12. HYPERBOLA

1. INTRODUCTION

A hyperbola is the locus of a point which moves in the plane in such a way that the ratio of its distance from a fixed point in the same plane to its distance from a fixed line is always constant which is always greater than unity.

The fixed point is called the focus, the fixed line is called the directrix. The constant ratio is generally denoted by e and is known as the eccentricity of the hyperbola. A hyperbola can also be defined as the locus of a point such that the absolute value of the difference of the distances from the two fixed points (foci) is constant. If S is the focus, ZZ' is the directrix and P is any point on the hyperbola as show in figure.

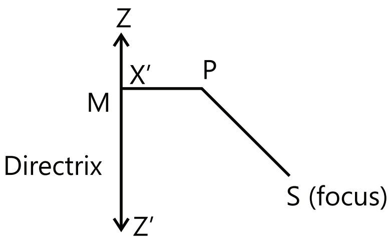


Figure 12.1

Then by definition, we have $\frac{SP}{PM} = e\ (e > 1)$.

Note: The general equation of a conic can be taken as $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

This equation represents a hyperbola if it is non-degenerate (i.e. eq. cannot be written into two linear factors)

$\Delta \neq 0$, $h^2 > ab$. Where $\left(\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}\right)$

CONCEPTS

1. The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ can be written in matrix form as

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + 2gx + 2fy + c = 0 \text{ and } \begin{bmatrix} x & y & 1 \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

Degeneracy condition depends on the determinant of the 3x3 matrix and the type of conic depends on the determinant of the 2x2 matrix.

2. Also the equation can be taken as the intersection of $z = ax^2 + 2hxy + by^2$ and the plane $z = -(2gx + 2fy + c)$

Vaibhav Gupta (JEE 2009, AIR 54)

2. STANDARD EQUATION OF HYPERBOLA

Let the center O of the hyperbola be at the origin O and the foci F_1 and F_2 be on the x-axis.

The coordinates of foci F_1 and F_2 are (–c, 0) and (c, 0).

By the definition of hyperbola,

Distance between a point P and focus F_1 – Distance between P and focus F_2 = constant (say 2a)

$PF_1 - PF_2 = 2a; \sqrt{(x+c)^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} = 2a$

$\Rightarrow \quad \sqrt{(x+c)^2+(y)^2} = 2a + \sqrt{(x-c)^2+(y)^2}$

Squaring both the sides, we get $\quad (x + c)^2 + y^2 = 4a^2 + 2(2a).\sqrt{(x-c)^2+(y)^2} + (x - c)^2 + y^2$

$\Rightarrow \quad x^2 + 2cx + c^2 + y^2 = 4a^2 + 4a\sqrt{(x-c)^2+(y)^2} + x^2 - 2cx + c^2 + y^2$

$\Rightarrow \quad 4cx = 4a^2 + 4a\sqrt{(x-c)^2+(y)^2} \quad \Rightarrow \quad cx = a^2 + a\,.\sqrt{(x-c)^2+(y)^2} \quad \Rightarrow \quad cx - a^2 = a\sqrt{(x-c)^2+(y)^2}$

Squaring again, we get

$c^2x^2 - 2a^2cx + a^4 = a^2[(x - c)^2 + y^2]$

$c^2x^2 - 2a^2cx + a^4 = a^2[x^2 - 2cx + c^2 + y^2] = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$

$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$

$\Rightarrow \quad (c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2) \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1$

$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad$ (taking $b^2 = c^2 - a^2$)

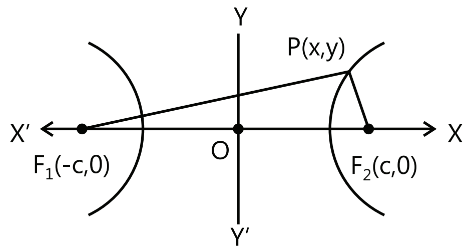


Figure 12.2

Hence, any point P(x, y) on the hyperbola satisfies the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

3. TERMS ASSOCIATED WITH HYPERBOLA

(a) Focus: The two fixed points are called the foci of the hyperbola and are denoted by F_1 and F_2. The distance between the two foci F_1 and F_2 is denoted by 2c.

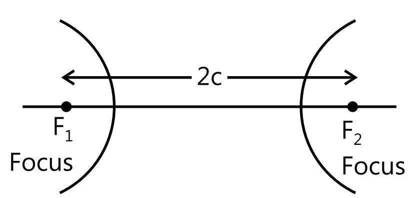


Figure 12.3

(b) Centre: The midpoint of the line joining the foci is called the center of the hyperbola.

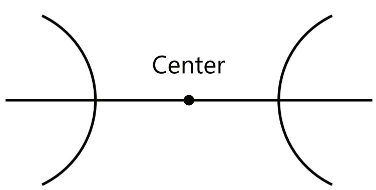


Figure 12.4

(c) Transverse-Axis: The line through the foci is called the transverse axis. Length of the transverse axis is 2a.

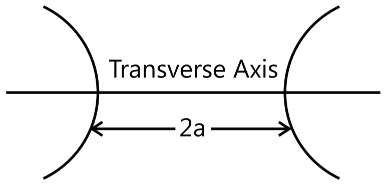


Figure 12.5

(d) Conjugate-Axis: The line segment through the center and perpendicular to the transverse axis is called the conjugate axis. Length of the conjugate axis is 2b.

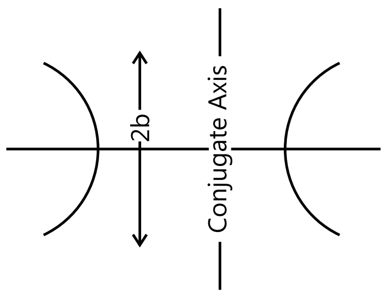


Figure 12.6

(e) Vertices: The points at which the hyperbola intersects the transverse axis are called the vertices of the hyperbola. The distance between the two vertices is denoted by 2a.

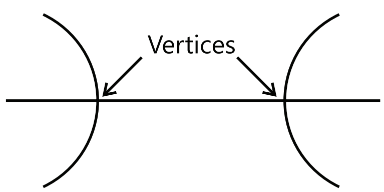


Figure 12.7

(f) Eccentricity: Eccentricity of the hyperbola is defined as $\frac{c}{a}$ and it is denoted by e. And e is always greater than 1 since c is greater than 1.

