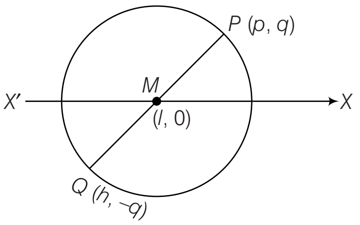
JEE Type Solved Examples :
Single Option Correct Type Questions

- This section contains **10 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Ex. 1 *Two distinct chords drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$, where $pq \neq 0$, are bisected by the X-axis. Then,*

(a) $|p| = |q|$ (b) $p^2 = 8q^2$ (c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$

Sol. (d)



Suppose chord bisect at $M(\lambda, 0)$, then other end point of chord is $(h, -q)$

where, $\lambda = \dfrac{p+h}{2}$

which lie on $x^2 + y^2 = px + qy$

or $h^2 + q^2 = ph - q^2$

$\Rightarrow$ $h^2 - ph + 2q^2 = 0$

for two distinct chords, $B^2 - 4AC > 0$

or $p^2 - 4 \cdot 1 \cdot 2q^2 > 0$

or $p^2 > 8q^2$

Ex. 2 *The values of λ for which the circle $x^2 + y^2 + 6x + 5 + \lambda(x^2 + y^2 - 8x + 7) = 0$ dwindles into a point are*

(a) $1 \pm \dfrac{\sqrt{2}}{3}$ (b) $2 \pm \dfrac{2\sqrt{2}}{3}$ (c) $2 \pm \dfrac{4\sqrt{2}}{3}$ (d) $1 \pm \dfrac{4\sqrt{2}}{3}$

Sol. (c) The given circle is

$$x^2 + y^2 + 6x + 5 + \lambda(x^2 + y^2 - 8x + 7) = 0$$

or $x^2(1+\lambda) + y^2(1-\lambda) + (6 - 8\lambda)x + (5 + 7\lambda) = 0$

$\Rightarrow$ $x^2 + y^2 + \left(\dfrac{6-8\lambda}{1+\lambda}\right)x + \left(\dfrac{5+7\lambda}{1+\lambda}\right) = 0$

This will dwindle into a point circle, then radius of the circle = 0

$$\sqrt{\left(\frac{3-4\lambda}{1+\lambda}\right)^2 + 0 - \left(\frac{5+7\lambda}{1+\lambda}\right)} = 0$$

$\Rightarrow$ $(3-4\lambda)^2 - (5+7\lambda)(1+\lambda) = 0$

$\Rightarrow$ $9 - 16\lambda^2 - 24\lambda - 5 - 5\lambda - 7\lambda - 7\lambda^2 = 0$

$\Rightarrow$ $9\lambda^2 - 36\lambda + 4 = 0$

$$\lambda = \frac{36 \pm \sqrt{(36)^2 - 4.9.4}}{2.9}$$

$\therefore$ $\lambda = 2 \pm \dfrac{4\sqrt{2}}{3}$

Ex. 3 *If $f(x+y) = f(x) \cdot f(y)$ for all x and y, $f(1) = 2$ and $\alpha_n = f(n), n \in N$, then the equation of the circle having (α_1, α_2) and (α_3, α_4) as the ends of its one diameter is*

(a) $(x-2)(x-8) + (y-4)(y-16) = 0$

(b) $(x-4)(x-8) + (y-2)(y-16) = 0$

(c) $(x-2)(x-16) + (y-4)(y-8) = 0$

(d) $(x-6)(x-8) + (y-5)(y-6) = 0$

Sol. (a) $\because$ $f(x+y) = f(x).f(y)$ ...(i)

$\because$ $f(1) = 2$

In Eq. (i), Put $x = y = 1$,

then $f(2) = f(1).f(1) = 2^2$

Now, in Eq. (i), $x = 1, y = 2$, then

$f(3) = f(1)f(2) = 2.2^2 = 2^3$

Hence, $f(n) = 2^n$

$\therefore$ $\alpha_n = f(n) = 2^n \forall n \in N$

$(\alpha_1, \alpha_2) \equiv (2, 4)$

and $(\alpha_3, \alpha_4) \equiv (8, 16)$

Equation of circle in diametric form is

$(x-2)(x-8) + (y-4)(y-16) = 0$

Ex. 4 *Two circles of radii a and b touching each other externally, are inscribed in the area bounded by $y = \sqrt{(1-x^2)}$ and the X-axis. If $b = \dfrac{1}{2}$, then a is equal to*

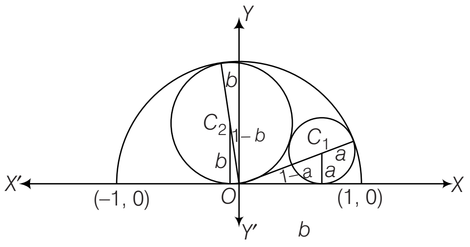
(a) $\dfrac{1}{4}$ (b) $\dfrac{1}{8}$

(c) $\dfrac{1}{2}$ (d) $\dfrac{1}{\sqrt{2}}$

Sol. (a) Let the centres of circles be C_1 and C_2, then

$C_1 \equiv (\sqrt{(1-2a)}, a)$

and $C_2 \equiv (\sqrt{(1-2b)}, b)$



Now, $C_1C_2 = a + b$

$$\Rightarrow \quad \left((\sqrt{(1-2a)})^2 + \left(a - \frac{1}{2}\right)^2 \right) = \left(a + \frac{1}{2}\right)^2 \qquad \left[\because b = \frac{1}{2}\right]$$

or $$1 - 2a + \left(a - \frac{1}{2}\right)^2 = \left(a + \frac{1}{2}\right)^2$$

or $$1 - 2a + a^2 + \frac{1}{4} - a = a^2 + \frac{1}{4} + a$$

or $$a = \frac{1}{4}$$

Ex. 5 *There are two circles whose equations are $x^2 + y^2 = 9$ and $x^2 + y^2 - 8x - 6y + n^2 = 0, n \in I$. If the two circles having exactly two common tangents, then the number of possible values of n is*

(a) 2 (b) 7 (c) 8 (d) 9

Sol. (d) Given circles are $S_1 : x^2 + y^2 - 9 = 0$

Its centre $C_1 : (0,0)$ and radius $r_1 = 3$

and $S_2 : x^2 + y^2 - 8x - 6y + n^2 = 0$

Its centre $C_2 : (4,3)$ and radius $r_2 = \sqrt{(25 - n^2)}$

Here, $25 - n^2 > 0 \Rightarrow -5 < n < 5$ ...(i)

For exactly two common tangents,

$$r_1 + r_2 > C_1C_2$$

$$\Rightarrow \quad 3 + \sqrt{(25 - n^2)} > \sqrt{(4^2 + 3^2)}$$

$$\Rightarrow \quad \sqrt{(25 - n^2)} > 2$$

$$\Rightarrow \quad 25 - n^2 > 4$$

or $$n^2 < 21$$

or $$-\sqrt{21} < n < \sqrt{21} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$-\sqrt{21} < n < \sqrt{21}$$

But $n \in I$. So, $n = -4, -3, -2, -1, 0, 1, 2, 3, 4$

Hence, number of possible values of n is 9.

Ex. 6 *Suppose $f(x, y) = 0$ is the equation of a circle such that $f(x, 1) = 0$ has equal roots (each equal to 2) and $f(1, x) = 0$ also has equal roots (each equal to zero). The equation of circle is*

(a) $x^2 + y^2 + 4x + 3 = 0$ (b) $x^2 + y^2 + 4y + 3 = 0$

(c) $x^2 + y^2 + 4x - 3 = 0$ (d) $x^2 + y^2 - 4x + 3 = 0$

Sol. (d) Let $f(x, y) = x^2 + y^2 + 2gx + 2fy + c$

$\Rightarrow \quad f(x, 1) = x^2 + 1 + 2gx + 2f + c \equiv (x - 2)^2$ (given)

then, $g = -2,\ 2f + c = 3$ ...(i)

Also, $f(1, x) = 1 + x^2 + 2g + 2fx + c \equiv (x - 0)^2$ (given)

then, $f = 0,\ 2g + c = -1$ ...(ii)

From Eqs. (i) and (ii), we get

$$g = -2, f = 0, c = 3$$

Thus, equation of circle is

$$x^2 + y^2 - 4x + 3 = 0$$

Ex. 7 *A variable circle C has the equation $x^2 + y^2 - 2(t^2 - 3t + 1)x - 2(t^2 + 2t)y + t = 0$, where t is a parameter. If the power of point (a, b) w.r.t. the circle C is constant, then the ordered pair (a, b) is*

(a) $\left(\frac{1}{10}, -\frac{1}{10}\right)$ (b) $\left(\frac{1}{10}, \frac{1}{10}\right)$

(c) $\left(-\frac{1}{10}, \frac{1}{10}\right)$ (d) $\left(-\frac{1}{10}, -\frac{1}{10}\right)$

Sol. (c) $\because C : x^2 + y^2 - 2(t^2 - 3t + 1)x - 2(t^2 + 2t)y + t = 0$

given power of circle = constant

$\therefore \quad a^2 + b^2 - 2(t^2 - 3t + 1)a - 2(t^2 + 2t)b + t = \text{constant}$

$\Rightarrow \quad -2(a + b)t^2 + (6a - 4b + 1)t + (a^2 + b^2 - 2a) = \text{constant}$

$\because$ Power of circle is constant, then

$a + b = 0$ and $6a - 4b + 1 = 0$

or $b = -a$, then $6a + 4a + 1 = 0$

$\therefore \quad a = -\frac{1}{10}, b = \frac{1}{10}$

Hence, required ordered pair is $\left(-\frac{1}{10}, \frac{1}{10}\right)$

Ex. 8 *If the radii of the circles $(x - 1)^2 + (y - 2)^2 = 1$ and $(x - 7)^2 + (y - 10)^2 = 4$ are increasing uniformly w.r.t. time as 0.3 unit/s and 0.4 unit/s respectively, then they will touch each other at t equals to*

(a) 45 s (b) 90 s

(c) 11 s (d) 135 s

Sol. (b) Given circles are $S_1 : (x - 1)^2 + (y - 2)^2 = 1$

Its centre $C_1 : (1,2)$ and radius $r_1 = 1$

and $S_2 : (x - 7)^2 + (y - 10)^2 = 4$

Its centre $C_2 : (7,10)$ and radius $r_2 = 2$

$\because \quad C_1C_2 = 10 > r_1 + r_2$

Hence, the two circles are separated.

