Session 2

Position of Two Points Relative to a Given Line, Position of a Point Which Lies Inside a Triangle, Equations of Lines Parallel and Perpendicular to a Given Line, Distance of a Point From a Line, Distance Between Two Parallel Lines, Area of Parallelogram,

Position of Two Points Relative to a Given Line

Theorem : The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same or opposite sides of the line $ax + by + c = 0$ according as

$$\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0 \text{ or } < 0 .$$

Proof : Let the line PQ be divided by the line $ax + by + c = 0$ in the ratio $\lambda : 1$ (internally) at the point R.

$\therefore$ The coordinates of R are $\left(\frac{x_1 + \lambda x_2}{1+\lambda}, \frac{y_1 + \lambda y_2}{1+\lambda}\right)$.

The point of R lies on the line $ax + by + c = 0$

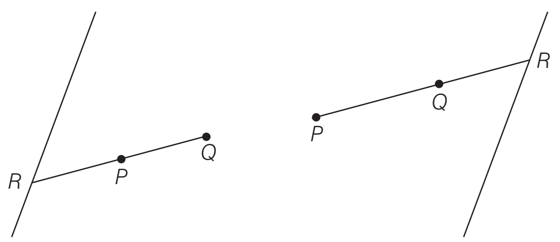
then $\quad a\left(\frac{x_1 + \lambda x_2}{1+\lambda}\right) + b\left(\frac{y_1 + \lambda y_2}{1+\lambda}\right) + c = 0$

$\Rightarrow \quad \lambda(ax_2 + by_2 + c) + (ax_1 + by_1 + c) = 0$

$\Rightarrow \quad \lambda = -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) \qquad (\because ax_2 + by_2 + c \neq 0)$

Case I : Let P and Q are on same side of the line $ax + by + c = 0$.

$\therefore$ R divides PQ externally.



$\therefore$ λ is negative

$\Rightarrow \quad -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) < 0$

$\Rightarrow \quad \left(\frac{ax_1 + by_1 + c}{a\,x_2 + by_2 + c}\right) > 0$

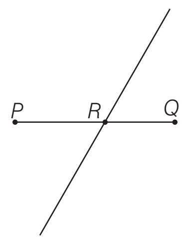
or $\quad \frac{f(x_1, y_1)}{f(x_2, y_2)} > 0$

where, $f(x, y) \equiv ax + by + c$.

Case II : Let P and Q are on opposite sides of the line $ax + by + c = 0$

$\therefore$ R divides PQ internally.

$\therefore$ λ is positive



$\Rightarrow \quad -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) > 0$

$\Rightarrow \quad \left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) < 0$

or $\quad \frac{f(x_1, y_1)}{f(x_2, y_2)} < 0$

where, $f(x, y) = ax + by + c$

Remarks

1. The side of the line where origin lies is known as origin side.
2. A point (α, β) will lie on origin side of the line $ax + by + c = 0$, if $a\alpha + b\beta + c$ and c have same sign.
3. A point (α, β) will lie on non-origin side of the line $ax + by + c = 0$, if $a\alpha + b\beta + c$ and c have opposite sign.

Example 46. Are the points (2, 1) and (–3, 5) on the same or opposite side of the line $3x - 2y + 1 = 0$?

Sol. Let $f(x, y) \equiv 3x - 2y + 1$

$$\therefore \quad \frac{f(2,1)}{f(-3,5)} = \frac{3(2) - 2(1) + 1}{3(-3) - 2(5) + 1} = -\frac{5}{18} < 0$$

Therefore, the two points are on the opposite sides of the given line.

Example 47. Is the point (2, –7) lies on origin side of the line $2x + y + 2 = 0$?

Sol. Let $f(x, y) \equiv 2x + y + 2$

$$\therefore \quad f(2, -7) = 2(2) - 7 + 2 = -1$$

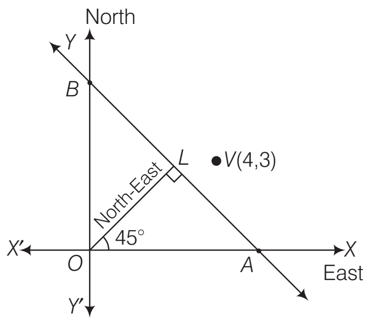
$$f(2, -7) < 0 \quad \text{and} \quad \text{constant } 2 > 0$$

Hence, the point $(2, -7)$ lies on non-origin side.

Example 48. A straight canal is at a distance of $4\frac{1}{2}$ km from a city and the nearest path from the city to the canal is in the north-east direction. Find whether a village which is at 3 km north and 4 km east from the city lies on the canal or not. If not, then on which side of the canal is the village situated ?

Sol. Let $O(0, 0)$ be the given city and AB be the straight canal.

Given, $OL = \frac{9}{2}$ km



$\therefore$ Equation of AB

i.e. Equation of canal is

$$x \cos 45° + y \sin 45° = \frac{9}{2}$$

or $$x + y = \frac{9}{\sqrt{2}} \quad \text{...(i)}$$

Let V be the given village, then $V \equiv (4, 3)$

Putting $x = 4$ and $y = 3$ in Eq. (i),

then $4 + 3 = \frac{9}{\sqrt{2}}$, i.e. $7 = \frac{9}{\sqrt{2}}$ which is impossible.

Hence, the given village V does not lie on the canal.

Also if $f(x, y) \equiv x + y - \frac{9}{\sqrt{2}}$

$$\therefore \quad \frac{f(4,3)}{f(0,0)} = \left(\frac{4 + 3 - \frac{9}{\sqrt{2}}}{0 + 0 - \frac{9}{\sqrt{2}}} \right) = -\left(\frac{7\sqrt{2} - 9}{9} \right) < 0$$

Hence, the village is on that side of the canal on which origin or the city lies.

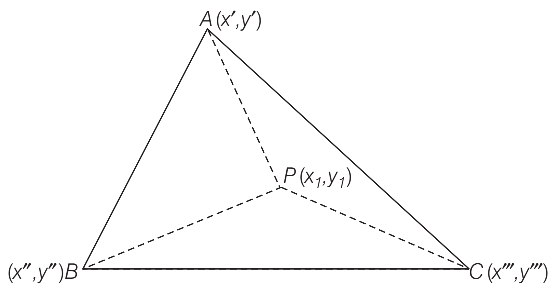
Position of a Point Which Lies Inside a Triangle

Let $P(x_1, y_1)$ be the point and equations of the sides of a triangle are

$$BC : a_1 x + b_1 y + c_1 = 0$$

$$CA : a_2 x + b_2 y + c_2 = 0$$

and $$AB : a_3 x + b_3 y + c_3 = 0$$



First find the coordinates of A, B and C say,

$$A \equiv (x', y');\ B \equiv (x'', y'') \text{ and } C \equiv (x''', y''')$$

and if coordinates of A, B, C are given, then find equations of BC, CA and AB.

If $P(x_1, y_1)$ lies inside the triangle, then P and A must be on the same side of BC, P and B must be on the same side of AC, P and C must be on the same side of AB , then

$$\frac{a_1 x_1 + b_1 y_1 + c_1}{a_1 x' + b_1 y' + c_1} > 0 \quad \text{...(i)}$$

$$\frac{a_2 x_1 + b_2 y_1 + c_2}{a_2 x'' + b_2 y'' + c_2} > 0 \quad \text{...(ii)}$$

and $$\frac{a_3 x_1 + b_3 y_1 + c_3}{a_3 x''' + b_3 y''' + c_3} > 0 \quad \text{...(iii)}$$

The required values of $P(x_1, y_1)$ must be intersection of these inequalities Eqs. (i), (ii) and (iii).

