इकाई–9 : निश्चित समाकलन

1. $\int_{-1}^{1} e^{|x|}dx$ का मान होगा–

(अ) $2e-2$ (ब) $2-2e$

(स) $2+2e$ (द) e^2-2 (अ)

हल: $\int_{-1}^{1} e^{|x|}dx = \int_{-1}^{0} e^{|x|}dx + \int_{0}^{1} e^{|x|}dx$

$= \int_{-1}^{0} e^{-x}dx + \int_{0}^{1} e^{x}dx$

$= \left[-e^{-x}\right]_{-1}^{0} + \left[e^{x}\right]_{0}^{1}$

$= \left[-e^{0}+e^{1}\right] + \left[e-e^{0}\right] = 2e-2$

2. $= \int_{0}^{\pi} |\cos x|\, dx = \int_{0}^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{\pi} |\cos x|\, dx$

(अ) 2 (ब)–2

(स) $\frac{1}{2}$ (द)उपर्युक्त में से कोई नहीं (अ)

हल: $= \int_{0}^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} -\cos x dx$

$= [\sin x]_{0}^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$

$= \left[\sin\frac{\pi}{2} - \sin 0\right] - \left[\sin \pi - \sin\frac{\pi}{2}\right]$

$= (1-0)-(0-1) = 2$

3. $\int_{-1}^{2} \frac{dx}{3x-2}$ का मान होगा–

(अ) $\frac{1}{3}\log\frac{4}{5}$ (ब) $3\log\frac{4}{5}$

(स) $\frac{1}{3}\log\frac{5}{4}$ (द) $3\log\frac{5}{4}$ (अ)

हल: $I = \int_{-1}^{2} \frac{dx}{3x-2} = \frac{1}{3}\left[\log(3x-2)\right]_{-1}^{2}$

$= \frac{1}{3}\left[\log|6-2| - \log|-3-2|\right]$

$= \frac{1}{3}\left[\log 4 - \log 5\right]$

$= \frac{1}{3}\log\frac{4}{5}$

4. $\int_{0}^{1} \frac{2x}{1+x^4} dx$ का मान ज्ञात कीजिए।

हल: $I = \int_{0}^{1} \frac{2x}{1+x^4} dx$

माना कि $x^2 = t \Rightarrow 2xdx = dt$

जब $x=0$ तब $t=0$

जब $x=1$ तब $t=1$

$I = \int_{0}^{1} \frac{dt}{1+t^2} = \left[\tan^{-1} t\right]_{0}^{1}$

$= \tan^{-1}(1) - \tan^{-1}(0)$

5. $\int_{4/\pi}^{2/\pi} \left(-\frac{1}{x^3}\right)\cos\left(\frac{1}{x}\right) dx$ को हल कीजिए।

हल: $I = \int_{4/\pi}^{2/\pi} \left(-\frac{1}{x^3}\right)\cos\left(\frac{1}{x}\right) dx$

माना $\frac{1}{x} = t$ जब $x = \frac{4}{\pi}$ तब $t = \frac{\pi}{4}$

$-\frac{1}{x^2}dx = dt$ $\quad x = \frac{2}{\pi}$ तब $t = \frac{\pi}{2}$

$\therefore I = \int_{\pi/4}^{\pi/2} t\cos t dt$

$t\int \cos t dt - \int\left\{\frac{d}{dt}(t)\int \cos t dt\right\}dt$

$= [t\sin t + \cos t]_{\pi/4}^{\pi/2} = \left[\left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - \left(\frac{\pi}{4}\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right)\right]$

$= \left(\frac{\pi}{2}+0\right) - \left(\frac{\pi}{4}\times\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$

$= \frac{\pi}{2} - \frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}}$

6. $I = \int_{\alpha}^{\beta} \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}, \beta > \alpha$ को हल कीजिए।

हलः $I=\int_{\alpha}^{\beta}\frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