(g) Directrix: Directrix is a line perpendicular to the transverse axis and cuts it at a distance of $\frac{a^2}{c}$ from the centre.

i.e. $x = \pm\frac{a^2}{c}$ or $y = \pm\frac{a^2}{c}$

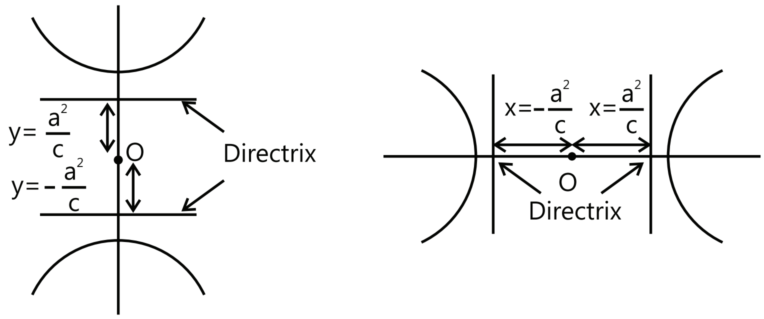


Figure 12.8

(h) Length of The Latus Rectum: The Latus rectum of a hyperbola is a line segment perpendicular to the transverse axis and passing through any of the foci and whose end points lie on the hyperbola. Let the length of LF be ℓ. Then, the coordinates of L are (c, ℓ)

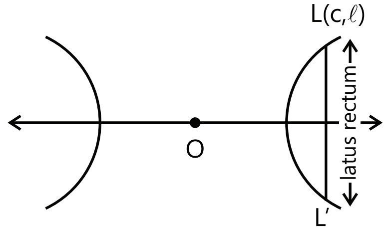


Figure 12.9

Since, L lies on hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Therefore, we have $\frac{c^2}{a^2}-\frac{\ell^2}{b^2}=1$

$\Rightarrow \frac{\ell^2}{b^2}=\frac{c^2}{a^2}-1=\frac{c^2-a^2}{a^2} \quad \Rightarrow \quad \ell^2=b^2\left(\frac{b^2}{a^2}\right)=\frac{b^4}{a^2} \Rightarrow \frac{b^2}{a}$

Latus rectum LL′ = LF + L′F = $\frac{b^2}{a}+\frac{b^2}{a}=\frac{2b^2}{a}$

(i) Focal Distance of a Point: Let P(x, y) be any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ as shown in figure. Then by definition,

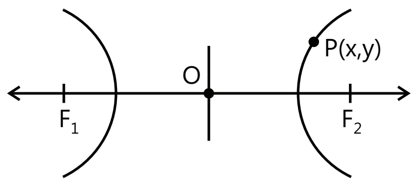


Figure 12.10

We have SP = e.PM and S¢P = e.PM′

$\Rightarrow$ SP = e.NK = e (CN – CK) = $e\left(x-\frac{a}{e}\right)$ = ex – a and S¢P = e(NK′)

= e(CN + CK′) = $e\left(x-\frac{a}{e}\right)$ = ex + a

$\Rightarrow$ S¢P – SP = (ex + a) – (ex – a) = 2a = length of transverse axis

Illustration 1: Find the equation of the hyperbola, where the foci are (±3, 0) and the vertices are (±2, 0). **(JEE MAIN)**

Sol: Use the relation $c^2 = a^2 + b^2$, to find the value of b and hence the equation of the hyperbola.

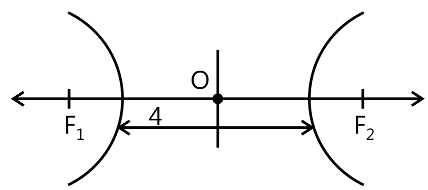


Figure 12.11

We have, foci = (±c, 0) = (±3, 0) $\Rightarrow$ c = 3

and vertices (±a, 0) = (±2, 0)

a = 2

But $c^2 = a^2 + b^2 \Rightarrow 9 = 4 + b^2 \Rightarrow b^2 = 9 - 4 = 5 \Rightarrow b^2 = 5$

Here, the foci and vertices lie on the x-axis, therefore the equation of the hyperbola is of the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \Rightarrow \qquad \frac{x^2}{4}-\frac{y^2}{5}=1$$

Illustration 2: Find the equation of the hyperbola, where the vertices are (0, ±5) and the foci are (0, ±8). **(JEE MAIN)**

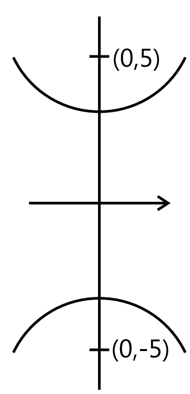


Figure 12.12

Sol: Similar to the previous question.

We have, vertices (0, ±a) = (0, ±5) $\Rightarrow$ a = 5

foci (0, ±c) = (0, ±8) $\Rightarrow$ c = 8

But, we know that $c^2 = a^2 + b^2 \Rightarrow 64 = 25 + b^2$

$\Rightarrow b^2 = 64 - 25 = 39$

Here, the foci and vertices lie on the y-axis, therefore the equation of hyperbola is of the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1.$ i.e., $\frac{y^2}{25}-\frac{x^2}{39}=1$

which is the required equation of the hyperbola.

Illustration 3: If circle c is a tangent circle to two fixed circles c_1 and c_2, then show that the locus of c is a hyperbola with c_1 and c_2 as the foci. **(JEE MAIN)**

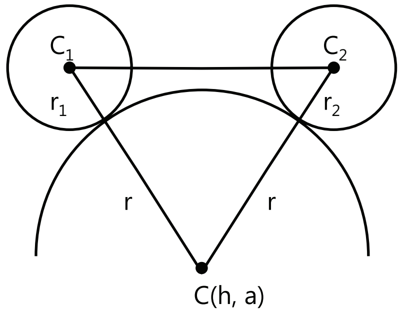


Figure 12.13

Sol: Refer to the definition of a hyperbola.

$cc_1 = r + r_1$; $cc_2 = r + r_1$

$cc_1 - cc_2 = r_1 - r_2$ = constant

Illustration 4: Find the equation of the hyperbola whose directrix is 2x + y = 1 and focus, (1, 2) and eccentricity $\sqrt{3}$. **(JEE MAIN)**

Sol: Use the definition of the hyperbola to derive the equation.