Aliter (Best Method) : First draw the exact diagram of the problem. If the point $P(x_1, y_1)$
move on the line $y = ax + b$ for all x_1, then

$$P \equiv (x_1, ax_1 + b)$$

and the portion DE of the line $y = ax + b$ (Excluding D and E) lies within the triangle. Now line $y = ax + b$ cuts any two sides out of three sides, then find coordinates of D and E.

$$D \equiv (\alpha, \beta)$$

and $\quad E \equiv (\gamma, \delta)$ (say)

then $\quad \alpha < x_1 < \gamma$

and $\quad \beta < ax_1 + b < \delta$

Example 49. For what values of the parameter t does the point $P(t, t+1)$ lies inside the triangle ABC where $A \equiv (0, 3), B \equiv (-2, 0)$ and $C \equiv (6, 1)$.

Sol. Equations of sides

$$BC : x - 8y + 2 = 0$$

$$CA : x + 3y - 9 = 0$$

and $\quad AB : 3x - 2y + 6 = 0$

Since, $P(t, t+1)$ lies inside the triangle ABC, then P and A must be on the same side of BC

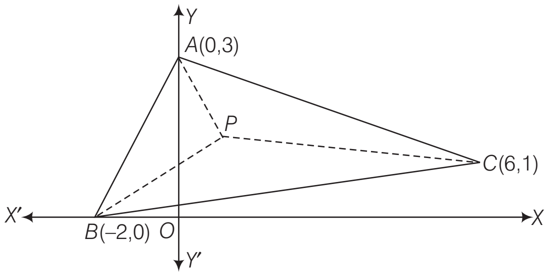
$$\therefore \quad \frac{\text{value of } (x - 8y + 2) \text{ at } P(t, t+1)}{\text{value of } (x - 8y + 2) \text{ at } A(0, 3)} > 0$$

i.e. $\quad \dfrac{t - 8(t+1) + 2}{0 - 24 + 2} > 0$

or $\quad \dfrac{-7t - 6}{-22} > 0$

or $\quad 7t + 6 > 0$

$$\therefore \quad t > -\frac{6}{7} \qquad \text{...(i)}$$



and P, B must be on the same side of CA

$$\therefore \quad \frac{\text{value of } (x + 3y - 9) \text{ at } P(t, t+1)}{\text{value of } (x + 3y - 9) \text{ at } B(-2, 0)} > 0$$

i.e. $\quad \dfrac{t + 3(t+1) - 9}{-2 + 0 - 9} > 0$

or $\quad \dfrac{4t - 6}{-11} > 0$

or $\quad 4t - 6 < 0$

$$\therefore \quad t < \frac{3}{2} \qquad \text{...(ii)}$$

and P, C must be on the same side of AB

$$\therefore \quad \frac{\text{value of } (3x - 2y + 6) \text{ at } P(t, t+1)}{\text{value of } (3x - 2y + 6) \text{ at } C(6, 1)} > 0$$

i.e. $\quad \dfrac{3t - 2(t+1) + 6}{18 - 2 + 6} > 0$

or $\quad \dfrac{t + 4}{22} > 0$

or $\quad t + 4 > 0$

$$\therefore \quad t > -4 \qquad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

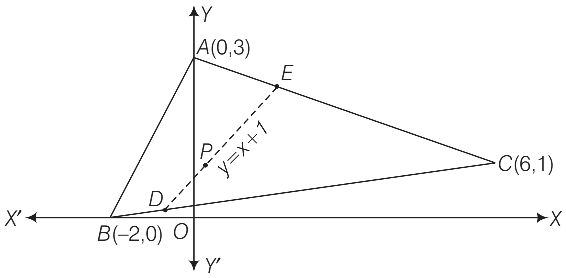
$$-\frac{6}{7} < t < \frac{3}{2}$$

i.e. $\quad t \in \left(-\dfrac{6}{7}, \dfrac{3}{2}\right)$

Aliter : First draw the exact diagram of ΔABC, the point $P(t, t+1)$ move on the line

$$y = x + 1$$

for all t.



Now, D and E are the intersection of

$$y = x + 1, x - 8y + 2 = 0$$

and $\quad y = x + 1, \quad x + 3y - 9 = 0$

respectively.

$$\therefore \quad D \equiv \left(-\frac{6}{7}, \frac{1}{7}\right)$$

and $\quad E \equiv \left(\dfrac{3}{2}, \dfrac{5}{2}\right)$

Thus, the points on the line $y = x + 1$ whose x-coordinates lies between $-\dfrac{6}{7}$ and $\dfrac{3}{2}$ lie within the triangle ABC.

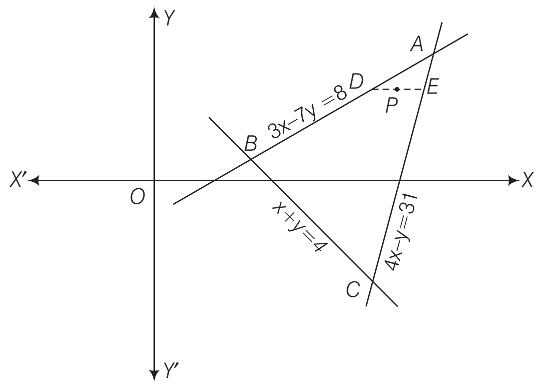
Hence, $\quad -\dfrac{6}{7} < t < \dfrac{3}{2}$

i.e. $\quad t \in \left(-\dfrac{6}{7}, \dfrac{3}{2}\right)$

Example 50. Find λ if $(\lambda, 2)$ is an interior point of ΔABC formed by $x + y = 4, 3x - 7y = 8$ and $4x - y = 31$.

Sol. Let $P \equiv (\lambda, 2)$

First draw the exact diagram of ΔABC, the point $P(\lambda, 2)$ move on the line $y = 2$ for all λ.



Now, D and E are the intersection of

$$3x - 7y = 8, y = 2$$

and $4x - y = 31, y = 2$ respectively.

$\therefore \quad D \equiv \left(\frac{22}{3}, 2\right)$ and $E \equiv \left(\frac{33}{4}, 2\right)$

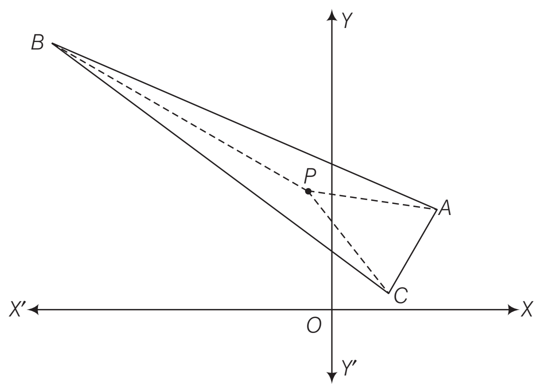
Thus, the points on the line $y = 2$ whose x-coordinates lies between $\frac{22}{3}$ and $\frac{33}{4}$ lie within the ΔABC.

