Application of Derivatives

Rate of Change of Quantities

- For a quantity y that varies with another quantity x and is given by the relation, say $y = f(x)$, the derivative of y, $\frac{dy}{dx}$, or $f'(x)$ represents the rate of change of y with respect to x.
- At a particular point, say x_0, $\left.\frac{dy}{dx}\right]_{x=x_0}$ or $f'(x_0)$ represents rate of change of y with respect to x at the point $x = x_0$.
- For example: The rate of change of volume of a cylindrical vessel with respect to its radius r, when $r = 8$ cm, can be calculated as:

$$V = \pi r^2 h$$
$$\frac{dV}{dr} = 2\pi rh$$
$$\left.\frac{dV}{dr}\right]_{r=8\text{ cm}} = 2\pi(8)h = 16\pi h$$

- If two quantities, say x and y, vary with respect to the same variable t, i.e. $x = f(t)$ and $y = g(t)$, then rate of change of y with respect to x is given by chain rule as:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}\left\{\text{provided } \frac{dx}{dt} \neq 0\right\}$$

- If y increases as x increases, then rate of change, i.e. $\frac{dy}{dx}$, is positive; and if y decreases as x increases, then rate of change, i.e. $\frac{dy}{dx}$, is negative.

Solved Examples

Example 1:

A balloon spherical in shape is deflating at the rate of 0.7 cm^3/s. Its surface area is decreasing at the rate of 1.4 cm^2/s. What is the radius of the balloon at that instant?

Solution:

Let the radius of the balloon be r cm.

Now, the balloon is spherical in shape.

$$\therefore V = \frac{4}{3}\pi r^3 \quad \dots (1)$$

According to the question, $\frac{dV}{dt} = 0.7 \text{cm}^3/\text{s}$

Differentiating (1) with respect to t, we obtain

$$\frac{dV}{dt} = \frac{4}{3} \times 3\pi r^2 \frac{dr}{dt}$$
$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$
$$\Rightarrow 0.7 = 4\pi r^2 \frac{dr}{dt}$$
$$\therefore \frac{dr}{dt} = \frac{0.7}{4\pi r^2} \quad \dots (2)$$

Surface area of balloon, $S = 4\pi r^2$

According to the question, $\frac{dS}{dt} = 1.4 \text{cm}^2/\text{s}$

$$\Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt} \quad \dots (3)$$

Substituting the value of $\frac{dr}{dt}$ from (2) in (3), we obtain

$$\frac{dS}{dt} = 8\pi r \cdot \frac{0.7}{4\pi r^2}$$
$$\Rightarrow 1.4 = \frac{1.4}{r}$$
$$\therefore r = \frac{1.4}{1.4} = 1 \text{ cm}$$

Thus, the radius of balloon at that instant is 1 cm.

Example 2:

The distance covered by an insect is given as a function of time as $x = at^2 - 3t + 6$.

If the speed of insect at $t = 2$ min is 13 cm/min, then what is the value of a?

Solution:

It is given that $x = at^2 - 3t + 6$

$$\frac{dx}{dt} = 2at - 3$$

$$\Rightarrow 13 = 4a - 3 \qquad \left\{\begin{array}{l} \frac{dx}{dt} = \text{speed} = 13 \text{ cm/min} \\ t = 2 \text{ min} \end{array}\right\}$$

