Inverse Trigonometric Function Chapter 13

INVERSE FUNCTION

Let f be a function defined from a set A to a set B, i.e. $f : A \to B$ and g be a function defined from the set B to the set A, i.e., $g : B \to A$; then the function g is said to be inverse of f if

$$g\{f(x)\} = x, \forall x \in A \text{ and the function } g \text{ is denoted by } f^{-1}.$$

Properties of inverse of a function :

(i) The inverse of bijection is unique.

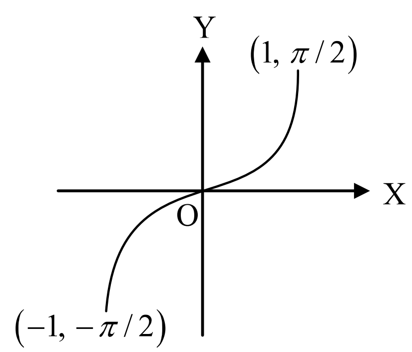
(ii) If $f : A \to B$ is bijection and $g : B \to A$ is inverse of f, then

$$f \, o \, g = I_B \text{ and } g \, o \, f = I_A$$

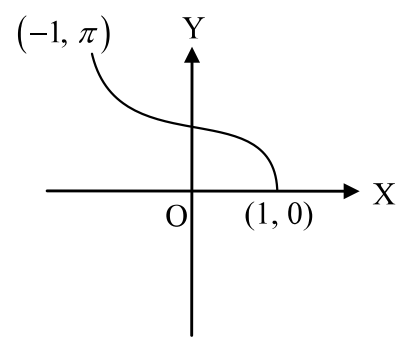
where, I_A and I_B are identity functions on the sets A and B respectively.

