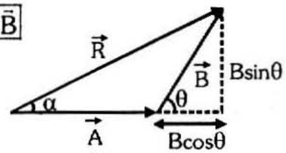
2 Vectors

- **Vector Quantities**

A physical quantity which requires magnitude and a particular direction, when it is expressed.

- **Triangle law of Vector addition** $\boxed{\vec{R}=\vec{A}+\vec{B}}$



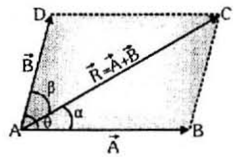
$R=\sqrt{A^2+B^2+2AB\cos\theta}$

$\tan\alpha=\dfrac{B\sin\theta}{A+B\cos\theta}$ If A = B then $R=2A\cos\dfrac{\theta}{2}$ & $\alpha=\dfrac{\theta}{2}$

R_{max} = A+B for θ=0° ; R_{min} = A–B for θ=180°

- **Parallelogram Law of Addition of Two Vectors**

If two vectors are represented by two adjacent sides of a parallelogram which are directed away from their common point then their sum (i.e. resultant vector) is given by the diagonal of the parallelogram passing away through that common point.

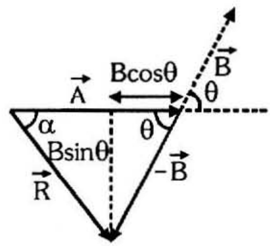


$\overline{AB}+\overline{AD}=\overline{AC}=\vec{R}$ or $\vec{A}+\vec{B}=\vec{R}\Rightarrow R=\sqrt{A^2+B^2+2AB\cos\theta}$

$\tan\alpha=\dfrac{B\sin\theta}{A+B\cos\theta}$ and $\tan\beta=\dfrac{A\sin\theta}{B+A\cos\theta}$

- **Vector subtraction**

$\boxed{\vec{R}=\vec{A}-\vec{B}}\Rightarrow\vec{R}=\vec{A}+(-\vec{B})$

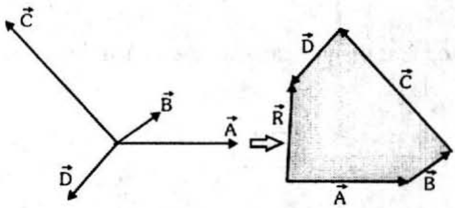


$R=\sqrt{A^2+B^2-2AB\cos\theta}$, $\tan\alpha=\dfrac{B\sin\theta}{A-B\cos\theta}$

If A = B then $R=2A\sin\dfrac{\theta}{2}$

- **Addition of More than Two Vectors (Law of Polygon)**

If some vectors are represented by sides of a polygon in same order, then their resultant vector is represented by the closing side of polygon in the opposite order.



$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$$

- **Rectangular component of a 3–D vector**

□ $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

Angle made with x-axis $\quad \cos\alpha = \dfrac{A_x}{A} = \dfrac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = \ell$

Angle made with y-axis $\quad \cos\beta = \dfrac{A_y}{A} = \dfrac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = m$

Angle made with z-axis $\quad \cos\gamma = \dfrac{A_z}{A} = \dfrac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = n$

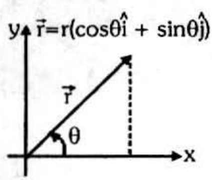
□ ℓ, m, n are called direction cosines

$$\ell^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{A_x^2 + A_y^2 + A_z^2}{\left(\sqrt{A_x^2 + A_y^2 + A_z^2}\right)^2} = 1$$

or $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

- **General Vector in x-y plane**

$$\vec{r} = x\hat{i} + y\hat{j} = r\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$



Examples

1. Construct a vector of magnitude 6 units making an angle of 60° with x-axis.

Sol. $\vec{r} = r(\cos 60\hat{i} + \sin 60\hat{j}) = 6\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) = 3\hat{i} + 3\sqrt{3}\hat{j}$

2. Construct an unit vector making an angle of 135° with x axis.

Sol. $\hat{r} = 1(\cos 135°\hat{i} + \sin 135°\hat{j}) = \frac{1}{\sqrt{2}}(-\hat{i} + \hat{j})$

- **Scalar product (Dot Product)**

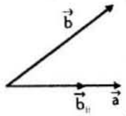
  - $\vec{A}.\vec{B} = AB\cos\theta \Rightarrow \boxed{\text{Angle between two vectors } \theta = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{AB}\right)}$

  - If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ & $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ then

    $\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$ and angle between $\vec{A}$ & $\vec{B}$ is given by

    $$\cos\theta = \frac{\vec{A}.\vec{B}}{AB} = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

  - $\hat{i}.\hat{i} = 1$, $\hat{j}.\hat{j} = 1$, $\hat{k}.\hat{k} = 1$, $\hat{i}.\hat{j} = 0$, $\hat{i}.\hat{k} = 0$, $\hat{j}.\hat{k} = 0$

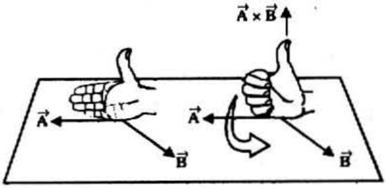


  - Component of vector $\vec{b}$ along vector $\vec{a}$, $\vec{b}_{||} = \left(\vec{b}\,.\,\hat{a}\right)\hat{a}$

  - Component of $\vec{b}$ perpendicular to $\vec{a}$, $\vec{b}_{\perp} = \vec{b} - \vec{b}_{||} = \vec{b} - \left(\vec{b}\cdot\hat{a}\right)\hat{a}$

- **Cross Product (Vector product)**

  - $\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$ where $\hat{n}$ is a vector perpendicular to $\vec{A}$ & $\vec{B}$ or their plane and its direction given by right hand thumb rule.



