Chapter 4

Inverse Trigonometric Functions

Ex 4.1

Question 1.
Find all the values of x such that
(i) $-10\pi \leq x \leq 10\pi$ and $\sin x = 0$
(ii) $-8\pi \leq x \leq 8\pi$ and $\sin x = -1$
Solution:
(i) $-10\pi \leq x \leq 10\pi$ and $\sin x = 0$
$\sin x = 0$, $\sin \theta = \sin \alpha$
$\sin x = \sin 0$, $\theta = n\pi + (-1)^n a$, $n \in R$
$x = n\pi + (-1)^n (0)$
$x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots\ldots \pm 10$
$n = \pm 1, \pm 2, \ldots \pm 10$

(ii) $-3\pi \leq x \leq 3\pi$ and $\sin x = -1$.

$\sin x = -1$

$\sin x = -\sin \frac{\pi}{2} = \sin(-\frac{\pi}{2})$

$x = (4n - 1)\frac{\pi}{2}$, $n = 0, \pm 1$

Question 2.
(i) $y = \sin 7x$

(ii) $y = -\sin(\frac{1}{3}x)$

(iii) $y = 4 \sin(-2x)$

Solution:

(i) $y = \sin 7x$

Period of the function $\sin x$ is 2π

Period of the function $\sin 7x$ is $\frac{2\pi}{7}$

The amplitude of $\sin 7x$ is 1.

(ii) $y = -\sin\frac{1}{3}x$

Period of $\sin x$ is 2π

So, period of $\sin\frac{1}{3}x$ is 6π and the amplitude is 1.

(iii) y = 4 sin(-2x) = -4 sin 2x
Period of sin x is 2π
π Period of sin 2x is π and the amplitude is 4.

Question 3.

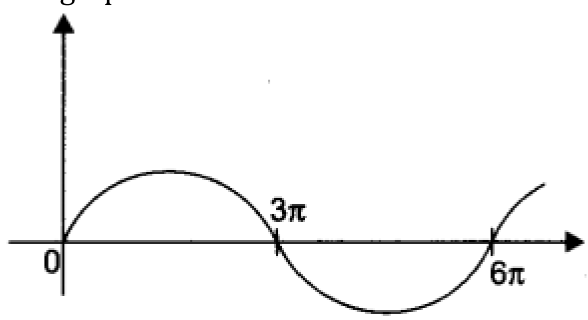
Sketch the graph of $y = \sin(\frac{1}{3}x)$ for $0 \le x < 6\pi$.

Solution:

The period of $\sin(\frac{1}{3}x)$ is 6π and the amplitude is 1.

x	0	π	2π	3π	4π	5π	6π
$\frac{1}{3}x$	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π

The graph is



Question 4.
Find the value of

(i) $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

(ii) $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$

Solution:

(i) $$\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$$

$$\therefore \quad \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

(ii) $$\sin\left(\frac{5\pi}{4}\right) = \sin\left(\pi + \frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = \sin\left(-\frac{\pi}{4}\right)$$

$$\therefore \quad \sin^{-1}\left(\sin\frac{5\pi}{4}\right) = \sin^{-1}\left(\sin\left(\frac{-\pi}{4}\right)\right) = \frac{-\pi}{4}$$

Question 5.
For that value of x does sin x = $\sin^{-1}$ x?
Solution:
Let y = $\sin^{-1}$ x
When y = 0 ⇒ 0 = $\sin^{-1}$ x
sin 0 = sin [sin-1(x)]
sin 0 = x
∴ x = 0
Only when x = 0, then sin x = $\sin^{-1}$(x)

Question 6.
Find the domain of the following
(i) $f(x)=\sin^{-1}\left(\frac{x^2+1}{2x}\right)$
(ii) $g(x)=2\sin^{-1}(2x-1)-\frac{\pi}{4}$
Solution:
(i) $f(x)=\sin^{-1}\left(\frac{x^2+1}{2x}\right)$
The range of sin-1 x is -1 to 1
$-1\leq\frac{x^2+1}{2x}\leq 1$
⇒ $\frac{x^2+1}{2x}\geq -1$ or $\frac{x^2+1}{2x}\leq 1$
⇒ $x^2 + 1 \geq -2x$ or $x^2 + 1 \leq 2x$
⇒ $x^2 + 1 + 2x \geq 0$ or $x^2 + 1 - 2x \leq 0$
⇒ $(x+ 1)^2 \geq 0$ or $(x - 1)^2 \leq 0$ which is not possible
⇒ -1 ≤ x ≤ 1 or
(ii) $g(x)=2\sin^{-1}(2x-1)-\frac{\pi}{4}$
-1 ≤ (2x – 1) ≤ 1
0 ≤ 2x ≤ 2
0 ≤ x ≤ 1
x ∈ [0, 1]

Question 7.
Find the value of $\sin^{-1}\left(\sin\frac{5\pi}{9}\cos\frac{\pi}{9}+\cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right)$

Solution:

$$\sin\frac{5\pi}{9}\cos\frac{\pi}{9}+\cos\frac{5\pi}{9}\sin\frac{\pi}{9} = \sin\left(\frac{5\pi}{9}+\frac{\pi}{9}\right) = \sin\frac{6\pi}{9}=\sin\frac{2\pi}{3}$$

$$= \sin\left(\pi-\frac{\pi}{3}\right)=\sin\frac{\pi}{3}$$

$$\sin^{-1}\left[\sin\frac{\pi}{3}\right] = \frac{\pi}{3}$$

Ex 4.2

Question 1.

Find all the values of x such that

(i) $-6\pi \le x \le 6\pi$ and $\cos x = 0$

(ii) $-5\pi \le x \le 5\pi$ and $\cos x = 1$

Solution:

(i) $\cos x = 0$

$\Rightarrow x = (2n + 1) \pm \frac{\pi}{2}$

$n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$

(ii) $\cos x = 1 = \cos 0$

$\Rightarrow x = 2n\pi \pm 0$

$n = 0, \pm 1, \pm 2$

Question 2.

State the reason for $\cos-1[\cos(-\frac{\pi}{6})] \neq -\frac{\pi}{6}$

Solution:

We know $\cos(-\pi) = \cos \pi$

so $\cos^{-1}\left[\cos\left(-\frac{\pi}{6}\right)\right] = \cos^{-1}\left[\cos\frac{\pi}{6}\right] = \frac{\pi}{6} \neq -\frac{\pi}{6}$

Question 3.

Is $\cos^{-1}(-x) = \pi - \cos^{-1} x$ true? Justify your answer.