Graphs of inverse trigonometric functions

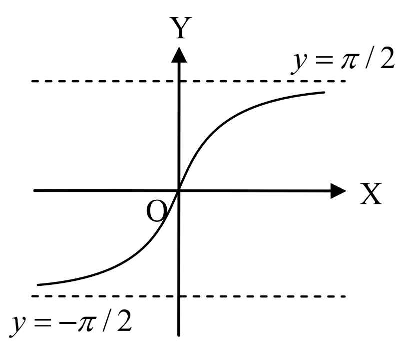
(i) Graph of $y = \sin^{-1} x$



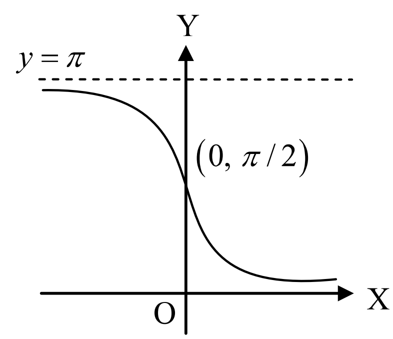
(ii) Graph of $y = \cos^{-1} x$



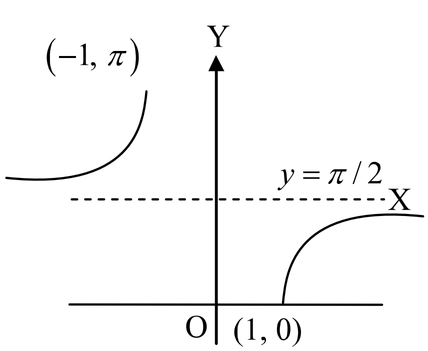
(iii) Graph of $y = \tan^{-1} x$



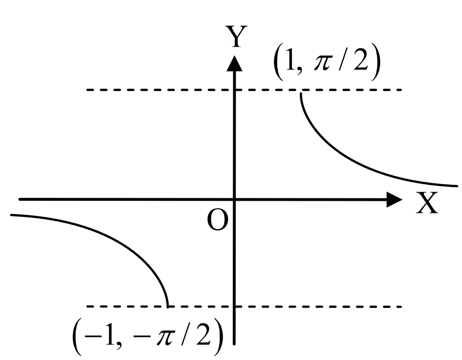
(iv) Graph of $y = \cot^{-1} x$



(v) Graph of $y = \sec^{-1} x$



(vi) Graph of $y = \text{cosec}^{-1} x$



DOMAIN AND RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

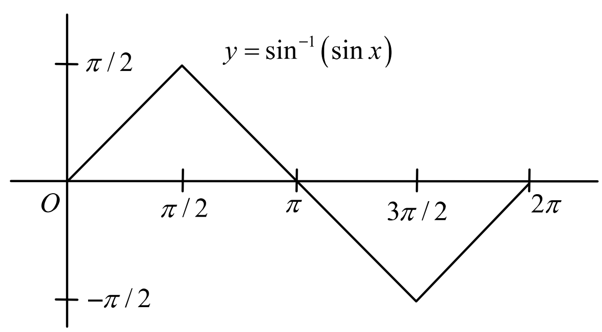
Function	Domain (D)	Range (R)
$\sin^{-1} x$	$-1 \le x \le 1$ or $[-1, 1]$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$-1 \le x \le 1$ or $[-1, 1]$	$0 \le \theta \le \pi$ or $[0, \pi]$
$\tan^{-1} x$	$x \in R$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ or $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	$x \in R$	$0 < \theta < \pi$ or $(0, \pi)$
$\sec^{-1} x$	$x \le -1$, or $1 \le x$ or $(-\infty, -1] \cup [1, \infty)$	$\theta \ne \frac{\pi}{2}, 0 \le \theta \le \pi$ or $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$\text{cosec}^{-1} x$	$x \le -1$, or $1 \le x$ or $(-\infty, -1] \cup [1, \infty)$	$\theta \ne 0, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ or $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

1. (i) $\sin^{-1}(\sin\theta) = \theta$ if and only if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and

$$\sin^{-1}(\sin x) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x \le \frac{3\pi}{2} \\ x - 2\pi, & \frac{3\pi}{2} < x \le 2\pi \end{cases}$$

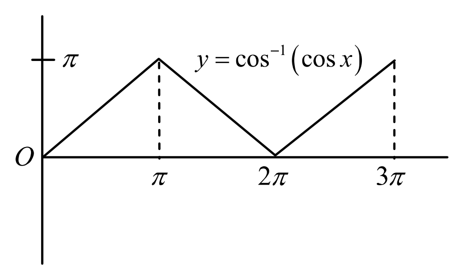
$\Rightarrow$ $f(x) = \sin^{-1}(\sin x)$ is periodic with period 2π.



(ii) $\cos^{-1}(\cos\theta) = \theta$ if and only if $0 \le \theta \le \pi$ and

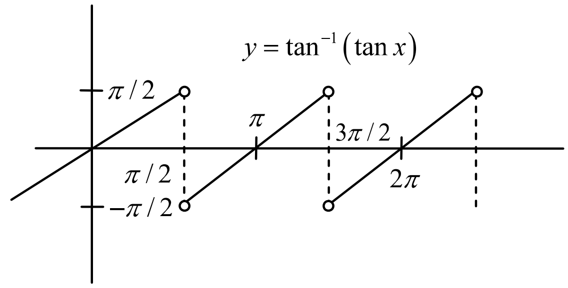
$$\cos^{-1}(\cos x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi < x \le 2\pi \end{cases}$$

$\Rightarrow$ $f(x) = \cos^{-1}(\cos x)$ is periodic with period 2π.



