Time Response Analysis of Control System

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The time response of a control system means as to how a system behaves in accordance with time when a specified input test signal is applied.

The time response of a control system is divided in two parts:

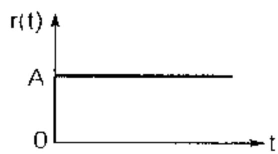
(i) Transient response

(ii) Steady state response

Transient part of time response reveals the nature of response and its speed, where as steady state part of time response reveals the accuracy of a control system.

Standard Input Test Signals

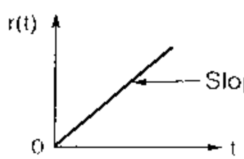
(a) Step function



$$r(t) = Au(t)$$

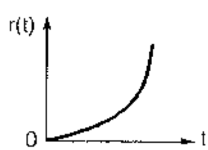
where, $$u(t) = \begin{cases} 1 \ ; & t \geq 0 \\ 0 \ ; & t < 0 \end{cases}$$

(b) Ramp function



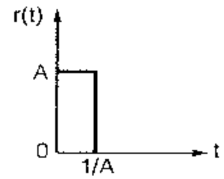
$$r(t) = Atu(t)$$

(c) Parabolic function



$$r(t) = \frac{At^2}{2}u(t)$$

(d) Impulse Function

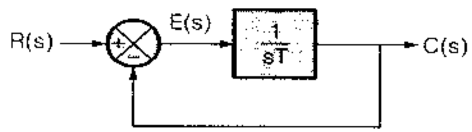


$$\delta(t) = \dot{u}(t)$$

$$\delta(t) = 0;\ t \neq 0$$

$$\lim_{\alpha \to 0} \int_{-\alpha}^{+\alpha} \delta(t)dt = 1$$

Time response of a first order control system

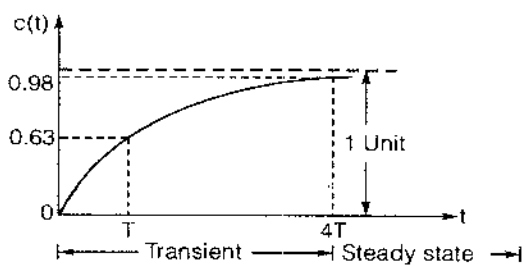


- Transfer Function

$$\frac{C(s)}{R(s)} = \frac{1}{sT + 1}$$

(i) Subjected to Unit Step Input Function

$$C(s) = \frac{1}{s(sT + 1)}$$



- Time response expression

$$c(t) = \mathcal{L}^{-1}C(s) = 1 - e^{-t/T}$$

- Error

$$e(t) = r(t) - c(t) = e^{-t/T}$$

- Steady state error

$$e_{ss} = \lim_{t \to \infty} e^{-t/T} = 0$$

Note: Lower the time constant, faster is the time response of a control system.

Subjected to Unit Ramp Input Function

- Time response expression

$$c(t) = (t - T + Te^{-t/T})$$

- Error

$$e(t) = r(t) - c(t) = (T - Te^{-t/T})$$

- Steady State Error

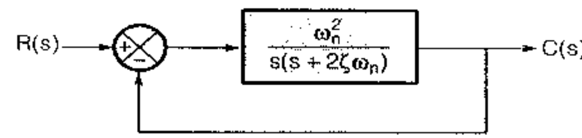
$$e_{ss} = \lim_{t \to \infty} (T - Te^{-t/T}) = T$$

Subjected to Unit Impulse Input Function

- Time response expression

$$c(t) = \frac{1}{T} e^{-t/T}$$

Time Response of Second Order Control System



- Transfer function

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where, ω_n = Natural frequency of oscillations
ζ = Damping ratio
$\zeta\omega_n$ = Damping factor

- Damped frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

where, ω_d = Damped frequency of oscillations

Subjected to Unit Step Input Function

- Time response expression

  Case-1: $\zeta < 1$ i.e. underdamped oscillations

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Note:

- Response settles within 2% of the desired value (1 unit) after damping out the oscillations in a time 4T (or $4/\zeta\omega_n$).