Solution:

$\cos^{-1}(-x) = \pi - \cos^{-1} x$

Take $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

$\cos^{-1}(-x) = \pi - \theta$

$\cos(\pi - \theta) = -\cos\theta = -x \in [-1, 1]$

$\pi - \cos^{-1}x = \pi - \theta$

$\pi - \cos^{-1} x = \cos^{-1}(-x)$ is true

Question 4.

Find the principal value of $\cos^{-1}\left(\frac{1}{2}\right)$

Solution:

Let $\cos^{-1}\left(\frac{1}{2}\right) = \pi$

$\Rightarrow \cos \pi = \frac{1}{2} = \cos \frac{\pi}{3}$

$\Rightarrow \pi = \frac{\pi}{3}$

$\Rightarrow \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Question 5.

Find the value of

(i) $2\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(\frac{1}{2}\right)$

(ii) $\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}(-1)$

(iii) $\cos^{-1}\left(\cos\frac{\pi}{7}\cos\frac{\pi}{17}-\sin\frac{\pi}{7}\sin\frac{\pi}{17}\right)$

Solution:

(*i*)
$$\cos^{-1}\frac{1}{2}=\frac{\pi}{3}$$
$$\sin^{-1}\frac{1}{2}=\frac{\pi}{6}$$

So $2\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(\frac{1}{2}\right)=2\left(\frac{\pi}{3}\right)+\frac{\pi}{6}=\frac{2\pi}{3}+\frac{\pi}{6}=\frac{4\pi+\pi}{6}=\frac{5\pi}{6}$

II Method:

We know $\cos^{-1}x+\sin^{-1}x=\frac{\pi}{2}$

So $2\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(\frac{1}{2}\right)=\cos^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{1}{2}$
$$=\cos^{-1}\left(\frac{1}{2}\right)+\frac{\pi}{2}=\frac{\pi}{3}+\frac{\pi}{2}=\frac{2\pi+3\pi}{6}=\frac{5\pi}{6}$$

(*ii*)
$$\cos^{-1}\left(-\frac{1}{2}\right)=\frac{\pi}{3}$$
$$\sin^{-1}(-1)=-\frac{\pi}{2}$$

So $\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}(-1)=\frac{\pi}{3}+\left(-\frac{\pi}{2}\right)=\frac{\pi}{3}-\frac{\pi}{2}=\frac{2\pi-3\pi}{6}=-\frac{\pi}{6}$

(*iii*) Let $\frac{\pi}{7}=A$ and $\frac{\pi}{17}=B$

Now $\cos A\cos B-\sin A\sin B=\cos(A+B)$
$$A+B=\frac{\pi}{7}+\frac{\pi}{17}=\frac{\pi(17+7)}{119}=\frac{24\pi}{119}$$

So $\cos^{-1}\left[\cos\frac{\pi}{7}\cos\frac{\pi}{17}-\sin\frac{\pi}{7}\sin\frac{\pi}{17}\right]=\cos^{-1}\left[\cos\left(\frac{24\pi}{119}\right)\right]=\frac{24\pi}{119}$

Question 6.
Find the domain of
(i) f(x)=sin−1(|x|−23)+cos−1(1−|x|4)
(ii) g(x) = sin-1 x + cos-1 x
Solution:

(i) $f(x) = \sin^{-1}\left(\frac{|x|-2}{3}\right) + \cos^{-1}\left(\frac{1-|x|}{4}\right)$

(ii) g(x) = $\sin^{-1}$ x + $\cos^{-1}$ x

Solution:

(i) $f(x) = \sin^{-1}\left(\frac{|x|-2}{3}\right) + \cos^{-1}\left(\frac{1-|x|}{4}\right)$ = U(x) + V(x) say

U(x):

$-1 < \frac{|x|-2}{3} < 1$

-3 < |x| – 2 < 3

-1 < |x| ≤ 5

V(x):

$-1 \leq \frac{1-|x|}{4} \leq 1$

-4 ≤ 1 – |x| ≤ 4

-5 ≤ -|x| ≤ 3

-3 ≤ |x| ≤ 5

from U(x) and V(x)

⇒ |x| ≤ 5

⇒ -5 ≤ lxl ≤ 5

(ii) g(x) = $\sin^{-1}$ x + $\cos^{-1}$ x

-1 ≤ x ≤ 1

Question 7.

For what value of x, the inequality $\frac{\pi}{2}$ < $\cos^{-1}$ (3x -1) < π holds?

Solution:

$\frac{\pi}{2}$ < $\cos^{-1}$ (3x -1) < π

⇒ cos $\frac{\pi}{2}$ < 3x – 1 < cos π

⇒ 0 < 3x – 1 < -1

⇒ 1 < 3x < 0

⇒ $\frac{1}{3}$ < x < 0

This inequality holds only if x < 0 or x > $\frac{1}{3}$

Question 8.
Find the value of

(i) $\cos\left(\cos^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{4}{5}\right)\right)$

(ii) $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)+\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$

Solution:

(*i*) We know $\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$

$$\therefore \quad \cos^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{4}{5}\right)\right] = \cos\frac{\pi}{2} = 0$$

(*ii*)

$$\cos\frac{4\pi}{3} = \cos\left(\pi+\frac{\pi}{3}\right) = -\cos\frac{\pi}{3}$$

$$\cos^{-1}\cos\frac{4\pi}{3} = -\frac{\pi}{3}$$

and

$$\cos\frac{5\pi}{4} = \cos\left(\pi+\frac{\pi}{4}\right) = -\cos\frac{\pi}{4}$$

$$\therefore \quad \cos^{-1}\cos\frac{5\pi}{4} = -\frac{\pi}{4}$$

so $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)+\cos^{-1}\cos\left(\frac{5\pi}{4}\right)$

$$= -\frac{\pi}{3}-\frac{\pi}{4} = -\frac{7\pi}{12}$$

Ex 4.3

Question 1.

(i) $\tan^{-1}(\sqrt{9-x^2})$

(ii) $\frac{1}{2}\tan^{-1}(1-x^2)-\frac{\pi}{4}$

Solution:

(i) $\tan^{-1}(\sqrt{9-x^2})$

$9 - x^2 \geq 0 \Rightarrow 9 \geq x^2$

$x^2 \leq 9 \Rightarrow x \leq \pm 3$

Domain [-3, 3]

Since tan x is an odd function and symmetric about the origin, $\tan^{-1}$ x should be an increasing function in its domain.

