}^{4a} \quad \text{½}$$
$$= 2\left[0 + \frac{16a^2}{2}\sin^{-1}1 - \frac{2a}{2}\sqrt{16a^2 - 4a^2} - \frac{16a^2}{2}\sin^{-1}\frac{1}{2}\right] \quad \text{½}$$
$$= 2\left[8a^2 \sin^{-1}1 - \sqrt{12}a^2 - 8a^2 \sin^{-1}\frac{1}{2}\right]$$
$$= 2\left[8 \times \frac{\pi}{2}a^2 - 2\sqrt{3}a^2 - \frac{4\pi}{3}a^2\right] \text{ sq. units} \quad \text{½}$$

So, area bounded by region *OABC* :

$$A = A_1 + A_2$$

$$\frac{16}{3}\sqrt{3}a^2 + 8\pi a^2 - 4\sqrt{3}a^2 - \frac{8\pi}{3}a^2 = 4\sqrt{3}a^2 + \frac{16}{3}\pi a^2 = \frac{4a^2}{3}(4\pi + 3\sqrt{3}) \text{ sq.unit.}$$

[CBSE Marking Scheme, 2013] ½

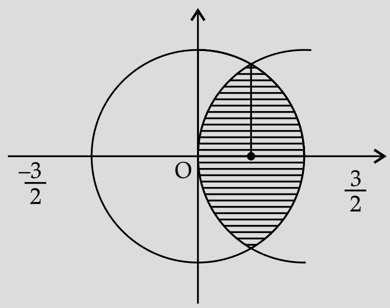
AI Q. 23. Find the area of the region $\{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\}$ using method of integration.

A [NCERT]

[O.D. Set III 2013][O.D. Set I, II, III Comptt. 2015]

[Delhi Comptt. 2017]

Sol. x-coordinate of point of intersection is,. $x = \frac{1}{2}$ **1**



1

Required area

$$= 2\left(\int_0^{\frac{1}{2}} 2\sqrt{x}dx + \int_{\frac{1}{2}}^{3/2} \sqrt{\frac{9}{4} - x^2}dx\right) \quad \mathbf{2}$$

$$= 2\left[\left\{\frac{4}{3}x^{3/2}\right\}_0^{\frac{1}{2}} + \left\{\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\frac{2x}{3}\right\}_{\frac{1}{2}}^{\frac{3}{2}}\right] \quad \mathbf{½+1}$$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right) \text{ or } \frac{\sqrt{2}}{6} + \frac{9}{4}\cos^{-1}\frac{1}{3} \quad \mathbf{½}$$

[CBSE Marking Scheme 2017]

Commonly Made Error

- Some students take 3 as a radius in this equation which is wrong.

Answering Tips

- First student should write this equation in standard form which is $x^2 + y^2 = \frac{9}{4}$. then radius will be $\frac{3}{2}$.

Alternative method :

Let $R = \{(x, y) : y^2 \leq 4x, 4x^2 + 4y^2 \leq 9\}$

$= [\{(x, y) : y^2 \leq 4x\} \cap \{(x, y) : 4x^2 + 4y^2 \leq 9]$

$\Rightarrow \quad R = R_1 \cap R_2,$

where $\quad R_1 = \{(x, y) : y^2 \leq 4x\}$

and $\quad R_2 = \{(x, y) : 4x^2 + 4y^2 \leq 9\}$ **1**

Region R_1 : Clearly $y^2 = 4x$ is the equation of the parabola with vertex at the origin and axis along x-axis. Since we are given that $y^2 \leq 4x$, so R_1 is the region lying inside the parabola $y^2 = 4x$.

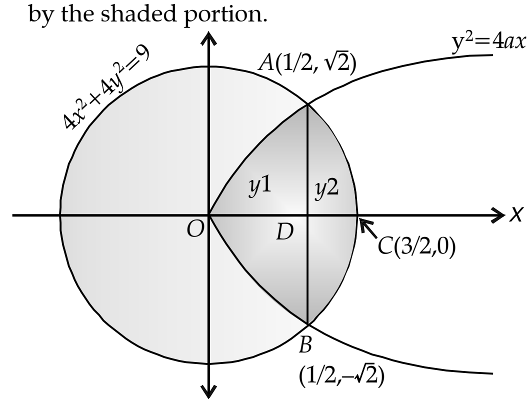
Region R_2 : We have

$$4x^2 + 4y^2 = 9 \text{ or } x^2 + y^2 = \left(\frac{3}{2}\right)^2 \quad \mathbf{1}$$

Clearly, it represents a circle with centre at the origin and radius $\frac{3}{2}$. It is given that $x^2 + y^2 \leq \frac{9}{4}$, so R_2 is the region lying inside the circle.

$$x^2 + y^2 = \left(\frac{3}{2}\right)^2$$

Thus the region R is the region bounded by the parabola $y^2 = 4x$ and the circle $x^2 + y^2 = (3/2)^2$ as shown by the shaded portion. **1**



Now $\quad y^2 = 4x \quad$ ...(i)

and $\quad 4x^2 + 4y^2 = 9 \quad$ ...(ii)

Put $y^2 = 4x$ from (i) into (ii), we get

$4x^2 + 16x = 9$

or $4x^2 + 16x - 9 = 0$

or $(2x + 9)(2x - 1) = 0$ **1**

or $\quad x = \frac{1}{2}$ or $x = -\frac{9}{2}$

From (i), $x = \frac{1}{2}$ or $y = \pm\sqrt{2}$ and $x = -\frac{9}{2}$ or y is imaginary, so the curves intersect at $(1/2, \sqrt{2})$ and $\left(\frac{1}{2}, -\sqrt{2}\right)$. So both the curves are symmetrical about x-axis.

So required area = 2 (Area of the shaded region lying about x-axis.)