Case-2: $\zeta = 0$ i.e. sustained (undamped) Oscillations

$$c(t) = (1 - \cos\omega_n t)$$

Case-3: $\zeta = 1$ i.e. critically Damped Oscillations

$$c(t) = [1 - e^{-\omega_n t}(1 + \omega_n t)]$$

Case-4: $\zeta > 1$ i.e. overdamped Oscillations

$$c(t) = 1 - \frac{e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}}{\left(2\sqrt{\zeta^2-1}\right)\left(\zeta-\sqrt{\zeta^2-1}\right)}$$

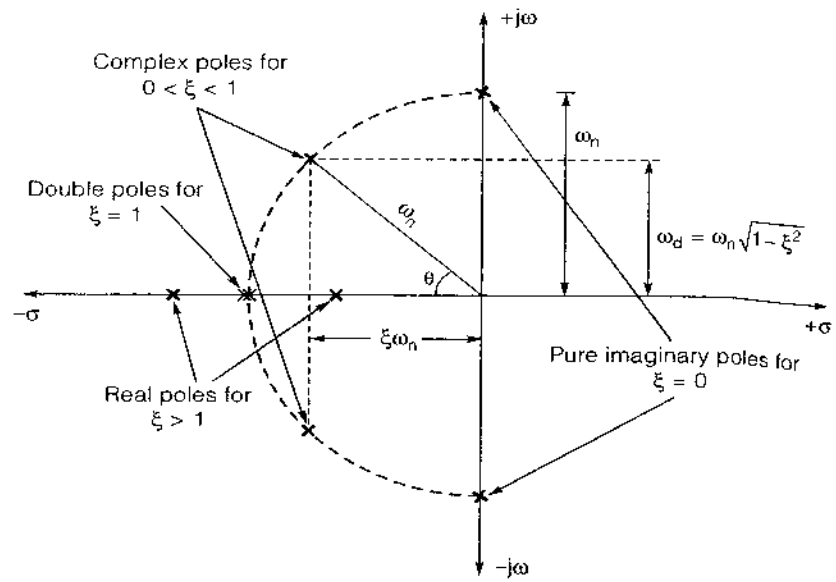
- Time constant of the response

$$T = \frac{1}{\left(\zeta - \sqrt{\zeta^2-1}\right)\omega_n}$$

Effect of ξ on Time Response

S.No.	Value of ξ	Type of closed poles	Nature of response	System classification
1.	$\xi = 0$	Purely imaginary	Oscillations with constant frequency and amplitude	Undamped
2.	$0 < \xi < 1$	Complex conjugate	Damped oscillation	Underdamped
3.	$\xi = 1$	Real, equal and negative	Critical and pure exponential	Critically damped
4.	$1 < \xi < \infty$	Real, unequal and negative	Purely exponential slow and sluggish	Overdamped

Pole Locations for Different Cases, for a Second Order System



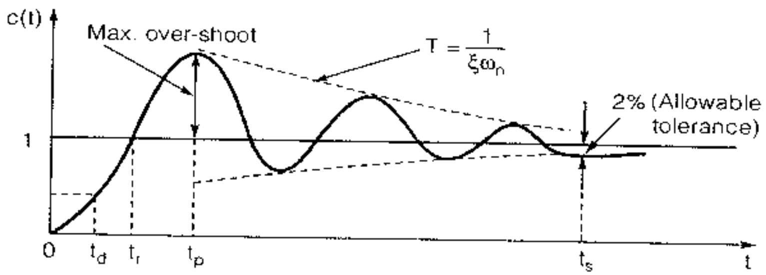
Note:

- As ξ increases from 0 to 1 then the frequency of oscillation reduces.
- For all positive value of ξ, system are stable.
- Pole away from origin called insignificant pole, and does not affect the stability of system.
- $\theta = \cos^{-1}\xi$.

Some Practical Example of Second Order System

Series RLC Circuit	Parallel RLC Circuit	Translatory System	Rotational System
Characteristic equation	Characteristic equation	Characteristic equation	Characteristic equation
$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$	$s^2 + \frac{f}{M}s + \frac{K}{M} = 0$	$s^2 + \frac{f}{J}s + \frac{K}{J} = 0$
$\omega_n = \frac{1}{\sqrt{LC}}$	$\omega_n = \frac{1}{\sqrt{LC}}$	$\omega_n = \sqrt{\frac{K}{M}}$	$\omega_n = \sqrt{\frac{K}{J}}$
$\xi = \frac{R}{2}\sqrt{\frac{C}{L}}$	$\xi = \frac{1}{2R}\sqrt{\frac{L}{C}}$	$\xi = \frac{f}{2\sqrt{KM}}$	$\xi = \frac{f}{2\sqrt{KJ}}$

Transient Response Specifications of Second Order Control System



1. Delay Time (t_d)

It is the time require for the response to rise from 0 to 50% of the final value.

