Question Set 9

APPLICATIONS OF DERIVATIVES

(Marks with option : 09)

9.1 GEOMETRICAL APPLICATION

Remember :

1. Let $y = f(x)$ be any curve and $P(a, f(a))$ be any point on it, then slope of the tangent to the curve at the point P is $f'(a)$. It is also called **gradient** of the curve at the point P.

   Hence, equation of the tangent at P is $y - f(a) = f'(a)(x - a)$.

2. Slope of the normal at $P(a, f(a))$ is $-\dfrac{1}{f'(a)}$, if $f'(a) \neq 0$ and equation of the normal at P is $y - f(a) = -\dfrac{1}{f'(a)}(x - a)$.

Solved Examples	2 marks each

Ex. 1. Find the equation of the tangent to the curve $y = 2x^3 - x^2 + 2$ at $\left(\dfrac{1}{2}, 2\right)$. ***(March '22)***

Solution : $y = 2x^3 - x^2 + 2$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(2x^3 - x^2 + 2)$$

$$= 2 \times 3x^2 - 2x + 0 = 6x^2 - 2x$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } (1/2,\ 2)} = 6\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)$$

$$= 6 \times \frac{1}{4} - 1 = \frac{1}{2}$$

$$= \text{slope of the tangent at } \left(\frac{1}{2}, 2\right)$$

$\therefore$ the equation of the tangent at $\left(\dfrac{1}{2}, 2\right)$ is

$$y - 2 = \frac{1}{2}\left(x - \frac{1}{2}\right)$$

$\therefore\ 2y - 4 = x - \frac{1}{2}$

$\therefore\ 4y - 8 = 2x - 1$

$\therefore\ 2x - 4y + 7 = 0$

Ex. 2. Find the equation of the normal to the curve $x^3 + 2x^2y - 9xy = 0$ at (2, 1).

Solution : $x^3 + 2x^2y - 9xy = 0$

Differentiating w.r.t. x, we get

$$3x^2 + 2\left[x^2\frac{dy}{dx} + y \times 2x\right] - 9\left[x\frac{dy}{dx} + y \times 1\right] = 0$$

$$\therefore\ 3x^2 + 2x^2\frac{dy}{dx} + 4xy - 9x\frac{dy}{dx} - 9y = 0$$

$$\therefore\ (2x^2 - 9x)\frac{dy}{dx} = -3x^2 - 4xy + 9y$$

$$\therefore\ \frac{dy}{dx} = \frac{-3x^2 - 4xy + 9y}{2x^2 - 9x}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }(2,\ 1)} = \frac{-3(2)^2 - 4(2)(1) + 9(1)}{2(2)^2 - 9(2)}$$

$$= \frac{-12 - 8 + 9}{8 - 18} = \frac{-11}{-10} = \frac{11}{10}$$

$=$ slope of the tangent at (2, 1)

$\therefore$ the slope of the normal at (2, 1)

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at }(2,\ 1)}} = \frac{-1}{\left(\frac{11}{10}\right)} = -\frac{10}{11}$$

$\therefore$ the equation of normal at (2, 1) is

$$y - 1 = -\frac{10}{11}(x - 2)$$

$\therefore\ 11y - 11 = -10x + 20 \qquad \therefore\ 10x + 11y - 31 = 0$

Examples for Practice	2 marks each

1. Find the equations of the tangents to the following curves at the given points :

(1) $y = x^2 + 4x + 1$ at $(-1,\ -2)$ (2) $y = x^3 - 2x^2 + 4$ at the point $x = 2$.

2. Find the equations of the normals to the following curves at the given points :

(1) $2x^2 + 3y^2 - 5 = 0$ at $(1, 1)$ (2) $y = x^2 + 2e^x + 2$ at $(0, 4)$.

ANSWERS

1. (1) $2x - y = 0$ (2) $4x - y = 4$.

2. (1) $3x - 2y - 1 = 0$ (2) $x + 2y - 8 = 0$.

Solved Examples	3 or 4 marks each

Ex. 3. Find the point on the curve $y = \sqrt{x-3}$, where the tangent is perpendicular to the line $6x + 3y - 5 = 0$.

Solution : Let the required point on the curve $y = \sqrt{x-3}$ be $P(x_1, y_1)$.

Differentiating $y = \sqrt{x-3}$ w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{x-3}) = \frac{1}{2\sqrt{x-3}} \cdot \frac{d}{dx}(x-3)$$

$$= \frac{1}{2\sqrt{x-3}} \times (1-0) = \frac{1}{2\sqrt{x-3}}$$

$\therefore$ slope of the tangent at (x_1, y_1)

$$= \left(\frac{dy}{dx}\right)_{\text{at } (x_1, y_1)} = \frac{1}{2\sqrt{x_1 - 3}}$$

Since this tangent is perpendicular to $6x + 3y - 5 = 0$ whose slope is $\frac{-6}{3} = -2$,

slope of the tangent $= \frac{-1}{-2} = \frac{1}{2}$.

$$\therefore \frac{1}{2\sqrt{x_1 - 3}} = \frac{1}{2}$$

$$\therefore \sqrt{x_1 - 3} = 1$$

$\therefore x_1 - 3 = 1$ $\therefore x_1 = 4$

Since (x_1, y_1) lies on $y = \sqrt{x-3}$, $y_1 = \sqrt{x_1 - 3}$

When $x_1 = 4$, $y_1 = \sqrt{4-3} = \pm 1$

Hence, the required points are $(4, 1)$ and $(4, -1)$.

Ex. 4. Find the equations of the normals to the curve $3x^2 - y^2 = 8$, which are parallel to the line $x + 3y = 4$.

Solution : Let $P(x_1, y_1)$ be the foot of the required normal to the curve $3x^2 - y^2 = 8$.

Differentiating $3x^2 - y^2 = 8$ w.r.t. x, we get

$$3 \times 2x - 2y \frac{dy}{dx} = 0$$

$$\therefore \ -2y \frac{dy}{dx} = -6x$$

$$\therefore \ \frac{dy}{dx} = \frac{3x}{y}$$

$$\therefore \ \left(\frac{dy}{dx}\right)_{\text{at } (x_1, y_1)} = \frac{3x_1}{y_1}$$

$= \text{slope of the tangent at } (x_1, y_1)$

$\therefore$ slope of the normal at P (x_1, y_1)

$$= m_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at } (x_1, y_1)}} = -\frac{y_1}{3x_1}$$

The slope of the line $x + 3y = 4$ is $m_2 = \dfrac{-1}{3}$

Since the normal at P (x_1, y_1) is parallel to the line $x + 3y = 4$, $m_1 = m_2$

$$\therefore \ -\frac{y_1}{3x_1} = -\frac{1}{3} \qquad \therefore \ y_1 = x_1$$

Since (x_1, y_1) lies on the curve $3x^2 - y^2 = 8$,

$3x_1^2 - y_1^2 = 8$

$\therefore \ 3x_1^2 - x_1^2 = 8$ ... $[\because \ y_1 = x_1]$

$\therefore \ 2x_1^2 = 8 \qquad \therefore \ x_1^2 = 4$

$\therefore \ x_1 = \pm 2$

When $x_1 = 2, y_1 = 2$

When $x_1 = -2, y_1 = -2$

$\therefore$ the coordinates of the point P are $(2, 2)$ or $(-2, -2)$ and the slope of the normal is $m_1 = m_2 = -\dfrac{1}{3}$.

$\therefore$ the equation of the normal at (2, 2) is

$$y-2=-\frac{1}{3}(x-2)$$

$\therefore\ 3y-6=-x+2$

$\therefore\ x+3y-8=0$

and the equation of the normal at $(-2,\ -2)$ is

$$y+2=-\frac{1}{3}(x+2)$$

$\therefore\ 3y-6=-x+2$

$\therefore\ x+3y+8=0$

Hence, the equations of the normals are

$x+3y-8=0$ and $x+3y+8=0$.