Let P(x, y) be any point on the hyperbola. Draw PM perpendicular from P on the directrix.

They by definition SP = e PM

$\Rightarrow (SP)^2 = e^2(PM)^2 \Rightarrow (x-1)^2 + (y-2)^2 = 3\left\{\frac{2x+y-1}{\sqrt{4+1}}\right\}^2 \Rightarrow 5(x^2 - y^2 - 2x - 4y + 5) = (4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow 7x^2 - 2y^2 + 12x + 14y - 22 = 0$

which is the required hyperbola.

Illustration 5: Find the equation of the hyperbola when the foci are at $(\pm 3\sqrt{5}, 0)$, and the latus rectum is of length 8. **(JEE ADVANCED)**

Sol: Use the formula for the length of the latus rectum to get a relation between a and b. Then use the foci and the relation between a and b to get the equation of the hyperbola.

Here foci are at $(\pm 3\sqrt{5}, 0)$ $\Rightarrow$ $c = 3\sqrt{5}$

Length of the latus rectum = $\frac{2b^2}{a} = 8$

$\Rightarrow$ $b^2 = 4a$ ... (i)

We know that

$c^2 = a^2 + b^2$

$(3\sqrt{5})^2 = a^2 + 4a$

$45 = a^2 + 4a$

$a^2 + 4a - 45 = 0$

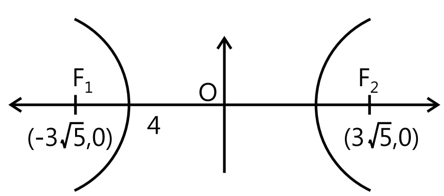


Figure 12.14

$(a + 9)(a - 5) = 0$

$a = -9, a = 5$ (a cannot be –ve)

Putting a = 5 in (i), we get

$b^2 = 5 \times 4 = 20 \Rightarrow b^2 = 20$

Since, foci lie on the x-axis, therefore the equation of the hyperbola is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{i.e.,} \quad \frac{x^2}{25} - \frac{y^2}{20} = 1$$

$$\Rightarrow \quad 20x^2 - 25y^2 = 500 \quad \Rightarrow \quad 4x^2 - 5y^2 = 100$$

Which is the required equation of hyperbola.

Illustration 6: Find the equation of the hyperbola when the foci are at $(0, \pm\sqrt{10})$, and passing through (2, 3)

(JEE ADVANCED)

Sol: Start with the standard equation of a hyperbola and use the foci and the point (2 , 3) to find the equation.

Here, foci are at $(0, \pm\sqrt{10})$

$\Rightarrow \quad c = \sqrt{10}$ Here the foci lie at the y-axis.

So the equation of the hyperbola is of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{... (i)}$$

Point (ii, iii) lies on (i).

$$\text{So} \quad \frac{9}{a^2} - \frac{4}{b^2} = 1 \quad \Rightarrow \quad \frac{9}{a^2} = 1 + \frac{4}{b^2} \quad \Rightarrow \quad \frac{9}{a^2} = \frac{b^2+4}{b^2} a^2 = \frac{9b^2}{b^2+4} \quad \text{... (ii)}$$

We know that

$c^2 = a^2 + b^2$

$$\Rightarrow \quad 10 = \frac{9b^2}{b^2+4} + b^2$$

$$\Rightarrow \quad \frac{9b^2 + b^4 + 4b^2}{b^2+4} = 10$$

$\Rightarrow \quad 10b^2 + 40 = b^4 + 13b^2$

$\Rightarrow \quad b^4 + 3b^2 - 40 = 0$

$\Rightarrow \quad (b^2 + 8)(b^2 - 5) = 0$

$\Rightarrow \quad b^2 + 8 = 0, b^2 - 5 = 0$

$\Rightarrow \quad b^2 = -8$ & $b^2 = 5$ ($b^2 = -8$ not possible)

$\Rightarrow \quad b^2 = 5$ in (ii), we get

$$a^2 = \frac{9 \times 5}{5+4} = \frac{45}{9} = 5$$

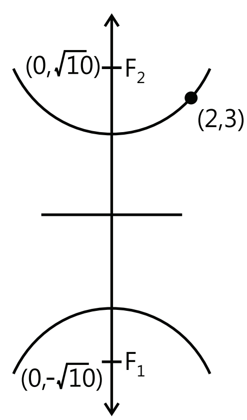


Figure 12.15

Again putting $a^2 = 5$ and $b^2 = 5$ in (i), we get

$$\frac{y^2}{5} - \frac{x^2}{5} = 1 \quad \Rightarrow \quad y^2 - x^2 = 5$$

Which is the required equation of the hyperbola.

Illustration 7: An ellipse and hyperbola are confocal i.e., having same focus and conjugate axis of hyperbola & minor axis of ellipse. If e_1 and e_2 are the eccentricities of the hyperbola and ellipse then find $\frac{1}{e_1^2}+\frac{1}{e_2^2}$.

(JEE ADVANCED)

Sol: Consider the standard equation of an ellipse and hyperbola by taking the eccentricity as e_1 and e_2 respectively. Find the relation between the eccentricities by using the condition that they have the same focus.

Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{A^2}+\frac{y^2}{B^2}=1 \Rightarrow$ $ae_1 = Ae_2$ and $B = b$

$\Rightarrow$ $B^2 = b^2$ $\Rightarrow$ $A^2(e_2^2 - 1) = a^2(1 - e_1)$

$\therefore$ $\frac{a^2e_1^2}{e_2^2}(e_2^2-1) = a^2(1 - e_1^2)$ $\therefore$ $\frac{1}{e_1^2}+\frac{1}{e_2^2} = 2$

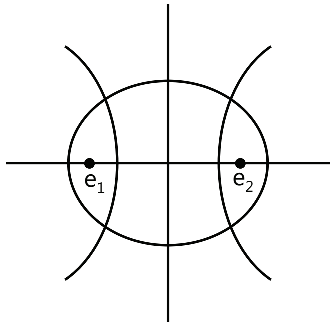


Figure 12.16

Illustration 8: Find the equation of a hyperbola if the distance of one of its vertices from the foci are 3 and 1. Find all the possible equations. **(JEE ADVANCED)**

Sol: Consider two cases when the major axis is parallel to the X – axis and the minor axis is parallel to the Y-axis and vice versa.