$$\Rightarrow 16 = 4a$$

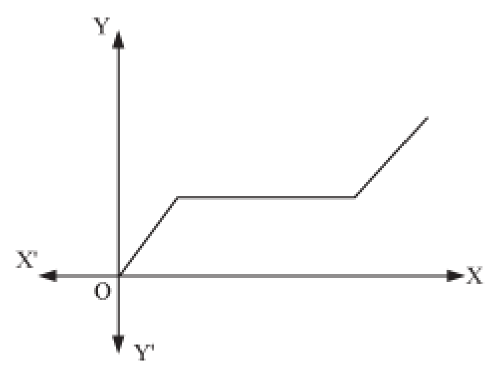
$$\Rightarrow a = 4$$

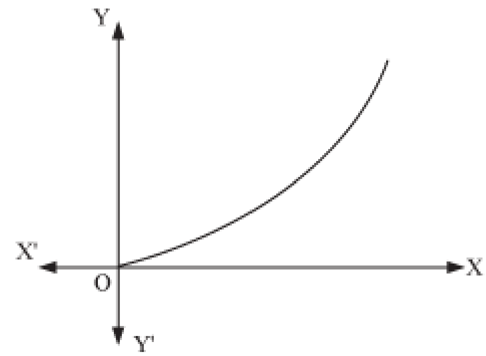
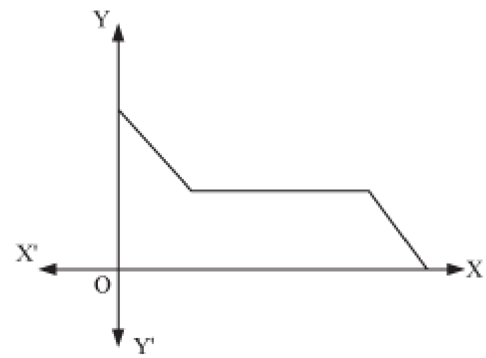
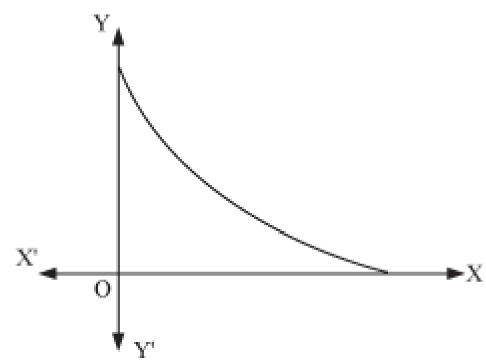
Thus, the value of a is 4.

Increasing and Decreasing Functions

Increasing-Decreasing Functions

- If f is a real valued function, I is an open interval contained in the domain and $x_1, x_2 \in I$, then f is said to be

	Condition	**Example**
Increasing on I	$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$	

Strictly increasing on I	$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$	
Decreasing on I	$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$	
Strictly Decreasing on I	$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$	

- A real valued function f is said to be increasing or decreasing at a point x_0 in the domain of f if there exists an interval $I = (x_0 - h, x_0 + h)$, $h > 0$ such that f is increasing or decreasing in I for $x_1, x_2 \in I$.

First Derivative Test for Increasing and Decreasing Functions

- If f is a continuous function in $[a, b]$ and differentiable in the open interval (a, b), then

- f is increasing in $[a, b]$ if $f'(x) > 0$ for each $x \in (a, b)$.
- f is decreasing in $[a, b]$ if $f'(x) < 0$ for each $x \in (a, b)$.
- f is neither increasing nor decreasing in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$.

For example, consider the function $f'(x) = x^3 - 6x^2 + 12x + 8$.
Now, $f'(x) = 3x^2 - 12x + 12$
$= 3(x^2 - 4x + 4)$
$= 3(x - 2)^2 \geq 0$ in every interval of **R**.
Therefore, f is increasing on **R**.

Solved Examples

Example 1

Find the intervals in which $f(x) = x^3 - 2x^2 - 4x + 7$ is increasing or decreasing.

Solution:

$f(x) = x^3 - 2x^2 - 4x + 7$

$f'(x) = 3x^2 - 4x - 4$

$f'(x) = 3x^2 - 6x + 2x - 4$

$f'(x) = 3x(x - 2) + 2(x - 2)$

$f'(x) = (3x + 2)(x - 2)$ … (1)

Putting $f'(x) = 0$, we obtain $x = -\frac{2}{3}$, 2

The points $-\frac{2}{3}$ and 2 divide real line into three intervals $\left(-\infty, -\frac{2}{3}\right), \left(-\frac{2}{3}, 2\right)$ and $(2, \infty)$.

We now find the sign of $f'(x)$ in these intervals.

Interval	**Sign of $f'(x)$**	**Nature of f**

$\left(-\infty, -\frac{2}{3}\right)$	(−) (−) > 0	Increasing
$\left(-\frac{2}{3}, 2\right)$	(+) (−) < 0	Decreasing
(2, ∞)	(+) (+) > 0	Increasing

Thus, $f(x)$ is increasing in the interval $\left(-\infty, -\frac{2}{3}\right)$ ∪ (2, ∞) and decreasing in $\left(-\frac{2}{3}, 2\right)$.

Example 2

Find the interval in which $f(x) = -\log x - x^2 + 3x$ is strictly increasing.

Solution:

$f(x) = -\log x - x^2 + 3x$

$f'(x) = -\frac{1}{x} - 2x + 3$

$f'(x) = \frac{-1-2x^2+3x}{x} = \frac{-\left(2x^2-3x+1\right)}{x} = \frac{-\left(2x-1\right)\left(x-1\right)}{x}$

Putting $f'(x) = 0$, we obtain $(2x - 1)(x - 1) = 0$

$\Rightarrow x = \frac{1}{2}, 1$

The intervals obtained are $\left(-\infty, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right)$ and $(1, \infty)$.