Ex. 5. If the line $y=4x-5$ touches the curve $y^2=ax^3+b$ at the point (2, 3), show that $7a+2b=0$.

Solution : $y^2=ax^3+b$

Differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx}=a\times 3x^2+0$$

$$\therefore\ \frac{dy}{dx}=\frac{3ax^2}{2y}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }(2,\ 3)}=\frac{3a(2)^2}{2(3)}=2a=\text{slope of the tangent at }(2,\ 3)$$

Since the line $y=4x-5$ touches the curve at the point (2, 3), slope of the tangent at (2, 3) is 4.

$\therefore\ 2a=4 \qquad \therefore\ a=2$

Since (2, 3) lies on the curve $y^2=ax^3+b$,

$(3)^2=a(2)^3+b \qquad \therefore\ 9=8a+b$

$\therefore\ 9=8(2)+b$ ... $[\because\ a=2]$

$\therefore\ b=-7$

$\therefore\ 7a+2b=7(2)+2(-7)=0.$

Examples for Practice	**3 or 4 marks each**

1. Show that the tangent to the curve $8y=(x-2)^2$ at the point $(-6,\ 8)$ is parallel to the tangent to the curve $y=x+(3/x)$ at the point (1, 4).

2. Find the points on the curve given by $y = x^3 - 6x^2 + x + 3$, where the tangents are parallel to the line $y = x + 5$.
3. Find the equation of the normal to the curve $x^2 + y^2 = 5$, where the tangent is parallel to the line $2x - y + 1 = 0$.
4. If the line $x + y = 0$ touches the curve $y^2 = ax^2 + b$ at $(1, -1)$, find a and b.

ANSWERS

2. (0, 3) and (4, -25)
3. $x + 2y = 0$
4. $a = \frac{2}{3}, b = \frac{1}{3}$.

9.2 RATE MEASURE

Remember :

1. If s is the displacement of a particle at time t, then the velocity, i.e. $v = \frac{ds}{dt}$ and the acceleration, i.e. $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.
2. If $y = f(x)$ is a differentiable function, where x and y are differentiable functions of t, then $\frac{dy}{dt} = f'(x) \cdot \frac{dx}{dt}$.

Solved Examples | **2 marks each**

Ex. 6. The displacement s of a particle at time t is given by $s = 2t^3 - 5t^2 + 4t - 3$. Find the time when the acceleration is 14 ft/sec².

Solution : $s = 2t^3 - 5t^2 + 4t - 3$

$$\therefore \text{ velocity} = v = \frac{ds}{dt} = \frac{d}{dt}(2t^3 - 5t^2 + 4t - 3)$$

$$= 2 \times 3t^2 - 5 \times 2t + 4 \times 1 - 0 = 6t^2 - 10t + 4$$

$$\text{and acceleration} = a = \frac{dv}{dt} = \frac{d}{dt}(6t^2 - 10t + 4)$$

$$= 6 \times 2t - 10 \times 1 + 0 = 12t - 10$$

Now, acceleration = 14, when $12t - 10 = 14$

i.e. when $12t = 24$, i.e. when $t = 2$

Hence, the time when acceleration is 14 ft/sec² is 2 seconds.

Ex. 7. The edge of a cube is decreasing at the rate of 0.6 cm/sec. Find the rate at which its volume is decreasing, when the edge of the cube is 2 cm.

Solution : Let x be the edge of the cube and V be its volume at any time t.

Then $V = x^3$

Differentiating both sides w.r.t. t, we get

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

Now, $\frac{dx}{dt} = 0.6$ cm/sec and $x = 2$ cm

$$\therefore \frac{dV}{dt} = 3(2)^2(0.6) = 7.2$$

Hence, the volume of the cube is decreasing at the rate of 7.2 cm^3/sec.

Ex. 8. A spherical soap bubble is expanding so that its radius is increasing at the rate of 0.02 cm/sec. At what rate is the surface area increasing, when its radius is 5 cm ?

Solution : Let r be the radius and S be the surface area of the soap bubble at any time t.

Then $S = 4\pi r^2$

Differentiating w.r.t. t, we get

$$\frac{dS}{dt} = 4\pi \times 2r \frac{dr}{dt}$$

$$\therefore \frac{dS}{dt} = 8\pi r \frac{dr}{dt} \quad \text{... (1)}$$

Now, $\frac{dr}{dt} = 0.02$ cm/sec and $r = 5$ cm

$\therefore$ (1) gives, $\frac{dS}{dt} = 8\pi(5)(0.02) = 0.8\pi$

Hence, the surface area of the soap bubble is increasing at the rate of 0.8π cm^2/sec.

Examples for Practice	2 marks each

1. The law of motion of a particle is given by $s = t^3 - 3t^2 + 6t + 1$, where s is the displacement of the particle at time t. Find its velocity and acceleration at $t = 1$.

2. The displacement s of a moving particle at a time t is given by $s = 5 + 20t - 2t^2$. Find its acceleration when the velocity is zero.

3. The displacement x of a particle at time t is given by $x = 160t - 16t^2$. Show that its velocity at $t = 1$ and $t = 9$ are equal in magnitude but opposite in directions.

4. A stone is dropped into a quite lake and waves in the form of a circle are generated, radius of the circular wave increases at the rate of 5 cm/sec. At the instant when the radius of the circular wave is 8 cm, how fast is the area enclosed increasing?

5. Water is being poured at the rate of 36 m^3/sec into a cylindrical vessel of base radius 3 metres. Find the rate at which water level is rising.

6. If each side of an equilateral triangle increases at the rate of $\sqrt{2}$ cm/sec, find the rate of increase of its area when its side is of length 3 cm.

ANSWERS

1. 3, 0 **2.** -4 units/sec^2 **4.** 80π cm^2/sec

5. $\left(\frac{4}{\pi}\right)$ m/sec **6.** $\frac{3\sqrt{6}}{2}$ cm^2/sec.

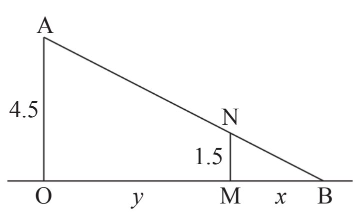
Solved Examples	3 or 4 marks each

Ex. 9. A man of height 1.5 metres walks towards a lamp post of height 4.5 metres, at the rate of $\left(\frac{3}{4}\right)$ metre/sec. Find the rate at which

(i) his shadow is shortening

(ii) the tip of shadow is moving.

Solution : Let OA be the lamp post, MN the man, MB $= x$ his shadow and OM $= y$ the distance of the man from lamp post at time t.



Then $\frac{dy}{dt} = \frac{3}{4}$ is the rate at which the man is moving towards the lamp post.

$\frac{dx}{dt}$ is the rate at which his shadow is shortening.

B is the tip of the shadow and it is at a distance of $x + y$ from the post.

$\therefore \frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt}$ is the rate at which the tip of the shadow is moving.

From the figure,

$$\frac{x}{1.5} = \frac{x+y}{4.5} \qquad \therefore\ 45x = 15x + 15y$$

$$\therefore\ 30x = 15y \qquad \therefore\ x = \frac{1}{2}y$$

$$\therefore\ \frac{dx}{dt} = \frac{1}{2} \cdot \frac{dy}{dt} = \frac{1}{2}\left(\frac{3}{4}\right) = \left(\frac{3}{8}\right) \text{ metre/sec}$$

and $\frac{dx}{dt} + \frac{dy}{dt} = \frac{3}{8} + \frac{3}{4} = \left(\frac{9}{8}\right)$ metres/sec.