Case I:

$ae - a = 1$

$ae + a = 3$

$\Rightarrow$ $e = 2$

$\Rightarrow$ $a = 1$

$\Rightarrow$ $b^2 = 3$

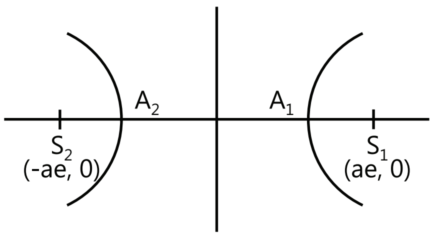


Figure 12.17

Equation of hyperbola is $\frac{x^2}{1}-\frac{y^2}{3} = 1$

Case II

$b(e - 1) = 1$

$b(e + 1) = 3$

$\Rightarrow$ $e = 2, b = 1, a^2 = 3$

Equation of hyperbola is $\frac{y^2}{3}-\frac{x^2}{1}=1$

4. CONJUGATE HYPERBOLA

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axis of a given hyperbola is called the conjugate hyperbola of the given hyperbola. The hyperbola conjugate to the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ is $\frac{y^2}{a^2}-\frac{x^2}{b^2} = 1$

The eccentricity of the conjugate hyperbola is given by a2 = b2(e2 – 1)

and the length of the latus rectum is $\frac{2a^2}{b}$

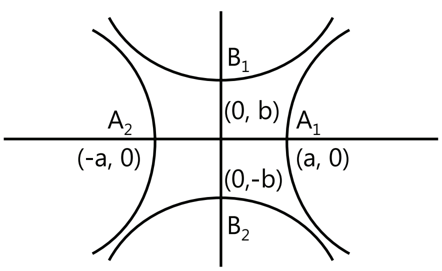
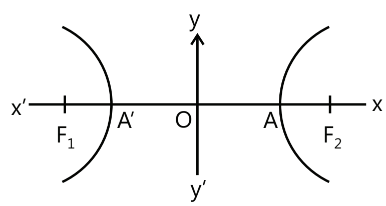
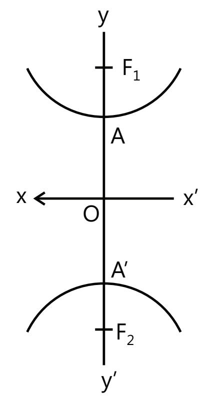


Figure 12.18

Condition of similarity: Two hyperbolas are said to be similar if they have the same value of eccentricity.

Equilateral hyperbola: If a = b or L(T.A.) = L(C.A) then it is an equilateral or rectangular hyperbola.

5. PROPERTIES OF HYPERBOLA/CONJUGATE HYPERBOLA

Equation of the Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Figure	 Figure 12.19	 Figure 12.20
Centre	(0, 0)	(0, 0)
Vertices	(±a, 0)	(0, ±b)
Transverse axis	2a	2b
Conjugate axis	2b	2a
Relation between a, b, c	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
Foci	(±c, 0)	(0, ±c)
Eccentricity	$e = \frac{c}{a}$	$e' = \frac{c}{b}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$

CONCEPTS

- If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate hyperbola then

$$\boxed{\frac{1}{e_1^2}+\frac{1}{e_2^2}=1}$$

$$e_1^2 = 1 + \frac{b^2}{a^2} \qquad e_2^2 = 1 + \frac{a^2}{b^2}$$

$$\therefore \quad \frac{1}{e_1^2}+\frac{1}{e_2^2} = 1$$

- The foci of a hyperbola and its conjugate hyperbola are CONCYCLIC and form vertices of square.

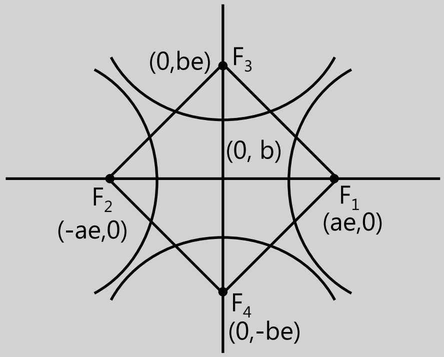


Figure 12.21

Anvit Tawar (JEE 2009, AIR 9)

6. AUXILIARY CIRCLE

A circle described on the transverse axis as diameter is an auxiliary circle and its equation is $x^2 + y^2 = a^2$

Any point of the hyperbola is $P \equiv (a \sec \theta, b \tan \theta)$

P, Q are called corresponding point and θ is eccentric angle of P.

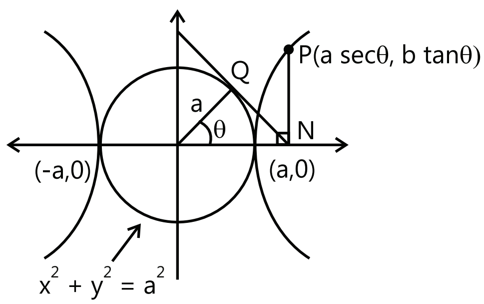


Figure 12.22

CONCEPTS

1. If O $\in$ $(0, \pi/2)$, P lies on upper right branch.
2. If O $\in$ $(\pi/2, \pi)$, P lies on upper left branch.
3. If O $\in$ $(\pi, 3\pi/2)$, P lies on lower left branch.
4. If O $\in$ $(3\pi/2, 2\pi)$, P lies on lower right branch.

Vaibhav Krishnan (JEE 2009, AIR 22)

7. PARAMETRIC COORDINATES

Let P(x, y) be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Draw PL perpendicular from P on OX and then a tangent LM from L to the circle described on A'A as diameter.

Then, x = CL = CM sec θ = a sec θ

Putting x = a sec θ in $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1, we obtain y = b tan θ

Thus, the coordinates of any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 are (a sec θ, b tan θ), where θ is the parameter such that $0 \le \theta \le 2\pi$. These coordinates are known as the parametric coordinates. The parameter θ is also called the eccentric angle of point P on the hyperbola.

The equation x = a sec θ and y = b tan θ are known as the parametric equations of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1.

Note: (i) The circle $x^2 + y^2 = a^2$ is known as the auxiliary circle of the hyperbola.

Let P (a sec θ_1, b tan θ_1) and Q (a sec θ_2, b tan θ_2) be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1.