∴ Domain is $(2n + 1)\frac{\pi}{2}$

(*ii*) We know $\frac{1}{2}\tan^{-1}x = \tan^{-1}\left(\frac{1-x}{1+x}\right)$

So $\frac{1}{2}\tan^{-1}(1-x^2) = \tan^{-1}\left[\frac{1-(1-x^2)}{1+(x-x^2)}\right] = \tan^{-1}\frac{x^2}{2-x^2}$

So $\frac{1}{2}\tan^{-1}(1-x^2)-\frac{\pi}{4} = \tan^{-1}\frac{x^2}{2-x^2} - \tan^{-1}1 = \tan^{-1}\left[\frac{\frac{x^2}{2-x^2}-1}{1+\left(\frac{x^2}{2-x^2}\right)(1)}\right]$

$= \tan^{-1}(x^2-1) = y$ (say)

The domain of y is (-∞, ∞) {x | x ∈ -1} and range is [-1, ∞) {y | y ≥ -1}
The domain for tan-1(x^2 – 1) is (2n + 1)π. Since tan x is an odd function.

Question 2.

Find the value of

(i) $\tan^{-1}\left(\tan\frac{5\pi}{4}\right)$

(ii) $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$

Solution:

(i) $$\tan\frac{5\pi}{4} = \tan\left(\pi+\frac{\pi}{4}\right) = \tan\frac{\pi}{4}$$

So $$\tan^{-1}\left(\tan\frac{5\pi}{4}\right) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$$

(ii) $$\tan\left(-\frac{\pi}{6}\right) = \tan\left(\pi-\frac{\pi}{6}\right) = \tan\frac{5\pi}{6}$$

So $$\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right) = \tan^{-1}\left(\tan\frac{5\pi}{6}\right) = \frac{5\pi}{6}$$

Question 3.
Find the value of

(i) $\tan\left(\tan^{-1}\frac{7\pi}{4}\right)$

(ii) $\tan(\tan^{-1}(1947))$

(iii) $\tan(\tan^{-1}(-0.2021))$

Solution:

we know that $\tan(\tan^{-1}(x)) = x$

(i) $\tan(\tan^{-1}(\frac{7\pi}{4})) = \frac{7\pi}{4}$

(ii) $\tan(\tan^{-1}(1947)) = 1947$

(iii) $\tan(\tan^{-1}(-0.2021)) = -0.2021$

Question 4.
Find the value of

(i) $\tan\left(\cos^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

(ii) $\sin\left(\tan^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right)$

(iii) $\cos\left(\sin^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right)$

Solution:

(i) $$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

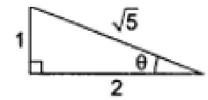
So $\quad \cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{3}-\left(\frac{-\pi}{6}\right)=\frac{\pi}{3}+\frac{\pi}{6} = \frac{2\pi+\pi}{6}=\frac{3\pi}{6}=\frac{\pi}{2}$

and $\quad \tan\left(\frac{\pi}{2}\right)=\infty$

(*ii*) $\sin\left(\tan^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{4}{5}\right)\right)$

$$\text{Let } \tan^{-1}\frac{1}{2}=\theta_1 \qquad \Rightarrow \tan\theta_1=\frac{1}{2}$$

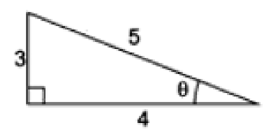
So $\quad \sin\theta_1=\frac{1}{\sqrt{5}}$



$$\text{Let } \cos^{-1}\frac{4}{5}=\theta_2$$

$\Rightarrow \quad \cos\theta_2=\frac{4}{5}$

$\Rightarrow \quad \tan\theta_2=\frac{3}{4}$



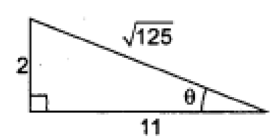
Now $\quad \left(\tan^{-1}\frac{1}{2}-\cos^{-1}\frac{4}{5}\right) = \tan^{-1}\frac{1}{2}-\tan^{-1}\frac{3}{4}$

$$= \tan^{-1}\left\{\frac{\frac{1}{2}-\frac{3}{4}}{1+\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)}\right\} = \tan^{-1}\left(\frac{\frac{2-3}{4}}{1+\frac{3}{8}}\right)=\tan^{-1}\left(\frac{-1}{4}\times\frac{8}{11}\right)$$

$$= \tan^{-1}\left(\frac{-2}{11}\right)$$

$$\text{Let } \tan^{-1}\left(-\frac{2}{11}\right)=\theta_3 \qquad \Rightarrow \tan\theta_3=\frac{2}{11}$$

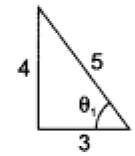
$\Rightarrow \quad \sin\theta_3=-\frac{2}{\sqrt{125}} \quad \Rightarrow \theta_3=\sin^{-1}\left(\frac{-2}{\sqrt{125}}\right)$



So $\sin\left(\tan^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{4}{5}\right)\right) = \sin\left[\sin^{-1}\left(-\frac{2}{\sqrt{125}}\right)\right]=\frac{-2}{\sqrt{125}}$

$$= \frac{-2}{5\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}=\frac{-2\sqrt{5}}{25}$$

(*iii*) $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)-\tan^{-1}\left(\frac{3}{4}\right)\right)$



Let $\sin^{-1}\frac{4}{5} = \theta_1$

$\Rightarrow \quad \sin\theta_1 = \frac{4}{5}$

$\Rightarrow \quad \tan\theta_1 = \frac{4}{3} \qquad \Rightarrow \theta_1 = \tan^{-1}\frac{4}{3}$

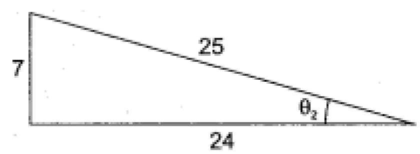
Now $\quad \sin^{-1}\left(\frac{4}{5}\right)-\tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}\frac{4}{3}-\tan^{-1}\frac{3}{4}$

$$= \tan^{-1}\left\{\frac{\frac{4}{3}-\frac{3}{4}}{1+\left(\frac{4}{3}\right)\left(\frac{3}{4}\right)}\right\} = \tan^{-1}\left[\frac{\frac{16-9}{12}}{1+\frac{12}{12}}\right] = \tan^{-1}\frac{7}{24}$$

Let $\quad \tan^{-1}\frac{7}{24} = \theta_2 \qquad \Rightarrow \tan\theta_2 = \frac{7}{24}$

$\Rightarrow \quad \cos\theta_2 = \frac{24}{25}$

$\Rightarrow \quad \theta_2 = \cos^{-1}\frac{24}{25}$



Now $\cos\left[\sin^{-1}\left(\frac{4}{5}\right)-\tan^{-1}\left(\frac{3}{4}\right)\right] = \cos\left[\cos^{-1}\frac{24}{25}\right] = \frac{24}{25}$

Ex 4.4

Question 1.