We now find the sign of $f'(x)$ in these intervals.

Interval	**Sign of $f'(x)$**	**Nature of f**

$\left(-\infty, \frac{1}{2}\right)$	(−)(+) = (−) < 0	Decreasing
$\left(\frac{1}{2}, 1\right)$	(+)(+) = (+) > 0	Increasing
(1, ∞)	(+)(+) = (+) > 0	Increasing

Thus, $f'(x)$ is strictly increasing in $\left(\frac{1}{2}, 1\right) \cup (1, \infty)$.

Tangents and Normals to the Curves

- The equation of tangent to the curve $y = f(x)$ at point (x_0, y_0) is given by $y - y_0 = f'(x_0)(x - x_0)$, where $f'(x_0)$ is the slope of tangent.
- If slope of tangent line is zero, then equation of tangent at point (x_0, y_0) is given by $y = y_0$.

  If slope of tangent line is not defined, then equation of tangent at point (x_0, y_0) is given by $x = x_0$.
- The equation of normal to the curve $y = f(x)$ at point (x_0, y_0) is given by:

$y - y_0 = -\dfrac{1}{f'(x_0)}(x - x_0)$ or $(y - y_0) f'(x_0) + (x - x_0) = 0$, where $\dfrac{-1}{f'(x_0)}$ is the slope of normal.

Solved Examples

Example 1:

Find the equation of tangent to the curve $y = 3x^3 - 2x + 6$ at the point $x = -1$.

Solution:

$y = 3x^3 - 2x + 6$

$y' = 9x^2 - 2$

$y']_{x=-1} = 9(-1)^2 - 2 = 7$

The equation of tangent to the curve $y = f(x)$ at point (x_0, y_0) is given by:

$(y - y_0) = f'(x_0)\ (x - x_0)$

At $x = -1$, $y = 3\ (-1)^3 - 2(-1) + 6 = 5$

∴ Equation of tangent at (−1, 5) is:

$(y - 5) = 7\{x - (-1)\}$

$y - 5 = 7(x + 1)$

$\Rightarrow y - 5 = 7x + 7$

$\Rightarrow 7x - y + 12 = 0$ is the required equation of the tangent.

Example 2:

Find the equation of normal to the curve $y = \cos x - 2\sin x$ at point $x = \frac{\pi}{2}$.

Solution:

$y = \cos x - 2\sin x$

$y' = -\sin x - 2\cos x$

$$y'\Big]_{x=\frac{\pi}{2}} = -1$$

The equation of normal to the curve $y = f(x)$ at point (x_0, y_0) is given by:

$(y - y_0)\, f'(x_0) + (x - x_0) = 0$

At $x = \frac{\pi}{2}$, we obtain $y = \cos\frac{\pi}{2} - 2\sin\frac{\pi}{2} = -2$

∴ Equation of normal at the point $\left(\frac{\pi}{2}, -2\right)$ is given by:

$$\{y - (-2)\}(-1) + \left(x - \frac{\pi}{2}\right) = 0$$

$$\Rightarrow (y + 2)(-1) + x - \frac{\pi}{2} = 0$$

$$\Rightarrow x - y - \left(\frac{\pi}{2} + 2\right) = 0 \text{ is the required eqation of the normal.}$$

Approximations Using Differentials

- For function $y = f(x)$, an increment in y (Δy) corresponding to an increment in x (Δx) is given by:

  $\Delta y = f(x + \Delta x) - f(x)$
  The following are defined for the increment:

- The differential of x, denoted by dx, is defined by $dx = \Delta x$.
- The differential of y, denoted by dy, is defined by $dy = f'(x)dx = \left(\frac{dy}{dx}\right)\Delta x$.
- The approximate values of certain quantities can be calculated using differentials.

For example, consider $\sqrt{25.8}$.
We take $y = \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$
Let $x = 25$ and $\Delta x = 0.8$
$\therefore \Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{25.8} - \sqrt{25} = \sqrt{25.8} - 5$
$\Rightarrow \sqrt{25.8} = 5 + \Delta y$ ... (1)
Now, dy is approximately equal to Δy and is given by,
$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(\Delta x) = \frac{0.8}{2 \times 5} = 0.08$
Thus, using (1), we obtain the approximate value of $\sqrt{25.8}$ as
$5 + \Delta y = 5 + 0.08 = 258$

Solved Examples

Example 1:

Find the approximate value of $\sqrt{0.0048}$.