The radii of the two circles at time t are $(1 + 0.3t)$ and $(2 + 0.4t)$

For the two circles touch each other, then

$$C_1C_2 = |(1+0.3t) \pm (2+0.4t)|$$

$\Rightarrow$ $10 = |3+0.7t|$ or $10 = |-1-0.1t|$

$\Rightarrow$ $0.7t + 3 = \pm 10$ or $-1-0.1t = \pm 10$

$\Rightarrow$ $t = 10$ or $t = 90$ $[\because t > 0]$

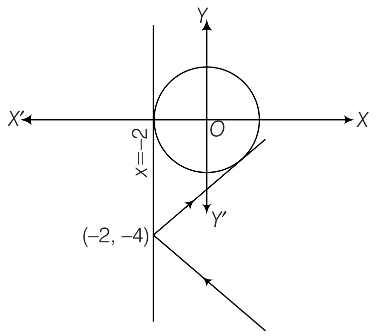
● **Ex. 9** *A light ray gets reflected from $x = -2$. If the reflected ray touches the circle $x^2 + y^2 = 4$ and the point of incident is $(-2, -4)$, then the equation of the incident ray is*

(a) $4y + 3x + 22 = 0$ (b) $3y + 4x + 20 = 0$

(c) $4y + 2x + 20 = 0$ (d) $y + x + 6 = 0$

Sol. (a) Any tangent of $x^2 + y^2 = 4$ is $y = mx \pm 2\sqrt{(1+m^2)}$.

If it passes through $(-2, -4)$, then $-4 = -2m \pm 2\sqrt{(1+m^2)}$



or $(m-2)^2 = 1 + m^2$

or $m = \infty$, $m = 3/4$

Hence, the slope of the reflected ray is 3/4.

Thus, the equation of the incident ray is

$$y + 4 = -\frac{3}{4}(x+2)$$

i.e. $4y + 3x + 22 = 0$

● **Ex. 10** *If a circle having centre at (α, β) radius r completely lies with in two lines $x + y = 2$ and $x + y = -2$, then, $\min.(|\alpha + \beta + 2|, |\alpha + \beta - 2|)$ is*

(a) greater than $\sqrt{2}r$

(b) less than $\sqrt{2}r$

(c) greater than $2r$

(d) less than $2r$

Sol. (a) Minimum distance of the centre from line > radius of circle i.e. $\min.\left\{\frac{|\alpha+\beta+2|}{\sqrt{2}}, \frac{|\alpha+\beta-2|}{\sqrt{2}}\right\} > r$

or $\min.\{|\alpha+\beta+2|, |\alpha+\beta-2|\} > \sqrt{2}r$

JEE Type Solved Examples :
More than One Correct Option Type Questions

- This section contains **5 multiple choice examples.** Each example has four choices (a), (b), (c) and (d) out of which MORE THAN ONE may be correct.

● **Ex. 11** *If point $P(x, y)$ is called a lattice point, if $x, y \in I$. Then, the total number of lattice points in the interior of the circle $x^2 + y^2 = a^2, a \neq 0$ cannot be*

(a) 202 (b) 203 (c) 204 (d) 205

Sol. (a, b, c) Given circle is $x^2 + y^2 = a^2$ ...(i)

Clearly (0, 0) will belong the interior of circle Eq. (i). Also, other points interior to circle Eq. (i) will have the coordinates of the form

$$(\pm\lambda, 0), (0, \pm\lambda), \text{ where } \lambda^2 < a^2$$

and $(\pm\lambda, \pm\mu)$ and $(\pm\mu, \pm\lambda)$, where $\lambda^2 + \mu^2 < a^2$ and $\lambda, \mu \in I$

$\therefore$ Number of lattice points in the interior of the circle will be of the form $1 + 4r + 8t$, where $r, t = 0, 1, 2, \ldots$

$\therefore$ Number of such points must be of the form $4n + 1$, where $n = 0, 1, 2, \ldots$

● **Ex. 12** *Let x, y be real variable satisfying $x^2 + y^2 + 8x - 10y - 40 = 0$. Let $a = \max.\{\sqrt{(x+2)^2 + (y-3)^2}\}$ and $b = \min.\{\sqrt{(x+2)^2 + (y-3)^2}\}$, then*

(a) $a + b = 18$ (b) $a - b = 4\sqrt{2}$

(c) $a + b = 4\sqrt{2}$ (d) $a \cdot b = 73$

Sol. (a, b, d) Given circle is

$$x^2 + y^2 + 8x - 10y - 40 = 0$$

The centre and radius of the circle are $(-4, 5)$ and 9, respectively.

Distance of the centre $(-4, 5)$ from $(-2, 3)$ is

$$\sqrt{(4+4)} = 2\sqrt{2}.$$

Therefore, $a = 2\sqrt{2} + 9$

and $b = -2\sqrt{2} + 9$

$\therefore$ $a + b = 18$, $a - b = 4\sqrt{2}$, $ab = 73$

Ex. 13 *The equation of the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$, are*

(a) $x = 0$

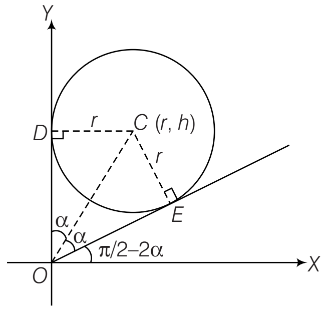
(b) $y = 0$

(c) $(h^2 - r^2)x - 2rhy = 0$

(d) $(h^2 - r^2)x + 2rhy = 0$

Sol. (a, c) The given equation is $(x - r)^2 + (y - h)^2 = r^2$

tangents are $x = 0$



and $$y = x\tan\left(\frac{\pi}{2} - 2\alpha\right) = x\cot 2\alpha$$

$$= \frac{x(1 - \tan^2\alpha)}{2\tan\alpha}$$

$$y = \frac{x\left(1 - \frac{r^2}{h^2}\right)}{2\left(\frac{r}{h}\right)} \qquad \left(\because \text{in } \Delta ODC, \tan\alpha = \frac{r}{h}\right)$$

or $(h^2 - r^2)x - 2rhy = 0$

Ex. 14 *Point M moved on the circle $(x - 4)^2 + (y - 8)^2 = 20$. Then it broke away from it and moving along a tangent to the circle cut the X-axis at point $(-2, 0)$. The coordinates of the point on the circle at which the moving point broke away is*

(a) $\left(\frac{42}{5}, \frac{36}{5}\right)$ (b) $\left(-\frac{2}{5}, \frac{44}{5}\right)$

(c) $(6, 4)$ (d) $(2, 4)$

Sol. (b, c) Given circle is

$$(x - 4)^2 + (y - 8)^2 = 20$$

or $$x^2 + y^2 - 8x - 16y + 60 = 0 \quad \text{...(i)}$$

Equation of chord of contact from $(-2, 0)$ is

$$-2 \cdot x + 0 \cdot y - 4(x - 2) - 8(y + 0) + 60 = 0$$

or $$3x + 4y - 34 = 0 \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$x^2 + \left(\frac{34 - 3x}{4}\right)^2 - 8x - 16\left(\frac{34 - 3x}{4}\right) + 60 = 0$$

or $$5x^2 - 28x - 12 = 0$$

or $$(x - 6)(5x + 2) = 0$$

or $$x = 6, -\frac{2}{5}$$

Therefore, the points are $(6, 4)$ and $\left(-\frac{2}{5}, \frac{44}{5}\right)$.

Ex. 15 *The equations of four circles are $(x \pm a)^2 + (y \pm a)^2 = a^2$. The radius of a circle touching all the four circles is*

(a) $(\sqrt{2} - 1)a$ (b) $2\sqrt{2}a$

(c) $(\sqrt{2} + 1)a$ (d) $(2 + \sqrt{2})a$

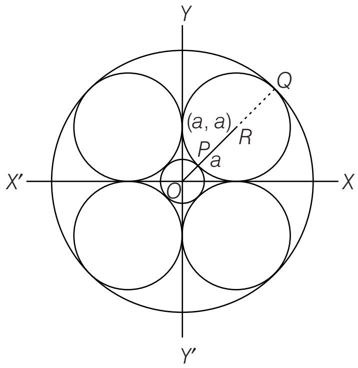
Sol. (a, c) Radius of inner circle $= OR - a$

$$= \sqrt{(a^2 + a^2)} - a$$

$$= a(\sqrt{2} - 1)$$

Radius of outer circle $= OR + RQ$

$$= a\sqrt{2} + a = a(\sqrt{2} + 1)$$



JEE Type Solved Examples :
Paragraph Based Questions

- This section contains **2 solved paragraphs** based upon each of the paragraph **3 multiple choice** questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Paragraph I
(Q. Nos. 16 to 18)

Consider the relation $4l^2 - 5m^2 + 6l + 1 = 0$, where $l, m \in R$.

16. The line $lx + my + 1 = 0$ touches a fixed circle whose equation is

(a) $x^2 + y^2 - 4x - 5 = 0$ (b) $x^2 + y^2 + 6x + 6 = 0$

(c) $x^2 + y^2 - 6x + 4 = 0$ (d) $x^2 + y^2 + 4x - 4 = 0$

17. Tangents PA and PB are drawn to the above fixed circle from the point P on the line $x + y - 1 = 0$. Then, the chord of contact AB passes through the fixed point

(a) $\left(\frac{1}{2}, -\frac{5}{2}\right)$ (b) $\left(\frac{1}{3}, \frac{4}{3}\right)$ (c) $\left(-\frac{1}{2}, \frac{3}{2}\right)$ (d) $\left(\frac{1}{2}, \frac{5}{2}\right)$

18. The number of tangents which can be drawn from the point $(2, -3)$ are

(a) 0 (b) 1 (c) 2 (d) 1 or 2

Sol.