Then the equation of the chord PQ is

$$y - b\tan q_1 = \frac{b\tan\theta_2 - b\tan\theta_1}{a\sec\theta_2 - a\sec\theta_1}(x - a\sec q_1) \Rightarrow \frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

Illustration 9: Find the eccentricity of the hyperbola whose latus rectum is half of its transverse axis. **(JEE MAIN)**

Sol: Establish the relation between a and b and then use the eccentricity formula.

Let the equation of the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Then transverse axis = 2a and latus rectum = $\frac{2b^2}{a}$

According to the question $\frac{2b^2}{a} = \frac{1}{2}(2a)$

$\Rightarrow$ $2b^2 = a^2$ $\Rightarrow 2a^2(e^2 - 1) = a^2$ $\Rightarrow 2e^2 - 2 = 1 \Rightarrow e^2 = 3/2$ $\therefore e = \sqrt{3/2}$

Illustration 10: If the chord joining two points (a sec θ_1, b tan θ_1) and (a sec θ_2, b tan θ_2) passes through the focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1, then prove that $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{1-e}{1+e}$. **(JEE ADVANCED)**

Sol: Obtain a relation between the two given eccentric angles by substituting the point in the equation of chord.

The equation of the chord joining (a sec θ_1, b tan θ_1) and (a sec θ_2, b tan θ_2) is

$$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

If it passes through the focus (ae, 0) then $e\cos\left(\frac{\theta_1-\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$

$$\Rightarrow \frac{\cos((\theta_1-\theta_2)/2)}{\cos((\theta_1+\theta_2)/2)} = 1/e$$

using componendo dividendo rule we get $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{1-e}{1+e}$.

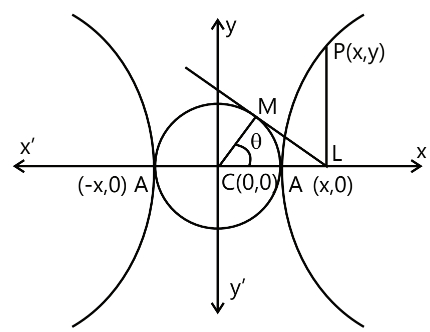


Figure 12.23

8. POINT AND HYPERBOLA

The point (x_1, y_1) lies outside, on or inside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 according to $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ '<' or '=' or '>' 0

Proof: Draw PL perpendicular to x-axis. Suppose it cuts the hyperbola at $Q(x_1, y_2)$.

Clearly, PL > QL

$\Rightarrow \quad y_1 > y_2 \quad \Rightarrow \quad \frac{y_1^2}{b^2} > \frac{y_2^2}{b^2} \quad \Rightarrow \quad -\frac{y_1^2}{b^2} < -\frac{y_2^2}{b^2} \quad \Rightarrow \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < \frac{x_1^2}{a^2} - \frac{y_2^2}{b^2} \quad \Rightarrow \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 1$

$\Rightarrow \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0 \qquad \left[\begin{array}{l} \because Q(x_1, y_2) \text{ lies on } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \\ \qquad \frac{x_1^2}{a^2} - \frac{y_2^2}{b^2} = 1 \end{array}\right]$

Thus the point (x_1, y_1) lies outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0$

Similarly, we can prove that the point (x_1, y_1) will lie inside or on the hyperbola according to

$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 > 0$ or, = 0.

P lies outside/on/inside $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0 / = 0 / > 0$

Illustration 11: Find the position of the points (7, –3) and (2, 7) relative to the hyperbola $9x^2 - 4y^2 = 36$.

(JEE MAIN)

Sol: Use the concept of position of a point w.r.t. the hyperbola.

The equation of the given hyperbola is $9x^2 - 4y^2 = 36$ or, $\frac{x^2}{4} - \frac{y^2}{9} = 1$. Now,

$\frac{7^2}{4} - \frac{(-3)^2}{9} - 1 = \frac{41}{4} > 0$ and, $\frac{2^2}{4} - \frac{7^2}{9} \Rightarrow 1 - \frac{49}{9} \Rightarrow 1 = \frac{-49}{9} < 0.$

Hence, the point (7, –3) lies inside the parabola whereas the point (2, 7) lies outside the hyperbola.

Illustration 12: Find the position of the point (5, –4) relative to the hyperbola $9x^2 - y^2 = 1$. **(JEE MAIN)**

Sol: Use the concept of position of a point

Since $9(5)^2 - (4)^2 = 1 = 225 - 16 - 1 = 208 > 0$. So the point (5, –4) inside the hyperbola $9x^2 - y^2 = 1$.

9. LINE AND HYPERBOLA

Consider a line y = mx + c and hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solving y = mx + c and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\Rightarrow \quad b^2x^2 - a^2(mx + c)^2 = a^2b^2 \quad \Rightarrow \quad (b^2 - a^2m^2)\, x^2 - 2a^2cmx - a^2(b^2 + c^2) = 0;$

D > 0 — Secant

D = 0 — Tangent; $c^2 = a^2m^2 - b^2$ (condition of tangency)

D < 0 — Neither secant nor tangent

$\Rightarrow$ $y = mx + \sqrt{a^2m^2 - b^2}$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

CONCEPTS

No. of tangents drawn to a hyperbola passing through a given point (h, k)

Let y = mx+c be tangent to the hyperbola

$\Rightarrow \quad c^2 = a^2m^2 - b^2$

Since line passes through (h, k)

$\Rightarrow \quad (k - mh)^2 = a^2m^2 - b^2$

$\Rightarrow \quad (h^2 - a^2)m^2 - 2hkm + k^2 + b^2 = 0$

Hence a maximum of 2 tangents can be drawn to the hyperbola passing through (h, k)

$m_1 + m_2 = \frac{2hk}{h^2 - a^2} \qquad m_1m_2 = \frac{k^2 + b^2}{h^2 - a^2}$

if $m_1m_2 = -1 \qquad \boxed{x^2 + y^2 = a^2 - b^2}$

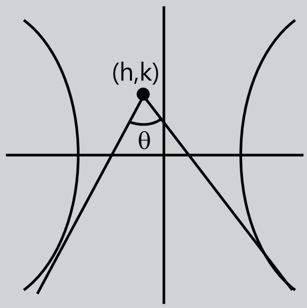


Figure 12.24

Shrikant Nagori (JEE 2009, AIR 30)

Illustration 13: Common tangent to $y^2 = 8x$ and $3x^2 - y^2 = 3$. **(JEE MAIN)**

Sol: Start with the standard equation of a tangent to a parabola and apply the condition for it be a tangent to $3x^2 - y^2 = 3$.

