Session 6

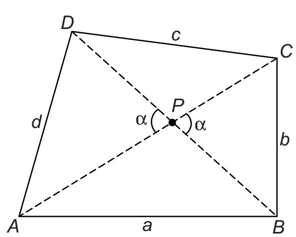
Quadrilaterals and Cyclic Quadrilaterals

Area of Quadrilateral

$ABCD$ is any quadrilateral where $AB = a$, $BC = b$, $CD = c$, $AD = d$ and $\angle DPA = \alpha$.

Let s denotes the area of quadrilateral, then

area of ΔDAC = area of ΔAPD + area of ΔDPC



$$= \frac{1}{2} DP \cdot AP \cdot \sin\alpha + \frac{1}{2} \cdot DP \cdot PC \cdot \sin(\pi - \alpha)$$

$$= \frac{1}{2} DP.(AP + PC)\sin\alpha$$

Area of $\Delta DAC = \frac{1}{2} DP \cdot AC \cdot \sin\alpha$ ...(i)

Similarly,

area of $\Delta ABC = \frac{1}{2} BP \cdot AC \cdot \sin\alpha$ ...(ii)

$\therefore$ s = area of ΔDAC + area of ΔABC

$$= \frac{1}{2} DP \cdot AC \cdot \sin\alpha + \frac{1}{2} BP \cdot AC \cdot \sin\alpha$$

[using Eqs. (i) and (ii)]

$$= \frac{1}{2}(DP + BP) \cdot AC \cdot \sin\alpha$$

$\Rightarrow$ $s = \frac{1}{2} BD \cdot AC \cdot \sin\alpha$

$\therefore$ **Area of quadrilateral = $\frac{1}{2}$ (product of the diagonals) $\times$ (sine of included angle).**

Again,

We can express the area of Δ in terms of sides and the sum of two opposite angles :

In ΔABD, $\quad BD^2 = a^2 + d^2 - 2ad\cos A \quad \ldots(\text{i})$

In ΔBCD, $\quad BD^2 = b^2 + c^2 - 2bc\cos C \quad \ldots(\text{ii})$

From Eqs. (i) and (ii),

$$a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos C$$

$$\therefore \quad a^2 + d^2 - b^2 - c^2 = 2ad\cos A - 2bc\cos C \quad \ldots(\text{iii})$$

Also, $\quad s = \Delta BAD + \Delta BCD$

$$= \frac{1}{2}ad\sin A + \frac{1}{2}bc\sin C$$

$$\Rightarrow \quad 4s = 2ad\sin A + 2bc\sin C \quad \ldots(\text{iv})$$

On squaring and adding Eqs. (iii) and (iv) both sides, we get

$$16s^2 + (a^2 + d^2 - b^2 - c^2)^2$$
$$= 4a^2d^2 + 4b^2c^2 - 8abcd\cos(A + C) \quad \ldots(\text{v})$$

Let $\quad A + C = 2\alpha$, then

$$\cos(A + C) = \cos 2\alpha = 2\cos^2\alpha - 1$$

From Eq. (v), we get

$$16s^2 = 4a^2d^2 + 4b^2c^2 - 8abcd(2\cos^2\alpha - 1) - (a^2 + d^2 - b^2 - c^2)^2$$

$$= \{2(ad + bc)\}^2 - (a^2 + b^2 - c^2 - d^2)^2 - 16abcd\cos^2\alpha$$

$$= \{(a + d)^2 - (b + c)^2\}\{(b + c)^2 - (a - d)^2\} - 16abcd\cos^2\alpha$$

$$= (a + d + b - c)(a + d - b + c)(b + c + a - d)(b + c - a + d) - 16abcd\cos^2\alpha$$

Let, $2s = a + b + c + d$

$\therefore$

$$16s^2 = (2s - 2a)(2s - 2b)(2s - 2c)(2s - 2d) - 16abcd\cos^2\alpha$$

$$s^2 = (s - a)(s - b)(s - c)(s - d) - abcd\cos^2\alpha$$

$$\Rightarrow \quad s = \sqrt{(s - a)(s - b)(s - c)(s - d) - abcd\cos^2\alpha}$$

where $2\alpha = A + C$ and $2s = a + b + c + d$

Thus, area of quadrilateral;

$$\frac{1}{2}BD \cdot AC \cdot \sin\alpha, \text{ where } \angle DPA = \alpha$$

or $\quad \boldsymbol{A = \sqrt{(s - a)(s - b)(s - c)(s - d) - abcd\cos^2\alpha}}$,

where $2\alpha = A + C$

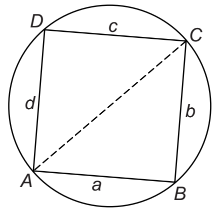
Area of Cyclic Quadrilateral

A quadrilateral is cyclic quadrilateral if its vertices lie on a circle.