16. (c) Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

The line $lx + my + 1 = 0$ touch circle Eq. (i), then

$$\frac{|-lg - mf + 1|}{\sqrt{(l^2 + m^2)}} = \sqrt{(g^2 + f^2 - c)}$$

$\Rightarrow$ $$(lg + mf - 1)^2 = (l^2 + m^2)(g^2 + f^2 - c)$$

or $$(f^2 - c)l^2 + (g^2 - c)m^2 - 2gflm + 2gl + 2fm - 1 = 0 \quad \text{...(ii)}$$

But the given condition is

$$4l^2 - 5m^2 + 6l + 1 = 0 \quad \text{...(iii)}$$

Comparing Eqs. (ii) and (iii), we get

$$\frac{f^2 - c}{4} = \frac{g^2 - c}{-5} = \frac{-2gf}{0} = \frac{g}{3} = \frac{2f}{0} = \frac{-1}{1}$$

Then, we get $g = -3$, $f = 0$, $c = 4$

Substituting these values in Eq. (i), the equation of the circle is

$$x^2 + y^2 - 6x + 4 = 0$$

17. (a) Let any point on the line $x + y - 1 = 0$ is

$$P(\lambda, 1 - \lambda), \lambda \in R.$$

Then, equation of AB is

$$\lambda x + (1 - \lambda)y - 3(x + \lambda) + 4 = 0$$

$\Rightarrow$ $$(-3x + y + 4) + \lambda(x - y - 3) = 0$$

for fixed point $-3x + y + 4 = 0$, $x - y - 3 = 0$

$\therefore$ $$x = \frac{1}{2}, y = -\frac{5}{2}$$

$\therefore$ Fixed point is $\left(\frac{1}{2}, \frac{-5}{2}\right)$

18. (c) Let $S \equiv x^2 + y^2 - 6x + 4 = 0$.

$\therefore$ $$S_1 = (2)^2 + (-3)^2 - 6(2) + 4$$
$$= 4 + 9 - 12 - 4$$
$$= 5 > 0$$

Therefore, point $(2, -3)$ lies outside the circle from which two tangents can be drawn.

Paragraph II
(Q. Nos. 19 to 21)

If α- chord of a circle be that chord which subtends an angle α at the centre of the circle.

19. If $x + y = 1$ is α-chord of $x^2 + y^2 = 1$, then α is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$

20. If slope of a $\frac{\pi}{3}$-chord of $x^2 + y^2 = 4$ is 1, then its equation is

(a) $x - y + \sqrt{6} = 0$ (b) $x - y + \sqrt{3} = 0$

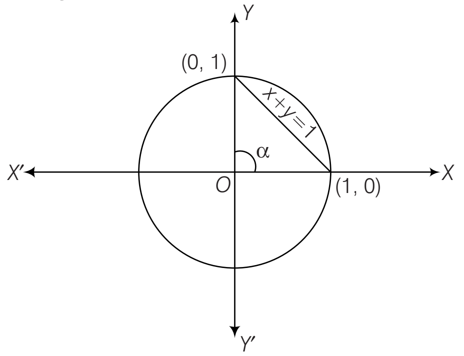
(c) $x - y - \sqrt{3} = 0$ (d) $x - y - 2\sqrt{3} = 0$

21. Distance of $\frac{2\pi}{3}$ — chord of $x^2 + y^2 + 2x + 4y + 1 = 0$ from the centre is

(a) $\frac{1}{\sqrt{2}}$ (b) 1 (c) $\sqrt{2}$ (d) 2

Sol.

19. (c) From figure

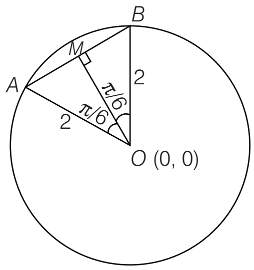


$$\alpha = \frac{\pi}{2}$$

20. (a) $\because$ Slope of chord is 1.

Let the equation of chord be $x - y + \lambda = 0$.

$\because \quad OM = 2\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$



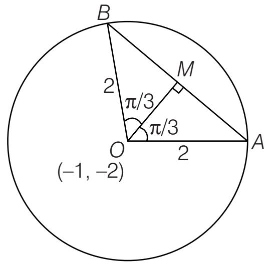
$\therefore \quad \frac{|0 - 0 + \lambda|}{\sqrt{2}} = \sqrt{3}$

$\Rightarrow \quad \lambda = \pm\sqrt{6}$

Hence, equation of chords are

$$x - y \pm \sqrt{6} = 0.$$

21. (b) From figure,



$$OM = 2\cos\left(\frac{\pi}{3}\right) = 1$$

JEE Type Solved Examples :

Single Integer Answer Type Questions

- This section contains **2 examples.** The answer to each example is a **single digit integer,** ranging from 0 to 9 (both inclusive).

Ex. 22 *A circle with centre in the first quadrant is tangent to $y = x + 10$, $y = x - 6$ and the Y-axis. Let (p, q) be the centre of the circle. If the value of $(p + q) = a + b\sqrt{a}$, when $a, b \in Q$, then the value of $|a - b|$ is*

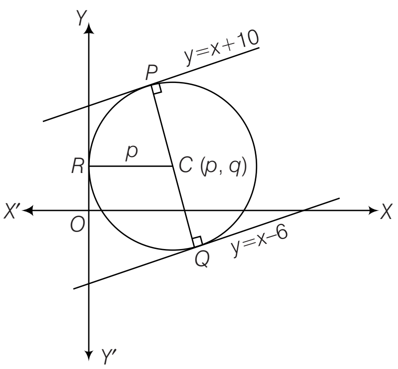
Sol. (6) $\because \quad CP = CR$

$\Rightarrow \quad \frac{|p - q + 10|}{\sqrt{2}} = p$

or $\quad p - q + 10 = p\sqrt{2} \quad$ ...(i)

and $\quad CP = CQ$

$\frac{p - q + 10}{\sqrt{2}} = -\left(\frac{p - q - 6}{\sqrt{2}}\right)$ or $p - q = -2 \quad$ ...(ii)



From Eqs. (i) and (ii), we get

$$p = 4\sqrt{2} \text{ and } q = 4\sqrt{2} + 2$$

Now, $\quad p + q = 2 + 8\sqrt{2} = a + b\sqrt{2} \quad$ (given)

$\therefore \quad a = 2, b = 8$

Hence, $|a - b| = |2 - 8| = 6$

Ex. 23 *If the circles $x^2 + y^2 + (3 + \sin\theta)x + 2\cos\phi y = 0$ and $x^2 + y^2 + (2\cos\phi)x + 2\lambda y = 0$ touch each other, then the maximum value of λ is*

Sol. (1) Since, both the circles are passing through the origin (0, 0), the equation of tangent at (0,0) of first circle will be same as that of the tangent at (0, 0) of second circle.

Equation of tangent at (0, 0) of first circle is

$$(3 + \sin\theta)x + (2\cos\phi)y = 0 \quad \text{...(i)}$$

Equaton of tangent at (0, 0) of second circle is

$$(2\cos\phi)x + 2\lambda y = 0 \quad \text{...(ii)}$$

Therefore, Eqs. (i) and (ii) must be identical, then

$$\frac{3 + \sin\theta}{2\cos\phi} = \frac{2\cos\phi}{2\lambda}$$

or $\quad \lambda = \frac{2\cos^2\phi}{(3 + \sin\theta)}$

or $\quad \lambda_{\max} = 1 \quad$ (when $\sin\theta = -1$ and $\cos\phi = 1$)

JEE Type Solved Examples :
Matching Type Questions

- This section contains **2 examples.** Examples 24 and 25 have four statements (A, B, C and D) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II.**

Ex. 24. *Consider the circles C_1 of radius a and C_2 of radius $b, b > a$ both lying the first quadrant and touching the coordinate axes.*

	Column I		Column II
(A)	C_1 and C_2 touch each other and $\frac{b}{a} = \lambda + \sqrt{\mu}$, $\lambda \in$ prime number and $\mu \in$ whole number, then	(p)	$\lambda + \mu$ is a prime number
(B)	C_1 and C_2 cut orthogonally and $\frac{b}{a} = \lambda + \sqrt{\mu}$, $\lambda \in$ prime number and $\mu \in$ whole number, then	(q)	$\lambda + \mu$ is a composite number
(C)	C_1 and C_2 intersect so that the common chord is longest and $\frac{b}{a} = \lambda + \sqrt{\mu}$, $\lambda \in$ prime number and $\mu \in$ whole number, then	(r)	$2\lambda + \mu$ is a perfect number
(D)	C_2 passes through the centre of C_1 and $\frac{b}{a} = \lambda + \sqrt{\mu}$, $\lambda \in$ prime number and $\mu \in$ whole number, then	(s)	$\|\lambda - \mu\|$ is a prime number

Sol. (A)→ (p, s); (B) → (p); (C) → (p, r, s); (D) → (q, r)

$\because \quad C_1 : x^2 + y^2 - 2ax - 2ay + a^2 = 0$

Centre : (a, a) and radius : a

and $\quad C_2 : x^2 + y^2 - 2bx - 2by + b^2 = 0$

Centre : (b, b) and radius : b

(A) $\because C_1$ and C_2 touch each other, then

$$\sqrt{2}(b - a) = b + a \Rightarrow \frac{b}{a} = (\sqrt{2} + 1)^2 = 3 + \sqrt{8}$$

$\Rightarrow \quad \lambda = 3, \mu = 8$

(B) $\because C_1$ and C_2 intersect orthogonally, then

$$2(b - a)^2 = b^2 + a^2$$

$\Rightarrow \quad a^2 + b^2 - 4ab = 0$

or $\quad \left(\frac{b}{a}\right)^2 - 4\left(\frac{b}{a}\right) + 1 = 0$

$\therefore \quad \frac{b}{a} = \frac{4 \pm \sqrt{(16 - 4)}}{2} = 2 + \sqrt{3}$

$\Rightarrow \quad \lambda = 2, \mu = 3]$

(C) $\because C_1$ and C_2 intersect, the common chord is

$$2(b - a)(x + y) = b^2 - a^2$$

given common chord is longest, then passes through (a, a)

$\Rightarrow \quad 2(b - a)(2a) = b^2 - a^2$

or $\quad (b - 3a)(b - a) = 0$

$\because \quad b - a \neq 0 \quad [b > a]$

$\therefore \quad b - 3a = 0$

or $\quad \frac{b}{a} = 3 \Rightarrow \lambda = 3, \mu = 0$

(D) $\because C_2$ passes through (a, a), then $a^2 + a^2 - 2ab - 2ab + b^2 = 0$

or $b^2 - 4ab + 2a^2 = 0$

or $\quad \left(\frac{b}{a}\right)^2 - 4\left(\frac{b}{a}\right) + 2 = 0$

or $\quad \frac{b}{a} = \frac{4 \pm \sqrt{(16 - 8)}}{2} = 2 + \sqrt{2}$

$\Rightarrow \quad \lambda = 2, \mu = 2$

Ex. 25. *Match the following*

	Column I		Column II
(A)	The circles $x^2 + y^2 + 2x + c = 0$ $(c > 0)$ and $x^2 + y^2 + 2y + c = 0$ touch each other, then the value of $2c$ is	(p)	1
(B)	The circles $x^2 + y^2 + 2x + 3y + c = 0$ $(c > 0)$ and $x^2 + y^2 - x + 2y + c = 0$ intersect orthogonally, then the value of $2c$ is	(q)	2
(C)	The circle $x^2 + y^2 = 9$ is contains the circle $x^2 + y^2 - 2x + 1 - c^2 = 0$ $(c > 0)$, then $2c$ can be	(r)	3
(D)	The circle $x^2 + y^2 = 9$ is contains in the circle $x^2 + y^2 - 2x + 1 - \frac{c^2}{4} = 0$ $(c > 0)$, then $(c - 6)$ can be	(s)	4