(iii) $\tan^{-1}(\tan\theta)=\theta$ if and only if $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$ and

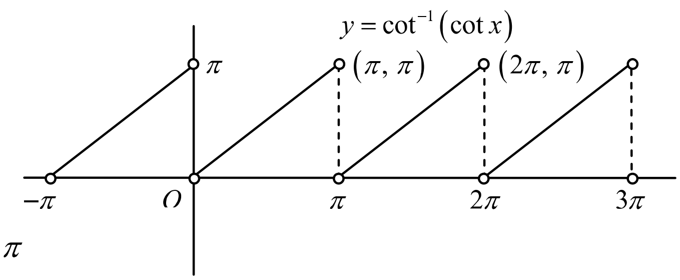
$$\tan^{-1}(\tan x)=\begin{cases} x, & 0\le x<\dfrac{\pi}{2} \\ x-\pi, & \dfrac{\pi}{2}<x<\dfrac{3\pi}{2} \\ x-2\pi, & \dfrac{3\pi}{2}<x\le 2\pi \end{cases}$$



$\Rightarrow$ $f(x)=\tan^{-1}(\tan x)$ is periodic with period π.

(iv) $\cot^{-1}(\cot\theta)=\theta$ if and only if $0<\theta<\pi$ and

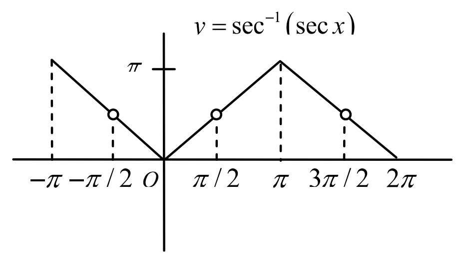
$$\cot^{-1}(\cot x)=\begin{cases} x+\pi, & -\pi<x<0 \\ x, & 0<x<\pi \\ x-\pi, & \pi<x<2\pi \end{cases}$$



$\Rightarrow$ $f(x)=\cot^{-1}(\cot x)$ is periodic with period π

(v) $\sec^{-1}(\sec\theta)=\theta$ if and only if $0\le\theta<\dfrac{\pi}{2}$ or $\dfrac{\pi}{2}<\theta\le\pi$ and

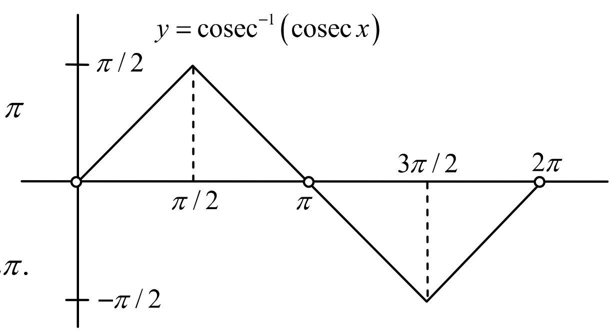
$$\sec^{-1}(\sec x)=\begin{cases} -x, & -\pi\le x\le 0, \quad x\ne-\dfrac{\pi}{2} \\ x, & 0<x\le\pi, \quad x\ne\dfrac{\pi}{2} \end{cases}$$



$\Rightarrow$ $f(x)=\sec^{-1}(\sec x)$ is periodic with period 2π.

(vi) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta)=\theta$ if and only if $-\dfrac{\pi}{2}\le\theta<0$ or $0<\theta\le\dfrac{\pi}{2}$ and

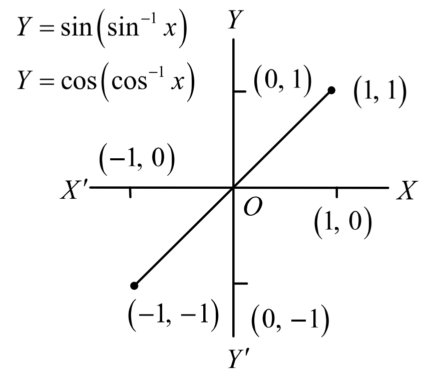
$$\operatorname{cosec}^{-1}(\operatorname{cosec} x)=\begin{cases} x, & 0<x\le\dfrac{\pi}{2}, \\ \pi-x, & \dfrac{\pi}{2}<x\le\dfrac{3\pi}{2} \quad x\ne\pi \\ x-2\pi, & \dfrac{3\pi}{2}<x<2\pi \end{cases}$$



$\Rightarrow$ $f(x)=\operatorname{cosec}^{-1}(\operatorname{cosec} x)$ is periodic with period 2π.