Let $ABCD$ be a cyclic quadrilateral such that $AB = a$, $BC = b$, $CD = c$ and $DA = d$.

Then, $\angle B + \angle D = 180°$ and $\angle A + \angle C = 180°$.

Let $2s = a + b + c + d$ be the perimeter of the quadrilateral.



Now, $\quad \Delta$ = area of cyclic quadrilateral $ABCD$

= area of ΔABC + area of ΔACD

$$= \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D$$

$$= \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin(\pi - B) = \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin B$$

$$\Rightarrow \quad \Delta = \frac{1}{2}(ab + cd)\sin B \quad \ldots(\text{i})$$

Using cosine formula in a ΔABC and ΔACD, we have

$$AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cdot \cos B$$

$$\Rightarrow \quad AC^2 = a^2 + b^2 - 2ab\cos B \quad \ldots(\text{ii})$$

and $\quad AC^2 = AD^2 + CD^2 - 2AD \cdot CD \cdot \cos D$

$$\Rightarrow \quad AC^2 = d^2 + c^2 - 2cd \cdot \cos(\pi - B) \quad \ldots(\text{iii})$$

From Eqs. (ii) and (iii), we have

$$a^2 + b^2 - 2ab\cos B = d^2 + c^2 - 2cd\cos B$$

$$\Rightarrow \quad 2(ab + cd)\cos B = a^2 + b^2 - c^2 - d^2 \quad \ldots(\text{iv})$$

$$\Rightarrow \quad 4(ab + cd)^2\cos^2 B = (a^2 + b^2 - c^2 - d^2)^2$$

$$\Rightarrow \quad 4(ab + cd)^2 \cdot (1 - \sin^2 B) = (a^2 + b^2 - c^2 - d^2)^2$$

$$\Rightarrow \quad 4(ab + cd)^2\sin^2 B = 4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$$

$$\Rightarrow \quad 4(ab + cd)^2 \cdot \sin^2 B = \{2(ab + cd) + (a^2 + b^2 - c^2 - d^2)\}\{2(ab + cd) - (a^2 + b^2 - c^2 - d^2)\}$$

$$\Rightarrow \quad 4(ab + cd)^2 \cdot \sin^2 B = \{(a + b)^2 - (c - d)^2\} \cdot \{(c + d)^2 - (a - b)^2\}$$

$$\Rightarrow \quad 4(ab + cd)^2 \cdot \sin^2 B = \{(a + b + c - d)(a + b - c + d)(c + d - a + b)(c + d + a - b)$$

$$\Rightarrow \quad 4(ab + cd)^2.\sin^2 B = (2s - 2d)(2s - 2c)(2s - 2b)(2s - 2a)$$

$$\Rightarrow \quad 16\Delta^2 = 16(s - a)(s - b)(s - c)(s - d) \quad [\text{using Eq. (i)}]$$

$$\Rightarrow \quad \Delta = \sqrt{(s - a)(s - b)(s - c)(s - d)} \quad \ldots(\text{v})$$

$\therefore$ From Eqs. (i), (iv) and (v),

Area of cyclic quadrilateral;

$$\boldsymbol{\Delta = \frac{1}{2}(ab + cd) \cdot \sin B, \ \Delta = \sqrt{(s - a)(s - b)(s - c)(s - d)}}$$

and $\quad \cos B = \dfrac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$

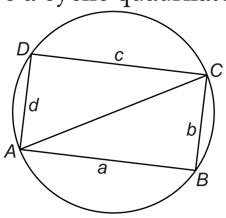
Ptolemy's Theorem

In a cyclic quadrilateral $ABCD$,

$$AC \cdot BD = AB \cdot CD + BC \cdot AD$$

i.e. in a cyclic quadrilateral the product of diagonals is equal to the sum of the products of the lengths of the opposite sides.