Solution:

Let $y = \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

Let $x = 0.0049$ and $\Delta x = -0.0001$

$$\therefore \Delta y = \sqrt{x+\Delta x} - \sqrt{x} = \sqrt{0.0048} - \sqrt{0.0049}$$
$$\Delta y = \sqrt{0.0048} - 0.07$$
$$\Rightarrow \sqrt{0.0048} = 0.07 + \Delta y \text{ ... (1)}$$

Now, dy is approximately equal to Δy and is given by:

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{\Delta x}{2\sqrt{x}} = \frac{-0.0001}{2\times 0.07} = -0.00071$$

Thus, from (1), we obtain

$$\sqrt{0.0048} = 0.07 - 0.00071 = 0.06929$$

Example 2:

Find the approximate value of $f(0.31)$, where $f(x) = 2x^2 + 3x - 5$.

Solution:

Let $x = 0.3$ and $\Delta x = 0.01$

$f(0.31) = f(x + \Delta x) = 2(x + \Delta x)^2 + 3(x + \Delta x) - 5$

Now, $\Delta y = f(x + \Delta x) - f(x)$

$\therefore f(x + \Delta x) = \Delta y + f(x) \approx f(x) + f'(x)\Delta x \qquad \{\because dy = \Delta y\}$

$\Rightarrow f(0.31) \approx (2x^2 + 3x - 5) + (4x + 3)\Delta x$

$= \{2(0.3)^2 + 3(0.3) - 5\} + \{4(0.3) + 3\}(0.01)$

$= \{2 \times 0.09 + 0.9 - 5\} + \{1.2 + 3\}(0.01)$

$= \{0.18 + 0.9 - 5\} + (4.2)(0.01)$

$= (1.08 - 5) + 0.042$

$= -3.92 + 0.042$

$= -3.878$

Thus, the approximate value of $f(0.31)$ is -3.878.

Maximum and Minimum Values of Functions Using Graphs

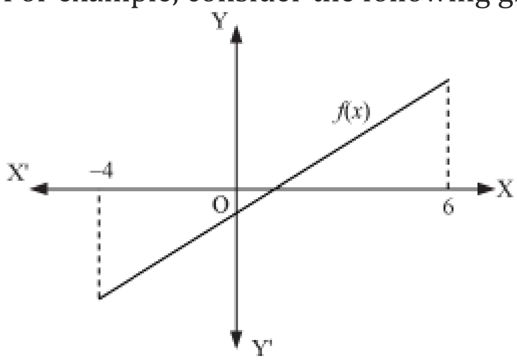
- Let f be a function defined on interval I, then

- f is said to have a maximum value in I if there exists a point c in I such that $f(c) > f(x)$, for all $x \in I$. $f(c)$ is the maximum value of f and c is the point of maximum value of f in I.

- f is said to have a minimum value in I if there exists a point c in I such that $f(c) < f(x)$ for all $x \in I$.$f(c)$ is the minimum value of f and c is the point of minimum value of f in I.
- f is said to have an extreme value in I if there exists a point c in I such that $f(c)$ is either the maximum value or the minimum value of f in I. $f(c)$ is the extreme value and c is known as the extreme point.

- The maximum and minimum values of functions can be found using graphs.

  For example, consider the following graph.



  In this graph, $f(6)$ is the maximum value of $f(x)$ and 6 is the point of maximum value; and $f(-4)$ is the minimum value of $f(x)$ and -4 is the point of minimum value.

Solved Examples

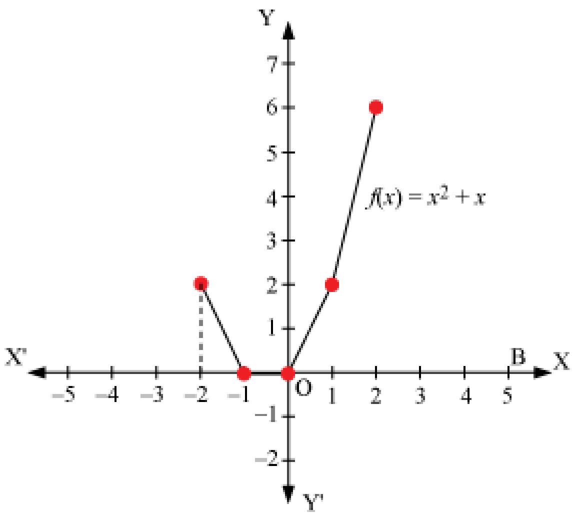
Example 1

Plot the graph of function $f(x) = x^2 + x$, $x \in [-2, 2]$ and find the maximum and minimum value of $f(x)$.