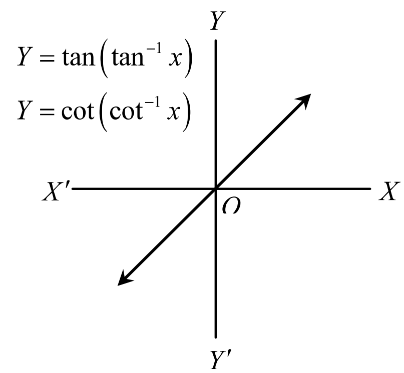
(vii) (a) $\sin\left(\sin^{-1} x\right) = x$ iff $-1 \le x \le 1$

(b) $\cos\left(\cos^{-1} x\right) = x$ iff $-1 \le x \le 1$



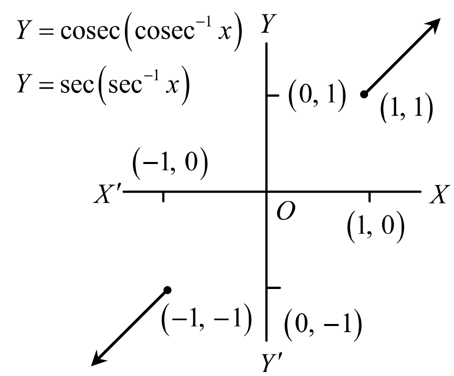
(viiii) (a) $\tan\left(\tan^{-1} x\right) = x$ for all x

(b) $\cot\left(\cot^{-1} x\right) = x$ for all x



(ix) (a) $\sec\left(\sec^{-1} x\right) = x$ iff $x \ge 1$ or $x \le -1$

(b) $\operatorname{cosec}\left(\operatorname{cosec}^{-1} x\right) = x$ iff $x \ge 1$ or $x \le -1$



2. (i) $\sin^{-1}(-x) = -\sin^{-1} x$, $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(ii) $\tan^{-1}(-x) = -\tan^{-1} x$, $\cot^{-1}(-x) = \pi - \cot^{-1} x$

(iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$ $\sec^{-1}(-x) = \pi - \sec^{-1} x$,

3. $$\begin{cases} \sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}, & \text{for all } x \in [-1, 1] \\ \tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}, & \text{for all } x \in R \\ \sec^{-1} x + \operatorname{cosec}^{-1} x = \dfrac{\pi}{2}, & \text{for all } x \in (-\infty, -1] \cup [1, \infty) \end{cases}$$

4. **Principal values for inverse circular functions.**

Principal values for $x \geq 0$	Principal values for $x < 0$
$0 \leq \sin^{-1} x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \sin^{-1} x < 0$
$0 \leq \cos^{-1} x \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1} x \leq \pi$
$0 \leq \tan^{-1} x < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1} x < 0$
$0 < \cot^{-1} x \leq \frac{\pi}{2}$	$\frac{\pi}{2} < \cot^{-1} x < \pi$
$0 \leq \sec^{-1} x < \frac{\pi}{2}$	$\frac{\pi}{2} < \sec^{-1} x \leq \pi$
$0 < \text{cosec}^{-1} x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \text{cosec}^{-1} x < 0$

5. **Conversion property :**

(i) $$\begin{cases} \sin^{-1} x = \cos^{-1}\sqrt{1-x^2}, & 0 \leq x \leq 1 \\ \sin^{-1} x = -\cos^{-1}\sqrt{1-x^2}, & -1 \leq x \leq 0 \end{cases}$$

(ii) $$\begin{cases} \sin^{-1} x = \cot^{-1}\frac{\sqrt{1-x^2}}{x}, & 0 < x \leq 1 \\ \sin^{-1} x = \cot^{-1}\frac{\sqrt{1-x^2}}{x} - \pi, & -1 \leq x < 0 \end{cases}$$