Sol. (A) → (p); (B) → (q); (C) → (p, q, r); (D) → (r, s)

(A) The circles

$$S_1 : (x + 1)^2 + y^2 = (\sqrt{(1 - c)})^2$$

Centre $C_1 : (-1, 0)$, radius $r_1 : \sqrt{(1 - c)}$

and $\quad S_2 : x^2 + (y + 1)^2 = (\sqrt{(1 - c)})^2$

Centre $C_2 : (0, -1)$, radius : $r_2 = \sqrt{(1 - c)}$

Now, $\quad C_1C_2 = \sqrt{2}$ and $r_1 = r_2$

$\therefore$ The circles will touch externally only and $C_1C_2 = r_1 + r_2$

$\Rightarrow \quad \sqrt{2} = 2\sqrt{(1 - c)}$ or $2c = 1$

(B) The circles $S_1:(x+1)^2+\left(y+\frac{3}{2}\right)^2=\left(\sqrt{\left(\frac{13}{4}-c\right)}\right)^2$

Centre $C_1:\left(-1,-\frac{3}{2}\right)$, radius $r_1:\sqrt{\left(\frac{13}{4}-c\right)}$

and $\quad S_2:\left(x-\frac{1}{2}\right)^2+(y+1)^2=\left(\sqrt{\left(\frac{5}{4}-c\right)}\right)^2$

Centre $C_2:\left(\frac{1}{2},-1\right)$, radius $r_2:\sqrt{\left(\frac{5}{4}-c\right)}$

For intersect orthogonally

$$(C_1C_2)^2=r_1^2+r_2^2$$

$$\Rightarrow \quad \left(\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2=\frac{13}{4}-c+\frac{5}{4}-c$$

or $\quad 2c=2$

(C) The circles

$$S_1:x^2+y^2=3^2$$

Centre $C_1:(0,0)$, radius $r_1:3$

and $\quad S_2:(x-1)^2+y^2=c^2$

Centre $C_2:(1,0)$, radius $r_2:c$

Now, S_2 will be contained in S_1, then

$$C_1C_2<r_1-r_2$$

or $\quad 1<3-c$ or $c<2\Rightarrow 2c<4$

(D) The circles

$$S_1:x^2+y^2=9$$

Centre $C_1:(0,0)$, radius $r_1:3$ and

$$S_2:(x-1)^2+y^2=\left(\frac{c}{2}\right)^2$$

Centre $C_2:(1,0)$, radius $r_2:\frac{c}{2}$

Now, S_1 will be contained in S_2,

then, $\quad r_2-r_1>C_1C_2$

$\Rightarrow \quad \frac{c}{2}-3>1$ or $c>8$

$\therefore \quad (c-6)>2$

JEE Type Solved Examples : Statement I and II Type Questions

- **Directions** (Ex. Nos. 26 and 27) are Assertion-Reason Type examples. Each of these examples contains two statements :
  Statement I (Assertion) and **Statement II** (Reason)

  Each of these examples also has four alternative choices only one of which is the correct answer. You have to select the correct choice as given below :

  (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
  (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
  (c) Statement I is true, Statement II is false
  (d) Statement I is false, Statement II is true

Ex. 26 *C_1 is a circle of radius 2 touching X-axis and Y-axis. C_2 is another circle of radius greater than 2 and touching the axes as well as the circle C_1.*

Statement I *Radius of Circle $C_2=\sqrt{2}(\sqrt{2}+1)(\sqrt{2}+2)$*

Statement II *Centres of both circles always lie on the line $y=x$.*

Sol. (c) $C_1:(x-2)^2+(y-2)^2=2^2$

$$C_2:(x-r)^2+(y-r)^2=r^2 \qquad (r>2)$$

According to question,

$$\sqrt{(r-2)^2+(r-2)^2}=r+2$$

$$(r-2)^2+(r-2)^2=(r+2)^2$$

$$r^2-12r+4=0$$

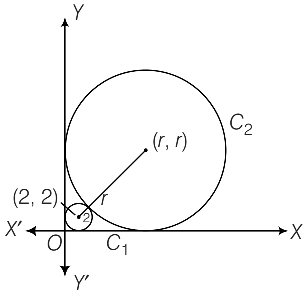
$$r=\frac{12\pm\sqrt{(144-16)}}{2}$$

$$=6\pm4\sqrt{2}$$

$$\therefore \quad r=6+4\sqrt{2} \qquad [\because r>2]$$

$$=2(\sqrt{2}+1)^2$$

$$=\sqrt{2}(\sqrt{2}+1)(2+\sqrt{2})$$



$\therefore$ Statement I is true and Statement II is always not true (where circles in II or IV quadrants)

Ex. 27 *From the point $P(\sqrt{2},\sqrt{6})$ tangents PA and PB are drawn to the circle $x^2+y^2=4$*

Statement I *Area of the quadrilateral OAPB (O being origin) is 4.*

Statement II *Tangents PA and PB are perpendicular to each other and therefore quadrilateral OAPB is a square.*

Sol. (a) Clearly, $P(\sqrt{2},\sqrt{6})$ lies on $x^2+y^2=8$, which is the director circle of $x^2+y^2=4$.

Therefore, tangents *PA* and *PB* are perpendicular to each other. So, *OAPB* is a square.

Hence, area of $OAPB=(\sqrt{S_1})^2=S_1$

$$=(\sqrt{2})^2+(\sqrt{6})^2-4=4$$

$\therefore$ Both statements are true and statement II is correct explanation of statement I.

Subjective Type Examples

- In this section, there are **16 subjective solved examples.**

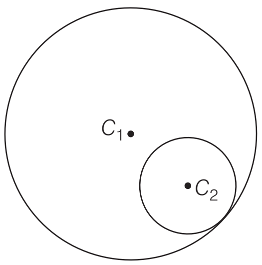
● **Ex. 28** *Find the equation of a circle having the lines $x^2 + 2xy + 3x + 6y = 0$ as its normals and having size just sufficient to contain the circle*

$$x(x-4) + y(y-3) = 0.$$

Sol. Given pair of normals is $x^2 + 2xy + 3x + 6y = 0$

or $(x+2y)(x+3) = 0$

$\therefore$ Normals are $x + 2y = 0$ and $x + 3 = 0$ the point of intersection of normals $x + 2y = 0$ and $x + 3 = 0$ is the centre of required circle, we get centre $C_1 \equiv (-3, 3/2)$ and other circle is



$$x(x-4) + y(y-3) = 0$$

or $x^2 + y^2 - 4x - 3y = 0$ ...(i)

its centre $C_2 \equiv (2, 3/2)$ and radius $r = \sqrt{4 + \frac{9}{4}} = \frac{5}{2}$

Since, the required circle just contains the given circle(i), the given circle should touch the required circle internally from inside.

$\Rightarrow$ radius of the required circle $= |C_1 - C_2| + r$

$$= \sqrt{(-3-2)^2 + \left(\frac{3}{2} - \frac{3}{2}\right)^2} + \frac{5}{2}$$

$$= 5 + \frac{5}{2} = \frac{15}{2}$$

Hence, equation of required circle is

$$(x+3)^2 + (y - 3/2)^2 = \left(\frac{15}{2}\right)^2$$

or $x^2 + y^2 + 6x - 3y - 54 = 0$

● **Ex. 29** *Let a circle be given by*

$$2x(x-a) + y(2y-b) = 0 \qquad (a \neq 0, b \neq 0)$$

Find the condition on a and b if two chords, each bisected by the X-axis, can be drawn to the circle from $(a, b/2)$.

Sol. The given circle is $2x(x-a) + y(2y-b) = 0$

or $x^2 + y^2 - ax - by/2 = 0$

Let AB be the chord which is bisected by X-axis at a point M. Let its coordinates be $M(h, 0)$

and let $S \equiv x^2 + y^2 - ax - by/2 = 0$

$\therefore$ Equation of chord AB is $T = S_1$

$$hx + 0 - \frac{a}{2}(x+h) - \frac{b}{4}(y+0) = h^2 + 0 - ah - 0$$

Since, its passes through $(a, b/2)$ we have

$$ah - \frac{a}{2}(a+h) - \frac{b^2}{8} = h^2 - ah$$

$$\Rightarrow \quad h^2 - \frac{3ah}{2} + \frac{a^2}{2} + \frac{b^2}{8} = 0$$

Now, there are two chords bisected by the X-axis, so there must be two distinct real roots of h.

$$\therefore \quad B^2 - 4AC > 0$$

$$\Rightarrow \quad \left(\frac{-3a}{2}\right)^2 - 4 \cdot 1 \cdot \left(\frac{a^2}{2} + \frac{b^2}{8}\right) > 0$$

$$\Rightarrow \quad a^2 > 2b^2.$$

Aliter : Given circle is

$$2x(x-a) + y(2y-b) = 0$$

or $x^2 + y^2 - ax - \frac{by}{2} = 0$ ...(i)

Let chords bisected at $M(h, 0)$ but given chords can be drawn $A\left(a, \frac{b}{2}\right)$ then chord cut the circle at $B(\lambda, -b/2)$

$\because$ Mid-point of ordinates of A and B is origin.

$\therefore$ $B(\lambda, b/2)$ lies on Eq. (i)

$$\therefore \quad \lambda^2 + \frac{b^2}{4} - a\lambda + \frac{b^2}{4} = 0$$

or $\lambda^2 - a\lambda + \frac{b^2}{2} = 0$

$\because$ λ is real

$\therefore$ $B^2 - 4AC > 0$ or $a^2 - 4 \cdot \frac{b^2}{2} > 0$ or $a^2 > 2b^2$

● **Ex. 30** *Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Identify the locus of the centre of C.*

Sol. Let the given circles C_1 and C_2 have centres O_1 and O_2 with radii r_1 and r_2, respectively. Let centre of circle C is at O radius is r.