Solution:

We start by plotting the graph of $f(x) = x^2 + x$ for different values of x in $[-2, 2]$.

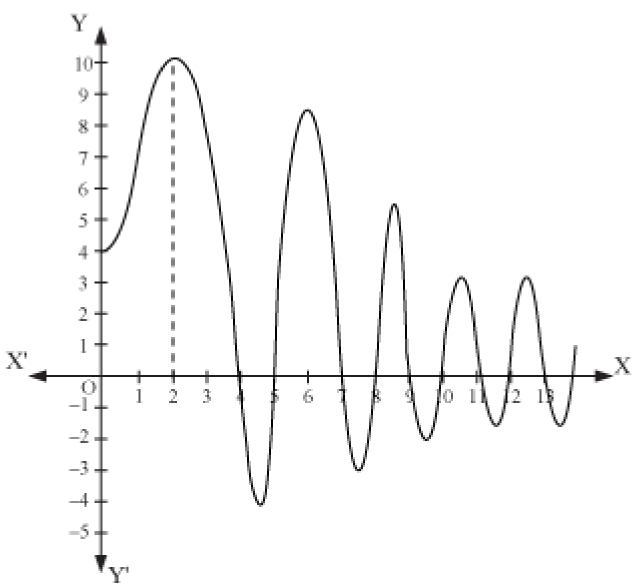
$f(x)$	2	0	0	2	6
x	−2	−1	0	1	2



It can be seen that the maximum value of $f(x)$ is at $x = 2$ i.e., $f(x) = 6$; and the minimum value is at $x = -1$, i.e., $f(-1) = 0$.

Example 2

From the following graph of $f(x)$, identify the maximum and minimum values of $f(x)$.



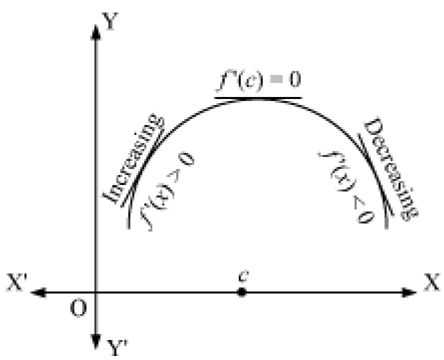
Solution:

It can be seen that the graph gets closer to the x-axis as the value of x increases. The maximum value attained is 10 (at $x = 2$), whereas $f(x)$ has a minimum value of – 4 (at $x = 4.5$).

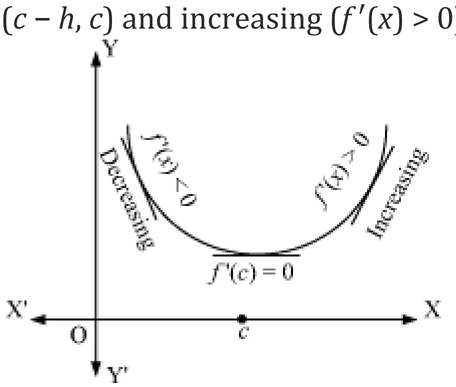
First and Second Derivative Tests for Finding Points of Local Maxima and Minima

Local Maxima and Local Minima

- If f is a real valued function and c is an interior point in the domain of f, then
- c is the point of local maxima if there exists $h > 0$ such that $f(c) > f(x)$ for all x in $(c - h, c + h)$. $f(c)$ is called the maximum value of f.
- c is the point of local minima if there exists $h > 0$ such that $f(c) < f(x)$ for all x in $(c - h, c + h)$. $f(c)$ is called the minimum value of f.
- Geometrically, local maxima and local minima can be obtained as
- If $x = c$ is a point of local maxima, then function f will be increasing $(f'(x) > 0)$ in the interval $(c - h, c)$ and decreasing $(f'(x) < 0)$ in the interval $(c, c + h)$.



- If $x = c$ is a point of local minima, then function f will be decreasing $(f'(x) < 0)$ in the interval $(c - h, c)$ and increasing $(f'(x) > 0)$ in the interval $(c, c + h)$.