$$t_d = \frac{1 + 0.7\xi}{\omega_n}$$

2. Rise Time (t_r)

It is the time required for the response to rise from

0 to 100% (underdamped) → for underdamped system

5 to 95% (critical damped) → for critically damped system

10 to 90% (over damped) → for overdamped system

of the final value

$$t_r = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_d} = \frac{\pi - \cos^{-1}\xi}{\omega_d} \text{ radian}$$

3. Peak Time (t_p)

It is the time required for the response to rise from zero to peaks of the time response.

$$t_p = \frac{n\pi}{\omega_d}$$

where, $n = 1, 2, 3, \ldots$

for $n = 1$, first overshoot

$n = 2$, first undershoot

4. Settling Time (t_s)

It is the time require for response to rise and reach to the tolerance band

For 2% tolerance band, $t_s = 4\tau = \frac{4}{\xi\omega_n}$

For 5% tolerance band, $t_s = 3\tau = \frac{3}{\xi\omega_n}$

5. Peak Over Shoot (M_p)

It gives the normalised difference between the steady state output to first peak of the time response.

$$\%M_p = e^{-\left(\frac{\pi\xi}{\sqrt{1-\xi^2}}\right)} \times 100\%$$

Note: ....

- The time period of the oscillation before reaching the steady state.

$$T_{oscillation} = \frac{2\pi}{\omega_d}$$

- Number of oscillation before reaching steady state is

$$N = \frac{T_{settling}}{T_{oscillation}}$$

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Types and Order of System

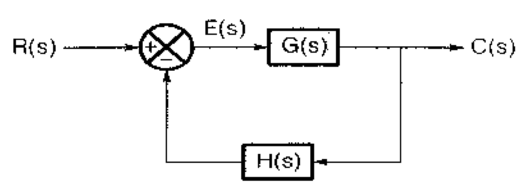
- Every Transfer function representing the control system has certain type and order.
- The steady state analysis depends on type of the system.
- The type of the system is obtain from open loop transfer function i.e. G(s) H(s).
- The number of open loop poles occuring at origin determines the type of the system.

Let $$G(s)H(s) = \frac{K(1+sT_a)}{s^P(1+sT_1)}$$

For
P = 0 for Type-0 system
P = 1 for Type-1 system
P = n for Type-n system

- Transient state analysis depends on order of the system.
- The order of system is obtain from closed loop transfer function i.e. $\frac{G(s)}{1+G(s)H(s)}$.
- The highest power of characteristic equation, i.e. 1 + G(s) H(s) = 0 determines the order of control system.

Steady State Error



$$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

Steady state error, $$e_{ss} = \lim_{t\to\infty} e(t)$$ (in time-domain)

$$e_{ss} = \lim_{s\to 0} sE(s)$$ (in s-domain)

$$e_{ss} = \lim_{s\to 0} s\frac{R(s)}{1+G(s)H(s)}$$

$$e_{ss} = \frac{\lim_{s\to 0} sR(s)}{1+\lim_{s\to 0} G(s)H(s)}$$

Note:

Steady state error depends on two factor:

(i) Type of input applied i.e. R(s).

(ii) Type of system i.e. G(s) H(s).

1. Steady State Error for Different Types of Input

(i) Step input

$$R(s) = \frac{A}{s}$$

$$e_{ss} = \frac{A}{1+K_p}$$

$$K_p = \lim_{s\to 0} G(s)H(s)$$

where, K_p = Static position error constant

(ii) Ramp input

$$R(s) = \frac{A}{s^2}$$

$$e_{ss} = \frac{A}{K_v}$$

$$K_v = \lim_{s\to 0} sG(s)H(s)$$

where, K_v = Static velocity error constant

(iii) Parabolic input

$$R(s) = \frac{A}{s^3}$$

$$\boxed{e_{ss} = \frac{A}{K_a}}$$

$$\boxed{K_a = \lim_{s \to 0} s^2 G(s)H(s)}$$

where, K_a = Static acceleration error constant

2. Steady State Error for Different Types of System

Type	Step input	Ramp input	Parabolic input
Type-0	$\frac{A}{1+K}$	∞	∞
Type-1	0	$\frac{A}{K}$	∞
Type-2	0	0	$\frac{A}{K}$

Where, K is the system gain.

Observation

(i) $e_{ss} \propto \frac{1}{K}$, so, as system gain increases, steady state error decreases.

(ii) For Linear Time Invariant (LTI) system, the maximum type number is two. Beyond type-2, the system exhibits non-linear characteristics.

Note:

- Steady state error is valid only for closed loop stable system'
- Steady state error are calculated to the closed loop system by using open loop transfer function.

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