Hence **(i)** the shadow is shortening at the rate of $\left(\frac{3}{8}\right)$ metre/sec, and

(ii) the tip of shadow is moving at the rate of $\left(\frac{9}{8}\right)$ metres/sec.

Ex. 10. The volume of a sphere increases at the rate of 20 cm³/sec. Find the rate of change of its surface area, when its radius is 5 cm. ***(Sept. '21)***

Solution : Let r be the radius, S be the surface area and V be the volume of the sphere at any time t.

Then $S = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$

Differentiating w.r.t. t, we get

$$\frac{dS}{dt} = 4\pi \times 2r\frac{dr}{dt} = 8\pi r\frac{dr}{dt}$$

and $\frac{dV}{dt} = \frac{4\pi}{3} \times 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$ ... (1)

From (1), $\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$

$$\therefore\ \frac{dS}{dt} = 8\pi r \times \frac{1}{4\pi r^2}\frac{dV}{dt}$$

$$\therefore \frac{dS}{dt} = \frac{2}{r} \cdot \frac{dV}{dt} \qquad \text{... (2)}$$

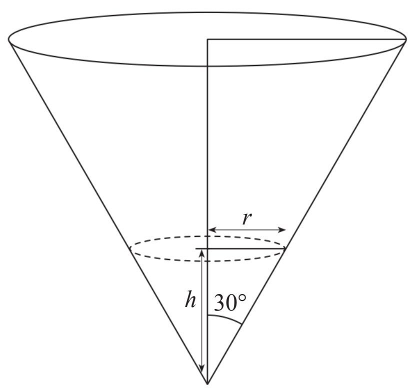
Now, $\frac{dV}{dt} = 20$ cm^3/sec and $r = 5$ cm

$\therefore$ (2) gives, $\frac{dS}{dt} = \frac{2}{5} \times 20 = 8$

Hence, the surface area of the sphere is changing at the rate of 8 cm^2/sec.

Ex. 11. If water is poured into an inverted hollow cone whose semi-vertical angle is 30°, so that its depth (measured along the axis) increases at the rate of 1 cm/sec. Find the rate at which the volume of water increasing when the depth is 2 cm.

Solution : Let r be the radius, h be the height, θ be the semi-vertical angle and V be the volume of the water at any time t.



Given : $\frac{dh}{dt} = 1$ cm/sec, $\theta = 30°$

Now, $V = \frac{1}{3}\,\pi r^2 h$

But, $\tan 30° = \frac{r}{h}$

$\therefore \frac{1}{\sqrt{3}} = \frac{r}{h} \qquad \therefore r = \frac{h}{\sqrt{3}}$

$\therefore V = \frac{1}{3}\,\pi \left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{\pi}{9}\,h^3$

Differentiating w.r.t. t, we get

$$\frac{dV}{dt} = \frac{\pi}{9} \times 3h^2 \frac{dh}{dt} = \frac{\pi}{3}\,h^2 \frac{dh}{dt}$$

When $h = 2$ cm, then

$$\frac{dV}{dt} = \frac{\pi}{3} \times (2)^2 \times 1 = \frac{4\pi}{3}$$

Hence, the volume of water is increasing at the rate of $\left(\frac{4\pi}{3}\right)$ cm^3/sec.

Examples for Practice	3 or 4 marks each

1. A man of 2 metres height walks at a uniform speed of 6 km/hr away from a lamp post of 6 metres high. Find the rate at which the length of his shadow increases.
2. A man of height 180 cm is moving away from a lamp post at the rate of 1.2 m/sec. If the height of the lamp post is 4.5 metres, find the rate at which (i) his shadow is lengthening (ii) the tip of his shadow is moving.
3. A ladder 10 metres long is leaning against a vertical wall. If the bottom of the ladder is pulled horizontally away from the wall at the rate of 1.2 metres per second, find how fast the top of the ladder is sliding down the wall, when the bottom is 6 metres away from the wall.
4. The volume of a spherical ball is increasing at the rate of 4π cc/sec. Find the rate of change of the surface area when the volume is $288\ \pi$ cc.
5. The surface area of a spherical balloon is increasing at the rate of $2\ \text{cm}^2/\text{sec}$. At what rate is the volume of the balloon is increasing, when the radius of the balloon is $6\ \text{cm}$?
6. A water tank in the form of an inverted cone is being emptied at the rate of 2 cubic feet per second. The height of the cone is 8 feet and the radius is 4 feet. Find the rate of change of the water level when the depth is 6 feet.
7. An aeroplane at an altitude of 1 km is flying horizontally at 800 km/hr, passes directly over an observer. Find the rate at which it is approaching the observer when it is 1250 metres away from him.

ANSWERS

1. 3 km/hr. **2.** (i) 0.8 m/sec (ii) 2 m/sec **3.** 0.9 metre/sec

4. $\frac{4\pi}{3}\ \text{cm}^2/\text{sec}$. **5.** $6\ \text{cm}^3/\text{sec}$ **6.** $\left(\frac{2}{9\pi}\right)$ ft/sec. **7.** 480 km/hr.

9.3 APPROXIMATIONS

Remember :

If a and $a+h$ belong to the domain of a differentiable function of x, then the approximate value of $f(a+h)$ is given by

$f(a+h) \doteqdot f(a) + h \cdot f'(a)$, where $f'(a) \neq 0$.

Solved Examples	2 or 3 marks each

Ex. 12. Find the approximate values of :

(1) $\sqrt{64.1}$ (2) $f(x) = x^3 - 3x + 5$ at $x = 1.99$.

Solution :

(1) Let $f(x) = \sqrt{x}$.

Then $f'(x) = \dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$

Take $a = 64$ and $h = 0.1$

Then $f(a) = f(64) = \sqrt{64} = 8$

Then $f'(a) = f'(64) = \dfrac{1}{2\sqrt{64}} = \dfrac{1}{16}$

The formula for approximation is

$f(a+h) \doteqdot f(a) + h \cdot f'(a)$

$\therefore \sqrt{64.1} = f(64 + 0.1)$

$\doteqdot f(64) + (0.1) \cdot f'(64)$

$\doteqdot 8 + (0.1) \times \dfrac{1}{16}$

$\doteqdot 8 + 0.00625 = 8.00625$

$\therefore \sqrt{64.1} \doteqdot 8.00625$

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(2) $f(x) = x^3 - 3x + 5$

$\therefore f'(x) = \dfrac{d}{dx}(x^3 - 3x + 5)$

$= 3x^2 - 3 \times 1 + 0 = 3x^2 - 3$

Take $a = 2$, $h = -0.01$

Then $f(a) = f(2) = (2)^3 - 3(2) + 5 = 8 - 6 + 5 = 7$

and $f'(a) = f'(2) = 3(2)^2 - 3 = 12 - 3 = 9$

The formula for approximation is

$f(a+h) \doteqdot f(a) + h \cdot f'(a)$

$\therefore f(1.99) = f(2 - 0.01) \doteqdot f(2) - (0.01) \cdot f'(2)$

$\doteqdot 7 - 0.01 \times 9 = 7 - 0.09 = 6.91$

$\therefore f(1.99) \doteqdot 6.91.$

Ex. 13. Find the approximate values of :

(1) $\cos(60^\circ\ 30')$, given that $1^\circ = 0.0175^c$, $\sqrt{3} = 1.732$

(2) $\tan^{-1}(1.001)$.