Find the principal value of

(i) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

(ii) $\cot^{-1}(\sqrt{3})$

(iii) $\operatorname{cosec}^{-1}(-\sqrt{2})$

Solution:

(i) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \theta$

$\sec\theta = \frac{2}{\sqrt{3}} = \sec\frac{\pi}{6}$

$\theta = \frac{\pi}{6}$

(ii) Let $\cot^{-1}(\sqrt{3}) = \theta$

$\Rightarrow \cot\theta = \sqrt{3}$

$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$

$\Rightarrow \theta = \frac{\pi}{6}$

(iii) Let $\operatorname{cosec}^{-1}(-\sqrt{2}) = \theta$

$\Rightarrow \operatorname{cosec}\theta = -\sqrt{2}$

$\Rightarrow \sin\theta = -\frac{1}{\sqrt{2}}$

$\Rightarrow \theta = -\frac{\pi}{4}$

Question 2.

Find the value of

(i) $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$

(ii) $\sin^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \cot^{-1}(2)$

(iii) $\cot^{-1}(1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sec^{-1}(-\sqrt{2})$

Solution:

(*i*) Let $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$\sec^{-1}(-2) = \theta \quad \Rightarrow \sec\theta = -2$

$\Rightarrow \quad \cos\theta = \frac{-1}{2} \quad \Rightarrow \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$$\therefore \tan^{-1}\left(\sqrt{3}\right) - \sec^{-1}(-2) = \frac{\pi}{3} - \left(\frac{2\pi}{3}\right) = -\frac{\pi}{3}$$

(*ii*) $\sin^{-1}(-1) = -\frac{\pi}{2}$

$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$$\therefore \sin^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \cot^{-1}(2) = -\frac{\pi}{2} + \frac{\pi}{3} + \cot^{-1}(2) = -\frac{\pi}{6} + \cot^{-1}(2)$$

(*iii*) $\cot^{-1}(1) = \frac{\pi}{4}$

$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

$\sec^{-1}\left(-\sqrt{2}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

So $\cot^{-1}(1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sec^{-1}\left(-\sqrt{2}\right) = \frac{\pi}{4} - \frac{\pi}{3} - \frac{3\pi}{4} = \frac{-5\pi}{6}$

Ex 4.5

Question 1.

(i) $\sin^{-1}(\cos \pi)$

(ii) $\tan^{-1}\left(\sin\left(-\frac{5\pi}{2}\right)\right)$

(iii) $\sin^{-1}[\sin 5]$

Solution:

(i) $\sin^{-1}(\cos \pi) = \sin^{-1}(-1) = -\frac{\pi}{2}$

(ii) $\tan^{-1}\left(\sin\frac{5\pi}{2}\right) = \tan^{-1}\left(-\sin\frac{\pi}{2}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$

(iii) $\sin^{-1}(\sin 5) = \sin^{-1}[\sin(5 - 2\pi)] = 5 - 2\pi$

Question 2.

(i) $\sin(\cos^{-1}(1 - x))$

(ii) $\cos(\tan^{-1}(3x - 1))$

(iii) $\tan\left(\sin^{-1}\left(x + \frac{1}{2}\right)\right)$

Solution:

(i) Let $\cos^{-1}(1 - x) = \theta$

$1 - x = \cos\theta$

We know $\sin^2\theta = 1 - \cos^2\theta$

$$\Rightarrow \quad \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(1-x)^2} = \sqrt{1-\left(1+x^2-2x\right)}$$

$$= \sqrt{2x-x^2}$$

$$\Rightarrow \quad \sin[\cos^{-1}(1-x)] = \sin\theta = \sqrt{2x-x^2}$$

(*ii*) Let $\tan^{-1}(3x-1) = \theta \quad \Rightarrow 3x - 1 = \tan\theta$

$$\text{So } \tan^2\theta = (3x-1)^2 = 9x^2 - 6x + 1$$

We know $1 + \tan^2\theta = \sec^2\theta = \dfrac{1}{\cos^2\theta}$

$$\Rightarrow \quad \frac{1}{\cos^2\theta} = 9x^2 - 6x + 1 + 1 = 9x^2 - 6x + 2$$

$$\text{So } \cos^2\theta = \frac{1}{9x^2-6x+2} \quad \Rightarrow \cos\theta = \frac{1}{\sqrt{9x^2-6x+2}}$$

$$\therefore \quad \cos[\tan^{-1}(3x-1)] = \frac{1}{\sqrt{9x^2-6x+2}}$$

(iii) Let $\sin^{-1}\left(x+\frac{1}{2}\right)=\theta \quad \Rightarrow x+\frac{1}{2}=\sin\theta$

$\Rightarrow \quad 2x+1=2\sin\theta \quad \Rightarrow \sin\theta=\frac{2x+1}{2}$

$$\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-\left(\frac{2x+1}{2}\right)^2}$$

$$=\sqrt{1-\frac{(4x^2+4x+1)}{4}}=\frac{1}{2}\sqrt{3-4x-4x^2}$$

Now $\quad \tan(\theta)=\dfrac{\sin\theta}{\cos\theta}=\dfrac{\frac{2x+1}{2}}{\frac{1}{2}\sqrt{3-4x-4x^2}}=\dfrac{2x+1}{\sqrt{3-4x-4x^2}}$

Question 3.

Find the value of

(i) $\sin^{-1}\left(\cos\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right)$

(ii) $\cot\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{4}{5}\right)$

(iii) $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

Solution:

(i) $\sin^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{3}$ and $\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$ and

so $\quad \sin^{-1}\left[\cos\left\{\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\}\right]=\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$

(ii) $\quad$ Let $\sin^{-1}\frac{4}{5}=\theta$

$\Rightarrow \quad \sin\theta=\frac{4}{5}$

$\Rightarrow \quad \cos\theta=\frac{3}{5} \quad \Rightarrow \theta=\cos^{-1}\frac{3}{5}$

$\therefore \quad \cot\left[\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{4}{5}\right]=\cot\left[\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{3}{5}\right]=\cot\left(\frac{\pi}{2}\right)=0$