माना $x-\alpha=t^2$

$dx=2tdt$

जब $x=\alpha$ तब $t=0$

$x=\beta$ तब $t=\sqrt{\beta-\alpha}$

$$I=\int_{0}^{\sqrt{\beta-\alpha}}\frac{2t}{\sqrt{t^2\left\{\beta-\left(\alpha+t^2\right)\right\}}}dt$$

$$=2\int_{0}^{\sqrt{\beta-\alpha}}\frac{dt}{\sqrt{(\beta-\alpha)-t^2}}$$

$$=2\left[\sin^{-1}\frac{t}{\sqrt{\beta-\alpha}}\right]_0^{\sqrt{\beta-\alpha}}$$

$$=2\left[\sin^{-1}(1)-\sin^{-1}(0)\right]$$

$$=\pi$$

7. $\int_{0}^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}dx$

हलः $I=\int_{0}^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}dx$

$\because (\sin x-\cos x)^2=\sin^2 x+\cos^2 x-2\sin x\cos x$

$(\sin x-\cos x)^2=1-\sin 2x$

$\sin 2x=1-(\sin x-\cos x)^2$

$$I=\int_{0}^{\pi/4}\frac{\sin x+\cos x}{9+16\left[1-(\sin x-\cos x)^2\right]}dx$$

$$=\int_{0}^{\pi/4}\frac{\sin x+\cos x}{25-16(\sin x-\cos x)^2}dx$$

$$=\frac{1}{16}\int_{0}^{\pi/4}\frac{\sin x+\cos x}{\frac{25}{16}-(\sin x-\cos x)^2}dx$$

$$=\frac{1}{16}\int_{0}^{\pi/4}\frac{\sin x+\cos x}{\left(\frac{5}{4}\right)^2-(\sin x-\cos x)^2}dx$$

माना $\sin x-\cos x=t$ जब $x=0$ तब $t=-1$

जब $x=\frac{\pi}{4}$ तब $t=0$

$(\cos x+\sin x)dx=dt$

$$=\frac{1}{16}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{4}\right)^2-t^2}$$

$$\because \int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$$

$$=\frac{1}{16}\left[\frac{1}{2\times\frac{5}{4}}\log\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right|\right]_{-1}^{0}$$

$$=\frac{1}{40}\left[\log\frac{5+4t}{5-4t}\right]_{-1}^{0}$$

$$=\frac{1}{40}\left[\log(1)-\log\frac{1}{9}\right]$$

$$=\frac{1}{40}\log 3^2=\frac{2}{40}\log|3|+c$$

$$=\frac{1}{20}\log|3|+c$$

8. $\int_{e}^{e^2}\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right]dx$ को हल कीजिए।

हलः माना $\log x=t \therefore x=e^t$

$\frac{1}{x}dx=dt$ जब $x=e$ तब $t=1$

$x=e^2$ तब $t=2$

$$I=\int_{1}^{2}\left(\frac{1}{t}-\frac{1}{t^2}\right)e^t dt$$

$$\because \left\{\int e^x f(x)+f'(x)\right\}dx=e^x f(x)+c$$

$$I=\left[e^t.\frac{1}{t}\right]_1^2=\frac{e^2}{2}-e$$

9. $\int_{0}^{3}\sqrt{\frac{x}{3-x}}dx$ को हल कीजिए।

हलः $I=\int_0^3\sqrt{\frac{x}{3-x}}dx$

$x=3\sin^2\theta$

$dx=3\times 2\sin\theta\cos\theta d\theta$

जब $x=0$ तब $\theta=0$

$x=3$ तब $\theta=\frac{\pi}{2}$

$$I=\int_0^{\pi/2}\sqrt{\frac{3\sin^2\theta}{3-3\sin^2\theta}}\times 6\sin\theta\cos\theta d\theta$$

$$=\int_0^{\pi/2}\frac{\sin\theta}{\cos\theta}\times 6\sin\theta\cos\theta d\theta$$

$$=6\int_0^{\pi/2}\sin^2\theta d\theta$$

$$=6\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$=3\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$=3\left[\frac{\pi}{2}-\frac{1}{2}\sin 2\times\frac{\pi}{2}-(0)\right]$$