Solution :

(1) Let $f(x) = \cos x$

Then $f'(x) = \dfrac{d}{dx}(\cos x) = -\sin x$

Take $a = 60^\circ = \dfrac{\pi}{3}$ and

$$h = 30' = \left(\frac{1}{2}\right)^\circ = \left(\frac{1}{2} \times 0.0175\right)^c = 0.00875^c$$

Then $f(a) = f\left(\dfrac{\pi}{3}\right) = \cos\dfrac{\pi}{3} = \dfrac{1}{2} = 0.5$

and $f'(a) = f'\left(\dfrac{\pi}{3}\right) = -\sin\dfrac{\pi}{3} = -\dfrac{\sqrt{3}}{2} = -\dfrac{1.732}{2} = -0.866$

The formula for approximation is

$$f(a+h) \doteqdot f(a) + h \cdot f'(a)$$

$$\begin{aligned}
\therefore\ \cos(60^\circ\ 30') &= f(60^\circ\ 30') = f\left(\frac{\pi}{3} + 0.00875\right) \\
&\doteqdot f\left(\frac{\pi}{3}\right) + 0.00875 \cdot f'\left(\frac{\pi}{3}\right) \\
&\doteqdot 0.5 + (0.00875)(-0.8660) \\
&\doteqdot 0.5 - 0.0075775 = 0.4924225
\end{aligned}$$

$\therefore\ \cos(60^\circ\ 30') \doteqdot 0.4924.$

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(2) Let $f(x) = \tan^{-1} x$

Then $f'(x) = \dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

Take $a = 1$ and $h = 0.001$

Then $f(a) = f(1) = \tan^{-1} 1 = \dfrac{\pi}{4}$

and $f'(a) = f'(1) = \dfrac{1}{1+1^2} = \dfrac{1}{2}$

The formula for approximation is

$f(a+h) \doteqdot f(a)+h \cdot f'(a)$

$$\therefore \tan^{-1}(1.001) = f(1.001) = f(1+0.001)$$

$$\doteqdot f(1)+(0.001)\cdot f'(1)$$

$$\doteqdot \frac{\pi}{4}+(0.001)\times\frac{1}{2} = \frac{\pi}{4}+0.0005$$

$$\therefore \tan^{-1}(1.001) \doteqdot \frac{\pi}{4}+0.0005.$$

Remark : The answer can also be given as :

$$\tan^{-1}(1.001) \doteqdot f(1)+(0.001)\cdot f'(1)$$

$$\doteqdot \frac{\pi}{4}+(0.001)\times\frac{1}{2}$$

$$\doteqdot \frac{3.1416}{4}+0.0005$$

$$\doteqdot 0.7854+0.0005 = 0.7859.$$

Ex. 14. Find the approximate values of :

(1) $e^{2.1}$, given that $e^2 = 7.389$

(2) $\log_{10}(1016)$, given that $\log_{10} e = 0.4343$.

Solution :

(1) Let $f(x) = e^x$

Then $f'(x) = \dfrac{d}{dx}(e^x) = e^x$

Take $a = 2$ and $h = 0.1$

Then $f(a) = f(2) = e^2 = 7.389$

and $f'(a) = f'(2) = e^2 = 7.389$

The formula for approximation is

$f(a+h) \doteqdot f(a)+ h \cdot f'(a)$

$$\therefore e^{2.1} = f(2.1) = f(2+0.1)$$

$$\doteqdot f(2)+(0.1)\cdot f'(2)$$

$$\doteqdot 7.389+0.1 \times 7.389$$

$$\doteqdot 7.389+0.7389 = 8.1279$$

$$\therefore e^{2.1} \doteqdot 8.1279.$$

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(2) Let $f(x) = \log_{10} x = \dfrac{\log_e x}{\log_e 10}$

$$= (\log_{10} e)(\log x) = (0.4343) \log x$$

Then $f'(x) = (0.4343) \cdot \dfrac{d}{dx} (\log x) = \dfrac{0.4343}{x}$

Take $a = 1000$ and $h = 16$.

Then $f(a) = f(1000) = \log_{10} 1000 = \log_{10} 10^3 = 3$

and $f'(a) = f'(1000) = \dfrac{0.4343}{1000}$

The formula for approximation is

$f(a+h) \doteqdot f(a) + h \cdot f'(a)$

$\therefore \log_{10} 1016 = f(1016) = f(1000 + 16)$

$$\doteqdot f(1000) + 16 \cdot f'(1000)$$

$$\doteqdot 3 + 16 \times \frac{0.4343}{1000}$$

$$\doteqdot 3 + 0.0069488 \doteqdot 3.006949$$

$\therefore \log_{10} 1016 \doteqdot 3.006949.$

Examples for Practice	**2 or 3 marks each**

Find the approximate values of :

1. (1) $\sqrt[3]{28}$ (2) $(3.98)^3$ (3) $\sqrt[5]{31.98}$

(4) $f(x) = x^3 + 5x^2 - 2x + 3$ at $x = 1.98$.

2. (1) $\sin(30° \, 30')$, given that $1° = 0.0175^c$ and $\cos 30° = 0.866$

(2) $\cos(29° \, 30')$, given that $1° = 0.0175^c$ and $\cos 30° = 0.866$

(3) $\tan(45° \, 40')$, given that $1° = 0.0175^c$.

3. (1) $\tan^{-1}(0.999)$ (2) $\cot^{-1}(0.999)$

(3) $\cos^{-1}(0.51)$, given that $\pi = 3.1416$, $\dfrac{2}{\sqrt{3}} = 1.1547$.

4. (1) $\log_e(101)$, given that $\log_e 10 = 2.3026$

(2) $\log_{10}(998)$, given that $\log_{10} e = 0.4343$

(3) $e^{1.005}$, given that $e = 2.7183$.

ANSWERS

1. (1) 3.03704 (2) 63.04 (3) 1.99975 (4) 26.4

2. (1) 0.575775 (2) 0.870375 (3) 1.02334

3. (1) $\frac{\pi}{4} - 0.005$ OR 0.7489 (2) $\frac{\pi}{4} + 0.0005$ OR 0.7859

(3) 1.035653

4. (1) 4.6152 (2) 2.9991314 (3) 2.73189.

9.4 ROLLE'S THEOREM

Remember :

1. **Rolle's Theorem :** If a function f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.
2. **Lagrange's Mean Value Theorem :** If a function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists at least one point $c \in (a, b)$ such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

Solved Examples | **2 or 3 marks each**

Ex. 15. Verify Rolle's theorem for the function

(1) $f(x) = x^2 - 5x + 9$, $x \in [1, 4]$

(2) $f(x) = \sin x + \cos x + 7$, $x \in [0, 2\pi]$.

Solution :

(1) The function f given as $f(x) = x^2 - 5x + 9$ is a polynomial function. Hence, it is continuous on $[1, 4]$ and differentiable on $(1, 4)$.

Now, $f(1) = 1^2 - 5(1) + 9 = 1 - 5 + 9 = 5$

and $f(4) = 4^2 - 5(4) + 9 = 16 - 20 + 9 = 5$

$\therefore f(1) = f(4)$

Thus, the function f satisfies all the conditions of the Rolle's theorem.

$\therefore$ there exists $c \in (1, 4)$ such that $f'(c) = 0$.

Now, $f(x) = x^2 - 5x + 9$

$\therefore f'(x) = \frac{d}{dx}(x^2 - 5x + 9)$

$= 2x - 5 \times 1 + 0 = 2x - 5$

$\therefore f'(c) = 2c - 5$

$\therefore f'(c) = 0$ gives, $2c - 5 = 0 \quad \therefore c = 5/2 \in (1, 4)$

Hence, the Rolle's theorem is verified.

---

(2) The functions $\sin x$, $\cos x$ and 7 are continuous and differentiable on their domains.