Hence, $\quad \frac{22}{3} < \lambda < \frac{33}{4}$

i.e. $\quad \lambda \in \left(\frac{22}{3}, \frac{33}{4}\right)$

Example 51. Determine all values of α for which the point (α, α^2) lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$ and $5x - 6y - 1 = 0$.

Sol. The coordinates of the vertices are



$$A\left(\frac{5}{4}, \frac{7}{8}\right), B(-7, 5) \text{ and } C\left(\frac{1}{3}, \frac{1}{9}\right)$$

$\because P(\alpha, \alpha^2)$ lies inside the ΔABC, then

(i) A and P must lie on the same side of BC

(ii) B and P must lie on the same side of CA

(iii) C and P must lie on the same side of AB, then

$$\frac{\frac{5}{2} + \frac{21}{8} - 1}{2\alpha + 3\alpha^2 - 1} > 0$$

$$\Rightarrow \quad \frac{33}{2\alpha + 3\alpha^2 - 1} > 0$$

or $\quad 3\alpha^2 + 2\alpha - 1 > 0$

$$\Rightarrow \quad (\alpha + 1)\left(\alpha - \frac{1}{3}\right) > 0$$

$$\Rightarrow \quad \alpha \in (-\infty, -1) \cup \left(\frac{1}{3}, \infty\right) \qquad \text{...(i)}$$

and $\quad \frac{-35 - 30 - 1}{5\alpha - 6\alpha^2 - 1} > 0$

$$\Rightarrow \quad 5\alpha - 6\alpha^2 - 1 < 0$$

$$\Rightarrow \quad \left(\alpha - \frac{1}{3}\right)\left(\alpha - \frac{1}{2}\right) > 0$$

$$\therefore \quad \alpha \in (-\infty, 1/3) \cup (1/2, \infty) \qquad \text{..(ii)}$$

and $\quad \frac{\frac{1}{3} + \frac{2}{9} - 3}{\alpha + 2\alpha^2 - 3} > 0$

$$\Rightarrow \quad \alpha + 2\alpha^2 - 3 < 0$$

$$\Rightarrow \quad (2\alpha + 3)(\alpha - 1) < 0$$

$$\therefore \quad \alpha \in (-3/2, 1) \qquad \text{...(iii)}$$

From Eq. (i), Eq. (ii) and Eq. (iii), we get

$$\alpha \in (-3/2, -1) \cup (1/2, 1).$$

Aliter : Let $P(\alpha, \alpha^2)$ first draw the exact diagram of ΔABC.

The point $P(\alpha, \alpha^2)$ move on the curve $y = x^2$ for all α.

Now, $\quad$ intersection of $y = x^2$

and $\quad 2x + 3y - 1 = 0$

or $\quad 2x + 3x^2 - 1 = 0$

$$\therefore \quad x = -1, x = \frac{1}{3}$$

Let intersection points

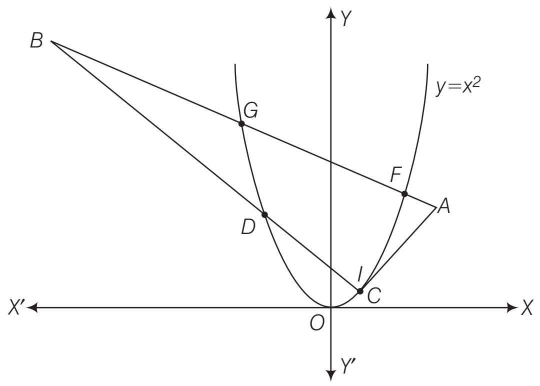
$$D \equiv (-1, 1) \text{ and } E \equiv \left(\frac{1}{3}, \frac{1}{9}\right)$$

intersection of $y = x^2$

and $\quad x + 2y - 3 = 0$

or $\quad x + 2x^2 - 3 = 0$

$$\therefore \quad x = 1, x = -3/2$$



Let intersection points

$$F \equiv (1, 1) \text{ and } G \equiv \left(-\frac{3}{2}, \frac{9}{4}\right)$$

and intersection of $y = x^2$ and $5x - 6y - 1 = 0$

or $\quad 5x - 6x^2 - 1 = 0$

$\therefore \quad x = \frac{1}{3}, x = \frac{1}{2}$

Let intersection points

$$H \equiv \left(\frac{1}{3}, \frac{1}{9}\right) \text{ and } I \equiv \left(\frac{1}{2}, \frac{1}{4}\right).$$

Thus the points on the curve $y = x^2$ whose x-coordinate lies between $-3/2$ & -1 and $\frac{1}{2}$ & 1 lies within the triangle ABC.

Hence, $\quad -\frac{3}{2} < \alpha < -1$ or $\frac{1}{2} < \alpha < 1$

i.e. $\quad \alpha \in \left(-\frac{3}{2}, -1\right) \cup \left(\frac{1}{2}, 1\right)$

Equations of Lines Parallel and Perpendicular to a Given Line

Theorem 1: The equation of line parallel to $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is some constant.

Proof : Let the equation of any line parallel to

$$ax + by + c = 0 \quad \text{...(i)}$$

be $\quad a_1 x + b_1 y + c_1 = 0 \quad$ ...(ii)

then $\quad \frac{a_1}{a} = \frac{b_1}{b} = k \quad$ (say)

$\therefore \quad a_1 = ak, b_1 = bk$

Then from Eq. (ii),

$$akx + bky + c = 0$$

Dividing it by k, then

$$ax + by + \frac{c}{k} = 0$$

or $\quad ax + by + \lambda = 0 \quad \left(\text{writing } \lambda \text{ for } \frac{c}{k}\right)$

Hence, any line parallel to $ax + by + c = 0$ is

$$ax + by + \lambda = 0$$

where λ is some constant.

Aliter : The given line is

$$ax + by + c = 0 \quad \text{...(i)}$$

Its slope $= -\frac{a}{b}$

Thus, any line parallel to Eq. (i) is given by

$$y = \left(-\frac{a}{b}\right) x + \lambda_1$$

$\Rightarrow \quad ax + by - b\lambda_1 = 0$

$\Rightarrow \quad ax + by + \lambda = 0 \quad$ (writing λ for $-b\lambda_1$)

where, λ is some constant.

***Corollary* :** The equation of the line parallel to $ax + by + c = 0$ and passing through (x_1, y_1) is

$$a(x - x_1) + b(y - y_1) = 0$$

Working Rule :

(i) Keep the terms containing x and y unaltered.
(ii) Change the constant.
(iii) The constant λ is determined from an additional condition given in the problem.

Theorem 2 : The equation of the line perpendicular to the line $ax + by + c = 0$ is

$bx - ay + \lambda = 0$, where λ is some constant.

Proof : Let the equation of any line perpendicular to

$$ax + by + c = 0 \quad \text{... (i)}$$

be $\quad a_1 x + b_1 y + c_1 = 0 \quad$ ...(ii)

then $\quad aa_1 + bb_1 = 0$

or $\quad aa_1 = -bb_1$

$\Rightarrow \quad \frac{a_1}{b} = \frac{b_1}{-a} = k \quad$ (say)

$\therefore \quad a_1 = bk, b_1 = -ak$

Then, from Eq. (ii), $bkx - aky + c_1 = 0$ dividing it by k, then

$$bx - ay + \frac{c_1}{k} = 0$$

or $$bx - ay + \lambda = 0 \qquad \left(\text{writing } \lambda \text{ for } \frac{c_1}{k}\right)$$

Hence, any line perpendicular to $ax + by + c = 0$ is

$$bx - ay + \lambda = 0$$

where, λ is some constant.