(iii) Let $\sin^{-1}\frac{3}{5} = \theta$

$\Rightarrow$ $\sin\theta = \frac{3}{5}$

$\Rightarrow$ $\tan\theta = \frac{3}{4}$

so $\theta = \tan^{-1}\frac{3}{4}$

so $\theta = \tan^{-1}\frac{3}{4}$

Now let $\cot^{-1}\frac{3}{2} = \theta$ $\Rightarrow \cot\theta = \frac{3}{2}$

$\Rightarrow$ $\tan\theta = \frac{2}{3}$ $\Rightarrow \theta = \tan^{-1}\frac{2}{3}$

$$\text{Now } \sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2} = \tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3} = \tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{9+8}{12}}{1-\frac{6}{12}}\right) = \tan^{-1}\left(\frac{17}{6}\right)$$

$$\text{so } \tan\left[\tan^{-1}\left(\frac{17}{6}\right)\right] = \frac{17}{6}$$

Question 4.
Prove that

(i) $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

(ii) $\sin^{-1}\frac{3}{5}-\cos^{-1}\frac{12}{13} = \sin^{-1}\frac{16}{65}$

Solution:

(i) $$\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24} = \tan^{-1}\left[\frac{\left(\frac{2}{11}\right)+\left(\frac{7}{24}\right)}{1-\frac{2}{11}\times\frac{7}{24}}\right] = \tan^{-1}\left[\frac{\frac{48+77}{11\times 24}}{1-\frac{14}{11\times 24}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{125}{264}}{\frac{250}{264}}\right] = \tan^{-1}\left(\frac{125}{250}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

(ii) Let $\cos^{-1}\frac{12}{13} = \theta \quad \Rightarrow \cos\theta = \frac{12}{13}$

so $$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

$\Rightarrow$ $$\theta = \sin^{-1}\frac{5}{13}$$

Now to find $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{5}{13}$

We know $$\sin^{-1}x-\sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2}-y\sqrt{1-x^2}\right]$$

LHS: $$\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{5}{13} = \sin^{-1}\left[\frac{3}{5}\sqrt{1-\frac{25}{169}}-\frac{5}{13}\sqrt{1-\frac{9}{25}}\right] = \sin^{-1}\left[\frac{3}{5}\times\frac{12}{13}-\frac{5}{13}\times\frac{4}{5}\right]$$

$$= \sin^{-1}\left(\frac{36}{65}-\frac{20}{65}\right) = \sin^{-1}\frac{16}{65} = \text{RHS.}$$

Question 5.

Prove that $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$

Solution:

$$\tan^{-1}x+\tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}(\text{A})$$

$$\text{Here A} = \frac{x+y}{1-xy}$$

So LHS: $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z = \tan^{-1}(\text{A})+\tan^{-1}z$

$$\tan^{-1}\left(\frac{\text{A}+z}{1-\text{A}z}\right) = \tan^{-1}\left[\frac{\frac{x+y}{1-xy}+z}{1-\frac{x+y}{1-xy}(z)}\right] = \tan^{-1}\left[\frac{\frac{x+y+z(1-xy)}{1-xy}}{\frac{1-xy-(x+y)z}{1-xy}}\right]$$

$$= \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-xz-yz}\right) = \text{RHS}$$

Question 6.

If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, show that $x + y + z = xyz$.

Solution:

Given $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right) = \pi$$

$$\Rightarrow \quad \frac{x+y+z-xyz}{1-xy-yz-zx} = \tan \pi = 0$$

$$\Rightarrow \quad x+y+z-xyz = 0 \qquad \Rightarrow x+y+z = xyz$$

Question 7.

Prove that $\tan^{-1} x + \tan^{-1} \frac{2x}{1-x^2} = \tan^{-1} \frac{3x-x^3}{1-3x^2}, |x| < \frac{1}{\sqrt{3}}$

Solution:

$$\text{LHS:} \quad \tan^{-1} x + \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\left[\frac{x+\frac{2x}{1-x^2}}{1-x\left(\frac{2x}{1-x^2}\right)}\right] = \tan^{-1}\left[\frac{\frac{x\left(1-x^2\right)+2x}{1-x^2}}{1-\frac{x(2x)}{1-x^2}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{x-x^3+2x}{\cancel{1-x^2}}}{\frac{1-x^2-2x^2}{\cancel{1-x^2}}}\right] = \tan^{-1}\frac{3x-x^3}{1-3x^2}$$

$$= \text{RHS}$$

Question 8.

Simplify: $\tan^{-1} \frac{x}{y} - \tan^{-1} \frac{x-y}{x+y}$

Solution:

$$\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{x-y}{x+y} = \tan^{-1}\left[\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right] = \tan^{-1}\left[\frac{\frac{x(x+y)-y(x-y)}{y(x+y)}}{1+\frac{x(x-y)}{y(x+y)}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{x^2+xy-xy+y^2}{y(x+y)}}{\frac{y(x+y)+x(x-y)}{y(x+y)}}\right] = \tan^{-1}\left(\frac{x^2+y^2}{xy+y^2+x^2-xy}\right)$$

$$= \tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Question 9.
Find the value of

(i) $\sin^{-1}\frac{5}{x}+\sin^{-1}\frac{12}{x}=\frac{\pi}{2}$

(ii) $2\tan^{-1}x=\cos^{-1}\frac{1-a^2}{1+a^2}-\cos^{-1}\frac{1-b^2}{1+b^2}$, a > 0, b > 0

(iii) 2 $\tan^{-1}$(cos x) = tarn^{-1} (2 cosec x)

(iv) $\cot^{-1}x - \cot^{-1}(x+2) = \frac{\pi}{12}$, x > 0

Solution:

(*i*) Given $\sin^{-1}\frac{5}{x}+\sin^{-1}\frac{12}{x} = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\frac{5}{x} = \frac{\pi}{2}-\sin^{-1}\frac{12}{x} = \cos^{-1}\frac{12}{x}$ $\left(\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right)$

so now $\sin^{-1}\frac{5}{x} = \cos^{-1}\frac{12}{x}$

Let $\sin^{-1}\frac{5}{x} = \theta$ | Let $\cos^{-1}\frac{12}{x} = \theta$

$\Rightarrow \sin\theta = \frac{5}{x}$ | $\Rightarrow \frac{12}{x} = \cos\theta$

But we know $\sin^2\theta+\cos^2\theta = 1$

$\Rightarrow \left(\frac{5}{x}\right)^2+\left(\frac{12}{x}\right)^2 = 1$

i.e. $\frac{25}{x^2}+\frac{144}{x^2} = 1$ $\Rightarrow x^2 = 169$

a $x = 13$

(*ii*) Let $2\tan^{-1}x = \cos^{-1}\frac{1-a^2}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2}$

Put $a = \tan\theta_1 \quad \Rightarrow \theta_1 = \tan^{-1}a$

Now $\frac{1-a^2}{1+a^2} = \frac{1-\tan^2\theta_1}{1+\tan^2\theta_1} = \cos 2\theta_1$