Proof: Let $ABCD$ be a cyclic quadrilateral, where



$$AC^2 = a^2 + b^2 - 2ab\cos B$$

and $$\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$$

$$\Rightarrow \quad AC^2 = a^2 + b^2 - 2ab \cdot \left\{\frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}\right\}$$

$$\Rightarrow \quad AC^2 = \frac{(a^2 + b^2)(ab + cd) - ab(a^2 + b^2 - c^2 - d^2)}{(ab + cd)}$$

$$\Rightarrow \quad AC^2 = \frac{(a^2 + b^2) \cdot cd + ab(c^2 + d^2)}{ab + cd}$$

$$\Rightarrow \quad AC^2 = \frac{(ac + bd) \cdot (ad + bc)}{ab + cd} \qquad \text{...(i)}$$

Similarly,

$$BD^2 = \frac{(ab + bd) \cdot (ac + bd)}{ad + bc} \qquad \text{...(ii)}$$

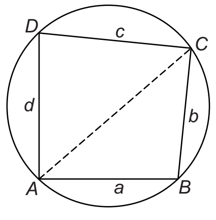
$$\Rightarrow \quad AC^2 \cdot BD^2 = (ac + bd)^2$$

$$\Rightarrow \quad AC \cdot BD = (ac + bd)$$

$$\Rightarrow \quad AC \cdot BD = AB \cdot CD + BC \cdot AD \text{[Ptolemy's theorem]}$$

Circum-radius of a Cyclic Quadrilateral

Let $ABCD$ be a cyclic quadrilateral. Then the circum-circle of the quadrilateral $ABCD$ is also the circum circle of ΔABC.



Hence, the circum-radius of the cyclic quadrilateral

$$ABCD = R = \text{Circum-radius of } \Delta ABC = \frac{AC}{2\sin B}$$

$$= \frac{AC(ab + cd)}{4\Delta} \qquad \left[\text{using } \Delta = \frac{1}{2}(ab + cd)\sin B\right]$$

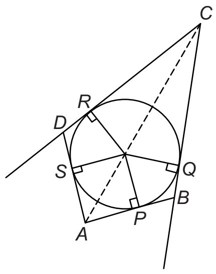
But, $$AC = \sqrt{\frac{(ac + bd)(ad + bc)}{(ab + cd)}} \qquad \text{[using Eq. (i)]}$$

Hence, $$\boldsymbol{R = \frac{1}{4\Delta}\sqrt{(ac + bd)(ad + bc)(ab + cd)}}$$

$$\therefore \quad \boldsymbol{R = \frac{1}{4}\sqrt{\frac{(ac + bd)(ad + bc)(ab + cd)}{(s - a)(s - b)(s - c)(s - d)}}}$$

Example 36. If the sides of a cyclic quadrilateral are 3, 3, 4, 4. Show that a circle can be inscribed in it.

Sol. By geometry,



$$AP = AS \qquad \text{...(i)}$$
$$BP = BQ \qquad \text{...(ii)}$$
$$DR = DS \qquad \text{...(iii)}$$
$$CR = CQ \qquad \text{...(iv)}$$

Adding all four equations, we get

$$AB + CD = AD + CB \qquad \text{...(v)}$$

Now, the sides of cyclic quadrilateral 3, 3, 4, 4 inscribe the circle in it, if it satisfy (v).

i.e. let $AB + CD = 3 + 4 = 7$, $AD + CB = 3 + 4 = 7$

i.e. 3, 3, 4, 4 i.e. $3 = AB$, $3 = BC$, $4 = CD$, $4 = CB$

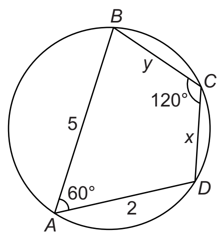
satisfy condition (v) or a circle can be inscribed.

Note

If sum of opposite side of a quadrilateral is equal, then and only then a circle can be inscribed in the quadrilateral.

Example 37. The two adjacent sides of a cyclic quadrilateral are 2 and 5 the angle between them is $60°$. If the area of the quadrilateral is $4\sqrt{3}$, then find the remaining two sides.