$$=3\left[\frac{\pi}{2}\right]=\frac{3\pi}{2}$$

10. $\int_{-\pi/4}^{\pi/4}x^5\cos^2 xdx$ का मान ज्ञात कीजिए।

हलः $I=\int_{-\pi/4}^{\pi/4}x^5\cos^2 xdx$

$\because f(x)=x^5\cos^2 x$

$f(-x)=(-x)^5\cos^2(-x)$

$=-x^5\cos^2 x=-f(x)$

यहां पर $f(-x)=-f(x)$

दिया गया फलन विषम है अतः P_7 से

$$\int_{-\pi/4}^{\pi/4}x^5\cos^2 xdx=0$$

11. $\int_0^1\log\left(\frac{1}{x}-1\right)dx$ को सरल कीजिए।

हलः $I=\int_0^1\log\left(\frac{1}{x}-1\right)dx=\int_0^1\log\left(\frac{1-x}{x}\right)dx$ ...1

P_4 से–

$$=\int_0^1\log\left(\frac{1-1+x}{1-x}\right)dx$$

$$=\int_0^1\log\left(\frac{x}{1-x}\right)dx$$ ...2

समीकरण 1 व 2 को जोड़ने पर

$$2I=\int_0^1\left\{\log\left(\frac{1-x}{x}\right)+\log\left(\frac{x}{1-x}\right)\right\}dx$$

$$2I=\int_0^1\log\left(\frac{1-x}{x}\times\frac{x}{1-x}\right)dx=\int_0^1\log 1\,dx=0$$

$2I=0\Rightarrow I=0$

12. $\int_0^{\pi/2}\log\sin 2xdx$ को हल कीजिए।

हलः माना कि $I=\int_0^{\pi/2}\log\sin 2xdx$

$2x=t\Rightarrow 2dx=dt$

$dx=\frac{1}{2}dt$

जब $x=0$ तब $t=0$

$x=\frac{\pi}{2}$ तब $t=\pi$

अतः $I=\int_0^{\pi}\log\sin t.\frac{1}{2}dt$

$$=\frac{1}{2}\int_0^{\pi}\log\sin tdt$$

P_1 से

$$I=\frac{1}{2}\int_0^{\pi}\log\sin xdx$$

P_7 से $\log\sin(\pi-x)=\log\sin x$

$$I=\frac{1}{2}\times 2\int_0^{\pi/2}\log\sin xdx$$

$$I = \int_0^{\pi/2} \log \sin x dx$$

$$I = \frac{-\pi}{2} \log 2$$

$$I = \frac{\pi}{2} \log \frac{1}{2}$$

13. $\int_0^{\pi} \log(1-\cos x) dx$ को हल कीजिए।

हलः $I = \int_0^{\pi} \log(1-\cos x) dx$ ...1

P_5 से

$$I = \int_0^{\pi} \log[1-\cos(\pi - x)] dx$$

$$= \int_0^{\pi} \log(1+\cos x) dx \quad ...2$$

समीकरण 1 + 2

$$2I = \int_0^{\pi} \log(1-\cos x)dx + \int_0^{\pi} \log(1+\cos x)dx$$

$$= \int_0^{\pi} \log\{(1-\cos x)(1+\cos x)\} dx$$

$$= \int_0^{\pi} \log(1-\cos^2 x)dx = \int_0^{\pi} \log \sin^2 x dx$$

$$2I = 2\int_0^{\pi} \log \sin x dx$$

$$I = \int_0^{\pi} \log \sin x dx$$

$$f(x) = \log \sin x$$
$$f(\pi - x) = \log \sin(\pi - x) = \log \sin x$$

P_7 से

$$I = 2\int_0^{\pi/2} \log \sin x dx$$

$$\because \int_0^{\pi/2} \log \sin x dx = \frac{-\pi}{2} \log 2$$

$$I = 2\left(\frac{-\pi}{2} \log 2\right) = -\pi \log 2$$

$$= \pi \log \frac{1}{2}$$