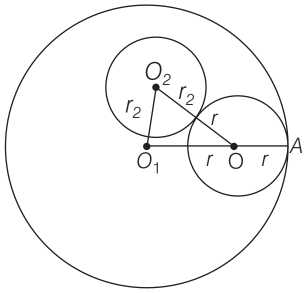
$\because$ $OO_2 = r + r_2$

$OO_1 = r_1 - r$

$\Rightarrow$ $OO_1 + OO_2 = r_1 + r_2$

which is greater than O_1O_2 as $O_1O_2 < r_1 + r_2$.

$\therefore$ Locus of O is an ellipse with foci O_1 and O_2.



Aliter :

Let $O_1 \equiv (0, 0)$, $O_2 \equiv (a, b)$ and $O \equiv (h, k)$

$\therefore \quad C_1 : x^2 + y^2 = r_1^2$

$$C_2 : (x - a)^2 + (y - b)^2 = r_2^2$$

$$C : (x - h)^2 + (y - k)^2 = r^2$$

$\Rightarrow \quad OO_2 = r + r_2$

$\Rightarrow \quad \sqrt{(h - a)^2 + (k - b)^2} = r + r_2 \quad \text{...(i)}$

and $\quad OO_1 = r_1 - r$

$\Rightarrow \quad \sqrt{(h^2 + k^2)} = r_1 - r \quad \text{...(ii)}$

On adding Eqs. (i) and (ii) we get

$$\sqrt{(h - a)^2 + (k - b)^2} + \sqrt{(h^2 + k^2)} = r_1 + r_2$$

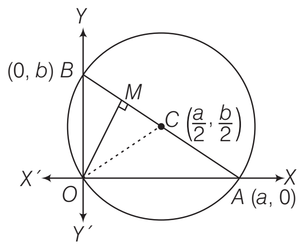
$\therefore$ Locus of O is $\sqrt{(x - a)^2 + (y - b)^2} + \sqrt{(x^2 + y^2)} = r_1 + r_2$

which represents an ellipse with foci are at (a, b) and $(0, 0)$.

● ***Ex. 31*** A *circle of constant radius r passes through the origin O, and cuts the axes at A and B. Show that the locus of the foot of the perpendicular from O to AB is*

$$(x^2 + y^2)^2 (x^{-2} + y^{-2}) = 4r^2$$

Sol. Let the coordinates of A and B are $(a, 0)$ and $(0, b)$.



$\therefore$ Equation of AB is $\quad \dfrac{x}{a} + \dfrac{y}{b} = 1 \quad \text{...(i)}$

Centre of circle lie on line AB, since AB is diameter of the circle $(\because \angle AOB = \pi / 2)$

$\therefore$ Coordinate of centre C is $C \equiv \left(\dfrac{a}{2}, \dfrac{b}{2}\right)$

Since, the radius of circle $= r$

$\therefore \quad r = AC = CB = OC$

$$= \sqrt{\left(0 - \frac{a}{2}\right)^2 + \left(0 - \frac{b}{2}\right)^2} = \sqrt{\frac{a^2 + b^2}{4}}$$

$\therefore \quad a^2 + b^2 = 4r^2 \quad \text{...(ii)}$

Equation of OM which is $\perp$ to AB is

$$ax - by = \lambda$$

It passes through $(0, 0)$

$\therefore \quad 0 = \lambda$

$\therefore$ Equation of OM is

$$ax - by = 0 \quad \text{...(iii)}$$

On solving Eq. (i) and Eq. (iii), we get

$$a = \frac{x^2 + y^2}{x} \text{ and } b = \frac{x^2 + y^2}{y}$$

Substituting the values of a and b in Eq. (ii), we get

$$(x^2 + y^2)^2 \left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 4r^2$$

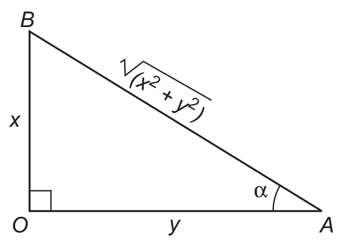
or $\quad (x^2 + y^2)^2 (x^{-2} + y^{-2}) = 4r^2$

which is the required locus.

Aliter :

$\because$ AB is the diameter of circle. If $\angle OAB = \alpha$, then

$$OA = 2r \cos \alpha, \; OB = 2r \sin \alpha$$



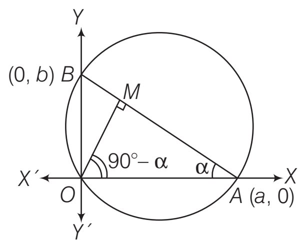
Equation of AB is

$$\frac{x}{2r \cos \alpha} + \frac{y}{2r \sin \alpha} = 1$$

$\Rightarrow \quad \dfrac{x}{\cos \alpha} + \dfrac{y}{\sin \alpha} = 2r \quad \text{...(i)}$

and equation of OM is $y = x \tan (90° - \alpha)$

$\Rightarrow \quad \cot \alpha = \dfrac{y}{x}$



$\therefore \quad \sin \alpha = \dfrac{x}{\sqrt{(x^2 + y^2)}}$

and $\quad \cos \alpha = \dfrac{y}{\sqrt{(x^2 + y^2)}}$

Then, from Eq. (i),

$$\frac{x}{y}\sqrt{(x^2+y^2)}+\frac{y}{x}\sqrt{(x^2+y^2)}=2r$$

$$\Rightarrow \quad \frac{(x^2+y^2)\sqrt{(x^2+y^2)}}{xy}=2r$$

On squaring, we have $(x^2+y^2)^2\,\dfrac{(x^2+y^2)}{x^2y^2}=4r^2$

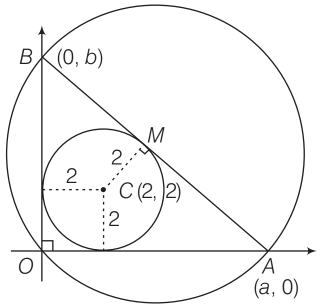
$$\Rightarrow \quad (x^2+y^2)^2\,(x^{-2}+y^{-2})=4r^2$$

● **Ex. 32** *The circle $x^2+y^2-4x-4y+4=0$ is inscribed in a triangle which has two of its sides along the coordinate axes. The locus of the circumcentre of the triangle is $x+y-xy+k(x^2+y^2)^{1/2}=0$. Find k.*

Sol. The given circle is $x^2+y^2-4x-4y+4=0$. This can be re-written as $(x-2)^2+(y-2)^2=4$ which has centre $C(2,2)$ and radius 2.

Let the equation of third side is

$$\frac{x}{a}+\frac{y}{b}=1 \qquad \text{(equation of } AB\text{)}$$



Length of perpendicular from (2, 2) on AB = radius = CM

$$\therefore \quad \frac{\left|\frac{2}{a}+\frac{2}{b}-1\right|}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}}=2$$

Since, origin and (2, 2) lie on the same side of AB

$$\therefore \quad -\frac{\left(\frac{2}{a}+\frac{2}{b}-1\right)}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}}=2$$

or $$\frac{2}{a}+\frac{2}{b}-1=-2\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)} \qquad \text{...(i)}$$

Since, $\angle AOB=\dfrac{\pi}{2}$

Hence, AB is the diameter of the circle passing through ΔOAB, mid-point of AB is the centre of the circle i.e. $\left(\dfrac{a}{2},\dfrac{b}{2}\right)$.

Let centre be $(h,k)\equiv\left(\dfrac{a}{2},\dfrac{b}{2}\right)$ then $a=2h$ and $b=2k$.

Substituting the values of a and b in Eq. (i), then

$$\frac{2}{2h}+\frac{2}{2k}-1=-2\sqrt{\left(\frac{1}{4h^2}+\frac{1}{4k^2}\right)}$$

$$\Rightarrow \quad \frac{1}{h}+\frac{1}{k}-1=-\sqrt{\left(\frac{1}{h^2}+\frac{1}{k^2}\right)}$$

or $$h+k-hk+\sqrt{(h^2+k^2)}=0$$

$\therefore$ Locus of $M(h,k)$ is

$$x+y-xy+\sqrt{(x^2+y^2)}=0$$

Hence, the required value of k is 1.

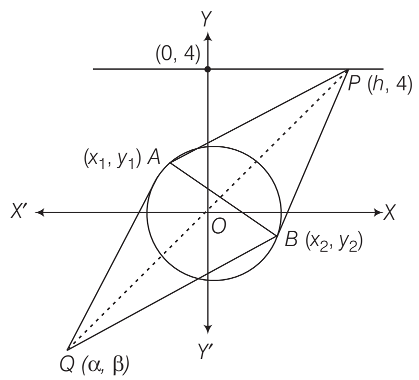
● **Ex. 33** *P is a variable on the line $y=4$. Tangents are drawn to the circle $x^2+y^2=4$ from P to touch it at A and B. The parallelogram PAQB is completed . Find the equation of the locus of Q.*

Sol. Let $P(h,4)$ be a variable point. Given circle is

$$x^2+y^2=4 \qquad \text{...(i)}$$

Draw tangents from $P(h,4)$ and complete parallelogram $PAQB$.

Equation of the diagonal AB which is chord of contact of $x^2+y^2=4$ is $hx+4y=4$ ...(ii)



Let coordinates of A and B are (x_1,y_1) and (x_2,y_2), respectively.