Aliter : The given line is

$$ax + by + c = 0 \qquad ...(i)$$

$$\text{Its slope} = -\frac{a}{b}$$

Slope of perpendicular line of Eq. (i) is $\frac{b}{a}$.

Thus any line perpendicular to Eq. (i) is given by

$$y = \left(\frac{b}{a}\right)x + \lambda_1$$

$\Rightarrow$ $$bx - ay + a\lambda_1 = 0$$

or $$bx - ay + \lambda = 0 \qquad (\text{writing } \lambda \text{ for } a\lambda_1)$$

where, λ is some constant.

***Corollary* 1 :** The equation of the line through (x_1, y_1) and perpendicular to $ax + by + c = 0$ is

$$b(x - x_1) - a(y - y_1) = 0$$

***Corollary* 2 :** Also equation of the line perpendicular to $ax + by + c = 0$ is written as

$$\frac{x}{a} - \frac{y}{b} + k = 0, \text{ where } k \text{ is some constant.}$$

Working Rule :

(i) Interchange the coefficients of x and y and changing sign of one of these coefficients.

(ii) Changing the constant term.

(iii) The value of λ can be determined from an additional condition given in the problem.

Example 52. Find the general equation of the line which is parallel to $3x - 4y + 5 = 0$. Also find such line through the point $(-1, 2)$.

Sol. Equation of any parallel to $3x - 4y + 5 = 0$ is

$$3x - 4y + \lambda = 0 \qquad ...(i)$$

which is general equation of the line.

Also Eq. (i) passes through $(-1, 2)$, then

$$3(-1) - 4(2) + \lambda = 0$$

$\therefore$ $$\lambda = 11$$

Then from Eq. (i) required line is

$$3x - 4y + 11 = 0$$

Example 53. Find the general equation of the line which is perpendicular to $x + y + 4 = 0$. Also find such line through the point $(1, 2)$.

Sol. Equation of any line perpendicular to $x + y + 4 = 0$ is

$$x - y + \lambda = 0 \qquad ...(i)$$

which is general equation of the line.

Also Eq. (i) passes through (1, 2), then

$$1 - 2 + \lambda = 0$$

$\therefore$ $$\lambda = 1$$

Then from Eq. (i), required line is

$$x - y + 1 = 0$$

Example 54. Show that the equation of the line passing through the point $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the line

$x \sec\theta + y \operatorname{cosec}\theta = a$ is

$x\cos\theta - y\sin\theta = a\cos 2\theta$

Sol. The given equation $x \sec\theta + y \operatorname{cosec}\theta = a$ can be written as

$$x\sin\theta + y\cos\theta = a\sin\theta\cos\theta \qquad ...(i)$$

$\therefore$ equation of perpendicular line of Eq. (i) is

$$x\cos\theta - y\sin\theta = \lambda \qquad ...(ii)$$

Also it is pass through $(a\cos^3\theta, a\sin^3\theta)$

$\therefore$ $$a\cos^3\theta\cdot\cos\theta - a\sin^3\theta\cdot\sin\theta = \lambda$$

$\Rightarrow$ $$\lambda = a(\cos^4\theta - \sin^4\theta)$$

$$= a(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$$

$$= a\cdot 1\cdot\cos 2\theta = a\cos 2\theta$$

From Eq. (ii), the required equation of the line is

$$x\cos\theta - y\sin\theta = a\cos 2\theta$$

Aliter : (From corollary (2) of Theorem (2)

Equation of any line perpendicular to the line

$$x\sec\theta + y\operatorname{cosec}\theta = a, \text{ is}$$

$$\frac{x}{\sec\theta} - \frac{y}{\operatorname{cosec}\theta} = k$$

or $$x\cos\theta - y\sin\theta = k \qquad ...(iii)$$

Also, it pass through $(a\cos^3\theta, a\sin^3\theta)$

$\therefore$ $$a\cos^3\theta\cdot\cos\theta - a\sin^3\theta\cdot\sin\theta = k$$

or $$k = a(\cos^4\theta - \sin^4\theta)$$

$$= a(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$$

$$= a\cdot 1\cdot\cos 2\theta$$

$$= a\cos 2\theta$$

From Eq. (iii), the required equation of the line is

$$x\cos\theta - y\sin\theta = a\cos 2\theta$$

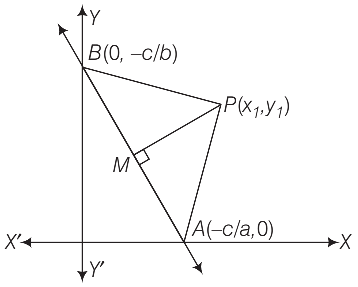
Distance of a Point From a Line

Theorem : The length of perpendicular from a point (x_1, y_1) to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

Proof : Given line is $ax + by + c = 0$

$$\Rightarrow \quad \frac{x}{\left(-\frac{c}{a}\right)} + \frac{y}{\left(-\frac{c}{b}\right)} = 1$$



Let the given line intersects the X-axis and Y-axis at A and B respectively, then coordinates of A and B are $\left(-\frac{c}{a}, 0\right)$ and $\left(0, -\frac{c}{b}\right)$ respectively.

Draw PM perpendicular to AB.

Now, Area of $\Delta\, PAB$

$$= \frac{1}{2}\left| x_1\left(0 + \frac{c}{b}\right) - \frac{c}{a}\left(-\frac{c}{b} - y_1\right) + 0\,(y_1 - 0)\right|$$

$$= \frac{1}{2}\left|\frac{c}{ab}\right| |ax_1 + by_1 + c| \qquad \text{...(i)}$$

Let $PM = p$

Also, area of $\Delta\, PAB$

$$= \frac{1}{2} \cdot AB \cdot PM = \frac{1}{2}\sqrt{\left\{\left(-\frac{c}{a} - 0\right)^2 + \left(0 + \frac{c}{b}\right)^2\right\}} \cdot p$$

$$= \frac{1}{2}\left|\frac{c}{ab}\right| \sqrt{(a^2 + b^2)} \cdot p \qquad \text{... (ii)}$$

From Eqs. (i) and (ii), we have

$$\frac{1}{2}\left|\frac{c}{ab}\right| \sqrt{(a^2 + b^2)} \cdot p = \frac{1}{2}\left|\frac{c}{ab}\right| |ax_1 + by_1 + c|$$

or

$$p = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

Aliter I : Let PM makes an angle θ with positive direction of X-axis.