Tangent to the parabola is of the form $y = mx + \frac{2}{m}$. For this line to be tangent to $\frac{x^2}{1} - \frac{y^2}{3} = 1 \qquad c^2 = a^2m^2 - b^2$

$\Rightarrow \quad \frac{4}{m^2} = m^2 - 1 \quad \Rightarrow \quad m^2 = 4 \quad \therefore \ \pm y = 2x + 1$ are the common tangents.

10. TANGENT

Point Form: The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$

Slope Form: The equation of tangents of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by

$y = mx \pm \sqrt{a^2m^2 - b^2}$

The coordinates of the points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$

Parametric Form: The equation of a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec\theta, b \tan\theta)$ is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

Note:

(i) The tangents at the point P $(a \sec \theta_1, b \tan \theta_1)$ and Q $(a \sec \theta_2, b \tan \theta_2)$ intersect at the point R $\left(\frac{a\cos((\theta_1 - \theta_2)/2)}{\cos((\theta_1 + \theta_2)/2)}, \frac{b\sin((\theta_1 + \theta_2)/2)}{\cos((\theta_1 + \theta_2)/2)}\right)$.

(ii) If $|\theta_1 + \theta_2| = \pi$, then the tangents at these points $(\theta_1$ & $\theta_2)$ are parallel.

(iii) There are two parallel tangents having the same slope m. These tangents touch the hyperbola at the extremities of a diameter.

(iv) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ upon any tangent is its auxiliary circle i.e. $x^2+y^2=a^2$ and the product of these perpendiculars is b^2.

(v) The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

(vi) The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as the diameter of the circle.

Illustration 14: Prove that the straight line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $a^2l^2 - b^2m^2 = n^2$. **(JEE MAIN)**

Sol: Apply the condition of tangency and prove the above result.

The given line is $lx + my + n = 0$ or $y = -l/m\, x - n/m$

Comparing this line with $y = Mx + c$ $\quad \therefore M = -l/m$ and $c = -n/m$ ....(i)

This line (i) will touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $c^2 = a^2M^2 - b^2$

$\Rightarrow \frac{n^2}{m^2}=\frac{a^2l^2}{m^2}-b^2$ or $a^2l^2 - b^2m^2 = n^2$. **Hence proved.**

Illustration 15: Find the equations of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$. **(JEE MAIN)**

Sol: Get the slope of the perpendicular line and use it to get the equation of the tangent.

Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y = 0$

$m \times 1 = -1$

$\Rightarrow m = -1$

Since $x^2 - 4y^2 = 36$ or $\frac{x^2}{36}-\frac{y^2}{9}=1$

Comparing this with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \therefore a^2 = 36$ and $b^2 = 9$

So the equation of the tangents are $y = (-1)x \pm \sqrt{36\times(-1)^2-9}$

$\Rightarrow y = -x \pm \sqrt{27} \Rightarrow x + y \pm 3\sqrt{3} = 0$

Illustration 16: If two tangents drawn from any point on hyperbola $x^2 - y^2 = a^2 - b^2$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ make angles θ_1 and θ_2 with the axis then $\tan\theta_1.\tan\theta_2$. **(JEE ADVANCED)**

Sol: Establish a quadratic in m, where m is the slope of the two tangents. Then use the sum and product of the roots to find $\tan\theta_1.\tan\theta_2$.

Let $c^2 = a^2 - b^2$

Any tangent to the ellipse $y = mx \pm \sqrt{a^2m^2+b^2}$

$c\tan\theta = mc\sec\theta \pm \sqrt{a^2m^2+b^2}$)

$c^2(\tan\theta - m\sec\theta)^2 = a^2m^2 + b^2$

$(c^2\sec^2\theta - a^2)m^2 + (......)m + c^2\tan^2\theta - b^2 = 0 \Rightarrow \tan\theta_1.\tan\theta_2$ =product of the roots.

$= \dfrac{c^2\tan^2\theta - b^2}{c^2\sec^2\theta - a^2} = 1.$

11. NORMAL

Point Form: The equation of the normal to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at (x_1, y_1) is $\dfrac{a^2x}{x_1} + \dfrac{b^2y}{y_1} = a^2 + b^2$.

Parametric Form: The equation of the normal at $(a\sec\theta, b\tan\theta)$ to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is $a\,x\cos\theta + b\,y\cot\theta = a^2 + b^2$.

Slope Form: The equation of a normal of slope m to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is given by

$y = mx \mp \dfrac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$ at the points $\left(\pm\dfrac{a^2}{\sqrt{a^2-b^2m^2}}, \mp\dfrac{b^2m}{\sqrt{a^2-b^2m^2}}\right)$

Note:

(i) At most four normals can be drawn from any point to a hyperbola.

(ii) Points on the hyperbola through which, normal through a given point pass are called co-normal points.

(iii) The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This illustrates the reflection property of the hyperbola as "**An incoming light ray**" aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common points.

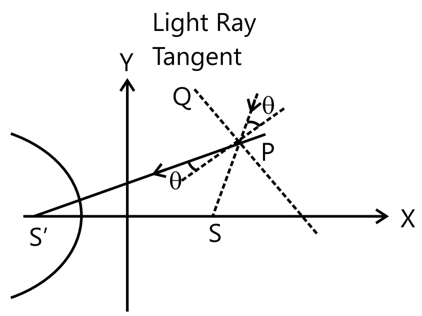


Figure 12.25

(iv) The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and the hyperbola $\dfrac{x^2}{a^2-k^2} - \dfrac{y^2}{k^2-b^2} = 1$ $(a > k > b > 0)$ are confocal and therefore orthogonal.

(v) The sum of the eccentric angles of co-normal points is an odd multiple of π.

(vi) If θ_1, θ_2 and θ_3 are eccentric angles of three points on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. The normals at which are concurrent, then $\sin(\theta_1 + \theta_2) + \sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) = 0$

(vii) If the normals at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ are concurrent, then $(x_1 + x_2 + x_3 + x_4)\left(\dfrac{1}{x_1} + \dfrac{1}{x_2} + \dfrac{1}{x_3} + \dfrac{1}{x_4}\right) = 4$.