Since, $A(x_1,y_1)$ and $B(x_2,y_2)$ lies on Eq. (ii)

$\therefore \quad hx_1+4y_1=4$ and $hx_2+4y_2=4$

$\therefore \quad h(x_1+x_2)+4(y_1+y_2)=8$ ...(iii)

Since, $PAQB$ is parallelogram

$\therefore$ Mid-point of AB = Mid-point of PQ

$$\Rightarrow \quad \frac{x_1+x_2}{2}=\frac{\alpha+h}{2}$$

and $$\frac{y_1+y_2}{2}=\frac{\beta+4}{2} \qquad \text{...(iv)}$$

Eliminating x from Eqs. (i) and (ii), then

$$\left(\frac{4-4y}{h}\right)^2 + y^2 = 4$$

$$\Rightarrow \quad 16 + 16y^2 - 32y + h^2y^2 = 4h^2$$

$$\Rightarrow \quad (16+h^2)\,y^2 - 32y + 16 - 4h^2 = 0$$

$$\therefore \quad y_1 + y_2 = \frac{32}{16+h^2} \quad \text{...(v)}$$

From Eqs. (iii) and (v), we get

$$x_1 + x_2 = \frac{8h}{16+h^2} \quad \text{...(vi)}$$

From Eqs. (iv) and (vi)

$$\beta + 4 = \frac{32}{16+h^2}$$

$$\text{or} \quad (16+h^2)(\beta+4) = 32 \quad \text{...(vii)}$$

From Eqs. (iv) and (vi)

$$\alpha + h = \frac{8h}{16+h^2}$$

$$\text{or} \quad (16+h^2)(\alpha+h) = 8h \quad \text{...(viii)}$$

Dividing Eq. (viii) by Eq. (vii), then

$$\frac{\alpha+h}{\beta+4} = \frac{h}{4} \quad \text{or} \quad h = \frac{4\alpha}{\beta}$$

Substituting the value of h in Eq. (vii) then

$$\left(16 + \frac{16\alpha^2}{\beta^2}\right)(\beta+4) = 32$$

$$\Rightarrow \quad (\alpha^2+\beta^2)(\beta+4) = 2\beta^2$$

Hence, locus of $Q\,(\alpha, \beta)$ is $(x^2+y^2)(y+4) = 2y^2$

● ***Ex. 34*** *Show that the circumcircle of the triangle formed by the lines $ax+by+c=0$; $bx+cy+a=0$ and $cx+ay+b=0$ passes through the origin if $(b^2+c^2)(c^2+a^2)(a^2+b^2) = abc\,(b+c)(c+a)(a+b)$.*

Sol. Equation of conic is

$$(bx+cy+a)(cx+ay+b) + \lambda\,(cx+ay+b)(ax+by+c) + \mu\,(ax+by+c)(bx+cy+a) = 0 \quad \text{...(i)}$$

where, λ and μ are constants.

Eq. (i) represents a circle if the coefficient of x^2 and y^2 are equal and the coefficient of xy is zero such that

$$bc + \lambda ca + \mu ab = ca + \lambda ab + \mu bc$$

$$\text{or} \quad (a-b)\,c + \lambda\,(b-c)\,a + \mu\,(c-a)\,b = 0 \quad \text{...(ii)}$$

$$\text{and} \quad (c^2+ab) + \lambda\,(a^2+bc) + \mu\,(b^2+ac) = 0 \quad \text{...(iii)}$$

on solving Eq. (ii) and Eq. (iii) by cross multiplication rule, we get

$$\frac{1}{(c^2-ab)(a^2+b^2)} = \frac{\lambda}{(a^2-bc)(b^2+c^2)} = \frac{\mu}{(b^2-ac)(c^2+a^2)}$$

$$\therefore \quad \lambda = \frac{(a^2-bc)(b^2+c^2)}{(c^2-ab)(a^2+b^2)}$$

$$\text{and} \quad \mu = \frac{(b^2-ac)(c^2+a^2)}{(c^2-ab)(a^2+b^2)} \quad \text{...(iv)}$$

and given, Eq. (i) passes through the origin then

$$ab + bc\lambda + ca\,\mu = 0 \quad \text{...(v)}$$

From Eqs. (iv) and (v), we get

$$ab + \frac{bc\,(a^2-bc)(b^2+c^2)}{(c^2-ab)(a^2+b^2)} + \frac{ca\,(b^2-ac)(c^2+a^2)}{(c^2-ab)(a^2+b^2)} = 0$$

$$\Rightarrow \quad (c^2-ab)(a^2+b^2)\,ab + (a^2-bc)(b^2+c^2)bc + (b^2-ca)(c^2+a^2)\,ca = 0$$

$$\Rightarrow \quad abc^2\,(a^2+b^2) + a^2bc\,(b^2+c^2) + b^2ca\,(c^2+a^2) = a^2b^2\,(a^2+b^2) + b^2c^2\,(b^2+c^2) + c^2a^2\,(c^2+a^2)$$

$$\Rightarrow \quad abc\,\{c\,(a^2+b^2) + a\,(b^2+c^2) + b\,(c^2+a^2)\} = a^2b^2\,(a^2+b^2) + b^2c^2\,(b^2+c^2) + c^2a^2\,(c^2+a^2)$$

$$\Rightarrow \quad abc\,\{(a+b)(b+c)(c+a) - 2abc\} = a^2b^2\,(a^2+b^2) + b^2c^2\,(b^2+c^2) + c^2a^2\,(c^2+a^2)$$

$$\Rightarrow \quad abc\,(a+b)(b+c)(c+a) = 2a^2b^2c^2 + a^2b^2\,(a^2+b^2) + b^2c^2\,(b^2+c^2) + c^2a^2\,(c^2+a^2)$$

$$\Rightarrow \quad abc\,(a+b)(b+c)(c+a) = (a^2+b^2)(b^2+c^2)(c^2+a^2)$$

Hence, $(a^2+b^2)(b^2+c^2)(c^2+a^2) = abc\,(a+b)(b+c)(c+a)$

● ***Ex. 35*** *If four points P, Q, R, S in the plane be taken and the square of the length of the tangents from P to the circle on QR as diameter be denoted by $\{P, QR\}$, show that*

$$\{P, RS\} - \{P, QS\} + \{Q, PR\} - \{Q, RS\} = 0$$

Sol. Let $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$, $R \equiv (x_3, y_3)$ and $S \equiv (x_4, y_4)$.

Equation of circle with RS as diameter is

$$(x-x_3)(x-x_4) + (y-y_3)(y-y_4) = 0$$

$$\therefore \quad \{P, RS\} = (x_1-x_3)(x_1-x_4) + (y_1-y_3)(y_1-y_4)$$

Now, equation of circle with QS as diameter is

$$(x-x_2)(x-x_4) + (y-y_2)(y-y_4) = 0$$

$$\therefore \quad \{P, QS\} = (x_1-x_2)(x_1-x_4) + (y_1-y_2)(y_1-y_4)$$

Equation of circle with PR as diameter is

$$(x - x_1)(x - x_3) + (y - y_1)(y - y_3) = 0$$

$\therefore \quad \{Q, PR\} = (x_2 - x_1)(x_2 - x_3) + (y_2 - y_1)(y_2 - y_3)$

Equation of circle with RS as diameter is

$$(x - x_3)(x - x_4) + (y - y_3)(y - y_4) = 0$$

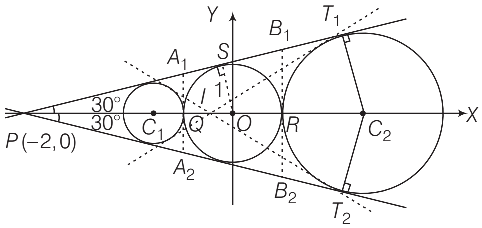
$\therefore \quad \{Q, RS\} = (x_2 - x_3)(x_2 - x_4) + (y_2 - y_3)(y_2 - y_4)$

Hence, $\{P, RS\} - \{P, QS\} + \{Q, PR\} - \{Q, RS\} = 0$

Ex. 36 *Let T_1, T_2 be two tangents drawn from $(-2, 0)$ on the circle $C : x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pair of tangents. Further, find the equations of all possible common tangents to these circles when taken two at a time.*

Sol. In figure $OS = 1$, $OP = 2$

$\therefore \quad \sin \angle SPO = \frac{1}{2} = \sin 30°$



$\therefore \quad \angle SPO = 30°$

$\because \quad PA_1 = PA_2 \Rightarrow \angle PA_1A_2 = \angle PA_2A_1$

$\Rightarrow \quad \Delta PA_1A_2$ is an equilateral triangle.

Therefore, centre C_1 is centroid of $\Delta PA_1A_2, C_1$ divides PQ in the ratio 2 : 1.

$\therefore \quad C_1 \equiv \left(-\frac{4}{3}, 0\right)$ and its radius $= C_1Q = \frac{1}{3}$

$\Rightarrow \quad C_1 : (x + 4/3)^2 + y^2 = \left(\frac{1}{3}\right)^2 \quad$ ...(i)

The other circle C_2 touches the equilateral triangle PB_1B_2 externally.

its radius is given by $= \frac{\Delta}{s - a}$, where $B_1B_2 = a$

$$= \frac{\frac{\sqrt{3}}{4}a^2}{\frac{3a}{2} - a} = \frac{\sqrt{3}}{2}a$$

but $\quad \tan 30° = \frac{a/2}{3} \Rightarrow a = \frac{6}{\sqrt{3}}$

$\therefore \quad$ Radius $= \frac{\sqrt{3}}{2} \cdot \frac{6}{\sqrt{3}} = 3$

$\Rightarrow$ coordinates of C_2 are (4, 0)

$\therefore$ Equation of $C_2 : (x - 4)^2 + y^2 = 3^2 \quad$ ...(ii)

Equations of common tangents to circle (i) and circle C are

$$x = -1, y = \pm \frac{1}{\sqrt{3}}(x + 2), \{T_1 \text{ and } T_2\}$$

and equations of common tangents to circle (ii) and circle C are

$$x = 1, y = \pm \frac{1}{\sqrt{3}}(x + 2) (\{T_1 \text{ and } T_2\}$$

To find the remaining two transverse common tangents to Eqs. (i) and (ii). If I divides C_1 and C_2 in the ratio $r_1 : r_2 = 1/3 : 3 = 1 : 9$.

Therefore coordinates of I are $(-4/5, 0)$.

Equation of any line through I is $y - 0 = m(x + 4/5)$. If it will touch Eq. (ii)

then $\quad \frac{|m(4 + 4/5) - 0|}{\sqrt{(1 + m^2)}} = 3$

$\Rightarrow \quad \left(\frac{24}{5}\right)^2 m^2 = 9(1 + m^2)$

$\Rightarrow \quad 64m^2 = 25 + 25m^2$

$\Rightarrow \quad 39m^2 = 25 \Rightarrow m = \pm \frac{5}{\sqrt{39}}$

Therefore, equations of transverse common tangents are

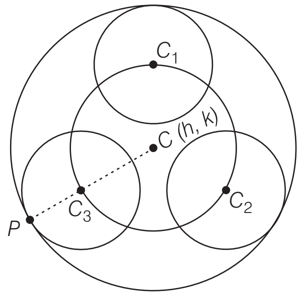
$$y = \pm \frac{5}{\sqrt{39}}(x + 4/5)$$

Ex. 37 *Find the equation of the circle of minimum radius which contains the three circles*

$$x^2 - y^2 - 4y - 5 = 0$$
$$x^2 + y^2 + 12x + 4y + 31 = 0$$
and $$x^2 + y^2 + 6x + 12y + 36 = 0$$

Sol. The coordinates of the centres and radii of three given circles are as given below :

$$C_1 \equiv (0, 2); r_1 = 3$$
$$C_2 \equiv (-6, -2); r_2 = 3$$
and $$C_3 \equiv (-3, -6); r_3 = 3$$



Let $C \equiv (h, k)$ be the centre of the circle passing through the centres $C_1(0, 2)$, $C_2(-6, -2)$ and $C_3(-3, -6)$.