$\therefore f(x) = \sin x + \cos x + 7$ is continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$

Now, $f(0) = \sin 0 + \cos 0 + 7 = 0 + 1 + 7 = 8$

and $f(2\pi) = \sin 2\pi + \cos 2\pi + 7 = 0 + 1 + 7 = 8$

$\therefore f(0) = f(2\pi)$

Thus, the function f satisfies all the conditions of Rolle's theorem.

$\therefore$ there exists $c \in (0, 2\pi)$ such that $f'(c) = 0$.

Now, $f(x) = \sin x + \cos x + 7$

$\therefore f'(x) = \frac{d}{dx}(\sin x + \cos x + 7)$

$= \cos x - \sin x + 0 = \cos x - \sin x$

$\therefore f'(c) = \cos c - \sin c$

$\therefore f'(c) = 0$ gives, $\cos c - \sin c = 0$

$\therefore \cos c = \sin c$

$\therefore c = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$

But $\frac{\pi}{4}, \frac{5\pi}{4} \in (0, 2\pi)$

$\therefore c = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

Hence, the Rolle's theorem is verified.

Ex. 16. Verify LMVT for the function

(1) $f(x) = x + \frac{1}{x}$, $x \in [1, 3]$

(2) $f(x) = \log x$, $x \in [1, e]$.

Solution :

(1) The function $f(x) = x + \frac{1}{x} = \frac{x^2 + 1}{x}$ is a rational function, which is continuous at all real numbers except when $x = 0$.

Hence, f is continuous on [1, 3] and differentiable on (1, 3).

Thus, the function f satisfies the conditions of Lagrange's mean value theorem.

$\therefore$ there exist $c \in (1,\ 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} \qquad \text{... (1)}$$

Now, $f(x) = x + \frac{1}{x}$

$\therefore f(3) = 3 + \frac{1}{3} = \frac{10}{3}$ and $f(1) = 1 + \frac{1}{1} = 2$

Also, $f'(x) = \frac{d}{dx}\left(x + \frac{1}{x}\right) = 1 - \frac{1}{x^2}$

$\therefore f'(c) = 1 - \frac{1}{c^2}$

$\therefore$ from (1), $1 - \frac{1}{c^2} = \frac{\frac{10}{3} - 2}{2} = \frac{4}{3 \times 2} = \frac{2}{3}$

$\therefore \frac{1}{c^2} = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore c^2 = 3 \qquad \mathrm{c} = \pm\sqrt{3}$

But $-\sqrt{3} \notin (1,\ 3)$

$\therefore c = \sqrt{3} \in [1,\ 3]$

Hence, Lagrange's mean value theorem is verified.

---

(2) The function f given as $f(x) = \log x$ is a logarithmic function which is continuous for all positive real numbers.

Hence, it is continuous on $[1,\ e]$ and differentiable on $(1,\ e)$.

Thus, the function f satisfies the conditions of Lagrange's mean value theorem.

$\therefore$ there exists $c \in (1,\ e)$ such that

$$f'(c) = \frac{f(e) - f(1)}{e - 1} \qquad \text{... (1)}$$

Now, $f(x) = \log x$

$\therefore f(1) = \log 1 = 0$ and $f(e) = \log e = 1$

Also, $f'(x) = \frac{d}{dx}(\log x) = \frac{1}{x}$

$\therefore f'(c) = \frac{1}{c}$

$\therefore$ from (1), $\frac{1}{c} = \frac{1-0}{e-1} = \frac{1}{e-1}$

$\therefore c = e - 1 \in (1, e)$

Hence, Lagrange's mean value theorem is verified.

Examples for Practice	**2 or 3 marks each**

1. Verify Rolle's theorem for the function

(1) $f(x) = x^2 - 4x + 3, x \in [1, 3]$

(2) $f(x) = x^2 - 4x + 10, x \in [0, 4]$

(3) $f(x) = \sin x - \cos x + 3, x \in [0, 2\pi]$.

2. Given an internal $[a, b]$ that satisfies hypothesis of Rolle's theorem for the function $f(x) = x^3 - 2x^2 + 3$. It is known that $a = 0$. Find the value of b.

3. Verify LMVT for the function

(1) $f(x) = \sqrt{x+4}$, $\quad x \in [0, 5]$

(2) $f(x) = x^2 - 3x - 1$, $\quad x \in \left[-\frac{11}{7}, \frac{13}{7}\right]$

(3) $f(x) = x^3 - 2x^2 - x + 2, x \in [1, 2]$.

4. Find c, if LMVT is applicable for $f(x) = (x-1)(x-2)(x-3), x \in [0, 4]$.

5. Verify Rolle's theorem for the function

$f(x) = e^x (\sin x - \cos x), x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

ANSWERS

1. (1) satisfied, $c = 2$ (2) satisfied, $c = 2$

(3) satisfied, $c = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

2. $b = 2$

3. (1) satisfied, $c = \frac{9}{4}$ (2) satisfied, $c = \frac{1}{7}$

(3) satisfied, $c = \frac{2+\sqrt{5}}{3}$

4. $2 \pm \frac{2}{\sqrt{3}}$ **5.** $c = \pi$.

9.5 INCREASING AND DECREASING FUNCTIONS

Remember :

A function f is said to be

(i) increasing in (a, b) if $f'(x) \geqslant 0$ for all $x \in (a, b)$

(ii) decreasing in (a, b) if $f'(x) \leqslant 0$ for all $x \in (a, b)$

(iii) strictly (or monotonically) increasing in (a, b) if $f'(x) > 0$ for all $x \in (a, b)$

(iv) strictly (or monotonically) decreasing in (a, b) if $f'(x) < 0$ for all $x \in (a, b)$.

Solved Examples | 2 marks each

Ex. 17. Test whether the following functions are increasing or decreasing :

(1) $f(x) = x^3 - 6x^2 + 12x - 16, x \in R$

(2) $f(x) = 2 - 3x + 3x^2 - x^3, x \in R$.

Solution :

(1) $f(x) = x^3 - 6x^2 + 12x - 16$

$$\therefore f'(x) = \frac{d}{dx}(x^3 - 6x^2 + 12x - 16)$$

$$= 3x^2 - 6 \times 2x + 12 \times 1 - 0$$

$$= 3x^2 - 12x + 12$$

$$= 3(x^2 - 4x + 4) = 3(x-2)^2 \geqslant 0 \text{ for all } x \in R$$

$\therefore f'(x) \geqslant 0$ for all $x \in R$

$\therefore f$ is increasing for all $x \in R$.

---

(2) $f(x) = 2 - 3x + 3x^2 - x^3$

$$\therefore f'(x) = \frac{d}{dx}(2 - 3x + 3x^2 - x^3)$$

$$= 0 - 3 \times 1 + 3 \times 2x - 3x^2$$

$$= -3 + 6x - 3x^2$$

$$= -3(x^2 - 2x + 1)$$

$$= -3(x-1)^2 \leqslant 0 \text{ for all } x \in R$$

$\therefore f'(x) \leqslant 0$ for all $x \in R$

$\therefore f$ is decreasing for all $x \in R$.

Ex. 18. Find the values of x for which $f(x) = x^3 + 12x^2 + 36x + 6$ is strictly increasing.

Solution : $f(x) = x^3 + 12x^2 + 36x + 6$

$$\therefore f'(x) = \frac{d}{dx}(x^3 + 12x^2 + 36x + 6)$$

$$= 3x^2 + 12 \times 2x + 36 \times 1 + 0$$

$$= 3x^2 + 24x + 36 = 3(x^2 + 8x + 12)$$

f is strictly decreasing if $f'(x) > 0$

i.e. if $3(x^2 + 8x + 12) > 0$

i.e. if $x^2 + 8x + 12 > 0$

i.e. if $x^2 + 8x > -12$

i.e. if $x^2 + 8x + 16 > -12 + 16$

i.e. if $(x + 4)^2 > 4$

i.e. if $x + 4 > 2$ or $x + 4 < -2$

i.e. if $x > -2$ or $x < -6$

i.e. f is strictly increasing, if $x < -6$ or $x > -2$

i.e. $x \in (-\infty, -6) \cup (-2, \infty)$.