(iii) $$\sin^{-1} x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \qquad |x| < 1$$

(iv) $$\begin{cases} \cos^{-1} x = \sin^{-1}\sqrt{1-x^2}, & 0 \leq x \leq 1 \\ \cos^{-1} x = \pi - \sin^{-1}\sqrt{1-x^2}, & -1 \leq x \leq 0 \end{cases}$$

(v) $$\begin{cases} \cos^{-1} x = \tan^{-1}\frac{\sqrt{1-x^2}}{x}, & 0 < x \leq 1 \\ \cos^{-1} x = \pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x}, & -1 \leq x < 0 \end{cases}$$

(vi) $$\cos^{-1} x = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \qquad |x| < 1$$

(vii) $$\begin{cases} \tan^{-1} x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}, & x \geq 0 \\ \tan^{-1} x = -\cos^{-1}\frac{1}{\sqrt{1+x^2}}, & x \leq 0 \end{cases}$$

(viii) $\begin{cases} \tan^{-1} x = \cot^{-1} \frac{1}{x}, & x > 0 \\ \tan^{-1} x = \cot^{-1} \frac{1}{x} - \pi, & x < 0 \end{cases}$

(ix) $\tan^{-1} x = \sin^{-1} \frac{x}{\sqrt{1+x^2}} \quad \forall\ x \in R$

(x) $\begin{cases} \cot^{-1} x = \sin^{-1} \frac{1}{\sqrt{1+x^2}}, & x \geq 0 \\ \cot^{-1} x = \pi - \sin^{-1} \frac{1}{\sqrt{1+x^2}}, & x < 0 \end{cases}$

(xi) $\begin{cases} \cot^{-1} x = \tan^{-1} \frac{1}{x}, & x > 0 \\ \cot^{-1} x = \pi + \tan^{-1} \frac{1}{x}, & x < 0 \end{cases}$

(xii) $\cot^{-1} x = \cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) \quad \forall x \in R$

6. **General values of inverse circular functions :** We know that if α is the smallest angle whose sine is x, then all the angles whose sine is x can be written as $n\pi + (-1)^n \alpha$, where $n \in I$. Therefore, the general value of $\text{Sin}^{-1}x$ can be taken as $n\pi + (-1)^n \alpha$.

Thus, we have $\text{Sin}^{-1}x = n\pi + (-1)^n \alpha, -1 \leq x \leq 1$ if $\sin\alpha = x$ and $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$.

Similarly, general values of other inverse circular functions are given as follows :

$\text{Cos}^{-1}x = 2n\pi \pm \alpha, -1 \leq x \leq 1$; If $\cos\alpha = x, 0 \leq \alpha \leq \pi$

$\text{Tan}^{-1}x = n\pi + \alpha, x \in R$; If $\tan\alpha = x, -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

$\text{Cot}^{-1}x = n\pi + \alpha, x \in R$ If $\cot\alpha = x, 0 < \alpha < \pi$

$\text{Sec}^{-1}x = 2n\pi \pm \alpha,\ x \leq -1$ or $x \geq 1$ If $\sec\alpha = x, 0 \leq \alpha \leq \pi$ and $\alpha \neq \frac{\pi}{2}$

$\text{Cosec}^{-1}x = n\pi + (-1)^n \alpha,\ x \leq -1$ or $x \geq 1$ If $\text{cosec}\ \alpha = x, -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ and $\alpha \neq 0$

Note : The first letter in all above inverse Trigonometric function are CAPITAL LETTER

Formulae for sum, difference of inverse trigonometric function

(1) $\begin{cases} \sin^{-1} x + \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}; & x \geq 0,\ y \geq 0 \text{ and } x^2 + y^2 \leq 1 \\ \sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}; & x \geq 0,\ y \geq 0 \text{ and } x^2 + y^2 \geq 1 \end{cases}$