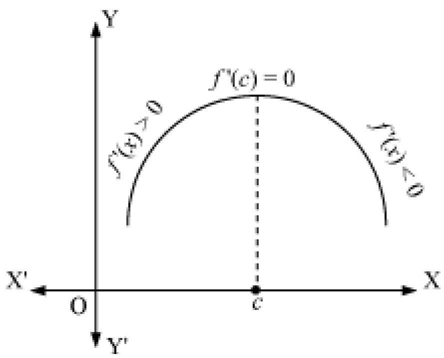


- Let f be a function defined on open interval I and $c \in I$ be any point. If f has local maxima or local minima at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c.
- Point c in the domain of f at which either $f'(c) = 0$ or f is not differentiable is known as the critical point of f.

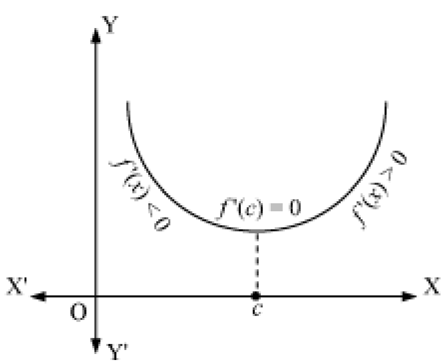
First Derivative Test

- Let f be a function defined on an open interval I and let it be continuous at critical point c on I. Then,

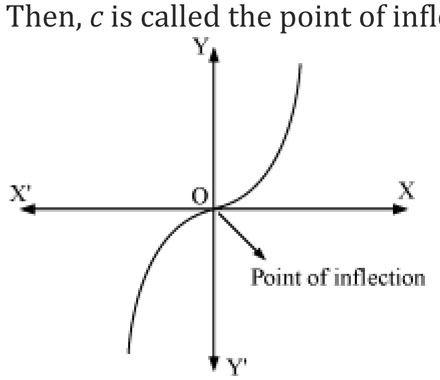
- If $f'(x)$ changes sign from positive to negative as x increases through c i.e., if $f'(x) > 0$ at every point close to and to the left of c, and $f'(x) < 0$ at every point close to and to the right of c, then c is the point of local maxima.



- If $f'(x)$ changes sign from negative to positive as x increases through c i.e., if $f'(x) < 0$ at every point close to and to the left of c, and $f'(x) > 0$ at every point close to and to the right of c, then c is the point of minima.
-



-
-
- If $f'(x)$ does not change sign as x increases through c, then c is neither the point of local maxima nor the point of local minima.
  Then, c is called the point of inflection.



- For example: Consider $f(x) = x^3 - 12x + 7$
  $f'(x) = 3x^2 - 12 = 3(x^2 - 4)$
  $\Rightarrow f'(x) = 0$ at $x = 2$ and $x = -2$
  Consider point $x = 2$

Let two points near $x = 2$ be 2.1 and 1.9.
$f'(2.1) = 3\{(2.1)^2 - 4\} = 3\{4.41 - 4\} = 1.23$
$f'(1.9) = 3\{(1.9)^2 - 4\} = 3\{3.61 - 4\} = -1.19$
$\therefore f'(x)$ changes sign from negative to positive at $x = 2$.
Thus, $x = 2$ is the point of local minima.
At point $x = -2$, consider points -2.1 and -1.9.
$f'(-2.1) = 3(-2.1 - 2)(-2.1 + 2) > 0$
$f'(-0.9) = 3(-1.9 - 2)(-1.9 + 2) < 0$
$\therefore f'(x)$ changes sign from positive to negative at $x = -2$.
Thus, $x = -2$ is the point of local maxima.

Second Derivative Test

- Let f be a function defined on an interval I and let $c \in I$. Let the second order derivative of f exist at c. Then,

- If $f'(c) = 0$ and $f''(c) < 0$, then c is the point of local maxima and $f(c)$ is the local maximum value.
- If $f'(c) = 0$ and $f''(c) > 0$, then c is the point of local minima and $f(c)$ is the local minimum value.
- If $f'(c) = 0$ and $f''(c) = 0$, then the test fails.

- For example: Consider $f(x) = 2x^3 - 9x^2 + 12x + 5$

$f'(x) = 6x^2 - 18x + 12$
$= 6(x^2 - 3x + 2)$
$= 6(x - 1)(x - 2)$
$f'(x) = 0$
$\Rightarrow 6(x - 1)(x - 2) = 0$
$\Rightarrow x = 1$ or $x = 2$

$f''(x) = 12x - 18$
Now, $f''(1) = 12(1) - 18$
$= -6$
$f''(2) = 12(2) - 18 = 6$
Thus, $x = 1$ is the point of local maxima and $x = 2$ is the point of local minima.