Illustration 17: How many real tangents can be drawn from the point (4, 3) to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$. Find the equation of these tangents and the angle between them. **(JEE MAIN)**

Sol: Use the concept of Position of a Point w.r.t. the hyperbola to find the number of real tangents.

Given point P = (4, 3)

Hyperbola $S \equiv \frac{x^2}{16}-\frac{y^2}{9} = 1 = 0$

$\because S_1 \equiv \frac{16}{16} - \frac{9}{9} - 1 = -1 < 0$

$\Rightarrow$ Point P $\equiv$ (4, 3) lies outside the hyperbola

$\therefore$ Two tangents can be drawn from the point P(4, 3). Equation of a pair of tangents is $SS_1 = T^2$.

$$\Rightarrow \left(\frac{x^2}{16}-\frac{y^2}{9}-1\right)(-1) \equiv \left(\frac{4x}{16}-\frac{3y}{9}-1\right)^2$$

$$\Rightarrow -\frac{x^2}{16}+\frac{y^2}{9}+1 = \frac{x^2}{16}+\frac{y^2}{9}+1-\frac{xy}{6}\ \frac{x}{2}+\frac{2y}{3} \Rightarrow 3x^2-4xy-12x+16y=0 \text{ and } \theta = \tan^{-1}\left(\frac{4}{3}\right)$$

Illustration 18: Find the equation of common tangents to hyperbolas

$$H_1: \frac{x^2}{a^2}-\frac{y^2}{b^2} = 1\ ;\ H_2: \frac{y^2}{a^2}-\frac{x^2}{b^2} = 1$$ **(JEE MAIN)**

Sol: Compare the equation of the common tangents to H_1 and H_2 and compare the two equations to find the value of m.

Tangent to H_1

$y = mx \pm \sqrt{a^2m^2-b^2}$

$H_2: \frac{x^2}{(-b^2)}-\frac{y^2}{(-a^2)} = 1$

$a^2m^2 - b^2 = (-b^2)\,m^2 - (-a^2)$

$\therefore \quad a^2(m^2-1) = b^2(1-m^2)$

$\quad m = \pm 1$

Equation of common tangents are $\pm y = x + \sqrt{a^2-b^2}$

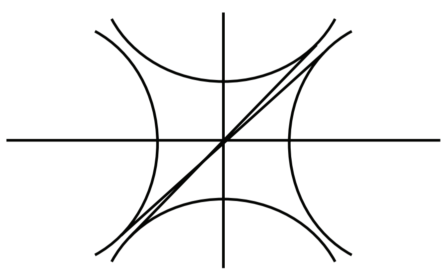


Figure 12.26

Illustration 19: If the normals at (x_r, y_r); r = 1, 2, 3, 4 on the rectangular hyperbola $xy = c^2$ meet at the point Q(h, k), prove that the sum of the ordinates of the four points is k. Also prove that the product of the ordinates is $-c^4$. **(JEE ADVANCED)**

Sol: Write the equation of the normal in the parametric form and then use the theory of equations.

Any point on the curve $xy = c^2$ is $\left(ct, \frac{c}{t}\right)$

The equation of the normal to the hyperbola at the point $\left(ct, \frac{c}{t}\right)$ is

$$y - \frac{c}{t} = \frac{-1}{\left(\frac{dy}{dx}\right)_{ct,\frac{c}{t}}}(x-ct).$$

Here, $xy = c^2$; or $y = \frac{c^2}{x'}$ $\therefore$ $\frac{dy}{dx} = \frac{-c^2}{x^2}$

$\therefore \quad \left(\frac{dy}{dx}\right)_{ct,\frac{c}{t}} = \frac{c^2}{c^2t^2} = -\frac{1}{t^2}$

$\therefore$ The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is

$y - \frac{c}{t} = t^2(x - ct)$ or $ty - c = t^3(x - ct)$ or $ct^4 - t^3x + ty - c = 0$

The normal passes through (h, k). So

$ct^4 - t^3h + tk - c = 0$ ... (i)

Let the roots of (i) be t_1, t_2, t_3, t_4. Then $x_r = ct$, $y_r = \frac{c}{t_r}$

$\therefore$ sum of ordinates $= y_1 + y_2 + y_3 + y_4$

$= \frac{c}{t_1} + \frac{c}{t_2} + \frac{c}{t_3} + \frac{c}{t_4} = c\frac{t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 + t_1t_2t_3}{t_1t_2t_3t_4}$

$= c \cdot \frac{-k/c}{-c/c} = k$, {from roots of the equation (i)} and, product of the ordinates

$= y_1y_2y_3y_4 = \frac{c}{t_1}\cdot\frac{c}{t_2}\cdot\frac{c}{t_3}\cdot\frac{c}{t_4} = \frac{c^4}{t_1t_2t_3t_4} = \frac{c^4}{-c/c} = -c^4.$

Hence proved.

Illustration 20: The perpendicular from the centre on the normal at any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meet at R. Find the locus of R. **(JEE ADVANCED)**

Sol: Solve the equation of the normal and the equation of line perpendicular to it passing through the origin.

Let (x_1, y_1) be any point on the hyperbola.

So, $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ ... (i)

The equation of the normal at (x_1, y_1) is $\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{y - y_1}{-\frac{y_1}{b^2}}$ or $\frac{x_1}{a^2}(y - y_1) + \frac{y_1}{b^2}(x - x_1) = 0$ ... (ii)

'm' of the normal $= -\frac{a^2y_1}{b^2x_1}$

$\therefore$ The equation of the perpendicular from the centre (0, 0) on (ii) is

$y = \frac{b^2x_1}{a^2y_1}.x$ ... (iii)

The intersection of (ii) and (iii) is R and the required locus is obtained by eliminating x_1, y_1 from (i), (ii) and (iii).

From (iii), $\frac{x_1}{a^2y} = \frac{y_1}{b^2x} = t$ (say)

Putting in (ii), $yt(y - b^2xt) + xt(x - a^2yt) = 0$

or $(x^2 + y^2)t - (a^2 + b^2)xyt^2 = 0$.

But $t \neq 0$ for then $(x_1, y_1) = (0, 0)$ which is not true.