Now, $\quad$ Area $(OADO) = \int_0^{1/2} y_1\, dx$

$$= \int_0^{1/2} 2\sqrt{x}\, dx \quad ½$$

and $\quad$ Area $(ADCA) = \int_{1/2}^{3/2} y_2\, dx$

$$= \int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2}\, dx \quad ½$$

Hence, required area :

$$A = 2\,[\text{Area } OADO + \text{Area } ADCA]$$

$$= 2\int_0^{1/2} 2\sqrt{x}dx + 2\int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2}dx$$

$$= 4 \times \frac{2}{3}\left[x^{3/2}\right]_0^{1/2}$$

$$+2\left[\frac{1}{2}x\sqrt{\frac{9}{4} - x^2} + \frac{1}{2}\cdot\frac{9}{4}\sin^{-1}\frac{2x}{3}\right]_{1/2}^{3/2} \quad ½$$

$$= \frac{8}{3}\left(\frac{1}{2\sqrt{2}} - 0\right)$$

$$+\left[\left\{\frac{9}{4}\sin^{-1}(1)\right\}-\left\{\frac{1}{2}\sqrt{2}+\frac{1}{2}\cdot\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

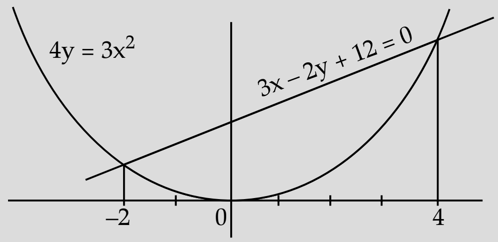
$$=\frac{2\sqrt{2}}{3}+\left[\frac{9}{8}\pi-\frac{1}{\sqrt{2}}-\frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$=\left[\frac{\sqrt{2}}{6}+\frac{9\pi}{8}-\frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right] \text{ sq. units}$$ ½

AI Q. 24. Find the area enclosed by the parabola $4y = 3x^2$ and the line $2y = 3x + 12$.

R&U **[O.D, 2017] [Foreign Set I, 2013] [NCERT] [O.D. Set I, II, III, Comptt. 2015]**

Sol.



1

$4y = 3x^2$ and $3x - 2y + 12 = 0$ or $4\left(\frac{3x+12}{2}\right) = 3x^2$

or $3x^2 - 6x - 24 = 0$ or $x^2 - 2x - 8 = 0$ or $(x - 4)(x + 2) = 0$

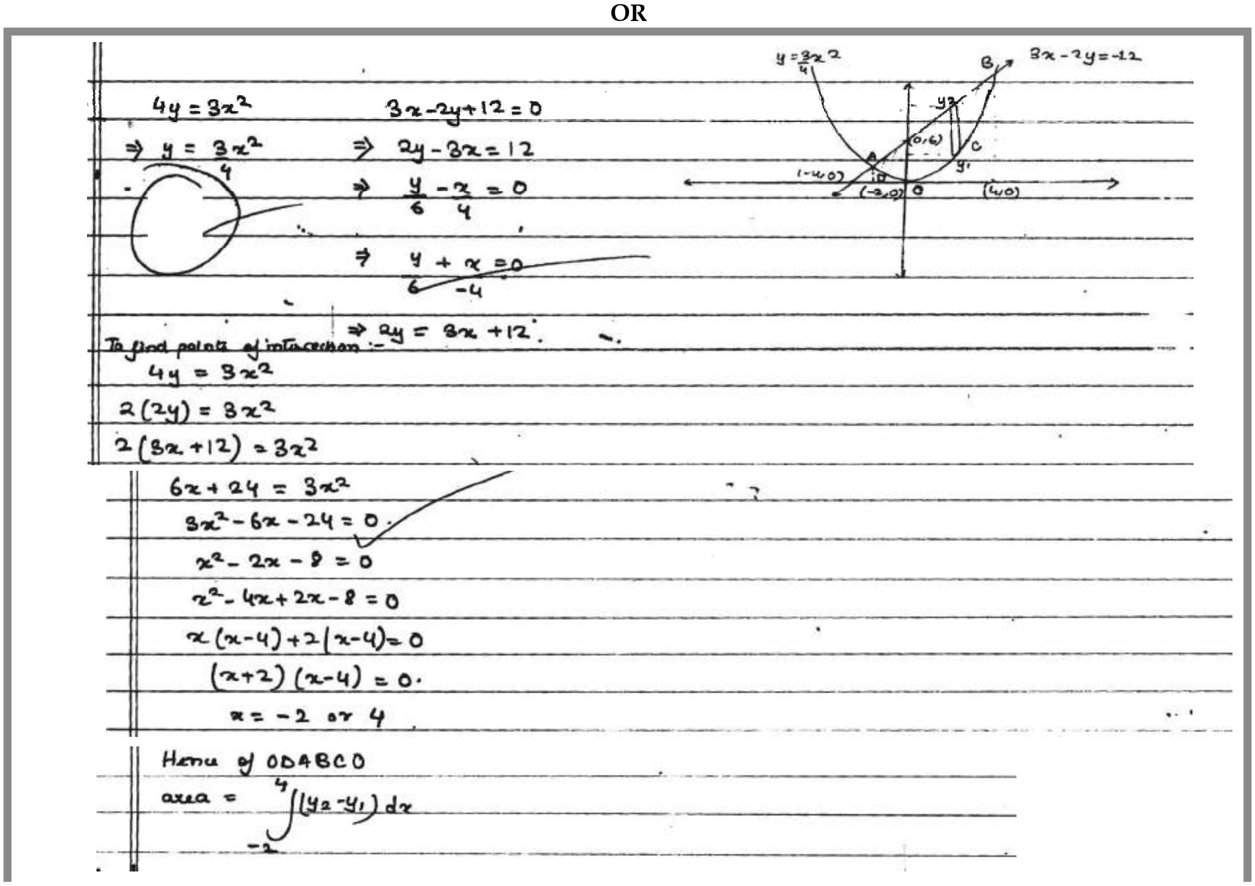
or x-coordinates of points of intersection are $x = -2$, $x = 4$ **1**

$\therefore$ Area $(A) = \int_{-2}^{4}\left[\frac{1}{2}(3x+12)-\frac{3}{4}x^2\right]dx$ **1½**

$$= \left[\frac{1}{2}\frac{(3x+12)^2}{6}-\frac{3}{4}\frac{x^3}{3}\right]_{-2}^{4}$$ **1½**