Solved Examples

Example 1

Find all the points of local maxima and local minima of the function $f(x) = \frac{2}{3}x^3 + 2x - 2x^2 + 7$ by first derivative test.

Solution

$$f(x) = \frac{2}{3}x^3 + 2x - 2x^2 + 7$$
$$\Rightarrow f'(x) = 2x^2 + 2 - 4x$$

For local maxima or minima, we have $f'(x) = 0$

$$\Rightarrow 2x^2 + 2 - 4x = 0$$
$$\Rightarrow 2(x-1)^2 = 0$$
$$\Rightarrow x = 1$$

Let two points near 1 be 0.9 and 1.1.

$f'(0.9) > 0$ and $f'(1.1) > 0$

Since, $f'(x)$ does not change sign as *x* increases through 1. Hence *x* =1 is neither a point of local maxima nor a point of local minima. In fact, *x* = 1 is a point of inflection.

Example 2

The cost of transportation of goods from a company to a market is given by *f*(*x*) = $\frac{x^3}{3} - 19x^2 - 80x + 100$. What is the minimum cost of transporting the goods?

Solution:
The cost of transportation is given by

$$f(x) = \frac{x^3}{3} - 19x^2 - 80x + 100$$
$$f'(x) = x^2 - 38x - 80$$
$$f'(x) = 0$$

$$\Rightarrow x^2 - 38x - 80 = 0$$
$$x^2 - 40x + 2x - 80 = 0$$
$$x(x-40) + 2x - 80 = 0$$
$$x(x-40) + 2(x-40) = 0$$
$$x = -2 \text{ and } x = 40$$
$$f''(x) = 2x - 38$$
$$f''(-2) = -4 - 38 = -42$$
$$f''(40) = 80 - 38 = 42$$

$\therefore x = 40$ is the point of minima.

$\Rightarrow$ Minimum value is $f(40) = \left|\dfrac{(40)^3}{3} - 19(40)^2 - 80(40) + 100\right| = 12166.67$

Thus, the minimum cost of transportation is Rs. 12166.67.

Absolute Maximum and Minimum Values of Functions in Closed Intervals

- The maximum or minimum value attained by a function in a closed interval is known as its absolute maximum or minimum value.

- For example, consider $f(x) = 3x - 2$ on $[-1, 0]$

  At $x = -1$, $f(-1) = -5$ i.e., it has the minimum value at $x = -1$ and its absolute minimum value is -5 (also called global minimum or least value).

  At $x = 0$, $f(0) = -2$ i.e., it has the maximum value at $x = 0$ and its absolute maximum value is -2 (also called global maximum or greatest value).

- Let f be a continuous function on an interval $I = [a, b]$, then the absolute maximum and minimum value of f exist and f attains it at least once in I.

- Let f be a differentiable function on a closed interval I and let c be any interior point of I, then $f'(c) = 0$, if f attains its absolute maximum or minimum value at c.

- To find absolute maximum or minimum value of a differentiable function,

- find all the critical points of f

- take the end points of interval

- calculate the value of f at all these points

- identify the maximum or minimum value out of the function values at these points

- For example, consider $f(x) = 3x^2 - 2x - 1$ on $[0, 1]$

Being a polynomial function, f is differentiable everywhere.

$$f'(x) = 6x - 2$$

$$f'(x) = 0 \Rightarrow x = \frac{1}{3}$$

Thus, the only critical point is $x = \frac{1}{3}$.

$$f(0) = -1$$

$$f\left(\frac{1}{3}\right) = 3 \times \left(\frac{1}{3}\right)^2 - 2 \times \frac{1}{3} - 1 = \frac{-4}{3}$$

$$f(1) = 3 - 2 - 1 = 0$$

Therefore, the absolute maximum value of *f* is 0 at $x = 1$ and absolute minimum value of *f* is $\frac{-4}{3}$ at $x = \frac{1}{3}$.

- To know the concept in a better way, consider the function $f(x) = 4x^3 - 5x^2 - 2x - 6$ on $[-3, 4]$.