Then, equation of PM in distance form will be

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = p \qquad (\because PM = p)$$

Therefore coordinates of M will be

$$(x_1 + p\cos\theta, y_1 + p\sin\theta)$$

Since, M lies on $ax + by + c = 0$, then

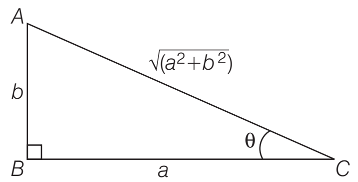
$$a\,(x_1 + p\cos\theta) + b\,(y_1 + p\sin\theta) + c = 0$$

or $\quad p\,(a\cos\theta + b\sin\theta) = -(ax_1 + by_1 + c) \qquad \text{...(iii)}$

Since, slope of $AB = -\frac{a}{b}$

$\therefore$ Slope of $PM = \frac{b}{a}$

$\therefore \quad \tan\theta = \frac{b}{a}$ ($\because$ PM makes an angle θ with positive direction of X-axis)



then $\quad \sin\theta = \frac{b}{\sqrt{(a^2 + b^2)}}$

and $\quad \cos\theta = \frac{a}{\sqrt{(a^2 + b^2)}}$

Now, from Eq. (iii),

$$p\left(\frac{a^2}{\sqrt{(a^2 + b^2)}} + \frac{b^2}{\sqrt{(a^2 + b^2)}}\right) = -(ax_1 + by_1 + c)$$

or $\quad p = -\frac{(ax_1 + by_1 + c)}{\sqrt{(a^2 + b^2)}}$

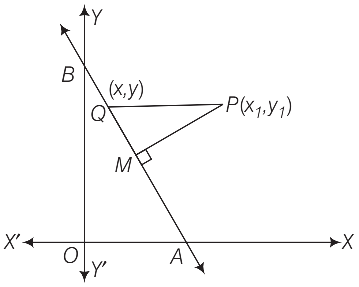
Since, p is positive

$$\therefore \quad p = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

Aliter II : Let $Q\,(x, y)$ be any point on the line

$$ax + by + c = 0$$

Hence, the length of perpendicular from P on AB will be least value of PQ.



Let $z = (PQ)^2$

$$= (x - x_1)^2 + (y - y_1)^2 \qquad \text{...(iv)}$$

$$= (x - x_1)^2 + \left(-\frac{c}{b} - \frac{ax}{b} - y_1\right)^2$$

$$\left(\begin{array}{l} \because\ ax + by + c = 0 \\ \therefore y = -\dfrac{c}{b} - \dfrac{ax}{b} \end{array}\right)$$

$\therefore$ $$\frac{dz}{dx} = 2(x - x_1) + 2\left(-\frac{c}{b} - \frac{ax}{b} - y_1\right)\left(-\frac{a}{b}\right)$$

and $$\frac{d^2z}{dx^2} = 2 + 2\left(-\frac{a}{b}\right)\left(-\frac{a}{b}\right) = 2\left(1 + \frac{a^2}{b^2}\right) = \text{positive}$$

$\because$ z is minimum

$\therefore$ PQ is also minimum.

For maximum or minimum, $\dfrac{dz}{dx} = 0$

$$2(x - x_1) + 2\left(-\frac{c}{b} - \frac{ax}{b} - y_1\right)\left(-\frac{a}{b}\right) = 0$$

or $$2(x - x_1) + 2(y - y_1)\left(-\frac{a}{b}\right) = 0 \qquad \left(\because y = -\frac{c}{b} - \frac{ax}{b}\right)$$

or $$\frac{(x - x_1)}{a} = \frac{(y - y_1)}{b} = \frac{a(x - x_1) + b(y - y_1)}{a \cdot a + b \cdot b}$$

$$= \frac{(ax + by + c) - (ax_1 + by_1 + c)}{(a^2 + b^2)}$$

(by law of proportion)

$$= \frac{0 - (ax_1 + by_1 + c)}{(a^2 + b^2)} \qquad (\because ax + by + c = 0)$$

$\Rightarrow$ $$(x - x_1) = -\frac{a(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

and $$(y - y_1) = -\frac{b(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

$\therefore$ From Eq. (iv),

$$PQ = \sqrt{(ax_1 + by_1 + c)^2 \, \frac{(a^2 + b^2)}{(a^2 + b^2)^2}} = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

$\because$ Least value of PQ is PM

$\therefore$ $$p = PM = \frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}$$

Aliter III : Let $M \equiv (h, k)$

Since, $M(h, k)$ lies on AB,

$$ah + bk + c = 0 \qquad \text{..(v)}$$

Now, AB and PM are perpendicular to each other, then

(slope of PM) $\times$ (slope of AB) $= -1$

$\Rightarrow$ $$\frac{y_1 - k}{x_1 - h} \times \left(-\frac{a}{b}\right) = -1$$

$\Rightarrow$ $$\frac{x_1 - h}{a} = \frac{y_1 - k}{b} = \frac{a(x_1 - h) + b(y_1 - k)}{a \,.\, a + b \,.\, b}$$

(by law of proportion)

$$= \frac{(ax_1 + by_1 + c) - (ah + bk + c)}{a^2 + b^2}$$

$$= \frac{ax_1 + by_1 + c}{a^2 + b^2} \qquad \text{[from Eq. (v)]}$$

$\therefore$ $$(PM)^2 = (x - x_1)^2 + (y - y_1)^2$$

$$= \left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)^2 (a^2 + b^2)$$

$\therefore$ Length of perpendicular

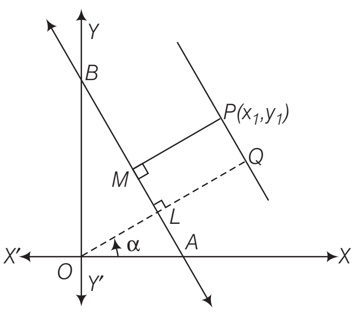
$$PM = \pm \frac{(ax_1 + by_1 + c)}{\sqrt{(a^2 + b^2)}}$$

Hence, $$PM = p = \frac{|ax_1 + by_1 + c)}{\sqrt{(a^2 + b^2)}}$$

Aliter IV : Equation of AB in normal form is

$$\frac{a}{\sqrt{(a^2 + b^2)}}\, x + \frac{b}{\sqrt{(a^2 + b^2)}}\, y = \frac{-c}{\sqrt{(a^2 + b^2)}}$$

$\Rightarrow$ $$OL = -\frac{c}{\sqrt{(a^2 + b^2)}}$$



Equation of line parallel to AB and passes through (x_1, y_1) is

$$a(x - x_1) + b(y - y_1) = 0$$

or $$ax + by = ax_1 + by_1$$

Normal form is

$$\frac{a}{\sqrt{(a^2+b^2)}}x + \frac{b}{\sqrt{(a^2+b^2)}}y = \frac{ax_1+by_1}{\sqrt{(a^2+b^2)}}$$

$$\Rightarrow \quad OQ = \frac{ax_1+by_1}{\sqrt{(a^2+b^2)}}$$

$$\therefore \quad PM = QL = OQ - OL = \frac{ax_1+by_1+c}{\sqrt{(a^2+b^2)}}$$

Hence, required perpendicular distance

$$p = \frac{|ax_1+by_1+c|}{\sqrt{(a^2+b^2)}}$$

Aliter V : The equation of line through $P(x_1, y_1)$ and perpendicular to $ax + by + c = 0$ is

$$b(x - x_1) - a(y - y_1) = 0 \qquad \text{...(vi)}$$

If this perpendicular meet the line $ax + by + c = 0$ in $M(x_2, y_2)$ then (x_2, y_2) lie on both the lines $ax + by + c = 0$ and Eq. (vi), then

$$b(x_2 - x_1) - a(y_2 - y_1) = 0, ax_2 + by_2 + c = 0$$

$$ax_2 + by_2 + c = a(x_2 - x_1) + b(y_2 - y_1) + ax_1 + by_1 + c = 0$$

or $$b(x_2 - x_1) - a(y_2 - y_1) = 0 \qquad \text{...(vii)}$$

and $$a(x_2 - x_1) + b(y_2 - y_1) = -(ax_1 + by_1 + c) \qquad \text{...(viii)}$$

On squaring and adding Eqs. (vii) and (viii), we get

$$(a^2+b^2)((x_2-x_1)^2 + (y_2-y_1)^2) = (ax_1+by_1+c)^2$$

or $$PM = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$= \frac{|ax_1+by_1+c|}{\sqrt{(a^2+b^2)}}$$