Sol. Let $AD = 2$, $AB = 5$, $\angle DAB = 60°$



Since, the quadrilateral is cyclic, $\angle BCD = 120°$

Area of

$$\Delta ABD = \frac{1}{2} \cdot 2 \cdot 5\sin 60° = 5 \cdot \frac{\sqrt{3}}{2} \qquad \text{...(i)}$$

Area of ΔBCD = Area of quadrilateral $ABCD$ − area of ΔABD

$$= 4\sqrt{3} - \frac{5\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

...(ii) [using Eq. (i)]

Let $CD = x$ and $BC = y$

Now, area of $\Delta BCD = \frac{1}{2}xy\sin 120°$

or $\qquad \dfrac{3\sqrt{3}}{2} = \dfrac{1}{2} xy \dfrac{\sqrt{3}}{2}$ $\qquad$ ...(iii) [using Eq. (ii)]

$\Rightarrow \qquad xy = 6$ $\qquad$ ...(iv)

Applying cosine rule in ΔBAD, we get

$$\cos 60^\circ = \frac{AD^2 + AB^2 - BD^2}{2AD \cdot AB}$$

or $\qquad \dfrac{1}{2} = \dfrac{2^2 + 5^2 - BD^2}{2 \cdot 2 \cdot 5}$

$\Rightarrow \qquad BD^2 = 19$

Applying cosine rule in ΔBCD, we get

$$\cos 120^\circ = \frac{x^2 + y^2 - 19}{2xy}$$

or $\qquad \dfrac{1}{2} = \dfrac{x^2 + y^2 - 19}{2xy}$

or $\qquad x^2 + y^2 + xy = 19$ $\qquad$ [using $xy = 6$]

or $\qquad x^2 + y^2 = 13$

Now, $\qquad x^2 + y^2 + 2xy = 13 + 12 = 25$

$\Rightarrow \qquad (x + y)^2 = 25$

$\Rightarrow \qquad x + y = 5$

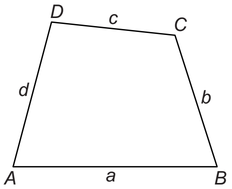
and $\qquad x^2 + y^2 - 2xy = 13 - 12$

$\Rightarrow \qquad x - y = \pm 1$

Solving we get, $(x = 3, y = 2)$ or $(x = 2, y = 3)$

Example 38. If a, b, c, d are the sides of a quadrilateral, then find the minimum value of $\dfrac{a^2 + b^2 + c^2}{d^2}$.

Sol. Here, $AB = a$, $BC = b$, $CD = c$ and $AD = d$ are the sides of quadrilateral $ABCD$.



And we know, $(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$

$\Rightarrow \qquad 2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$

$\Rightarrow \qquad 3(a^2 + b^2 + c^2) \geq (a^2 + b^2 + c^2) + 2(ab + bc + a)$

[i.e., adding $(a^2 + b^2 + c^2)$ to both sides]

$\Rightarrow \qquad 3(a^2 + b^2 + c^2) \geq (a + b + c)^2$

[$\because$ sum of any three sides of quadrilateral is greater than fourth]

$\Rightarrow \qquad 3(a^2 + b^2 + c^2) \geq (a + b + c)^2 > d^2$ $\qquad$ [$\because a + b + c > d$]

$\Rightarrow \qquad \dfrac{a^2 + b^2 + c^2}{d^2} > \dfrac{1}{3}$

$\therefore$ Minimum value of $\dfrac{a^2 + b^2 + c^2}{d^2}$ is $\dfrac{1}{3}$.

Exercise for Session 6

1. The area of a cyclic quadrilateral $ABCD$ is $\dfrac{(3\sqrt{3})}{4}$. The radius of the circle circumscribing cyclic quadrilateral is 1. If $AB = 1$, $BD = \sqrt{3}$, then find $BC \cdot CD$.

2. If two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the third side is 3, then find the remaining fourth side.

3. The ratio of the area of a regular polygon of n sides inscribed in a circle to that of the polygon of same number of sides circumscribing the same circle is 3 : 4. Then, the value of n is

4. A right angled trapezium is circumscribed about a circle. Find the radius of the circle. If the lengths of the bases (i.e. parallel sides) are equal to a and b.

5. If A, B, C, D are the angles of quadrilateral, then find $\dfrac{\Sigma \tan A}{\Sigma \cot A}$.

Answers

Exercise for Session 6

1. 2 **2.** 2 **3.** 6 **4.** $\frac{ab}{a+b}$ **5.** $\pi \tan A$