$= 45 - 18 = 27$ sq. units **[CBSE Marking Scheme 2017] 1**

OR



$4y = 3x^2$

$\Rightarrow y = \frac{3x^2}{4}$

$3x - 2y + 12 = 0$

$\Rightarrow 2y - 3x = 12$

$\Rightarrow \frac{y}{6} - \frac{x}{4} = 0$

$\Rightarrow \frac{y}{6} + \frac{x}{-4} = 0$

$\Rightarrow 2y = 3x + 12$

To find points of intersection :-

$4y = 3x^2$

$2(2y) = 3x^2$

$2(3x + 12) = 3x^2$

$6x + 24 = 3x^2$

$3x^2 - 6x - 24 = 0$

$x^2 - 2x - 8 = 0$

$x^2 - 4x + 2x - 8 = 0$

$x(x-4) + 2(x-4) = 0$

$(x+2)(x-4) = 0$

$x = -2$ or 4

Hence of ODABCO

area $= \int_{-2}^{4}(y_2 - y_1)\,dx$

$$= \int_{-2}^{4} \frac{3x+12}{2} - \frac{3x^2}{4}\, dx.$$

$$= \left[\frac{3}{2}\frac{x^2}{2} + 6x - \frac{3}{4}\frac{x^3}{3}\right]_{-2}^{4}$$

$$= \left[\frac{3x^2}{4} + 6x - \frac{x^3}{4}\right]_{-2}^{4}$$

$$= \left[\frac{3(16)}{4} + 6(4) - \frac{64}{4}\right] - \left[\frac{3\times 4}{4} - 12 - \frac{(-8)}{4}\right]$$

$$= [12 + 24 - 16] - [3 - 12 + 2]$$

$$= 20 + 7$$

$$= 27 \text{ sq. units.}$$

[Topper's Answer 2017]

Alternative Method :

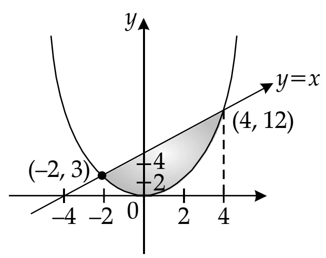
Given the equation of parabola

$$4y = 3x^2$$

or $$y = \frac{3x^2}{4} \quad ...(i)$$

and the line $3x - 2y + 12 = 0$ ...(ii)

or $$\frac{3x+12}{2} = y \quad 1$$



2

The line intersects the parabola at (– 2, 3) and (4, 12).

Hence the required area will be the shaded region.

$$\text{Required area} = \int_{-2}^{4} \frac{3x+12}{2}dx - \int_{-2}^{4}\frac{3x^2}{4}dx \quad 1$$

$$= \left[\frac{3}{4}x^2 + 6x - \frac{x^3}{4}\right]_{-2}^{4} \quad 1$$

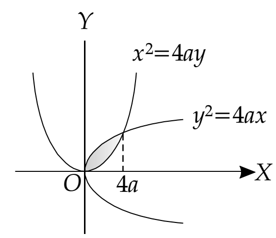
$$= (12 + 24 - 16) - (3 - 12 + 2)$$
$$= 20 + 7$$
$$= 27 \text{ sq. units.} \quad 1$$

Q. 25. Find the area of the region bounded by the two parabolas $y^2 = 4ax$ and $x^2 = 4ay$, when $a > 0$.

R&U [O.D 2009] [Foreign Set III 2013]

Sol. The curves $y^2 = 4ax$ and $x^2 = 4ay$ intersect at points, where

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$



or $$\frac{x^4}{16a^2} = 4ax$$

or $$x^4 = 64a^3x \quad 1$$

or $$x(x^3 - 64a^3) = 0$$

or $$x = 0 \text{ or } x = 4a \quad 2$$

We plot the curves on same system of axes to get the required region.

$$\therefore \text{ The enclosed area} = \int_0^{4a}\left(\sqrt{4ax} - \frac{x^2}{4a}\right)dx \quad 1$$

$$= \left[2\sqrt{a}\,\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{12a}\right]_0^{4a} \quad 1$$

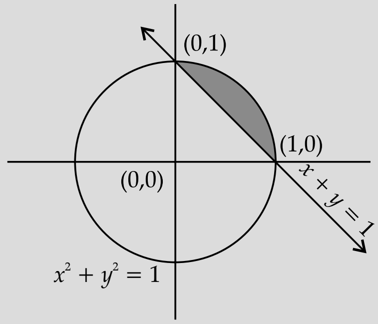
$$= \frac{4}{3}\sqrt{a}(4a)^{\frac{3}{2}} - \frac{(4a)^3}{12a} - 0$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3} \text{ sq. units.} \quad 1$$

AI Q. 26. Find the area of the region $\{(x, y) : x^2 + y^2 \le 1 \le (x + y)\}$. **A [O.D 2011, O.D Comptt. 2010] [S.Q.P. 2013] [Foreign 2017]**

Sol.



For correct figure
For correct shading } **1½**

$$A = \int_0^1 (\sqrt{1-x^2} - (1-x))dx$$ **1½**

$$= \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 - \left[x - \frac{x^2}{2}\right]_0^1$$ **2**

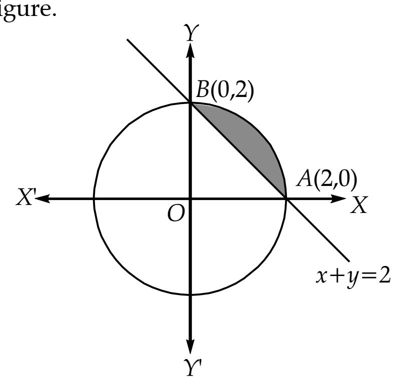
$$= \frac{1}{2}\sin^{-1}(1) - \frac{1}{2} = \left(\frac{\pi}{4} - \frac{1}{2}\right) \text{sq. units}$$ **1**

[CBSE Marking Scheme 2017]

Alternative Method :

Given region $\{(x, y) : (x^2 + y^2) \leq 1 \leq (x + y)\}$

or The given region is bounded inside the circle $x^2 + y^2 = 1$ and above the line $x + y = 1$ as shown in the figure. **1**



1

$\therefore$ Required area of the shaded portion

$$= \int_0^1 \sqrt{1-x^2}dx - \int_0^1 (1-x)\,dx$$ **1**

$$= \int_0^1 \sqrt{1-x^2}dx - \int_0^1 (1-x)\,dx$$ **1**

$$= \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 - \left[x - \frac{x^2}{2}\right]_0^1$$ **1**

$$= \left[0 + \frac{1}{2}\sin^{-1}(1) - 0 - 0\right] - \left[1 - \frac{1}{2} - 0 + 0\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2} \text{ sq. units.}$$ **1**

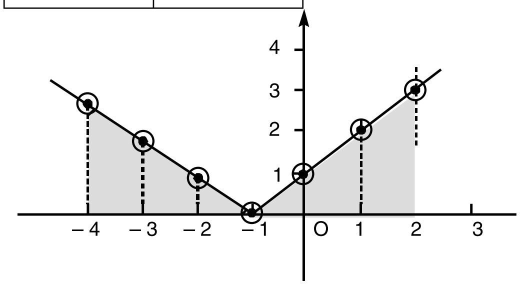
Q. 27. Draw the graph of $y = |x + 1|$ and using integration find the area below $y = |x + 1|$ above x-axis and between $x = -4$ to $x = 2$.