Hence, length of perpendicular

$$PM = p = \frac{|ax_1+by_1+c|}{\sqrt{(a^2+b^2)}}$$

***Corollary* 1 :** The length of perpendicular from the origin to the line $ax + by + c = 0$ is

$$\frac{|a\cdot 0 + b\cdot 0 + c|}{\sqrt{(a^2+b^2)}} \quad \text{i.e.} \quad \frac{|c|}{\sqrt{(a^2+b^2)}}$$

***Corollary* 2 :** The length of perpendicular from (x_1, y_1) to the line $x\cos\alpha + y\sin\alpha = p$ is

$$\frac{|x_1\cos\alpha + y_1\sin\alpha - p|}{\sqrt{(\cos^2\alpha+\sin^2\alpha)}} = |x_1\cos\alpha + y_1\sin\alpha - p|$$

Working Rule :

(i) Put the point (x_1, y_1) for (x, y) on the LHS while the RHS is zero.

(ii) Divide LHS after Eq. (i) by $\sqrt{(a^2+b^2)}$, where a and b are the coefficients of x and y respectively.

Example 55. Find the sum of the abscissas of all the points on the line $x + y = 4$ that lie at a unit distance from the line $4x + 3y - 10 = 0$.

Sol. Any point on the line $x + y = 4$ can be taken as $(x_1, 4 - x_1)$. As it is at a unit distance from the line $4x + 3y - 10 = 0$, we get

$$\frac{|4x_1 + 3(4 - x_1) - 10|}{\sqrt{(4^2+3^2)}} = 1$$

$$\Rightarrow \quad |x_1 + 2| = 5 \Rightarrow x_1 + 2 = \pm 5$$

$$\Rightarrow \quad x_1 = 3 \quad \text{or} \quad -7$$

$\therefore$ Required sum $= 3 - 7 = -4$.

Example 56. If p and p' are the length of the perpendiculars from the origin to the straight lines whose equations are $x\sec\theta + y\,\mathrm{cosec}\,\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$, then find the value of $4p^2 + p'^2$.

Sol. We have, $p = \dfrac{|-a|}{\sqrt{(\sec^2\theta + \mathrm{cosec}^2\theta)}}$

$$\therefore \quad p^2 = \frac{a^2}{\sec^2\theta + \mathrm{cosec}^2\theta} = \frac{a^2\sin^2\theta\cos^2\theta}{1}$$

$$\Rightarrow \quad 4p^2 = a^2\sin^2 2\theta \qquad \text{...(i)}$$

and $$p' = \frac{|-a\cos 2\theta|}{\sqrt{(\cos^2\theta+\sin^2\theta)}} = |-a\cos 2\theta|$$

$$\therefore \quad (p')^2 = a^2\cos^2 2\theta \qquad \text{...(ii)}$$

$\therefore$ Adding Eqs. (i) and (ii), we get

$$4p^2 + p'^2 = a^2$$

Example 57. ***If p is the length of the perpendicular from the origin to the line $\frac{x}{a} + \frac{y}{b} = 1$, then prove that***

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}.$$

Sol. p = length of perpendicular from origin to

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$= \frac{|0 + 0 - 1|}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}} = \frac{1}{\sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}}$$

or $$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} \quad \text{or} \quad \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$$

Example 58. Prove that no line can be drawn through the point $(4, -5)$ so that its distance from $(-2, 3)$ will be equal to 12.

Sol. Suppose, if possible.

Equation of line through $(4, -5)$ with slope of m is

$$y + 5 = m(x - 4)$$

$$\Rightarrow \quad mx - y - 4m - 5 = 0$$

Then, $\dfrac{|m(-2) - 3 - 4m - 5|}{\sqrt{m^2 + 1}} = 12$

$$\Rightarrow \quad |-6m - 8| = 12\sqrt{(m^2 + 1)}$$

On squaring, $(6m + 8)^2 = 144(m^2 + 1)$

$$\Rightarrow \quad 4(3m + 4)^2 = 144(m^2 + 1)$$

$$\Rightarrow \quad (3m + 4)^2 = 36(m^2 + 1)$$

$$\Rightarrow \quad 27m^2 - 24m + 20 = 0 \qquad \text{...(i)}$$

Since, the discriminant of Eq. (i) is $(-24)^2 - 4 \cdot 27 \cdot 20 = -1584$ which is negative, there is no real value of m. Hence no such line is possible.

Distance between Two Parallel Lines

Let the two parallel lines be

$$ax + by + c = 0 \quad \text{and} \quad ax + by + c_1 = 0$$

The distance between the parallel lines is the perpendicular distance of any point on one line from the other line.

Let (x_1, y_1) be any point on $ax + by + c = 0$

$$\therefore \quad ax_1 + by_1 + c = 0 \qquad \text{...(i)}$$

Now, perpendicular distance of the point (x_1, y_1) from the line $ax + by + c_1 = 0$ is

$$\frac{|ax_1 + by_1 + c_1|}{\sqrt{(a^2 + b^2)}} = \frac{|c_1 - c|}{\sqrt{(a^2 + b^2)}} \qquad \text{[from Eq. (i)]}$$

This is required distance between the given parallel lines.

Aliter I : The distance between the lines is

$$d = \frac{\lambda}{\sqrt{(a^2 + b^2)}}$$

(i) $\lambda = |c_1 - c|$, if both the lines are on the same side of the origin.

(ii) $\lambda = |c_1| + |c|$, if the lines are on the opposite side of the origin.

Aliter II : Find the coordinates of any point on one of the given lines, preferably putting $x = 0$ or $y = 0$. Then the perpendicular distance of this point from the other line is the required distance between the lines.

Example 59. Find the distance between the lines $5x - 12y + 2 = 0$ and $5x - 12y - 3 = 0$.

Sol. The distance between the lines

$$5x - 12y + 2 = 0 \quad \text{and} \quad 5x - 12y - 3 = 0 \text{ is}$$

$$\frac{|2 - (-3)|}{\sqrt{(5)^2 + (-12)^2}} = \frac{5}{13}$$

Aliter I : The constant term in both equations are 2 and -3 which are of opposite sign. Hence origin lies between them.

$\therefore$ Distance between lines is $\dfrac{|2| + |-3|}{\sqrt{(5)^2 + (-12)^2}} = \dfrac{5}{13}$

Aliter II : Putting $y = 0$ in $5x - 12y - 3 = 0$ then $x = \dfrac{3}{5}$

$\therefore \quad \left(\dfrac{3}{5}, 0\right)$ lie on $5x - 12y - 3 = 0$

Hence, distance between the lines

$5x - 12y + 2 = 0$ and $(5x - 12y - 3 = 0)$

$=$ Distance from $\left(\dfrac{3}{5}, 0\right)$ to the line $5x - 12y + 2 = 0$

$$= \frac{\left|5 \times \frac{3}{5} - 0 + 2\right|}{\sqrt{5^2 + (-12)^2}} = \frac{5}{13}$$

Example 60. Find the equations of the line parallel to $5x - 12y + 26 = 0$ and at a distance of 4 units from it.