Similarly, put $b = \tan\theta_2 \quad \Rightarrow \theta_2 = \tan^{-1}b$

Now $\frac{1-b^2}{1+b^2} = \cos 2\theta_1$

so $\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \cos^{-1}(\cos 2\theta_1) = 2\theta_1 = 2\tan^{-1}a$

and $\cos^{-1}\left(\frac{1-b^2}{1+b^2}\right) = \cos^{-1}(\cos 2\theta_2) = 2\theta_2 = 2\tan^{-1}b$

Now $2\tan^{-1}x = 2\tan^{-1}a - 2\tan^{-1}b$

$\Rightarrow$ ($\div$ by 2) $\tan^{-1}x = \tan^{-1}a - \tan^{-1}b$

i.e. $\tan^{-1}x = \tan^{-1}\left(\frac{a-b}{1+ab}\right) \quad \Rightarrow x = \frac{a-b}{1+ab}$

(*iii*) $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\,x)$

$$\text{LHS} = 2\tan^{-1}(\cos x) = \tan^{-1}(\cos x) + \tan^{-1}(\cos x)$$

$$= \tan^{-1}\left(\frac{\cos x + \cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right)$$

$$\text{RHS} = \tan^{-1}(2\,\text{cosec}\,x)$$

Now LHS = RHS

$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right) = \tan^{-1}(2\,\text{cosec}\,x)$

$\Rightarrow \quad \frac{2\cos x}{\sin^2 x} = 2\,\text{cosec}\,x = \frac{2}{\sin x}$

$2\cot x = 2$

$\Rightarrow \quad \cot x = 1 \quad \Rightarrow x = \frac{\pi}{4}$

$$(iv) \quad \tan^{-1}\frac{1}{x} - \tan^{-1}\frac{1}{x+2} = \frac{\pi}{12}$$

$$= \tan^{-1}\left[\frac{\frac{1}{x} - \frac{1}{x+2}}{1+\left(\frac{1}{x}\right)\left(\frac{1}{x+2}\right)}\right] = \tan^{-1}\left[\frac{\frac{x+2-x}{x(x+2)}}{\frac{x(x+2)+1}{x(x+2)}}\right]$$

$$= \tan^{-1}\frac{2}{x^2+2x+1} = \frac{\pi}{12}$$

$$\Rightarrow \quad \frac{2}{x^2+2x+1} = \tan\frac{\pi}{12} = \tan 15^\circ = \tan(45^\circ - 30^\circ)$$

$$= \frac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ} = \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}$$

$$\Rightarrow \quad \frac{2}{x^2+2x+1} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{3-1}{\left(\sqrt{3}+1\right)^2}$$

$$\Rightarrow \quad \frac{2}{x^2+2x+1} = \frac{2}{\left(\sqrt{3}+1\right)^2}$$

$$\Rightarrow \quad (x+1)^2 = \left(\sqrt{3}+1\right)^2 \quad \Rightarrow \quad x = \sqrt{3}$$

Question 10.

Find the number of solution of the equation $\tan^{-1}(x - 1) + \tan^{-1} x + \tan^{-1} (x + 1) = \tan^{-1}(3x)$.

Solution:

$\tan^{-1}(x - 1) + \tan^{-1} x + \tan^{-1} (x + 1) = \tan^{-1}(3x)$

$\tan^{-1}(x - 1) + \tan^{-1} (x + 1) = \tan^{-1} 3x - \tan^{-1} x$

$$\text{LHS:} \quad \tan^{-1}\frac{(x-1)+(x+1)}{1-(x-1)(x+1)} = \tan^{-1}\frac{2x}{1-(x^2-1)} = \tan^{-1}\frac{2x}{2-x^2} \quad \text{........ } \mathbf{(1)}$$

$$\text{RHS:} \quad \tan^{-1} 3x - \tan^{-1} x = \tan^{-1}\frac{3x-x}{1+(3x)(x)} = \tan^{-1}\frac{2x}{1+3x^2} \quad \text{....... } \mathbf{(2)}$$

LHS = RHS

$$\Rightarrow \tan^{-1}\frac{2x}{2-x^2} = \tan^{-1}\frac{2x}{1+3x^2}$$

$$\Rightarrow \frac{2x}{2-x^2} = \frac{2x}{1+3x^2}$$

$$\Rightarrow 2 - x^2 = 1 + 3x^2$$

$$\Rightarrow 4x^2 = 1$$

$$\Rightarrow x^2 = \frac{1}{4}$$

$$\Rightarrow x = \pm\frac{1}{2}$$

So, the equation has 2 solutions.

Ex 4.6

Choose the correct or the most suitable answer from the given four alternatives.

Question 1.

The value of $\sin^{-1}(\cos x)$, $0 \leq x \leq \pi$ is ....

(a) $\pi - x$

(b) $x - \frac{\pi}{2}$

(c) $\frac{\pi}{2} - x$

(d) $\pi - x$

Solution:

(c) $\frac{\pi}{2} - x$

Hint:

$$\text{We know } \cos x = \sin\left(\frac{\pi}{2} - x\right)$$

$$\text{So } \sin^{-1}(\cos x) = \sin^{-1}\left[\sin\left(\frac{\pi}{2} - x\right)\right] = \frac{\pi}{2} - x$$

Question 2.

If $\sin^{-1} + \sin^{-1} y = \frac{2\pi}{3}$; then $\cos^{-1}x + \cos^{-1}y$ is equal to ....

(a) $\frac{2\pi}{3}$

(b) $\frac{\pi}{3}$

(c) $\frac{\pi}{6}$

(d) π

Solution:

(b) $\frac{\pi}{3}$

Hint:

$$\text{We know } \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\text{and } \sin^{-1} y + \cos^{-1} y = \frac{\pi}{2}$$

Adding them

$$\text{so } (\sin^{-1} x + \sin^{-1} y) + (\cos^{-1} x + \cos^{-1} y) = \pi$$

i.e. $$\frac{2\pi}{3}+\left(\cos^{-1}x+\cos^{-1}y\right)=\pi$$

$\Rightarrow$ $$\cos^{-1}x+\cos^{-1}y=\pi-\frac{2\pi}{3}=\frac{\pi}{3}$$

Question 3.

$\sin^{-1}\frac{3}{5}-\cos^{-1}\frac{12}{13}+\sec^{-1}\frac{5}{3}-\operatorname{cosec}^{-1}\frac{13}{12}=$ ....

(a) 2π
(b) π
(c) 0
(d) $\tan^{-1}\frac{12}{65}$

Solution:

(c) 0

Hint:

We know $$\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{3}{5}=\frac{\pi}{2}$$

and $$\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{12}{13}=\frac{\pi}{2}$$

so $$\frac{\pi}{2}-\frac{\pi}{2}=0$$

Question 4.