A [Delhi Set I Comptt. 2012]

Sol. $y = |x + 1|$ ½

y	x
1	0
0	– 1
2	+ 1
3	+ 2
1	– 2
2	– 3

½



½

Required area is given by integral of curve :

$$A = \int_{-4}^{2} |x+1|\,dx$$ **1**

$|x + 1| = 0$ at $x = -1$ ½

At $x > -1$, $|x + 1| = x + 1$

At $x < -1$, $|x + 1| = -(x + 1)$ ½

So, $A = \int_{-4}^{-1} -(x+1)\,dx + \int_{-1}^{2} (x+1)\,dx$ ½

$$= \left[-\left(\frac{x^2}{2} + x\right)\right]_{-4}^{-1} + \left[\left(\frac{x^2}{2} + x\right)\right]_{-1}^{2}$$ **1**

$$= -\left[\left(\frac{1}{2} - 1\right) - \left(\frac{16}{2} - 4\right)\right] + \left[\left(\frac{4}{2} + 2\right) - \left(\frac{1}{2} - 1\right)\right]$$ ½

$$= 4\frac{1}{2} + 4\frac{1}{2} = 9 \text{ sq. units.}$$ ½

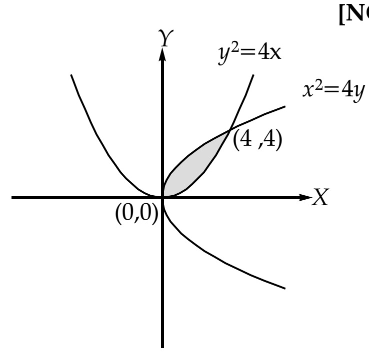
Q.28. Using integration, find the area of the triangle *ABC*, where *A* is (2, 3), *B* is (4, 7) and *C* is (6, 2).

R&U [Delhi Set II Comptt. 2012]

Sol. Try yourself like Q.10 long Answer Type Question-II.

Q. 29. Using integration, find the area of the region bounded by the two parabolas $y^2 = 4x$ and $x^2 = 4y$. **R&U [Delhi Set III Comptt. 2012]**
[NCERT Exemplar]

Sol.



For intersection points, substitute **1**

$$y^2 = 4x \text{ or } y = 2\sqrt{x}$$

$$x^2 = 4y$$

or $x^2 = 4 \times 2\sqrt{x}$ $(\because y = 2\sqrt{x})$

or $x^4 = 64x$ ½

or $x(x^3 - 64) = 0$

or $x = 0$

and $x = 4$ **1**

When $x = 0$, $y^2 = 4 \times 0$ or $y = 0$

When $x = 4$, $y^2 = 4 \times 4$ or $y = 4$

$\therefore$ Points of intersection are (0, 0) and (4, 4). **1**

Given, $y^2 = 4x$ or $y = 2\sqrt{x} = f(x)$

and $y = \frac{1}{4}x^2 = g(x)$

where $f(x) \geq g(x)$ in (0, 4),

$\therefore$ Required Area $= \int_0^4 [f(x) - g(x)]\,dx$

$$= \int_0^4 [2\sqrt{x} - \frac{1}{4}x^2)]\,dx \quad ½$$

$$= \left[\frac{4}{3}x^{3/2} - \frac{1}{12}x^3\right]_0^4 \quad \mathbf{1}$$

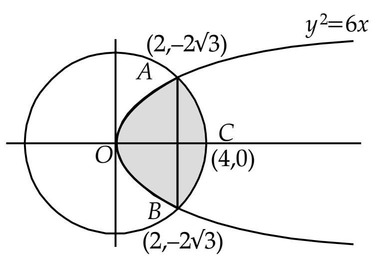
$$= \frac{4}{3} \times 4^{3/2} - \frac{1}{12} \times 4^3$$

$$= \frac{16}{3} \text{ sq. units} \quad \mathbf{1}$$

AI **Q. 30. Using integration, find the area of the circle $x^2 + y^2 = 16$ which is common to the parabola $y^2 = 6x$.** R&U **[NCERT] [O.D. Set III Comptt. 2012]**

[O.D 2016]

Sol.



1

$$x^2 + y^2 = 16$$

or $y^2 = 16 - x^2$ ...(i)

and $y^2 = 6x$ ...(ii) **1**

Solving eqns. (i) and (ii), we get points of intersection $(2, 2\sqrt{3})$ and $(2, -2\sqrt{3})$. ½

Substituting these values of x in eq. (ii).

Since both curves are symmetrical about x-axis.

Hence the required area

$$= 2\int_0^2 y_{(\text{parabola})}\,dx + 2\int_2^4 y_{(\text{circle})}\,dx \quad ½$$

$$= 2\int_0^2 \sqrt{6x}\,dx + 2\int_2^4 \sqrt{16 - x^2}\,dx$$

$$= \left[2 \times \sqrt{6} \times \frac{x^{3/2}}{3/2}\right]_0^2 + 2 \times \left[\frac{1}{2}x\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

½

$$= 2\sqrt{6} \times \frac{2}{3} \times \left[2^{3/2}\right]$$

$$+ 2\left[\frac{1}{2} \times 4 \times 0 - \frac{1}{2} \times 2 \times \sqrt{12} + 8\sin^{-1}1 - 8\sin^{-1}\frac{1}{2}\right]$$

$$= \frac{16}{\sqrt{3}} + 2\left[-2\sqrt{3} + 8 \times \frac{\pi}{2} - 8 \times \frac{\pi}{6}\right] \quad ½$$

$$= \frac{16}{\sqrt{3}} - 4\sqrt{3} + 8\pi - \frac{8\pi}{3}$$

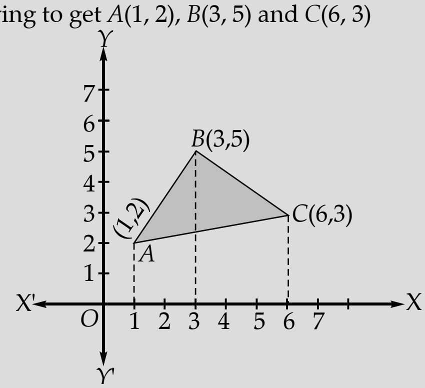
$$= \frac{16 - 12}{\sqrt{3}} + 8\pi \times \frac{2}{3} \quad \mathbf{1}$$

$$= \frac{4}{\sqrt{3}} + \frac{16\pi}{3} \text{ sq. units} \quad \mathbf{1}$$

Q. 31. Using the method of integration, find the area of the region bounded by the lines $3x - 2y + 1 = 0$, $2x + 3y - 21 = 0$ and $x - 5y + 9 = 0$. R&U **[Delhi Set I, 2012]**