Sol. Equation of any line parallel to $5x - 12y + 26 = 0$ is

$$5x - 12y + \lambda = 0 \qquad \text{...(i)}$$

Since, the distance between the parallel lines is 4 units, then

$$\frac{|\lambda - 26|}{\sqrt{(5)^2 + (-12)^2}} = 4$$

or $\quad |\lambda - 26| = 52 \quad$ or $\quad \lambda - 26 = \pm 52$

or $\quad \lambda = 26 \pm 52 \quad \therefore \quad \lambda = -26$ or 78

Substituting the values of λ in Eq. (i), we get

$$5x - 12y - 26 = 0$$

and $\quad 5x - 12y + 78 = 0$

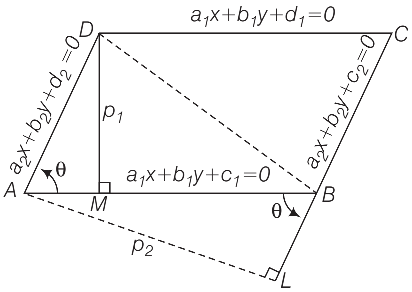
Area of Parallelogram

Theorem : Area of parallelogram $ABCD$ whose sides AB, BC, CD and DA are represented by $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_1x + b_1y + d_1 = 0$ and $\quad a_2x + b_2y + d_2 = 0$ is

$$\frac{p_1p_2}{\sin\theta} \quad \text{or} \quad \frac{|c_1 - d_1||c_2 - d_2|}{\left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|}$$

where, p_1 and p_2 are the distances between parallel sides and θ is the angle between two adjacent sides.

Proof : Since, p_1 and p_1 are the distances between the pairs of parallel sides of the parallelogram and θ is the angle between two adjacent sides, then



Area of parallelogram $ABCD$

$$= 2 \times \text{Area of } \Delta ABD$$
$$= 2 \times \frac{1}{2} \times AB \times p_1$$
$$= AB \times p_1$$
$$= \frac{p_2}{\sin\theta} \times p_1 \qquad \left(\because \text{ in } \Delta ABL, \sin\theta = \frac{p_2}{AB}\right)$$
$$= \frac{p_1 p_2}{\sin\theta} \qquad \text{...(i)}$$

Now, $p_1 =$ Distance between parallel sides AB and DC

$$= \frac{|c_1 - d_1|}{\sqrt{(a_1^2 + b_1^2)}}$$

and $p_2 =$ Distance between parallel sides AD and BC

$$= \frac{|c_2 - d_2|}{\sqrt{(a_2^2 + b_2^2)}}$$

$$\text{Also, } \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\frac{a_1}{b_1} - \left(-\frac{a_2}{b_2}\right)}{1 + \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)}\right|$$

$$\left(\begin{array}{l}\because\ m_1 = \text{slope of } AB = -\dfrac{a_1}{b_1} \\ \text{and } m_2 = \text{slope of } AD = -\dfrac{a_2}{b_2}\end{array}\right)$$

$$= \left|\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right|$$

$$\therefore \qquad \sin\theta = \frac{|a_1 b_2 - a_2 b_1|}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

Now, substitute the values of p_1, p_2 and $\sin\theta$ in Eq. (i)

$$\therefore \text{ Area of parallelogram } ABCD = \frac{|c_1 - d_1||c_2 - d_2|}{|a_1 b_2 - a_2 b_1|}$$
$$= \frac{|c_1 - d_1||c_2 - d_2|}{\left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|}$$

Corollaries :

1. If $p_1 = p_2$, then $ABCD$ becomes a rhombus

$$\therefore \text{ Area of rhombus } ABCD = \frac{p_1^2}{\sin\theta}$$
$$= \frac{(c_1 - d_1)^2}{|a_1 b_2 - a_2 b_1| \sqrt{\left(\dfrac{a_1^2 + b_1^2}{a_2^2 + b_2^2}\right)}}$$

2. If d_1 and d_2 are the lengths of two perpendicular diagonals of a rhombus, then

$$\text{Area of rhombus} = \frac{1}{2} d_1 d_2$$

3. Area of the parallelogram whose sides are $y = mx + a$, $y = mx + b$, $y = nx + c$ and $y = nx + d$ is $\dfrac{|a - b||c - d|}{|m - n|}$.

Example 61. Show that the area of the parallelogram formed by the lines $x + 3y - a = 0$, $3x - 2y + 3a = 0$, $x + 3y + 4a = 0$ and $3x - 2y + 7a = 0$ is $\frac{20}{11} a^2$ sq units.

Sol. Required area of the parallelogram

$$= \frac{|-a - 4a|\,|3a - 7a|}{\left|\begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix}\right|} = \frac{20}{11} a^2 \text{ sq units}$$

Example 62. Show that the area of the parallelogram formed by the lines

$$x\cos\alpha + y\sin\alpha = p,\ x\cos\alpha + y\sin\alpha = q,$$
$$x\cos\beta + y\sin\beta = r,\ x\cos\beta + y\sin\beta = s \text{ is}$$
$$|(p - q)(r - s)\operatorname{cosec}(\alpha - \beta)|.$$

Sol. The equation of sides of the parallelogram are

$$x\cos\alpha + y\sin\alpha - p = 0,$$
$$x\cos\alpha + y\sin\alpha - q = 0,$$
$$x\cos\beta + y\sin\beta - r = 0,$$
$$\text{and} \qquad x\cos\beta + y\sin\beta - s = 0$$

$\therefore$ Required area of the parallelogram

$$= \frac{|-p-(-q)||-r-(-s)|}{\left|\begin{vmatrix}\cos\alpha & \sin\alpha \\ \cos\beta & \sin\beta\end{vmatrix}\right|} = \frac{|p-q||r-s|}{|\sin(\beta-\alpha)|}$$

$$= |(p-q)(r-s)\operatorname{cosec}(\alpha-\beta)|$$

Example 63. Prove that the diagonals of the parallelogram formed by the lines

$$\frac{x}{a}+\frac{y}{b}=1, \frac{x}{b}+\frac{y}{a}=1, \frac{x}{a}+\frac{y}{b}=2 \text{ and } \frac{x}{b}+\frac{y}{a}=2$$

are at right angles. Also find its area ($a \neq b$).

Sol. The distance between the parallel sides

$$\frac{x}{a}+\frac{y}{b}=1 \quad \text{and} \quad \frac{x}{a}+\frac{y}{b}=2$$

is $$\frac{|2-1|}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}} = \frac{1}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}} = p_1 \quad \text{(say)}$$

and the distance between the parallel sides

$$\frac{x}{b}+\frac{y}{a}=1 \quad \text{and} \quad \frac{x}{b}+\frac{y}{a}=2$$

is $$\frac{|2-1|}{\sqrt{\left(\frac{1}{b^2}+\frac{1}{a^2}\right)}} = \frac{1}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}} = p_2 \quad \text{(say)}$$

Here, $p_1 = p_2$.

$\therefore$ Parallelogram is a rhombus.

But we know that diagonals of rhombus are perpendicular to each other.