Ex. 19. Find the values of x for which the function $f(x) = x^3 - 9x^2 + 24x + 12$ is strictly decreasing.

Solution : $f(x) = x^3 - 9x^2 + 24x + 12$

$$\therefore f'(x) = \frac{d}{dx}(x^3 - 9x^2 + 24x + 12)$$

$$= 3x^2 - 9 \times 2x + 24 \times 1 + 0$$

$$= 3x^2 - 18x + 24 = 3(x^2 - 6x + 8)$$

f is strictly decreasing if $f'(x) < 0$

i.e. if $3(x^2 - 6x + 8) < 0$

i.e. if $x^2 - 6x + 8 < 0$

i.e. if $x^2 - 6x < -8$

i.e. if $x^2 - 6x + 9 < -8 + 9$

i.e. if $(x - 3)^2 < 1$

i.e. if $-1 < x - 3 < 1$

i.e. if $-1+3<x-3+3<1+3$

i.e. if $2<x<4$

i.e., if $x \in (2, 4)$

$\therefore f$ is strictly decreasing if $x \in (2, 4)$.

Ex. 20. Show that $f(x)=x-\cos x$ is increasing for all x.

Solution : $f(x)=x-\cos x$

$$\therefore f'(x)=\frac{d}{dx}(x-\cos x)$$

$$=1-(-\sin x)=1+\sin x$$

Now, $-1 \leqslant \sin x \leqslant 1$ for all $x \in R$

$\therefore -1+1 \leqslant 1+\sin x \leqslant 1$ for all $x \in R$

$\therefore 0 \leqslant f'(x) \leqslant 1$ for all $x \in R$

$\therefore f'(x) \geqslant 0$ for all $x \in R$

$\therefore f$ is increasing for all x.

Examples for Practice	2 marks each

1. Test whether the following functions are increasing or decreasing :

(1) $f(x)=x^3+6x^2+12x-5, x \in R$

(2) $f(x)=\cos x, 0<x<\pi$.

2. Find the values of x such that $f(x)=2x^3-3x^2-12x+6$ is strictly increasing function.

3. Find the values of x such that $f(x)=2x^3-15x^2-144x-7$ is strictly decreasing function.

4. Find the values of x if $f(x)=\dfrac{x}{x^2+1}$ is

(1) strictly increasing function (2) strictly decreasing function.

5. Show that $f(x)=3x+\dfrac{1}{3x}$ is increasing in $\left(\dfrac{1}{3}, 1\right)$ and decreasing in $\left(\dfrac{1}{9}, \dfrac{1}{3}\right)$.

6. Show that function $f(x)=\tan x$ is increasing in $\left(0, \dfrac{\pi}{2}\right)$. *(March '22)*

ANSWERS

1. (1) increasing for all $x \in R$. (2) decreasing when $0<x<\pi$

2. $x<-1$ or $x>2$

3. $-3<x<8$

4. (1) $-1<x<1$ (2) $x<-1$ or $x>1$.

9.6 MAXIMA AND MINIMA

Remember :

Procedure for finding Maxima or Minima :

Second Derivative Test :

(1) Find $f'(x)$ and $f''(x)$.

(2) Find the roots of the equation $f'(x) = 0$.

(3) If c is a root of this equation, find $f''(c)$.

   (i) If $f''(c) < 0$, then f has a maximum at $x = c$ and $f(c)$ is the maximum value

   (ii) If $f''(c) > 0$, then f has a minimum at $x = c$ and $f(c)$ is the minimum value.

(4) Use the same procedure for the other roots of $f'(x) = 0$, as in (3).

Solved Examples	3 or 4 marks each

Ex. 21. Examine the function $f(x) = x^3 - 9x^2 + 24x$ for maxima and minima. ***(Sept. '21)***

Solution : $f(x) = x^3 - 9x^2 + 24x$

$$\therefore f'(x) = \frac{d}{dx}(x^3 - 9x^2 + 24x)$$

$$= 3x^2 - 9 \times 2x + 24 \times 1$$

$$= 3x^2 - 18x + 24$$

$$\text{and } f''(x) = \frac{d}{dx}(3x^2 - 18x + 24)$$

$$= 3 \times 2x - 18 \times 1 + 0 = 6x - 18$$

$f'(x) = 0$ gives $3x^2 - 18x + 24 = 0$

$\therefore x^2 - 6x + 8 = 0 \quad \therefore (x-2)(x-4) = 0$

$\therefore$ the roots of $f'(x) = 0$ are $x_1 = 2$ and $x_2 = 4$.

(a) $f''(2) = 6(2) - 18 = -6 < 0$

$\therefore$ by the second derivative test, f has maximum at $x = 2$ and maximum value of f at $x = 2$

$= f(2) = (2)^3 - 9(2)^2 + 24(2) = 8 - 36 + 48 = 20$

(b) $f''(4) = 6(4) - 18 = 6 > 0$

$\therefore$ by the second derivative test, f has minimum at $x = 4$ and minimum value of f at $x = 4$

$= f(4) = (4)^3 - 9(4)^2 + 24(4)$

$= 64 - 144 + 96 = 16.$

Hence, the function f has maximum value 20 at $x = 2$ and minimum value 16 at $x = 4$.

Ex. 22. Show that $f(x) = \dfrac{\log x}{x}$, $x \neq 0$ is maximum at $x = e$.

Solution : $f(x) = \dfrac{\log x}{x}$

$$\therefore\ f'(x) = \frac{d}{dx}\left(\frac{\log x}{x}\right)$$

$$= \frac{x\dfrac{d}{dx}(\log x) - \log x\dfrac{d}{dx}(x)}{x^2}$$

$$= \frac{x\left(\dfrac{1}{x}\right) - (\log x)(1)}{x^2} = \frac{1 - \log x}{x^2} \text{ and}$$

$$f''(x) = \frac{d}{dx}\left(\frac{1 - \log x}{x^2}\right)$$

$$= \frac{x^2\dfrac{d}{dx}(1 - \log x) - (1 - \log x)\dfrac{d}{dx}(x^2)}{x^4}$$

$$= \frac{x^2\left(0 - \dfrac{1}{x}\right) - (1 - \log x) \times 2x}{x^4}$$

$$= \frac{-x - 2x + 2x\log x}{x^4}$$

$$= \frac{x(2\log x - 3)}{x^4}$$

$$\therefore\ f''(x) = \frac{2\log x - 3}{x^3}$$

Now, $f'(x) = 0$, if $\dfrac{1 - \log x}{x^2} = 0$

i.e. if $1 - \log x = 0$, i.e. if $\log x = 1 = \log e$

i.e. if $x = e$

and $f''(e) = \dfrac{2\log e - 3}{e^3} = \dfrac{-1}{e^3} < 0$ ... $[\because\ \log e = 1]$

$\therefore$ by the second derivative test, $f(x)$ is maximum at $x = e$.

Ex. 23. A wire of length 36 metres is bent in the form of a rectangle. Find its dimensions if the area of the rectangle is maximum. ***(March '22)***

Solution : Let x metres and y metres be the length and breadth of the rectangle.

Then its perimeter is $2(x+y)=36$

$\therefore\ x+y=18 \qquad \therefore\ y=18-x$

Area of the rectangle $= xy = x(18-x)$

Let $f(x)=x(18-x)=18x-x^2$

$\therefore\ f'(x)=\dfrac{d}{dx}(18x-x^2)=18-2x$

and $f''(x)=\dfrac{d}{dx}(18-2x)=0-2\times 1=-2$

Now, $f'(x)=0$, if $18-2x=0$

i.e. if $x=9$

and $f''(9)=-2<0$

$\therefore$ by the second derivative test, f has maximum value at $x=9$.