Sol. Let AB be $3x - 2y + 1 = 0$, BC be $2x + 3y - 21 = 0$ and CA be $x - 5y + 9 = 0$

Solving to get $A(1, 2)$, $B(3, 5)$ and $C(6, 3)$ **1½**



½

Area of ABC

$$= \frac{1}{2}\int_1^3 (3x+1)dx + \frac{1}{3}\int_3^6 (21-2x)dx - \frac{1}{5}\int_1^6 (x+9)dx$$

1½

$$= \frac{1}{2}\left[\frac{3x^2}{2} + x\right]_1^3 + \frac{1}{3}\left[21x - x^2\right]_3^6 - \frac{1}{5}\left[\frac{x^2}{2} + 9x\right]_1^6 \quad \mathbf{1½}$$

$$= 14 + 20 - \frac{25}{2} = \frac{43}{2} \text{ sq. units.} \quad \mathbf{1}$$

[CBSE Marking Scheme 2012]

Commonly Made Error

- Sometimes students do not apply correct limits or consider area twice the result.

Answering Tip

- Learn to apply limits correctly to avoid errors.

Q. 32. Using the method of integration, find the area of the region bounded by the lines $3x - y - 3 = 0$, $2x + y - 12 = 0$ and $x - 2y - 1 = 0$

R&U **[Delhi Set II, 2012]**

Sol. Let given equation of lines are

$AB: \quad 3x - y - 3 = 0$ ...(i)

$BC: \quad 2x + y - 12 = 0$ ...(ii)

$CA: \quad x - 2y - 1 = 0$ ...(iii)

Solving eqns. (i) and (ii), we get

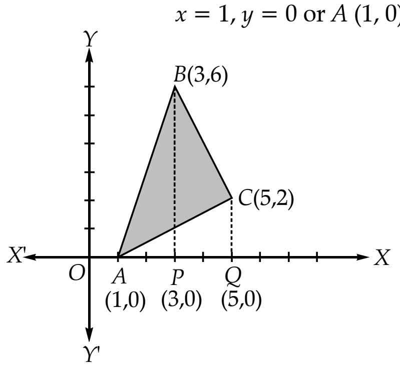
$x = 3, y = 6$ or B (3, 6) ½

Solving eqns. (ii) and (iii), we get

$x = 5, y = 2$ or C (5, 2) ½

Solving eqns. (i) and (iii), we get

$x = 1, y = 0$ or A (1, 0) ½



1

Required area of ΔABC = Area of ΔABP + Area of trapezium $BCQP$ – Area of ΔACQ **1**

$$= \int_{AB} y\,dx + \int_{BC} y\,dx - \int_{AC} y\,dx$$

$$= \int_1^3 (3x-3)dx + \int_3^5 (12-2x)dx - \int_1^5 \frac{1}{2}(x-1)dx$$

$$= 3\left[\frac{x^2}{2} - x\right]_1^3 + \left[12x - x^2\right]_3^5 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^5 \qquad 1$$

$$= 3\left[\left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right] + [(60 - 25) - (36 - 9)] - \frac{1}{2}\left[\left(\frac{25}{2} - 5\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$= 3[2] + [8] - \frac{1}{2}[8] \qquad ½$$

$= 6 + 8 - 4 = 10$ sq. units **1**

Q. 33. Using the method of integration, find the area of the region bounded by the lines :

$$\mathbf{5x - 2y - 10 = 0,}$$

$$\mathbf{x + y - 9 = 0}$$

and $\mathbf{2x - 5y - 4 = 0}$ **R&U [Delhi Set III, 2012]**

Sol. Let given equation of lines are :

$AB: \quad 5x - 2y - 10 = 0$ ...(i)

$BC: \quad x + y - 9 = 0$ ...(ii)

$CA: \quad 2x - 5y - 4 = 0$ ...(iii)

Solving eqns. (i) and (iii), we get

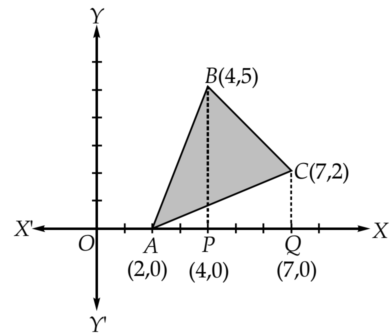
$x = 2, y = 0$ or A (2, 0) ½

Solving eqns. (i) and (ii), we get

$x = 4, y = 5$ or B (4, 5) ½

Solving eqns. (ii) and (iii), we get

$x = 7, y = 2$ or B (7, 2) ½



1

Required area of ΔABC

= Area of ΔABP + Area of trapezium $BCQP$ – Area of ΔACQ ½

$$= \int_{AB} y\,dx + \int_{BC} y\,dx - \int_{AC} y\,dx$$

$$= \int_2^4 \frac{1}{2}(5x-10)\,dx + \int_4^7 (9-x)\,dx - \int_2^7 \frac{1}{5}(2x-4)\,dx \qquad 1$$

$$= \frac{5}{2}\left[\frac{x^2}{2} - 2x\right]_2^4 + \left[9x - \frac{x^2}{2}\right]_4^7 - \frac{1}{5}\left[x^2 - 4x\right]_2^7 \qquad 1$$

$$= \frac{5}{2}[(8-8) - (2-4)] + \left[\left(63 - \frac{49}{2}\right) - (36 - 8)\right] - \frac{1}{5}[(49 - 28) - (4 - 8)]$$

$$= \frac{5}{2}[2] + \frac{21}{2} - \frac{1}{5}[25]$$

$= 5 + \frac{21}{2} - 5 = 10.5$ sq. units. **1**

Q. 34. Find the area of the region in the first quadrant enclosed by x-axis, the line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$. **R&U [Foreign Set I, II, III, 2012] [NCERT]**

Sol. Obviously $x^2 + y^2 = 4$ is a circle having centre at (0, 0) and radius 2 units.