$\therefore$ Area of the rhombus $$= \frac{|(-1+2)(-1+2)|}{\left|\begin{vmatrix}\frac{1}{a} & \frac{1}{b} \\ \frac{1}{b} & \frac{1}{a}\end{vmatrix}\right|}$$

$$= \frac{a^2b^2}{|b^2-a^2|} (a \neq b)$$

Example 64. Show that the four lines $ax \pm by \pm c = 0$ enclose a rhombus whose area is $\frac{2c^2}{|ab|}$.

Sol. The given lines are

$$ax + by + c = 0 \quad \text{...(i)}$$
$$ax + by - c = 0 \quad \text{...(ii)}$$
$$ax - by + c = 0 \quad \text{...(iii)}$$
and $$ax - by - c = 0 \quad \text{...(iv)}$$

Distance between the parallel lines Eqs. (i) and (ii) is $\frac{2c}{\sqrt{(a^2+b^2)}} = p_1$ (say) and distance between the parallel lines Eqs. (iii) and (iv) is

$$\frac{2c}{\sqrt{(a^2+b^2)}} = p_2 \quad \text{(say)}$$

Here, $p_1 = p_2$

$\therefore$ it is a rhombus.

$\therefore$ Area of rhombus $$= \frac{|(c+c)(c+c)|}{\left|\begin{vmatrix}a & b \\ a & -b\end{vmatrix}\right|} = \frac{4c^2}{|-2ab|} = \frac{2c^2}{|ab|}$$

Exercise for Session 2

1. The number of lines that are parallel to $2x + 6y - 7 = 0$ and have an intercept 10 between the coordinate axes is

(a) 1 (b) 2 (c) 4 (d) infinitely many

2. The distance between the lines $4x + 3y = 11$ and $8x + 6y = 15$ is

(a) $\frac{7}{2}$ (b) $\frac{7}{5}$ (c) $\frac{7}{10}$ (d) $\frac{9}{10}$

3. If the algebraic sum of the perpendicular distances from the points (2, 0), (0, 2) and (1, 1) to a variable straight line is zero, then the line passes through the point

(a) (1, 1) (b) (–1, 1) (c) (–1, –1) (d) (1, –1)

4. If the quadrilateral formed by the lines $ax + by + c = 0$, $a'x + b'y + c' = 0$, $ax + by + c' = 0$ and $a'x + b'y + c' = 0$ have perpendicular diagonals, then

(a) $b^2 + c^2 = b'^2 + c'^2$ (b) $c^2 + a^2 = c'^2 + a'^2$ (c) $a^2 + b^2 = a'^2 + b'^2$ (d) None of these

5. The area of the parallelogram formed by the lines $3x - 4y + 1 = 0$, $3x - 4y + 3 = 0$, $4x - 3y - 1 = 0$ and $4x - 3y - 2 = 0$, is

(a) $\frac{1}{7}$ sq units (b) $\frac{2}{7}$ sq units (c) $\frac{3}{7}$ sq units (d) $\frac{4}{7}$ sq units

6. Area of the parallelogram formed by the lines $y = mx$, $y = mx + 1$, $y = nx$ and $y = nx + 1$ equals

(a) $\frac{|m+n|}{(m-n)^2}$ (b) $\frac{2}{|m+n|}$ (c) $\frac{1}{|m+n|}$ (d) $\frac{1}{|m-n|}$

7. The coordinates of a point on the line $y = x$ where perpendicular distance from the line $3x + 4y = 12$ is 4 units, are

(a) $\left(\frac{3}{7}, \frac{5}{7}\right)$ (b) $\left(\frac{3}{2}, \frac{3}{2}\right)$ (c) $\left(-\frac{8}{7}, -\frac{8}{7}\right)$ (d) $\left(\frac{32}{7}, -\frac{32}{7}\right)$

8. A line passes through the point $(2, 2)$ and is perpendicular to the line $3x + y = 3$, then its y-intercept is

(a) $-\frac{2}{3}$ (b) $\frac{2}{3}$ (c) $-\frac{4}{3}$ (d) $\frac{4}{3}$

9. If the points (1, 2) and (3, 4) were to be on the opposite side of the line $3x - 5y + a = 0$, then

(a) $7 < a < 11$ (b) $a = 7$ (c) $a = 11$ (d) $a < 7$ or $a > 11$

10. The lines $y = mx$, $y + 2x = 0$, $y = 2x + k$ and $y + mx = k$ form a rhombus if m equals

(a) –1 (b) $\frac{1}{2}$ (c) 1 (d) 2

11. The points on the axis of x, whose perpendicular distance from the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is a

(a) $\frac{b}{a}(a \pm \sqrt{(a^2 + b^2)}, 0)$ (b) $\frac{a}{b}(b \pm \sqrt{(a^2 + b^2)}, 0)$

(c) $\frac{b}{a}(a + b, 0)$ (d) $\frac{a}{b}(a \pm \sqrt{(a^2 + b^2)}, 0)$

12. The three sides of a triangle are given by $(x^2 - y^2)(2x + 3y - 6) = 0$. If the point $(-2, a)$ lies inside and $(b, 1)$ lies outside the triangle, then

(a) $a \in \left(2, \frac{10}{3}\right); b \in (-1, 1)$ (b) $a \in \left(-2, \frac{10}{3}\right); b \in \left(-1, \frac{9}{2}\right)$

(c) $a \in \left(1, \frac{10}{3}\right); b \in (-3, 5)$ (d) None of these

13. Are the points (3, 4) and (2, –6) on the same or opposite sides of the line $3x - 4y = 8$?

14. If the points (4, 7) and $(\cos\theta, \sin\theta)$, where $0 < \theta < \pi$, lie on the same side of the line $x + y - 1 = 0$, then prove that θ lies in the first quadrant.

15. Find the equations of lines parallel to $3x - 4y - 5 = 0$ at a unit distance from it.

16. Show that the area of the parallelogram formed by the lines $2x - 3y + a = 0$, $3x - 2y - a = 0$, $2x - 3y + 3a = 0$ and $3x - 2y - 2a = 0$ is $\frac{2a^2}{5}$ sq units.

17. A line 'L' is drawn from $P(4, 3)$ to meet the lines $L_1 : 3x + 4y + 5 = 0$ and $L_2 : 3x + 4y + 15 = 0$ at point A and B respectively. From 'A' a line, perpendicular to L is drawn meeting the line L_2 at A_1. Similarly from point 'B' a line, perpendicular to L is drawn meeting the line L_1 at B_1. Thus a parallelogram AA_1BB_1 is formed. Find the equation (s) of 'L' so that the area of the parallelogram AA_1BB_1 is least.

18. The vertices of a ΔOBC are $O(0, 0)$, $B(-3, -1)$, $C(-1, -3)$. Find the equation of the line parallel to BC and intersecting the sides OB and OC and whose perpendicular distance from the origin is $\frac{1}{2}$.

Answers

Exercise for Session 2

1. (b) **2.** (c) **3.** (a) **4.** (c) **5.** (b) **6.** (d)
7. (c,d) **8.** (d) **9.** (d) **10.** (d) **11.** (b) **12.** (d)

13. The two points are on the opposite side of the given line.

15. $3x - 4y = 0$ and $3x - 4y - 10 = 0$

17. $7x + y - 31 = 0$ **18.** $2x + 2y + \sqrt{2} = 0$