Permutations & Combinations

1. FUNDAMENTAL PRINCIPLE OF COUNTING

Multiplication Principle : If an operation can be performed in ‘m’ different ways; following which a second operation can be performed in ‘n’ different ways, then the two operations in succession can be performed in $m \times n$ ways. This can be extended to any finite number of operations.

Addition Principle : If an operation can be performed in ‘m’ different ways and another operation, which is independent of the first operation, can be performed in ‘n’ different ways. Then either of the two operations can be performed in $(m + n)$ ways. This can be extended to any finite number of mutually exlusive operations.

2. FACTORIALS

If n is a natural number then the product of all natural numbers upto n is called factorial n and it is denoted by $n\,!$ or $\lfloor n$.

3. PERMUTATION

Each of the different arrangements which can be made by taking some or all of a number of given things is called a permutation.

4. COUNTING FORMULAE FOR PERMUTATIONS

(I) Without Repetition :

(i) The number of permutations of n different things, taking r at a time is denoted by ${}^{n}P_{r}$ or P(n, r)

(ii) The number of arrangements of n different objects taken all at a time is ${}^{n}P_{n} = n\,!$

(II) With Repetition :

(i) The number of permutations of n things taken all at a time when p are alike of one kind, q are alike of second kind and r are alike of a third kind and the rest $n - (p + q + r)$ are all different is

$$\frac{n!}{p!\; q!\; r!}$$

(ii) The number of permutations of n different things taken r at a time when each thing may be repeated any number of times is n^r.

(III) Number of permutations under certain conditions :

- The number of permutation of n different things taken all together when r particular things are to be placed at some r given places

$$= {}^{n-r}P_{n-r} = \lfloor n-r$$

- The number of permutations of n different things taken r at a time when m particular things are to be placed at m given places $= {}^{n-m}P_{r-m}$. $(m < r)$
- Number of permutations of n different things, taken r at a time, when a particular thing is to be always included in each arrangement, is.

$$r\,.\,{}^{n-1}P_{r-1}$$

- Number of permutation of n different things, taken r at a time, when a particular thing is never taken in each arrangement is ${}^{n-1}P_{r}$
- Number of permutations of n different things, taken all at a time, when m specified things always come together is $m\,! \times (n - m + 1)!$

- Number of permutations of n different things, taken all at a time, when m specified things never come together is $n! - (m! \times (n - m + 1)!)$

(IV) Circular Permutations :

(i) Arrangement around a circular table :

The number of circular permutations of n different things taken all at a time is $(n - 1)!$, if clockwise and anticlockwise orders are taken as different.

(ii) Arrangement of beads or flowers (all different) around a circular necklace or garland :

The number of circular permutations of n different things taken all at a time is $\frac{1}{2}(n - 1)!$, if clockwise and anticlockwise orders are taken as not different.

(iii) Number of circular permutations of n different things taken r at a time :

- Case I : If clockwise and anticlockwise orders are taken as different, then the required number of circular permutations = $({}^nP_r)/r$.
- Case II : If clockwise and anticlockwise orders are taken as not different, then the required number of circular permutations = $({}^nP_r)/(2r)$.

(iv) Restricted Circular Permutations

When there is a restriction in a circular permutation then first of all we shall perform the restricted part of the operation and then perform the remaining part treating it similar to a linear permutation.

5. COMBINATION

Each of the different groups or selections which can be made by some or all of a number of given things without reference to the order of the things in each group is called a combination.

6. COUNTING FORMULAE FOR COMBINATION

(I) Selections of objects without repetition

The number of combinations of n differnt things taken r at a time is denoted by ${}^nC_r$ or C (n, r) or $\binom{n}{r}$

(II) Selections of objects with repetition :

The total number of selections of r things from n differents things when each thing may be repeated any number of times is ${}^{n+r-1}C_r$

(III) Restricted Selections / Arrangements :

(i) The number of combinations of n different things taken r at a time,

- when k particular objects occur is ${}^{n-k}C_{r-k}$.
- If k particular objects never occur is ${}^{n-k}C_r$.

(ii) The number of arrangements of n distinct objects taken r at a time so that k particular object are

- always included = ${}^{n-k}C_{r-k} \cdot r!$
- never included = ${}^{n-k}C_r \cdot r!$

(iii) The number of combinations of n objects, of which p are indentical, taken r at a time is

- ${}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + \ldots\ldots\ldots + {}^{n-p}C_0$ if $r < p$.
- ${}^{n-p}C_r + {}^{n-p}C_{r-1} + {}^{n-p}C_{r-2} + \ldots\ldots\ldots + {}^{n-p}C_{r-p}$ if $r > p$.