For graph of line $x = \sqrt{3}y$

x	0	1
y	0	1·73

For intersecting point of given circle and line putting $x = \sqrt{3}y$ in $x^2 + y^2 = 4$, we get

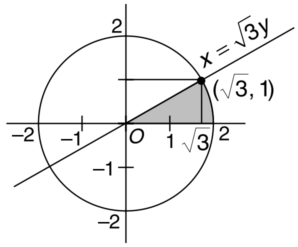
$$\left(\sqrt{3}y\right)^2 + y^2 = 4$$

or $\quad 3y^2 + y^2 = 4$

or $\quad 4y^2 = 4$

or $\quad y = \pm 1$ **1**

$\therefore \quad x = \pm\sqrt{3}$



1

Intersecting points are $(\sqrt{3}, 1), (-\sqrt{3}, 1)$.

Shaded region is required region.

Now required area = $\int_0^{\sqrt{3}} \frac{x}{\sqrt{3}} dx + \int_{\sqrt{3}}^{2} \sqrt{4-x^2} dx$ **1**

$= \frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{\sqrt{3}} + \left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{\sqrt{3}}^{2}$ ½

$= \frac{1}{2\sqrt{3}}(3-0) + \left[2\sin^{-1}1 - \left(\frac{\sqrt{3}}{2} + 2\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$ **1**

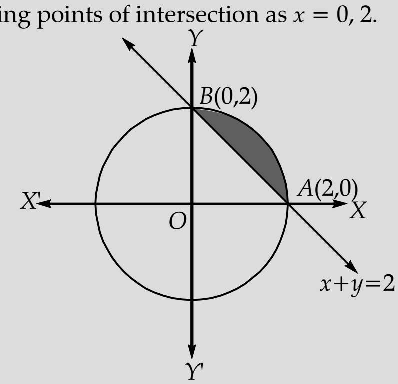
$= \frac{\sqrt{3}}{2} + \left[2\frac{\pi}{2} - \frac{\sqrt{3}}{2} - \frac{2\pi}{3}\right]$ ½

$= \frac{\sqrt{3}}{2} + \pi - \frac{\sqrt{3}}{2} - \frac{2\pi}{3}$

$= \pi - \frac{2\pi}{3} = \frac{\pi}{3}$ sq. units **1**

Q. 35. Find the area of the region $\{(x, y) : x^2 + y^2 \le 4, x + y \ge 2\}$. R&U **[O.D. Set I, II, III, 2012]**

Sol. Finding points of intersection as $x = 0, 2$. **1**



1

$A = \int_0^2 \sqrt{4-x^2}dx - \int_0^2 (2-x)dx$ **1**

$= \left[\frac{x}{2}\times\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2$ **1**

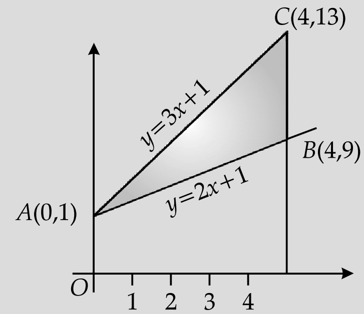
$= \left(0 + 2\cdot\frac{\pi}{2}\right) - (4-2)$ **1**

$= (\pi - 2)$ sq. units **1**

[CBSE Marking Scheme 2012]

Q. 36. Using integration, find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$. R&U **[Delhi Set I, II, III, 2012] [O.D Comptt. 2011] [Delhi 2011]**

Sol.



1

Getting the points of intersection as $A(0, 1)$, $B(4, 9)$ and $C(4, 13)$ **1½**

Area of $\Delta ABC = \int_0^4 (3x+1)dx - \int_0^4 (2x+1)dx$ **2**

$= \int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4$ **1**

= 8 sq. units. ½

[CBSE Marking Scheme 2012]

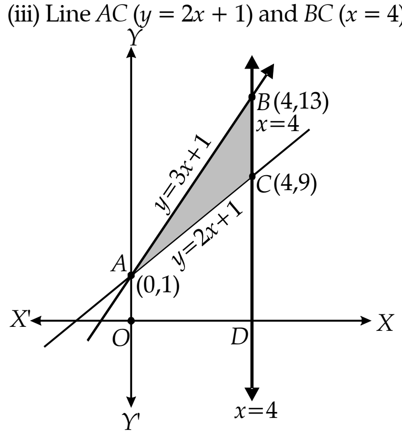
Alternative Method :

Points of intersections of :

(i) Line AB ($y = 3x + 1$) and AC ($y = 2x + 1$) is $A(0, 1)$. ½

(ii) Line AB ($y = 3x + 1$) and BC ($x = 4$) is B (4, 13). ½

(iii) Line AC ($y = 2x + 1$) and BC ($x = 4$) is C (4, 9). ½



1

Required area of shaded portion

= Area of trapezium $AODB$

– Area of trapezium $AODC$ ½

$= \int_{AB} y\ dx - \int_{AC} y\,dx$ **1**

$= \int_0^4 (3x+1)\,dx - \int_0^4 (2x+1)\,dx$ **1**

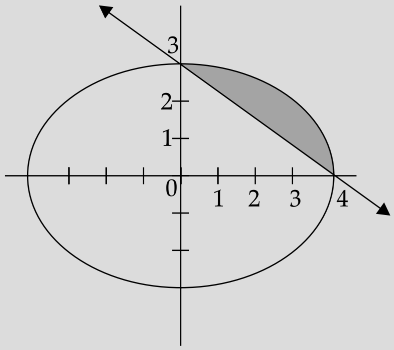
$= \left[\frac{3x^2}{2} + x\right]_0^4 - \left[x^2 + x\right]_0^4$

= (24 + 4) – (16 + 4)