If $\sin^{-1}x = 2\sin^{-1}\alpha$ has a solution, then ....

(a) $|\alpha|\le\frac{1}{\sqrt{2}}$ (b) $|\alpha|\ge\frac{1}{\sqrt{2}}$ (c) $|\alpha|<\frac{1}{\sqrt{2}}$ (d) $|\alpha|>\frac{1}{\sqrt{2}}$

Solution:

(a) $|\alpha| \le \frac{1}{\sqrt{2}}$

Hint:

$$\sin^{-1}x=2\sin^{-1}\alpha$$

Range of $\sin^{-1}x$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

i.e. $$-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}$$

$\Rightarrow$ $$-\frac{\pi}{2}\le2\sin^{-1}\alpha\le\frac{\pi}{2}\quad\Rightarrow-\frac{\pi}{4}\le\sin^{-1}\alpha\le\frac{\pi}{4}$$

$$\Rightarrow \quad \sin^{-1}\left(-\frac{\pi}{4}\right) \le \alpha \le \sin\left(\frac{\pi}{4}\right) \quad \Rightarrow -\frac{1}{\sqrt{2}} \le \alpha \le \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad |\alpha| = \frac{1}{\sqrt{2}}$$

Question 5.

$\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ is valid for ……………..

(a) $-\pi \le x \le 0$

(b) $0 \le x \le \pi$

(c) $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(d) $-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$

Solution:

(b) $0 \le x \le \pi$

Question 6.

If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$, the value of $x^{2017} + y^{2018} + z^{2019} - \frac{9}{x^{101}+y^{101}+z^{101}}$ is ………………

(a) 0

(b) 1

(c) 2

(d) 3

Solution:

(a) 0

Hint:

The maximum value of $\sin^{-1} x$ is $\frac{\pi}{2}$ and $\sin^{-1} 1 = \frac{\pi}{2}$

Here it is given that

$\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$

$\Rightarrow x = y = z = 1$

and so $1 + 1 + 1 - \frac{9}{1+1+1} = 3 - 3 = 0$

Question 7.

If $\cot^{-1} x = \frac{2\pi}{5}$ for some $x \in R$, the value of $\tan^{-1} x$ is ....

(a) $-\frac{\pi}{10}$

(b) $\frac{\pi}{5}$

(c) $\frac{\pi}{10}$

(d) $-\frac{\pi}{5}$

Solution:

(c) $\frac{\pi}{10}$

Hint:

$$\text{Given } \cot^{-1} x = \frac{2\pi}{5} \quad \Rightarrow x = \cot\frac{2\pi}{5}$$

We know $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

so $\tan^{-1} x = \frac{\pi}{2} - \frac{2\pi}{5} = \frac{\pi}{10}$

Question 8.

The domain of the function defined by $f(x) = \sin^{-1} \sqrt{x-1}$ is ....

(a) [1, 2]
(b) [-1, 1]
(c) [0, 1]
(d) [-1, 0]

Solution:

(a) [1, 2]

Hint:

The domain for $\sin^{-1} x$ is [0, 1]

So $\sqrt{x-1} = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$

$\sqrt{x-1} = 1 \Rightarrow x - 1 = 0 \Rightarrow x = 2$

$\therefore$ The domain is [1, 2]

Question 9.

If $x = \frac{1}{5}$, the value of $\cos(\cos^{-1} x + 2 \sin^{-1} x)$ is ....

(a) $-\sqrt{\frac{24}{25}}$ (b) $\sqrt{\frac{24}{25}}$ (c) $\frac{1}{5}$ (d) $-\frac{1}{5}$

Solution:

(d) $-\frac{1}{5}$

Hint:

$$\text{Now } \cos\left[\cos^{-1}\frac{1}{5}+2\sin^{-1}\frac{1}{5}\right] = \cos\left[\cos^{-1}\frac{1}{5}+\sin^{-1}\frac{1}{5}+\sin^{-1}\frac{1}{5}\right]$$

$$= \cos\left[\frac{\pi}{2}+\sin^{-1}\frac{1}{5}\right] = \cos\left[\frac{\pi}{2}+\frac{\pi}{2}-\cos^{-1}\frac{1}{5}\right]$$

$$= \cos\left[\pi-\cos^{-1}\frac{1}{5}\right] = \cos\cos^{-1}\left(-\frac{1}{5}\right) = -\frac{1}{5}$$

Question 10.

$\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)$ is equal to ………..

(a) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ (b) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$ (c) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ (d) $\tan^{-1}\left(\frac{1}{2}\right)$

Solution:

(d) $\tan^{-1}\frac{1}{2}$

Hint:

$$\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9} = \tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right) = \tan^{-1}\left(\frac{\frac{9+8}{36}}{1-\frac{2}{36}}\right).$$

$$= \tan^{-1}\left(\frac{17}{34}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

Question 11.

If the function $f(x) = \sin^{-1}(x^2 - 3)$, then x belongs to …………..

(a) [1, -1]

(b) $[\sqrt{2}, 2]$

(c) $[-2,-\sqrt{2}] \cup [\sqrt{2},2]$

(d) $[-2,-\sqrt{2}] \cap [\sqrt{2},2]$

Solution:

(c) $[-2, -\sqrt{2}] \cup [\sqrt{2}, 2]$

Hint:

$f(x) = \sin^{-1}(x^2 - 3)$

Domain of $\sin^{-1}(x)$ is $[-1, 1]$

$\Rightarrow -1 \le x^2 - 3 \le 1 \Rightarrow 2 \le x^2 \le 4$

$\Rightarrow \sqrt{2} \le x \le 2 \Rightarrow \sqrt{2} \le |x| \le 2$

$x \in [-2, -\sqrt{2}] \cup [\sqrt{2}, 2]$

Question 12.

If $\cot^{-1} 2$ and $\cot^{-1} 3$ are two angles of a triangle, then the third angle is ………….

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

Solution:

(b) $\frac{3\pi}{4}$

Hint:

Let $\cot^{-1} 2 = \theta_1 \quad \Rightarrow \cot\theta_1 = 2$ so $\quad \tan\theta_1 = \frac{1}{2}$

Let $\cot^{-1} 3 = \theta_2 \quad \Rightarrow \cot\theta_2 = 3$ so $\quad \tan\theta_2 = \frac{1}{3}$

$$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}}$$

$$\Rightarrow \quad \tan(\theta_1 + \theta_2) = \frac{\frac{3+2}{6}}{1 - \frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1 \quad \Rightarrow \theta_1 + \theta_2 = \frac{\pi}{4}$$

Let θ_3 be the third angle

so $\quad \theta_1 + \theta_2 + \theta_3 = 180 = \pi$

$$\Rightarrow \quad \theta_3 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Question 13.