(2) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}; \quad x \geq 0,\ y \geq 0$

(3) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\}; \qquad x\ge 0,\ y\ge 0$

(4) $$\begin{cases}\cos^{-1}x-\cos^{-1}y=\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\}; & x\ge 0,\ y\ge 0,\ x\le y\\ \cos^{-1}x-\cos^{-1}y=-\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\}; & x\ge 0,\ y\ge 0,\ x\ge y\end{cases}$$

(5) $$\begin{cases}\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & x\ge 0,\ y\ge 0 \text{ and } xy<1\\ \tan^{-1}x+\tan^{-1}y=\pi/2\ ; & x>0,\ y>0 \text{ and } xy=1\\ \tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & x\ge 0,\ y\ge 0 \text{and } xy>1\end{cases}$$

(6) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\dfrac{x-y}{1+xy}\right); \qquad x\ge 0,\ y\ge 0$

Inverse trigonometric ratios of multiple angles

1. $$\begin{cases}2\sin^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right) & \text{If } -\dfrac{1}{\sqrt{2}}\le x\le\dfrac{1}{\sqrt{2}}\\ 2\sin^{-1}x=\pi-\sin^{-1}\left(2x\sqrt{1-x^2}\right) & \text{If } \dfrac{1}{\sqrt{2}}<x\le 1\\ 2\sin^{-1}x=-\pi+\sin^{-1}\left(2x\sqrt{1-x^2}\right) & \text{If } -1\le x\le\dfrac{-1}{\sqrt{2}}\end{cases}$$

2. $$\begin{cases}3\sin^{-1}x=\sin^{-1}\left(3x-4x^3\right), & \text{If } \dfrac{-1}{2}\le x\le\dfrac{1}{2}\\ 3\sin^{-1}x=\pi-\sin^{-1}\left(3x-4x^3\right), & \text{If } \dfrac{1}{2}<x\le 1\\ 3\sin^{-1}x=-\pi-\sin^{-1}\left(3x-4x^3\right), & \text{If } -1\le x<-\dfrac{1}{2}\end{cases}$$

3. $$\begin{cases}2\cos^{-1}x=\cos^{-1}\left(2x^2-1\right), & \text{If } 0\le x\le 1\\ 2\cos^{-1}x=2\pi-\cos^{-1}\left(2x^2-1\right), & \text{If } -1\le x\le 0\end{cases}$$

4. $$\begin{cases}3\cos^{-1}x=\cos^{-1}\left(4x^3-3x\right), & \text{If } \dfrac{1}{2}\le x\le 1\\ 3\cos^{-1}x=2\pi-\cos^{-1}\left(4x^2-3x\right), & \text{If } -\dfrac{1}{2}\le x\le\dfrac{1}{2}\ ,\\ 3\cos^{-1}x=2\pi+\cos^{-1}\left(4x^3-3x\right), & \text{If } -1\le x\le-\dfrac{1}{2}\end{cases}$$

5. $$\begin{cases} 2\tan^{-1}x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{If } -1 < x < 1 \\ 2\tan^{-1}x = \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{If } x > 1 \\ 2\tan^{-1}x = -\pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{If } x < -1 \end{cases}$$

6. $$\begin{cases} 2\tan^{-1}x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{If } -1 \le x \le 1 \\ 2\tan^{-1}x = \pi + \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{If } x > 1 \\ 2\tan^{-1}x = -\pi + \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{If } x < -1 \end{cases}$$

7. $$\begin{cases} 2\tan^{-1}x = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{If } 0 \le x \\ 2\tan^{-1}x = -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{If } x \le 0 \end{cases}$$

8. $$\begin{cases} 3\tan^{-1}x = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{If } \dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ 3\tan^{-1}x = \pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{If } x > \dfrac{1}{\sqrt{3}} \\ 3\tan^{-1}x = -\pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{If } x < -\dfrac{1}{\sqrt{3}} \end{cases}$$