= 8 sq. units. **1**

Q. 37. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and the straight line $3x + 4y = 12$.

R&U **[Outside Delhi Set I, II, III, Comptt. 2016]**

Sol.  **1**

Getting the points of intersection as (4, 0), (0, 3). **1**

$\therefore$ Required area

$= \int_0^4 \frac{3}{4}\sqrt{16-x^2}\,dx - \frac{1}{4}\int_0^4 (12-3x)dx$ **1½**

$= \left[\frac{3}{4}\left[\frac{x}{2}\sqrt{16-x^2} + 8\sin^{-1}\frac{x}{4}\right] - \frac{1}{4}\left(12x - \frac{3x^2}{2}\right)\right]_0^4$ **1½**

$= \left(\frac{3}{4}\cdot 8\cdot\frac{\pi}{2} - 6\right)$

$= (3\pi - 6)$ sq. units **1**

[CBSE Marking Scheme 2016]

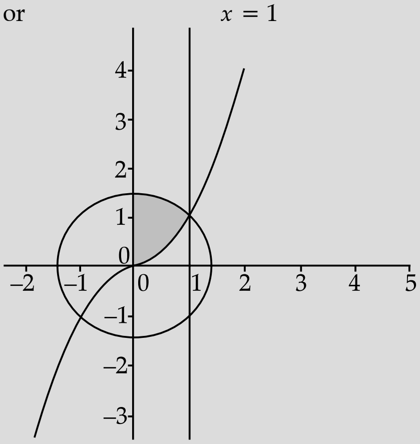
Q. 38. Using the method of integration, find the area of the triangular region whose vertices are (2, – 2), (4, 3) and (1, 2). R&U **[Outside Delhi 2016]**

Sol. Try yourself similar to Q.10 long Answer Type Question-II.

Q. 39. Using integration, find the area in the first quadrant bounded by the curve $y = x|x|$, the circle $x^2 + y^2 = 2$ and the y-axis. A **[SQP 2017-18]**

Sol. $y = x\,|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2 & x < 0 \end{cases}$ **1**

Solving $y = x^2$, $x^2 + y^2 = 2$ simultaneously,
$y + y^2 - 2 = 0$ or $(y + 2)(y - 1) = 0$ or $y = 1$
($y = x^2$ lies in quadrant I). **½**
or $x = 1$

 **1**

The required area = the shaded area

$= \int_0^1 (\sqrt{2-x^2} - x^2)dx$ **2**

$= \frac{1}{2}\left[x\sqrt{2-x^2} + 2\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1 - \frac{1}{3}[x^3]_0^1$

$= \left(\frac{1}{6} + \frac{\pi}{4}\right)$ sq units. **1+½**

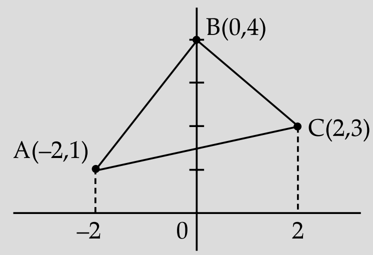
[CBSE Marking Scheme 2017-18]

Q. 40. Using the method of integration, find the area of the triangle *ABC*, coordinates of whose vertices are *A*(4, 1), B(6, 6) and *C*(8, 4). A **[OD 2017]**

Sol. Try yourself similar to Q.10 long Answer Type Question-II.

Q. 41. Using integration, find the area of region bounded by the triangle whose vertices are (– 2, 1), (0, 4) and (2, 3). A **[Delhi 2017]**

Sol.



1

Equation of $AB : y = \frac{3}{2}x + 4$

Equation of $BC : y = 4 - \frac{x}{2}$

Equation of $AC : y = \frac{1}{2}x + 2$ **1½**

Required area $= \int_{-2}^0 \left(\frac{3}{2}x + 4\right)dx + \int_0^2 \left(4 - \frac{x}{2}\right)dx$

$- \int_{-2}^2 \left(\frac{1}{2}x + 2\right)dx$ **1**

$= \left[\frac{3x^2}{4} + 4x\right]_{-2}^0 + \left[4x - \frac{x^2}{4}\right]_0^2 - \left[\frac{x^2}{4} + 2x\right]_{-2}^2$ **1½**

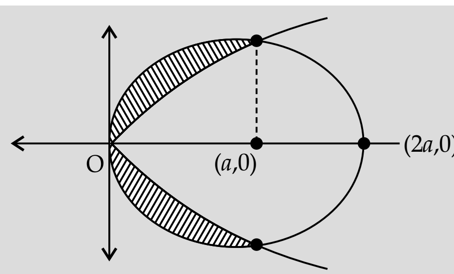
$= 5 + 7 - 8$ **1**

$= 4$ sq. units

[CBSE Marking Scheme 2017]

Q. 42. Using Integration, find the area of the following region :
$\{(x, y) : y^2 \geq ax, x^2 + y^2 \leq 2ax, a > 0\}$
A **[Delhi Comptt. 2017]**

Sol.



1

x-coordinate of point of intersection is, $x = a$ **1**

Required area $= 2\left[\int_0^a (\sqrt{a^2 - (x-a)^2} - \sqrt{a}\sqrt{x})dx\right]$ **2**

$$= 2\left[\frac{x-a}{2}\sqrt{a^2-(x-a)^2}+\frac{a^2}{2}\sin^{-1}\frac{x-a}{a}-2\sqrt{a}\,\frac{x^{\frac{3}{2}}}{3}\right]_0^a$$ **1**

$$= 2[0 + 0 - \frac{2}{3}\sqrt{a}\,a^{3/2} - \frac{a^2}{2}\ \sin^{-1}(-1)]$$

$$= \left(\frac{\pi}{2}-\frac{4}{3}\right)a^2$$

[CBSE Marking Scheme 2017] 1

Q. 43. Using method of integration find the area of the triangle *ABC*, co-ordinates of whose vertices are *A*(1, – 2), *B*(3, 5) and *C*(5, 2). A **[OD Comptt. 2017]**

Sol. Try yourself similar to Q.10 long Answer Type Question-II.

Q. 44. Using integration, find the area of the region bounded by the curves $y = \sqrt{4-x^2}$, $x^2 + y^2 - 4x = 0$ and the *X*-axis. A **[Foreign, 2016]**

Sol. Give, equation of curves are

$$y = \sqrt{4-x^2} \quad \text{...(i)}$$

and $$x^2 + y^2 - 4x = 0 \quad \text{...(ii)}$$

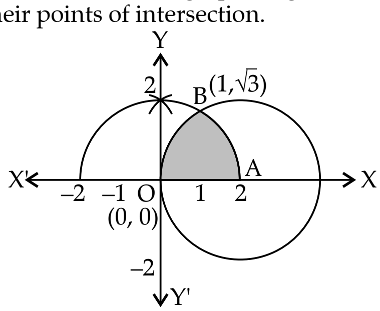
Consider the curve

$$y = \sqrt{4-x^2} \Rightarrow y^2 = 4 - x^2$$

or $x^2 + y^2 = 4$, which represents a circle with centre (0, 0) and radius 2 units.