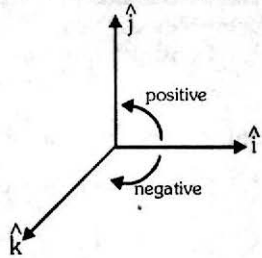
  - $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}(A_yB_z - A_zB_y) - \hat{j}(A_xB_z - B_xA_z) + \hat{k}(A_xB_y - B_xA_y)$

  - $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

- $(\vec{A}\times\vec{B}).\vec{A} = (\vec{A}\times\vec{B}).\vec{B} = 0$
- $\hat{i}\times\hat{i}=\vec{0}$, $\hat{j}\times\hat{j}=\vec{0}$, $\hat{k}\times\hat{k}=\vec{0}$
- $\hat{i}\times\hat{j}=\hat{k}$; $\hat{j}\times\hat{k}=\hat{i}$,

  $\hat{k}\times\hat{i}=\hat{j}$; $\hat{j}\times\hat{i}=-\hat{k}$

  $\hat{k}\times\hat{j}=-\hat{i}$, $\hat{i}\times\hat{k}=-\hat{j}$



- **Differentiation**

- $\frac{d}{dt}(\vec{A}.\vec{B}) = \frac{d\vec{A}}{dt}.\vec{B} + \vec{A}.\frac{d\vec{B}}{dt}$
- $\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$

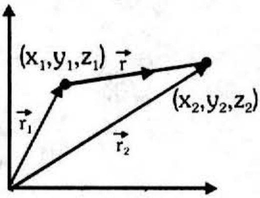
- **When a particle moved from (x_1, y_1, z_1) to (x_2, y_2, z_2) then its displacement vector**

$$\vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

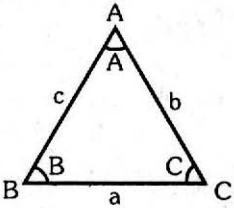
$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

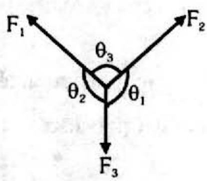
Magnitude

$$r = |\vec{r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



- **Lami's theorem**

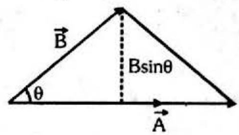




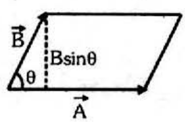
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_3}{\sin\theta_3}$$

- **Area of triangle** Area $= \frac{|\vec{A}\times\vec{B}|}{2} = \frac{1}{2}AB\sin\theta$



- **Area of parallelogram** Area $= |\vec{A}\times\vec{B}| = AB\sin\theta$



- **For Parallel vectors** $\vec{A}\times\vec{B} = \vec{0}$
- **For perpendicular vectors** $\vec{A}.\vec{B} = 0$
- **For coplanar vectors** $\vec{A}.(\vec{B}\times\vec{C}) = 0$

Examples of dot products :

- Work, $W = \vec{F}.\vec{d} = Fd\cos\theta$ where $F \to$ force, $d \to$ displacement
- Power, $P = \vec{F}.\vec{v} = Fv\cos\theta$ where $F \to$ force, $v \to$ velocity
- Electric flux, $\phi_E = \vec{E}.\vec{A} = EA\cos\theta$ where $E \to$ electric field, $A \to$ Area
- Magnetic flux, $\phi_B = \vec{B}.\vec{A} = BA\cos\theta$ where $B \to$ magnetic field, $A \to$ Area
- Potential energy of dipole in where $p \to$ dipole moment,

  uniform field, $U = -\vec{p}.\vec{E}$ where $E \to$ Electric field

Examples of cross products :

- Torque $\vec{\tau} = \vec{r} \times \vec{F}$ where $r \to$ position vector, $F \to$ force
- Angular momentum $\vec{J} = \vec{r} \times \vec{p}$ where $r \to$ position vector, $p \to$ linear momentum
- Linear velocity $\vec{v} = \vec{\omega} \times \vec{r}$ where $r \to$ position vector, $\omega \to$ angular velocity
- Torque on dipole placed in electric field $\vec{\tau} = \vec{p} \times \vec{E}$

  where $p \to$ dipole moment, $E \to$ electric field

KEY POINTS :

- **Tensor :** A quantity that has different values in different directions is called tensor.

  Ex. Moment of Inertia

  In fact tensors are merely a generalisation of scalars and vectors; a scalar is a zero rank tensor, and a vector is a first rank tensor.
- Electric current is not a vector as it does not obey the law of vector addition.
- A unit vector has no unit.
- To a vector only a vector of same type can be added and the resultant is a vector of the same type.
- A scalar or a vector can never be divided by a vector.