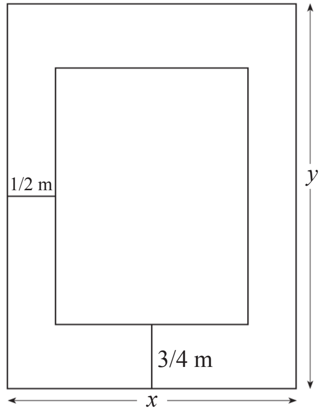
When $x=9$, $y=18-9=9$

$\therefore\ x=9$ cm, $y=9$ cm

$\therefore$ the rectangle is a square of side 9 metres.

Ex. 24. A rectangular sheet of paper has its area 24 sq metres. The margins at the top and bottom are 75 cm each and at the sides 50 cm each. What are the dimensions of the paper, if the area of the printed space is maximum?

Solution :



Let the lengths at the bottom and side be x m and y m respectively. Then the area of the sheet of paper $= xy = 24$.

$\therefore\ y = 24/x.$

After leaving the margins, the dimensions of the rectangular space left for printing are $(x-1)$ metres and $\left(y - \frac{3}{2}\right)$ m, i.e. $\left(\frac{24}{x} - \frac{3}{2}\right)$ metres.

$\therefore$ area of the space for printing

$$= (x-1)\left(\frac{24}{x} - \frac{3}{2}\right) = \frac{51}{2} - \frac{24}{x} - \frac{3x}{2} = f(x) \qquad \text{... (Say)}$$

$$\therefore\ f'(x) = \frac{24}{x^2} - \frac{3}{2} \quad \text{and} \quad f''(x) = -\frac{48}{x^3}$$

Now, $f'(x) = 0$, when $\frac{24}{x^2} - \frac{3}{2} = 0$,

i.e. when $\frac{24}{x^2} = \frac{3}{2}$, i.e. when $x^2 = 16$,

i.e. when $x = 4$ or $x = -4$

Since x is the length of a side, it is not negative

$\therefore\ x \neq -4 \quad \therefore\ x = 4$ and

$$f''(4) = -\frac{48}{(4)^3} < 0$$

$\therefore$ by the second derivative test, f has a maximum, when $x = 4$.

When $x = 4$, $y = \frac{24}{x} = \frac{24}{4} = 6$.

$\therefore$ the area of the printed space is maximum, when the dimensions of the rectangular sheet of paper are 4 metres and 6 metres.

Ex. 25. A wire of length l is cut into two parts. One part is bent into a circle and the other into a square. Prove that the sum of the areas of the circle and the square is the least, if the radius of the circle is half of the side of the square.

Solution : Let r be the radius of the circle and x be the length of the side of the square. Then (circumference of the circle) + (perimeter of the square) $= l$

$$\therefore\ 2\pi r + 4x = l \qquad \therefore\ r = \frac{l - 4x}{2\pi}$$

$A =$ (area of the circle) + (area of the square)

$$= \pi r^2 + x^2$$

$$= \pi\left(\frac{l-4x}{2\pi}\right)^2 + x^2 = x^2 + \frac{1}{4\pi}(l-4x)^2 = f(x) \qquad \text{... (Say)}$$

Then $f'(x) = 2x + \dfrac{1}{4\pi} \times 2\,(l-4x)(-4)$

$$= 2x - \frac{2}{\pi}(l-4x)$$

and $f''(x) = 2 - \dfrac{2}{\pi}(-4) = 2 + \dfrac{8}{\pi}$

Now, $f'(x) = 0$ when $2x - \dfrac{2}{\pi}(l-4x) = 0$

i.e. when $2\pi x - 2l + 8x = 0$

i.e. when $2(\pi+4)x = 2l$

i.e. when $x = \dfrac{l}{\pi+4}$

and $f''\left(\dfrac{l}{\pi+4}\right) = 2 + \dfrac{8}{\pi} > 0$

$\therefore$ by the second derivative test, f has a minimum, when $x = \dfrac{l}{\pi+4}$.

For this value of x,

$$r = \frac{l - 4\left(\dfrac{l}{\pi+4}\right)}{2\pi} = \frac{\pi l + 4l - 4l}{2\pi(\pi+4)} = \frac{l}{2(\pi+4)} = \frac{x}{2}$$

This shows that the sum of the areas of circle and square is least, when

radius of the circle $= \left(\dfrac{1}{2}\right)$ side of the square.

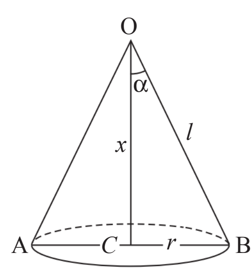
Ex. 26. The slant side of a right circular cone is *l*. Show that the semivertical angle of the cone of maximum volume is $\tan^{-1}(\sqrt{2})$.

Solution : Let the height of the cone be x. If the radius of its base is r, then $r^2 = l^2 - x^2$.

$\therefore$ the volume of the cone

$$= \frac{1}{3}\pi r^2 x = \frac{\pi}{3}(l^2 - x^2)x$$

$$= \frac{\pi}{3}(l^2 x - x^3) = f(x) \qquad \text{... (Say)}$$



$\therefore f'(x) = \dfrac{\pi}{3}(l^2 - 3x^2)$ and $f''(x) = -2\pi x$

Now, $f'(x) = 0$, when $\frac{\pi}{3}(l^2 - 3x^2) = 0$,

i.e. when $3x^2 = l^2$, i.e. when $x = \pm l/\sqrt{3}$.

Since x is not negative, $x = l/\sqrt{3}$ and

$$f''\left(\frac{l}{\sqrt{3}}\right) = -2\pi\left(\frac{l}{\sqrt{3}}\right) < 0$$

$\therefore$ by the second derivative test, f has a maximum, when $x = l/\sqrt{3}$.

$\therefore$ when the volume of the cone is maximum, its height

$= x = l/\sqrt{3}$ and the radius of its base

$$= r = \sqrt{l^2 - x^2} = \sqrt{l^2 - (l^2/3)} = \frac{l\sqrt{2}}{\sqrt{3}}.$$

Let α be the semi-vertical angle of the cone.

$$\text{Then } \tan\alpha = \frac{r}{x} = \frac{l\sqrt{2}/\sqrt{3}}{l/\sqrt{3}} = \sqrt{2}$$

$\therefore\ \alpha = \tan^{-1}\sqrt{2}$.

Hence, the semi-vertical angle of the cone of maximum volume is $\tan^{-1}(\sqrt{2})$.

Ex. 27. Show that among rectangles of given area, the square has least perimeter.

Solution : Let x be the length and y be the breadth of the rectangle whose area is A sq units (which is given as constant).

$$\text{Then } xy = A \qquad \therefore\ y = \frac{A}{x} \qquad \text{... (1)}$$

Let P be the perimeter of the rectangle.

$$\text{Then } P = 2(x + y) = 2\left(x + \frac{A}{x}\right) \qquad \text{... [By (1)]}$$

$$\therefore\ \frac{dP}{dx} = 2 \cdot \frac{d}{dx}\left(x + \frac{A}{x}\right) = 2\,[1 + A\,(-1)\,x^{-2}] = 2\left(1 - \frac{A}{x^2}\right)$$

$$\text{and } \frac{d^2P}{dx^2} = 2\,\frac{d}{dx}\left(1 - \frac{A}{x^2}\right) = 2\,[0 - A\,(-2)\,x^{-3}] = \frac{4A}{x^3}$$

Now, $\frac{dP}{dx} = 0$ gives $2\left(1 - \frac{A}{x^2}\right) = 0$

$\therefore\ x^2 - A = 0 \qquad \therefore\ x^2 = A$

$$\therefore\ x = \sqrt{A} \qquad \text{... } [\because\ x > 0]$$

and $\left(\dfrac{d^2P}{dx^2}\right)_{\text{at } x=\sqrt{A}} = \dfrac{4A}{(\sqrt{A})^3} > 0$

$\therefore$ P is minimum when $x = \sqrt{A}$.