Now, consider the curve $x^2 + y^2 - 4x = 0$, which also represents a circle with centre (2, 0) and radius 2 units. **1**

Now, let us sketch the graph of given curves and find their points of intersection.



1

On substituting the value of y from Eq. (i) in Eq. (ii), we get

$x^2 + (4 - x^2) - 4x = 0$ or $4 - 4x = 0$ or $x = 1$

On substituting $x = 1$ in Eq. (i), we get $y = \sqrt{3}$

Thus, the point of intersection is $(1, \sqrt{3})$.

Clearly, required area

= Area of shaded region *OABO*

$$= \int_0^1 y_{(\text{second circle})}\,dx + \int_1^2 y_{(\text{first circle})}\,dx$$ **1**

$$= \int_0^1 \sqrt{4x-x^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx$$ **½**

$$= \int_0^1 \sqrt{-(x^2-4x)}\,dx + \int_1^2 \sqrt{2^2-x^2}\,dx$$

$$= \int_0^1 \sqrt{-[x^2-2(2)(x)+4-4]}\,dx$$

$$+\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_1^2$$ **½**

$$= \int_0^1 \sqrt{4-(x-2)^2}\,dx$$

$$+\left[2\sin^{-1}(1)-\left\{\frac{1}{2}\sqrt{3}+2\sin^{-1}\left(\frac{1}{2}\right)\right\}\right]$$

$$= \left[\frac{(x-2)}{2}\sqrt{4x-x^2}+2\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$$

$$+\left[2\cdot\frac{\pi}{2}-\frac{\sqrt{3}}{2}-2\cdot\frac{\pi}{6}\right]$$

$$= \left[\left\{\frac{-\sqrt{3}}{2}+2\sin^{-1}\left(\frac{-1}{2}\right)\right\}-\{2\sin^{-1}(-1)\}\right]$$

$$+\left(\pi-\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$$ **1**

$$= -\frac{\sqrt{3}}{2}-2\sin^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}(1)$$

$$+\frac{2\pi}{3}-\frac{\sqrt{3}}{2} \quad [\because \sin^{-1}(-x) = -\sin^{-1}x]$$

$$= 2\cdot\frac{\pi}{2}-2\cdot\frac{\pi}{6}+\frac{2\pi}{3}-\sqrt{3}$$

$$= \pi-\frac{\pi}{3}+\frac{2\pi}{3}-\sqrt{3}$$

$$= \pi+\frac{\pi}{3}-\sqrt{3} = \left(\frac{4\pi}{3}-\sqrt{3}\right)\text{sq units}$$ **1**

Q. 45. Using integration, find the area of the region bounded by the line $x - y + 2 = 0$, the curve $x = \sqrt{y}$ and *Y*-axis. A **[Foreign 2015]**

Sol. Given curves are

$$x - y + 2 = 0 \quad \text{...(i)}$$

and $$x = \sqrt{y} \quad \text{...(ii)}$$

Consider $x = \sqrt{y}$ or $x^2 = y$, which represents the parabola whose vertex is (0, 0) and axis is *Y*-axis.

Now, the point of intersection of Eqs. (i) and (ii) is given by $x = \sqrt{x+2}$ **1**

or $x^2 = x + 2$

or $x^2 - x - 2 = 0$

or $(x - 2)(x + 1) = 0$

or $x = -1, 2$

But $x = -1$ does not satisfy the Eq. (ii).

$\therefore$ $x = 2$

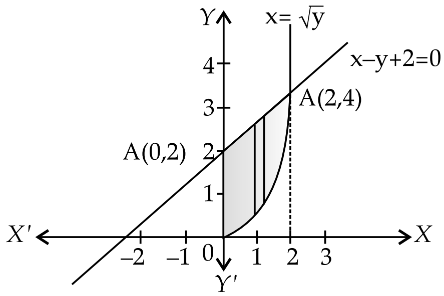
Now, putting $x = 2$ in Eq. (ii), we get

$2 = \sqrt{y}$ or $y = 4$ **1**

Hence, the point of intersection is (2, 4).

But we have actual equation of parabola is $x = \sqrt{y}$, it means a semi-parabola which is one right side of *Y*-axis.

The graph of given curve area shown below :



2

Clearly, area of bounded region

= Area of region $OABO$

$= \int_0^2 [y_{(\text{line})} - y_{(\text{parabola})}]dx$ **1**

$= \int_0^2 (x+2)dx - \int_0^2 x^2 dx$

$= \left[\frac{x^2}{2} + 2x\right]_0^2 - \left[\frac{x^3}{3}\right]_0^2$

$= \left[\frac{4}{2} + 4 - 0\right] - \left[\frac{8}{3} - 0\right]$

$= 6 - \frac{8}{3} = \frac{18-8}{3} = \frac{10}{3}$ sq units **1**

Q. 46. Using integration, find the area of the region bounded by the triangle whose vertices are (1, 3), (2, 5) and (3, 4). A **[Delhi 2009C]**

Sol. Try yourself similar to Q 10 Long Answer Type-I

Q. 47. Using integration, find the area of ΔABC, the coordinates of whose vertices are $A(2, 5)$, $B(4, 7)$ and $C(6, 2)$.

[Delhi 2011; All India 2010C]

Sol. Try yourself similar to Q 10 Long Answer Type-I

Q. 48. Sketch the graph of $y = |x + 3|$ and evaluate the area under the curve $y = |x + 3|$ above X-axis and between $x = -6$ to $x = 0$. A **[All India 2011]**

Sol. Try yourself similar to Q 27 Long Answer Type-I

Q. 49. Find the area of the region bounded by the region enclosed by the curves $(x - 6)^2 + y^2 = 36$ and $x^2 + y^2 = 36$. **[All India 2009C]**

Sol. Try yourself similar to Q 18 Long Answer Type-I

Q. 50. Find the area of the region enclosed by the parabola $y^2 = x$ and the line $x + y = 2$.

A **[Delhi 2009]**

Sol. Try yourself similar to Q 24 Long Answer Type-I