Then, $\quad CC_1 = CC_2 = CC_3$

$\Rightarrow \quad (CC_1)^2 = (CC_2)^2 = (CC_3)^2$

$\Rightarrow \quad (h-0)^2+(k-2)^2=(h+6)^2+(k+2)^2$

$$=(h+3)^2+(k+6)^2$$

$\Rightarrow \quad -4k+4=12h+4k+40=6h+12k+45$

$\Rightarrow \quad 12h+8k+36=0$

or $\quad 3h+2k+9=0 \quad$ ...(i)

and $\quad 6h-8k-5=0 \quad$ ...(ii)

On solving Eqs. (i) and (ii), we get $h=-\frac{31}{18}, k=-\frac{23}{12}$

Now, $CP=CC_3+C_3P=CC_3+3$

$$=\sqrt{\left(-3+\frac{31}{18}\right)^2+\left(-6+\frac{23}{12}\right)^2}+3=\left(\frac{5}{36}\sqrt{(949)}+3\right)$$

Hence, equation of required circle is

$$\left(x+\frac{31}{18}\right)^2+\left(y+\frac{23}{12}\right)^2=\left(3+\frac{5}{\sqrt{36}}\sqrt{949}\right)^2$$

Remark

If radii of three given circles are distinct say $r_1<r_2<r_3$ then the radius of the required circle will be equal to (CC_1 or CC_2 or CC_3) + r_3 ($\because CC_1=CC_2=CC_3$)

Ex. 38 *Find the point P on the circle $x^2+y^2-4x-6y+9=0$ such that*

(i) $\angle POX$ is minimum,

(ii) *OP is maximum, when O is the origin and OX is the X-axis.*

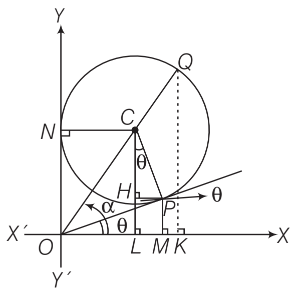
Sol. Given circle is

$$x^2+y^2-4x-6y+9=0$$

or $\quad (x-2)^2+(y-3)^2=2^2 \quad$ .... (i)

Its centre is $C\equiv(2,3)$ and radius $r=2$

Eq. (i) Let OP and ON be the two tangents from O to the circle Eq. (i), then $\quad OP=ON=3$



then $\angle POX$ is minimum when OP is tangent to the circle Eq. (i) at P

Let $\angle POX=\theta$

$\therefore \quad P\equiv(OP\cos\theta, OP\sin\theta)$

i.e. $\quad P\equiv(3\cos\theta, 3\sin\theta) \quad$ ...(ii)

From figure, $OM=OL+LM=NC+HP=NC+CP\sin\theta$

$\Rightarrow \quad OP\cos\theta=NC+CP\sin\theta$

$\Rightarrow \quad 3\cos\theta=2+2\sin\theta$

$\Rightarrow \quad 9(1-\sin^2\theta)=4(1+\sin\theta)^2$

$\Rightarrow \quad 9(1-\sin\theta)=4(1+\sin\theta) \quad (\because \sin\theta\neq-1)$

$\therefore \quad \sin\theta=\frac{5}{13}$ and $\cos\theta=\frac{12}{13}$

From Eq. (ii), $P\equiv\left(3\times\frac{12}{13}, 3\times\frac{5}{13}\right)$ i.e. $P\equiv\left(\frac{36}{13},\frac{15}{13}\right)$

Eq. (ii) OP will be maximum, if P becomes the point extended part of OC cuts the circle. Let this point be Q

then maximum value of $OP=OQ=OC+CQ=(\sqrt{13}+2)$

Let $\quad \angle COX=\alpha$

then, $\quad Q\equiv(OQ\cos\alpha, OQ\sin\alpha)$

$$\equiv((2+\sqrt{13})\cos\alpha, (2+\sqrt{13})\sin\alpha) \quad \text{... (iii)}$$

Now, in ΔCOL, $\quad \cos\alpha=\frac{OL}{OC}=\frac{NC}{OC}=\frac{2}{\sqrt{13}}$

$\therefore \quad \sin\alpha=\frac{3}{\sqrt{13}}$

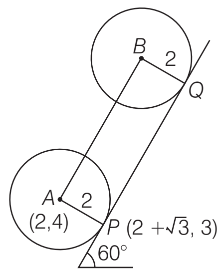
Now, from Eq. (iii), $Q\equiv\left(2+\frac{4}{\sqrt{13}}, 3+\frac{6}{\sqrt{13}}\right)$

Ex. 39 *The circle $x^2+y^2-4x-8y+16=0$ rolls up the tangent to it at $(2+\sqrt{3},3)$ by 2 units, assuming the X-axis as horizontal, find the equation of the circle in the new position.*

Sol. Given circle is

$$x^2+y^2-4x-8y+16=0 \quad \text{...(i)}$$

Let $\quad P\equiv(2+\sqrt{3},3)$



Equation of tangent to the circle Eq. (i) at $P(2+\sqrt{3},3)$ is

$$(2+\sqrt{3})x+3y-2(x+2+\sqrt{3})-4(y+3)+16=0$$

or $\quad \sqrt{3}x-y-2\sqrt{3}=0 \quad$ ...(ii)

Let A and B be the centres of the circles in old and new positions, then

$$B\equiv(2+2\cos 60°, 4+2\sin 60°)$$

($\because AB$ makes an angle $60°$ with X-axis)

or $\quad B\equiv(3, 4+\sqrt{3})$

and $\quad$ radius $= \sqrt{2^2 + 4^2 - 16} = 2$

$\therefore$ Equation of the required circle is

$$(x-3)^2 + (y - 4 - \sqrt{3})^2 = 2^2$$

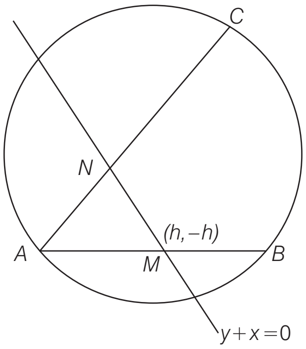
or $$x^2 + y^2 - 6x - 2(4 + \sqrt{3})\,y + 24 + 8\sqrt{3} = 0$$

Ex. 40 *Find the intervals of the values of 'a' for which the line $y + x = 0$ bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle*

$$2x^2 + 2y^2 - (1 + \sqrt{2}\,a)x - (1 - \sqrt{2}a)y = 0.$$

Sol. The point $A\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ lies on the given circle as its coordinate satisfy the equation of the circle. Let AB and AC are two chords drawn from A. Let M and N are the mid-points of AB and AC.

Let coordinate of M be $(h, -h)$ and coordinate of B is (α, β), then



$$h = \frac{\alpha + \frac{1+\sqrt{2}a}{2}}{2}$$

and $$-h = \frac{\beta + \frac{1-\sqrt{2}a}{2}}{2}$$

$\therefore$ $$\alpha = 2h - \frac{1-\sqrt{2}a}{2}$$

and $$\beta = -2h - \frac{1-\sqrt{2}a}{2}$$

Since, $B(\alpha, \beta)$ lies on the given circle, we have

$\Rightarrow$ $$2\left[2h - \frac{1+\sqrt{2}a}{2}\right]^2 + 2\left[-2h - \frac{1-\sqrt{2}a}{2}\right]^2 - (1+\sqrt{2}a)\left[2h - \frac{1+\sqrt{2}a}{2}\right] - (1-\sqrt{2}a)\left[-2h - \frac{1-\sqrt{2}a}{2}\right] = 0$$

$\Rightarrow$ $$16h^2 - 4h(1+\sqrt{2}a) + 4h(1-\sqrt{2}a) + \frac{(1+\sqrt{2}a)^2}{2} + \frac{(1-\sqrt{2}a)^2}{2} - 2h(1+\sqrt{2}a) + \frac{(1+\sqrt{2}a)^2}{2} + 2h(1-\sqrt{2}a) + \frac{(1-\sqrt{2}a)^2}{2} = 0$$

$\Rightarrow$ $$16h^2 - 12\sqrt{2}\,ah + (1+\sqrt{2}a)^2 + (1-\sqrt{2}a)^2 = 0$$

$\Rightarrow$ $$16h^2 - 12\sqrt{2}\,ah + 2 + 4a^2 = 0$$

or $$8h^2 - 6\sqrt{2}\,ah + 1 + 2a^2 = 0$$

Hence, for two real and different values of h, we must have

$$(-6\sqrt{2}a)^2 - 4\cdot 8(1 + 2a^2) > 0$$

or $$72a^2 - 32(1 + 2a^2) > 0$$

$\Rightarrow$ $$8a^2 - 32 > 0$$

$\Rightarrow$ $$a^2 - 4 > 0$$

$$(a+2)(a-2) > 0$$

Hence, the required value of a (from wavy curve)

$$a \in (-\infty, -2) \cup (2, \infty)$$

Aliter : Equation of chord AB whose mid-point is $(h, -h)$ is

$$T = S_1$$

$$2xh - 2yh - (1+\sqrt{2}a)\left(\frac{x+h}{2}\right) - (1-\sqrt{2}a)\left(\frac{y-h}{2}\right) = 2h^2 + 2h^2 - (1+\sqrt{2}a)\,h + (1-\sqrt{2}a)\,h$$

$\Rightarrow$ $$4xh - 4yh - (1+\sqrt{2}a)(x+h) - (1-\sqrt{2}a)(y-h) = 8h^2 - 2(1+\sqrt{2}a)\,h + 2(1-\sqrt{2}a)\,h$$

$\Rightarrow$ $$x[4h - (1+\sqrt{2}a)] - y[4h + (1-\sqrt{2}a)] - h(1+\sqrt{2}a) + h(1-\sqrt{2}a) = 8h^2 - 2(1+\sqrt{2}a)\,h + 2(1-\sqrt{2}a)\,h$$

or $$8h^2 - (1+\sqrt{2}a)\,h + (1-\sqrt{2}a)\,h - x[4h - (1+\sqrt{2}a)] + y[4h + (1-\sqrt{2}a)] = 0$$