If $x = \sqrt{A}$, then $y = \dfrac{A}{x} = \dfrac{A}{\sqrt{A}} = \sqrt{A}$

$\therefore$ $x = y$

$\therefore$ rectangle is a square.

Hence, among rectangles of given area, the square has least perimeter.

Ex. 28. A box with a square base is to have an open top. The surface area of the box is 192 sq cm. What should be its dimensions in order that the volume is as large as possible ?

Solution : Let x cm be the length of the side of the square base and y cm be the height of the box. Since the box has an open top (i.e. it is open at the top), it has 5 faces and hence its surface area

$=$ (area of the base) $+$ 4 (area of a side face) $= x^2 + 4xy$

$\therefore$ $x^2 + 4xy = 192$ $\quad\quad$ $\therefore$ $y = \dfrac{192 - x^2}{4x}$

Let V be the volume of the box. Then

$$V = x^2y = x^2\left(\frac{192 - x^2}{4x}\right) = \frac{192x - x^3}{4} = f(x) \quad \text{... (Say)}$$

$$\text{Then } f'(x) = \frac{192 - 3x^2}{4} = \frac{3(64 - x^2)}{4} = \frac{3(8 + x)(8 - x)}{4}$$

$$\text{and } f''(x) = \frac{d}{dx}\left(\frac{192}{4} - \frac{3x^2}{4}\right) = \frac{d}{dx}\left(\frac{192}{4}\right) - \frac{3}{4}\frac{d}{dx}(x^2) = 0 - \frac{3}{4}(2x) = -\frac{3x}{2}.$$

Now, $f'(x) = 0$, when $\dfrac{3(8 + x)(8 - x)}{4} = 0$, i.e. when $x = 8$ or $x = -8$

Since x is the length of the side, it is not negative. $\quad$ $\therefore$ $x \neq -8$.

$\therefore$ $x = 8$, and $f''(8) = -\dfrac{3 \times 8}{2} = -12 < 0$

$\therefore$ by the second derivative test, V is maximum, when $x = 8$.

When $x = 8$, $y = \dfrac{192 - x^2}{4x} = \dfrac{192 - 8^2}{4 \times 8} = 4$.

$\therefore$ when the volume of the box is maximum, the length of the side of the square base is 8 cm and its height is 4 cm.

Examples for Practice	3 or 4 marks each

1. Examine the following functions for maxima and minima :

(1) $f(x) = 2x^3 - 21x^2 + 36x - 20$ (2) $f(x) = x^2 + \dfrac{16}{x^2}$

(3) $f(x) = x^3 - 3x^2 - 24x + 5$ (4) $f(x) = x^2 e^x$.

2. Show that $f(x) = x^x$ is minimum when $x = \dfrac{1}{e}$.

3. Divide the number 20 into two parts such that sum of their squares is minimum.

4. A wire of length 120 cm is bent to form a rectangle. Find its dimensions if the area of the rectangle is maximum.

5. A manufacturer can sell x items at the rate of ₹ $(330 - x)$ each. The cost of producing x items is ₹ $(x^2 + 10x + 12)$. How many items must be sold so that his profit is maximum?

6. An open box is to be cut out of piece of square card of side 18 cm by cutting of equal squares from the corners and turning up the sides. Find the maximum volume of the box.

7. Two sides of a triangle are given. Find the angle between them such that the area of the triangle is maximum.

8. Find the maximum volume of right circular cylinder, if the sum of its radius and height is 6 units.

ANSWERS

1.

	Max. at	Max. value	Min. at	Min. value
(1)	1	-3	6	-128
(2)	–	–	± 2	8
(3)	-2	33	4	-75
(4)	-2	$\dfrac{4}{e^2}$	0	0

3. 10 and 10 **4.** Length = breadth = 30 cm

5. 80 **6.** 432 cc **7.** $\dfrac{\pi}{2}$ **8.** 32π cu units.

MULTIPLE CHOICE QUESTIONS	2 marks each

Select and write the most appropriate answer from the given alternatives in each of the following questions :

1. The equation of the tangent to the curve $y = 3x^2 - x + 1$ at the point (1, 3) is

(a) $y = 5x + 2$ (b) $y = 5x - 2$

(c) $y = \frac{1}{5}x + 2$ (d) $y = \frac{1}{5}x - 2$

2. If the tangent at (1, 1) on $y^2 = x(2 - x)^2$ meets the curve again at P, then P is

(a) (4, 4) (b) $(-1, 2)$

(c) (3, 6) (d) $\left(\frac{9}{4}, \frac{3}{8}\right)$

3. The approximate value of tan (44° 30') given that $1° = 0.0175^c$ is

(a) 0.8952 (b) 0.9528 (c) 0.9285 (d) 0.9825

4. If the function $f(x) = ax^3 + bx^2 + 11x - 6$ satisfies the conditions of Rolle's theorem in [1, 3] and $f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$, then values of a and b are respectively

(a) $1, -6$ (b) $-2, 1$ (c) $-1, -6$ (d) $-1, 6$

5. Let $f(x) = x^3 - 6x^2 + 9x + 18$, then $f(x)$ is strictly decreasing in

(a) $(-\infty, 1)$ (b) $[3, \infty)$

(c) $(-\infty, 1] \cup [3, \infty)$ (d) (1, 3). ***(Sept. '21)***

6. If the function $f(x) = \log x$, $x \in [1, e]$ satisfies all the conditions of LMVT, then the value of c is

(a) 1 (b) e (c) $\frac{1}{e}$ (d) $e - 1$.

7. The radius of a circular blot of oil is increasing at the rate of 2 cm/min. The rate of change of its circumference is

(a) 4 cm/sec (b) 4π cm/sec

(c) 2π cm/sec (d) π cm/sec.

8. Two numbers x and y such that $x + y = 2$ and $x^3 \cdot y$ is maximum are

(a) $\frac{1}{3}, \frac{5}{3}$ (b) $\frac{3}{2}, \frac{1}{2}$ (c) 1, 1 (d) $\frac{6}{5}, \frac{4}{5}$.

9. The maximum value of the function $f(x) = \dfrac{\log x}{x}$ is

(a) e (b) $\dfrac{1}{e}$ (c) e^2 (d) $\dfrac{1}{e^2}$. ***(March '22)***

10. If $x = -1$ and $x = 2$ are the extreme points of $y = \alpha \log x + \beta x^2 + x$, then

(a) $\alpha = -6,\ \beta = \dfrac{1}{2}$ (b) $\alpha = -6,\ \beta = -\dfrac{1}{2}$

(c) $\alpha = 2,\ \beta = -\dfrac{1}{2}$ (d) $\alpha = 2,\ \beta = \dfrac{1}{2}$.

ANSWERS

1. (b) $y = 5x - 2$ **2.** (d) $\left(\dfrac{9}{4}, \dfrac{3}{8}\right)$

3. (d) 0.9825 **4.** (a) 1, -6

5. (d) (1, 3) **6.** (d) $e - 1$

7. (b) 4π cm/sec **8.** (b) $\dfrac{3}{2}, \dfrac{1}{2}$

9. (b) $\dfrac{1}{e}$ **10.** (c) $\alpha = 2,\ \beta = -\dfrac{1}{2}$.