$\sin^{-1}(\tan\frac{\pi}{4}) - \sin^{-1}(\sqrt{\frac{3}{x}}) = \frac{\pi}{6}$. Then x is a root of the equation ……………

(a) $x^2 - x - 6 = 0$
(b) $x^2 - x - 12 = 0$
(c) $x^2 + x - 12 = 0$
(d) $x^2 + x - 6 = 0$

Solution:
(b) $x^2 - x - 12 = 0$

Hint:

$$\sin^{-1}(1) - \sin^{-1}\left(\frac{\sqrt{3}}{\sqrt{x}}\right) = \frac{\pi}{6}$$

$$\Rightarrow \quad \frac{\pi}{2} - \sin^{-1}\frac{\sqrt{3}}{\sqrt{x}} = \frac{\pi}{6} \qquad \Rightarrow \sin^{-1}\frac{\sqrt{3}}{\sqrt{x}} = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{\sqrt{x}} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \Rightarrow \sqrt{x} = \frac{2\sqrt{3}}{\sqrt{3}} = 2 \qquad \Rightarrow x = 4$$

Question 14.

$\sin^{-1}(2\cos^2 x - 1) + \cos^{-1}(1 - 2\sin^2 x) =$ ……………

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

Solution:
(a) $\frac{\pi}{2}$

Hint:
$2 \cos 2x - 1 = \cos 2x$
$1 - 2 \sin 2 x = \cos 2x$
$\therefore \sin^{-1} x(\cos 2x) + \cos^{-1}(\cos 2x) = \frac{\pi}{2}$ $(\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2})$

Question 15.

If cot−1(sinα−−−−√)+tan−1(sinα−−−−√) = u, then cos 2u is equal to …………..

(a) $\tan^2 \alpha$
(b) 0
(c) -1
(d) $\tan 2\alpha$

Solution:
(c) -1

Hint:
$\cot^{-1} x + \tan^{-1} x = \frac{\pi}{2} \Rightarrow u = \frac{\pi}{2}$ so $2u = \pi$
$\therefore \cos 2u = \cos \pi = -1$

Question 16.

If |x| ≤ 1, then $2\tan^{-1} x - \sin^{-1} \frac{2x}{1+x^2}$ is equal to ....

(a) $\tan^{-1} x$
(b) $\sin^{-1} x$
(c) 0
(d) π

Solution:
(c) 0
Hint:

Let $x = \tan\theta$ so $\frac{2x}{1+x^2} = \sin 2\theta$.

Now $2\tan^{-1}(\tan\theta) - \sin^{-1}(\sin 2\theta) = 2\theta - 2\theta = 0$

Question 17.

The equation $\tan^{-1} x - \cot^{-1} x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ has ....

(a) no solution
(b) unique solution
(c) two solutions
(d) infinite number of solutions

Solution:
(b) unique solution
Hint:

$$\tan^{-1} x - \cot^{-1} x = \tan^{-1}\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \tan^{-1} x - \left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{\pi}{6}$$

$$\tan^{-1} x + \tan^{-1} x = \frac{\pi}{2} + \frac{\pi}{6}$$

$$2\tan^{-1} x = \frac{2\pi}{3} \qquad \Rightarrow x = \tan\frac{\pi}{3} = \sqrt{3}$$

⇒ It has a unique Solution

Question 18.

If $\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is equal to ....

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{5}}$ (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{\sqrt{3}}{2}$

Solution:

(b) $\frac{1}{\sqrt{5}}$

Hint:

$\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$

$$\Rightarrow \quad \sin^{-1} x = \frac{\pi}{2} - \cot^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{2}$$

$$\sin^{-1} x = \sin^{-1}\left(\frac{\frac{1}{2}}{\sqrt{1+\left(\frac{1}{2}\right)^2}}\right)$$

$$\Rightarrow \quad \sin^{-1} x = \sin^{-1}\left(\frac{\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right) = \frac{1}{\sqrt{5}}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$\Rightarrow \quad x = \frac{1}{\sqrt{5}}$$

Question 19.

If $\sin^{-1}\frac{x}{5} + \operatorname{cosec}^{-1}\frac{5}{4} = \frac{\pi}{2}$, then the value of x is ....

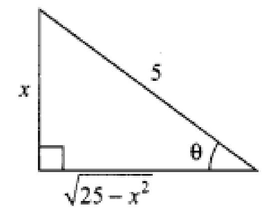
(a) 4
(b) 5
(c) 2
(d) 3

Solution:
(d) 3
Hint:

$$\sin^{-1}\frac{x}{5} + \operatorname{cosec}^{-1}\frac{5}{4} = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}\frac{x}{5} = \frac{\pi}{2} - \operatorname{cosec}^{-1}\frac{5}{4} = \sec^{-1}\frac{5}{4}$$

$$\Rightarrow \quad \sin^{-1}\frac{x}{5} = \sec^{-1}\frac{5}{4} = \cos^{-1}\frac{4}{5}$$



$$\text{Let} \quad \sin^{-1}\frac{x}{5} = \theta$$

$$\Rightarrow \quad \sin\theta = \frac{x}{5}$$

$$\text{so} \quad \cos\theta = \frac{\sqrt{25-x^2}}{5}$$

$$\Rightarrow \quad \theta = \cos^{-1}\frac{\sqrt{25-x^2}}{5} = \cos^{-1}\frac{4}{5} \text{ (given)}$$

$$\Rightarrow \quad \frac{\sqrt{25-x^2}}{5} = \frac{4}{5} \qquad \Rightarrow \sqrt{25-x^2} = 4$$

$$\Rightarrow \quad 25 - x^2 = 16 \qquad \Rightarrow x^2 = 9 \qquad \Rightarrow x = 3$$

Question 20.
sin(tan^{-1}), |x| < 1 is equal to …………

(a) $\frac{x}{\sqrt{1-x^2}}$ (b) $\frac{1}{\sqrt{1-x^2}}$ (c) $\frac{1}{\sqrt{1+x^2}}$ (d) $\frac{x}{\sqrt{1+x^2}}$

Solution:
(d) $\frac{x}{\sqrt{1+x^2}}$

Hint:

$$\sin(\tan^{-1} x) = \sin\left(\sin^{-1}\frac{x}{\sqrt{1+x^2}}\right) = \frac{x}{\sqrt{1+x^2}}$$

$$= \frac{x}{\sqrt{1+x^2}}$$