(IV) Selections from distinct objects :

- The number of ways of selecting one or more out of n different things
$${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - 1.$$
- The number of ways of selecting zero or more out of n different things
$${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n.$$

(V) Selections from identical objects

- The number of combination of n identical things taking $r(r < n)$ at a time is 1.
- The number of ways of selecting r things out of n alike things is $n + 1$ (where $r = 0, 1, 2, ..., n$).
- The number of ways in which a selection of atleast one thing can be made from $(p+q)$ things of which p all are alike and q all are alike $= (p+1)(q+1)-1$
- The number of ways to select some or all out of $(p + q + r)$ things where p are alike of first kind, q are alike of second kind and r are alike of third kind is

$$= (p + 1)(q + 1)(r + 1) - 1$$

(VI) Selection when both identical and distinct objects are present :

If out of $(p + q + r + t)$ things, p are alike of one kind, q are alike of second kind, r are alike of third kind and t are different, then the total number of combinations is

$$(p + 1)(q + 1)(r + 1)\,2^t - 1$$

7. DIVISION AND DISTRIBUTION OF OBJECTS

The number of ways in which $(m + n)$ different things can be divided into two groups which contain m and n things respectively is

$$^{m+n}C_m\ ^{n}C_n = \frac{(m+n)!}{m!\,n!},\ m \neq n$$

Particular cases :

When $m = n$, then total number of combination is

- $\frac{(2m)!}{(m!)^2}$ when order of groups is considered.
- $\frac{(2m)!}{2!\,(m!)^2}$ when order of groups is not considered.
- The number of ways in which $(m + n + p)$ different things can be divided into three groups which contain m, n and p things respectively is

$$^{m+n+p}C_m .\ ^{n+p}C_n .\ ^{p}C_p = \frac{(m+n+p)!}{m!\,n!\,p!},\ m \neq n \neq p$$

- Suppose mn distinct objects are to be divided into m groups, each containing n objects and the order of groups is not important, then the number of ways of doing this is given by $\frac{(mn)!}{m! \times (n!)^m}$
- If, lower, the order of groups is important then the number of ways is given by $\frac{(mn)!}{(n!)^m}$

Particular cases :

When $m = n = p$, then total number of combination is

- $\frac{(3m)!}{(m!)^3}$ when order of groups is considered.
- $\frac{(3m)!}{3!\,(m!)^3}$ when order of groups is not considered.
- Total number of ways to divide n identical things among r persons when there is no restriction is

$$^{n+r-1}C_{r-1}$$

- Total number of ways to divide n identical things among r persons so that each gets atleast one is

$$^{n-1}C_{r-1}$$

8. DEARRANGEMENT THEOREM

Any change in the given order of the thing is called a Dearrangement.

(i) If n items are arranged in a row, then the number of ways in which they can be rearranged so that no one of them occupies the place assigned to it is

$$n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+....+(-1)^n\frac{1}{n!}\right]$$

(ii) If n things are arranged at n places then the number of ways to rearrange such that exactly r things are at right places is

$$\frac{n!}{r!}\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+....+(-1)^{n-r}\frac{1}{(n-r)!}\right]$$

9. SUM OF NUMBERS

(i) For given n different digits a_1, a_2, a_3 ...., a_n the sum of the digits in the unit place of all numbers fromed (if numbers are not repeated) is

$(a_1 + a_2 + a_3 + ... + a_n)\ (n – 1)!$

i.e. (sum of the digits) (n – 1) !

(ii) Sum of the total numbers which can be formed with given n different digits a_1, a_2, a_3 .... a_n is

$(a_1 + a_2 + a_3 + ... + a_n)\ (n – 1)!$. (111 .... n times) (If nos. are not repeated)

10. IMPORTANT RESULTS ABOUT POINTS

If there are n points in a plane of which m(< n) are collinear, then

- Total number of different straight lines obtained by joining these n points is ${}^nC_2 - {}^mC_2 + 1$
- Total number of different triangles formed by joining these n points is ${}^nC_3 - {}^mC_3$
- Number of diagonals in polygon of n sides is ${}^nC_2 - n$ i.e. $\frac{n(n-3)}{2}$
- If m parallel lines in a plane are intersected by a family of other n parallel lines, then total number of parallelograms so formed is ${}^mC_2 \times {}^nC_2$.

11. SOLUTION OF EQUATION

$$x_1 + x_2 + x_3 + ... + x_r = n$$

- The number of non-negative integral solutions of

$$x_1 + x_2 + x_3 + ... + x_r = n \qquad ...(1)$$

= the number of ways of distributing n identical objects among r groups $= {}^{n+r-1}C_{r-1}$

- The number of positive integral solutions of equation (1)

$= {}^{n-r}C_{r-1}$

12. EXPONENT OF PRIME p IN n!

Exponent of a prime p in n! is denoted by E_p (n!) and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + ... + \left[\frac{n}{p^k}\right]$$

where $p^k < n < p^{k+1}$ and [.] denotes the greatest integer function.