It passes through $A\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$, then

$$8h^2 - 2\sqrt{2}ah - \left(\frac{1+\sqrt{2}a}{2}\right)[4h - (1+\sqrt{2}a)] + \left(\frac{1-\sqrt{2}a}{2}\right)[4h + (1-\sqrt{2}a)] = 0$$

or $$8h^2 - 2\sqrt{2}ah - 2h(1+\sqrt{2}a) + \frac{(1+\sqrt{2}a)^2}{2} + 2h(1-\sqrt{2}a) + \frac{(1-\sqrt{2}a)^2}{2} = 0$$

or $$8h^2 - 6\sqrt{2}ah + 1 + 2a^2 = 0$$

Hence, for two real and different values of h, we must have

$$(-6\sqrt{2}a)^2 - 4\cdot 8\cdot(1+2a^2) > 0$$

or $$a^2 - 4 > 0$$

$\therefore$ $$(a+2)(a-2) > 0$$

$\therefore$ $$a \in (-\infty, -2) \cup (2, \infty)$$

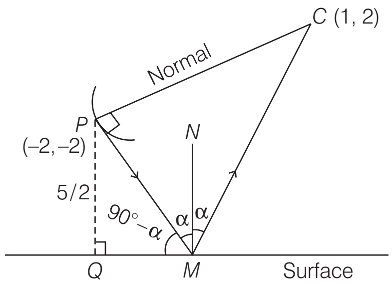
Ex. 41 *A ball moving around the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ in anti-clockwise direction leaves it tangentially at the point $P(-2, -2)$. After getting reflected from a straight line, it passes through the centre of the circle. Find the equation of the straight line if its perpendicular distance from P is $5/2$. You can assume that the angle of incidence is equal to the angle of reflection.*

Sol. Radius of the circle $= CP = \sqrt{9+16} = 5$

Let the equation is of surface is $y = mx + c$

given $$PQ = \frac{5}{2}$$

$\therefore$ $$\frac{-2m+2+c}{\sqrt{(1+m^2)}} = \pm\frac{5}{2} \qquad \text{...(i)}$$



Tangent at P strikes it at the point M and after reflection passes through the centre $C(1, 2)$.

Let MN be the normal at M.

$$\angle PMN = \angle NMC = \alpha$$

In ΔPCM, $$\tan 2\alpha = \frac{PC}{PM}$$

$\Rightarrow$ $$\tan 2\alpha = \frac{5}{PM}$$

$\Rightarrow$ $$PM = 5\cot 2\alpha \qquad \text{...(ii)}$$

and in ΔPQM

$$\sin(90° - \alpha) = \frac{5/2}{PM}$$

$\therefore$ $$PM = \frac{5}{2\cos\alpha} \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), $5\cot 2\alpha = \dfrac{5}{2\cos\alpha}$

$\Rightarrow$ $$2\cot 2\alpha \cos\alpha = 1$$

$\Rightarrow$ $$\frac{2\cos 2\alpha}{\sin 2\alpha}.\cos\alpha = 1$$

$\Rightarrow$ $$\frac{2(1-2\sin^2\alpha)\cos\alpha}{2\sin\alpha\cos\alpha} = 1$$

$\Rightarrow$ $$1 - 2\sin^2\alpha = \sin\alpha$$

$\Rightarrow$ $$2\sin^2\alpha + \sin\alpha - 1 = 0$$

$\Rightarrow$ $$(2\sin\alpha - 1)(\sin\alpha + 1) = 0$$

$\Rightarrow$ $$\sin\alpha \neq -1$$

$\therefore$ $$\sin\alpha = \frac{1}{2}$$

$\therefore$ $$\alpha = 30°$$

Tangent at $P(-2, -2)$ is

$$-2x - 2y - (x-2) - 2(y-2) - 20 = 0$$

$\Rightarrow$ $$3x + 4y + 14 = 0$$

Slope of $PM = -3/4$

$\because$ $\angle PMQ = 90° - \alpha = 90° - 30° = 60°$

$\therefore$ $\tan 60° = \left|\dfrac{m + 3/4}{1 - 3m/4}\right|$, $\sqrt{3} = \dfrac{4m+3}{4-3m}$

$\therefore$ $$m = \frac{4\sqrt{3}-3}{4+3\sqrt{3}}$$

From Eq. (i) $\pm\dfrac{5}{2} = \dfrac{2(1-m)+c}{\sqrt{1+m^2}}$

we get $$c = \frac{11+2\sqrt{3}}{4+3\sqrt{3}} \quad \text{or} \quad \frac{-39+2\sqrt{3}}{4+3\sqrt{3}}$$

c being intercept on Y-axis made by surface is clearly–ve. Hence, the required line is

$$y = \left(\frac{4\sqrt{3}-3}{4+3\sqrt{3}}\right)x + \left(\frac{-39+2\sqrt{3}}{4+3\sqrt{3}}\right)$$

$\Rightarrow$ $$(4\sqrt{3}-3)x - (4+3\sqrt{3})y - (39-2\sqrt{3}) = 0.$$

Ex. 42 *Find the limiting points of the circles $(x^2 + y^2 + 2gx + c) + \lambda(x^2 + y^2 + 2fy + d) = 0$ and show that the square of the distance between them is*

$$\frac{(c-d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2}{f^2 + g^2}$$

Sol. The given circles are

$$(x^2 + y^2 + 2gx + c) + \lambda(x^2 + y^2 + 2fy + d) = 0$$

$\Rightarrow$ $$x^2 + y^2 + \frac{2g}{1+\lambda}x + \frac{2f\lambda}{1+\lambda}y + \frac{(c+\lambda d)}{1+\lambda} = 0$$

Centre of the circle $\left(\dfrac{-g}{1+\lambda}, \dfrac{-f\lambda}{1+\lambda}\right)$

Equating the radius of this circle to zero, we get

$$\frac{g^2}{(1+\lambda)^2} + \frac{f^2\lambda^2}{(1+\lambda)^2} - \frac{(c+\lambda d)}{(1+\lambda)} = 0$$

$$\Rightarrow \quad g^2 + f^2\lambda^2 - (c+\lambda d)(1+\lambda) = 0$$

$$\Rightarrow \quad (f^2 - d)\lambda^2 - (c+d)\lambda + g^2 - c = 0$$

Let the roots be λ_1 and λ_2

then $\lambda_1 + \lambda_2 = \dfrac{(c+d)}{(f^2-d)}, \lambda_1\lambda_2 = \dfrac{g^2-c}{f^2-d}$

$$\therefore \quad (\lambda_1 - \lambda_2) = \sqrt{(\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2}$$

$$= \sqrt{\frac{(c+d)^2}{(f^2-d)^2} - \frac{4(g^2-c)}{(f^2-d)}}$$

$$= \frac{\sqrt{(c+d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2}}{(f^2-d)} \quad \text{...(i)}$$

$$\therefore \quad \lambda_1 = \frac{(c+d) + \sqrt{(c-d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2}}{2(f^2-d)} \quad \text{...(ii)}$$

and $\lambda_2 = \dfrac{(c+d) - \sqrt{\{(c-d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2\}}}{2(f^2-d)}$ ...(iii)

Hence, limiting points are

$$\left(\frac{-g}{1+\lambda_1}, \frac{-f\lambda_1}{1+\lambda_1}\right) \text{ and } \left(\frac{-g}{1+\lambda_2}, \frac{-f\lambda_2}{1+\lambda_2}\right)$$

Substituting the values of λ_1 and λ_2 from Eqs. (ii) and (iii) square of the distance between limiting points

$$= \left(\frac{-g}{1+\lambda_1} + \frac{g}{1+\lambda_2}\right)^2 + \left(\frac{-f\lambda_1}{1+\lambda_1} + \frac{f\lambda_2}{1+\lambda_2}\right)^2$$

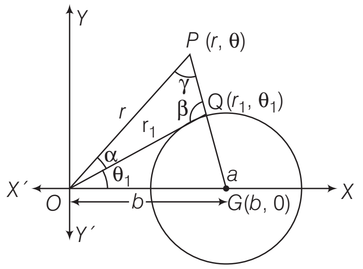
$$= \frac{(g^2+f^2)(\lambda_1-\lambda_2)^2}{[1+(\lambda_1+\lambda_2)+\lambda_1\lambda_2]^2}$$

$$= \frac{(g^2+f^2)\dfrac{\{(c-d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2\}}{(f^2-d)^2}}{\left(\dfrac{g^2+f^2}{f^2-d}\right)^2}$$

$$\Rightarrow \quad \frac{[(c-d)^2 - 4f^2g^2 + 4cf^2 + 4dg^2]}{(g^2+f^2)}$$

● ***Ex. 43*** *One vertex of a triangle of given species is fixed and another moves along circumference of a fixed circle. Prove that the locus of the remaining vertex is a circle and find its radius.*

Sol. Let OPQ be a triangle of given species. Then the angles α, β, γ will be fixed.



Let the polar coordinates of Q be (r_1, θ_1), we have to find the locus of $P(r, \theta)$. In $\angle OCQ$

$$\cos\theta_1 = \frac{r_1^2 + b^2 - a^2}{2r_1 b} \quad \text{...(i)}$$

$$\because \quad \theta = \alpha + \theta_1, \quad \therefore \theta_1 = \theta - \alpha \quad \text{...(ii)}$$

using sine rule in ΔOPQ

$$\frac{r}{\sin\beta} = \frac{r_1}{\sin\gamma}$$

$$\therefore \quad r_1 = \frac{r\sin\gamma}{\sin\beta} \quad \text{...(iii)}$$

Substituting the values of θ_1 and r_1 from Eqs. (ii)and (iii) in Eq. (i)

$$2b\frac{r\sin\gamma}{\sin\beta}\cos(\theta-\alpha) = \frac{r^2\sin^2\gamma}{\sin^2\beta} + b^2 - a^2$$

$$\Rightarrow \quad \frac{a^2\sin^2\beta}{\sin^2\gamma} = r^2 + \frac{b^2\sin^2\beta}{\sin^2\gamma} - 2b\frac{r\sin\beta}{\sin\gamma}\cos(\theta-\alpha)$$

This is an equation of circle in polar form with radius $\dfrac{\sin\beta}{\sin\gamma}$.