  Being $f(x)$ a polynomial function, it is differentiable anywhere.

$$f'(x) = \frac{d}{dx}\left(4x^3 - 5x^2 - 2x - 6\right) = 12x^2 - 10x - 2 = 2\left(6x^2 - 5x - 1\right)$$

$$f'(x) = 0$$

$$\Rightarrow 6x^2 - 5x - 1 = 0$$

$$\Rightarrow (6x + 1)(x - 1) = 0$$

$$\Rightarrow x = -\frac{1}{6},\ 1$$

Thus, the only critical point are $x = -\frac{1}{6}$ and 1.

These critical points lie in the interval $[-3, 4]$.

Now, the value of the function at the extremities and the critical points can be calculated as:

$$f(-3) = 4\times(-3)^3 - 5\times(-3)^2 - 2(-3) - 6 = -108 - 45 + 6 - 6 = -153$$

$$f\left(-\frac{1}{6}\right) = 4\times\left(-\frac{1}{6}\right)^3 - 5\times\left(-\frac{1}{6}\right)^2 - 2\left(-\frac{1}{6}\right) - 6 = -\frac{1}{54} - \frac{5}{36} + \frac{1}{3} - 6 = -\frac{629}{108}$$

$$f(1) = 4\times1^3 - 5\times1^2 - 2\times1 - 6 = 4 - 5 - 2 - 6 = -9$$

$$f(4) = 4\times4^3 - 5\times4^2 - 2\times4 - 6 = 256 - 80 - 8 - 6 = 162$$

Among $-153, -\frac{629}{108}, -9,$ and 162; 162 and -153 are the greatest and the smallest values respectively.

Therefore, the absolute maximum value of f is 162 at $x = 4$ and the absolute minimum value of f is −153 at $x = -3$.

Solved Examples

Example 1:

What is the absolute maximum and minimum values of the function $f(x) = 2x^3 + 5x^2 - 4x - 3$ on the interval [−4, 0]?

Solution:

$$f(x) = 2x^3 + 5x^2 - 4x - 3$$

$$f'(x) = 6x^2 + 10x - 4$$

f is differentiable everywhere.

$$f'(x) = 0$$

$$\Rightarrow 6x^2 + 10x - 4 = 0$$

$$\Rightarrow 3x^2 + 5x - 2 = 0$$

$$3x^2 + 6x - x - 2 = 0$$

$$3x(x+2) - 1(x+2) = 0$$

$$\Rightarrow x = \frac{1}{3},\ -2$$

Thus, the critical points are $x = \frac{1}{3}$ and $x = -2$

$$f(-4) = 2(-4)^3 + 5(-4)^2 - 4(-4) - 3 = -35$$

$$f(-2) = 2(-2)^3 + 5(-2)^2 - 4(-2) - 3 = 9$$

$$f\left(\frac{1}{3}\right) = 2\left(\frac{1}{3}\right)^3 + 5\left(\frac{1}{3}\right)^2 - 4\left(\frac{1}{3}\right) - 3 = \frac{-110}{27}$$

$$f(0) = 2(0)^3 + 5(0)^2 - 4(0) - 3 = -3$$

Thus, f has absolute maximum value at $x = -2$ and the absolute maximum value is 9. The absolute minimum value is −35 at $x = -4$

Example 2:

Find the absolute maximum and minimum value of function $f(x)=\frac{x^2}{64}+\frac{2}{\sqrt{x}}+2$ on [1, 36].

Solution:

$$f(x)=\frac{x^2}{64}+\frac{2}{\sqrt{x}}+2$$

$$f'(x)=\frac{2x}{64}-\frac{2}{2x^{\frac{3}{2}}}$$

$$\Rightarrow f'(x)=0$$

$$\Rightarrow \frac{2x}{64}-\frac{2}{2x^{\frac{3}{2}}}=0$$

$$\Rightarrow \frac{2x}{64}=\frac{2}{2x^{\frac{3}{2}}}$$

$$\Rightarrow x^{\frac{5}{2}}=32$$

$$\Rightarrow x=(32)^{\frac{2}{5}}$$

$$\Rightarrow x=4$$

$$f(1)=\frac{1}{64}+2+2=\frac{257}{64}$$

$$f(4)=\frac{(4)^2}{64}+\frac{2}{\sqrt{4}}+2=\frac{13}{4}$$

$$f(36)=\frac{(36)^2}{64}+\frac{2}{\sqrt{36}}+2=\frac{271}{12}$$

Thus, the absolute maximum value of f is $\frac{271}{12}$ at $x = 36$ and absolute minimum value is

$\frac{13}{4}$ at $x = 4$