$\therefore \quad t = \frac{x^2 + y^2}{xy(a^2 + b^2)}$; $\quad \therefore x_1 = \frac{x^2 + y^2}{xy(a^2 + b^2)}a^2y = \frac{a^2(x^2 + y^2)}{x(a^2 + b^2)}$

and $y_1 = \frac{x^2+y^2}{xy(a^2+b^2)}b^2x = \frac{b^2(x^2+y^2)}{y(a^2+b^2)}$

$\therefore$ from (i), $\frac{1}{a^2}.\frac{a^4(x^2+y^2)^2}{x^2(a^2+b^2)} - \frac{1}{b^2}.\frac{b^4(x^2+y^2)^2}{y^2(a^2+b^2)^2} = 1$

or $\{x^2 + y^2)^2.\left(\frac{a^2}{x^2}-\frac{b^2}{y^2}\right) = (a^2 + b^2)^2.$

Illustration 21: A normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ meets the axes in M and N and lines MP and NP are drawn perpendicular to the axes meeting at P. Prove that the locus of P is the hyperbola $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$.

(JEE ADVANCED)

Sol: Find the co-ordinates of the point M and N and then eliminate the parameter between the ordinate and abscissae.

The equation of normal at the point Q(a sec ϕ, b tan ϕ) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ is

$ax\cos\phi + by\cot\phi = a^2 + b^2$ ... (i)

The normal (i) meets the x-axis in M $\left(\frac{a^2+b^2}{a}\sec\varphi, 0\right)$ and y-axis in N $\left(0, \frac{a^2+b^2}{b}\tan\varphi\right)$

$\therefore$ Equation of MP, the line through M and perpendicular to axis, is

$x = \left(\frac{a^2+b^2}{a}\right)\sec\varphi$ or $\sec\phi = \frac{ax}{(a^2+b^2)}$ ... (ii)

and the equation of NP, the line through N and perpendicular to the y-axis is

$y = \left(\frac{a^2+b^2}{b}\right)\tan\phi$ or $\tan\phi = \frac{by}{(a^2+b^2)}$ ... (iii)

The locus of the point is the intersection of MP and NP and will be obtained by eliminating ϕ from (ii) and (iii), so we have $\sec^2\phi - \tan^2\phi = 1$

$\Rightarrow \frac{a^2x^2}{(a^2+b^2)^2} - \frac{b^2y^2}{(a^2+b^2)^2} = 1$ or $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$ is the required locus of P.

Illustration 22: Prove that the length of the tangent at any point of hyperbola intercepted between the point of contact and the transverse axis is the harmonic mean between the lengths of perpendiculars drawn from the foci on the normal at the same point. **(JEE ADVANCED)**

Sol: Proceed according to the question to prove the above statement.

$\frac{P_1}{P} = \frac{S_1G}{TG} = \frac{e^2x_1 - ae}{e^2x_1 - a\cos\theta} = \frac{ae^2 - ae\cos\theta}{ae^2 - a\cos^2\theta}$

$\therefore \frac{P_1}{P} = \frac{e(e-\cos\theta)}{(e-\cos\theta)(e+\cos\theta)} \Rightarrow \frac{P}{P_1} = \frac{e+\cos\theta}{e} = 1 + \frac{\cos\theta}{e}$

Similarly we get $\frac{P}{P_1} = 1 - \frac{\cos\theta}{e}$ $\quad \therefore \frac{P}{P_1} + \frac{P}{P_2} = 2 \Rightarrow \frac{1}{P_1} + \frac{P}{P_2} = \frac{2}{P}$

Hence Proved.

12. DIRECTOR CIRCLE

The locus of the intersection point of tangents which are at right angles is known as the Director Circle of the hyperbola. The equation to the director circle is: $x^2 + y^2 = a^2 - b^2$.

If $b^2 < a^2$ this circle is real.

If $b^2 = a^2$ (rectangular hyperbola) the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no pair of tangents at right angles can be drawn to the curve.

Or we can say that

If L(T.A) > L(C.A) $\Rightarrow$ circle is real.

If L(T.A) < L(C.A) $\Rightarrow$ No real locus, Imaginary circle.

If L(T.A) = L(C.A) $\Rightarrow$ point circle

13. CHORD

13.1 Chord of Contact

It is defined as the line joining the point of intersection of tangents drawn from any point. The equation to the chord of contact of tangent drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

13.2 Chord Bisected at a Given Point

The equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, bisected at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ $(T = S_1')$, where T and S_1' have their usual meanings.

13.3 Chord of Hyperbola (Parametric Form)

Note: Chord of ellipse $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$

For a hyperbola it is

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$$

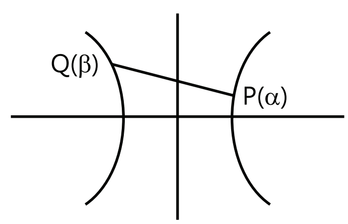


Figure 12.27

Passing through (d, 0) $\frac{d}{a}\cos\left(\frac{\alpha-\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$

$$\frac{d}{a} = \frac{\cos((\alpha+\beta)/2)}{\cos((\alpha-\beta)/2)}$$

$$\frac{d+a}{d-a} = \frac{-2\cos\alpha/2\ \cos\beta/2}{2\cos\alpha/2\ \sin\beta/2}$$

$$\frac{a-d}{a+d} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

if $\quad d = ae \Rightarrow \frac{1-e}{1+e} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$

CONCEPTS

Point of intersection of tangents at $P(\alpha)$ and $Q(\beta)$ can be obtained by comparing COC with the chord at $P(\alpha)$ & $Q(\beta)$

Equation of PQ

COC $\Rightarrow \dfrac{xh}{a^2} - \dfrac{yk}{b^2} = 1$

PQ $\Rightarrow \dfrac{x}{a}\cos\left(\dfrac{\alpha-\beta}{2}\right) - \dfrac{y}{b}\sin\left(\dfrac{\alpha+\beta}{2}\right) = \cos\left(\dfrac{\alpha+\beta}{2}\right)$

$\therefore\ h = a\dfrac{\cos((\alpha-\beta)/2)}{\cos((\alpha+\beta)/2)},\ k = b\dfrac{\sin((\alpha+\beta)/2)}{\cos((\alpha+\beta)/2)}$

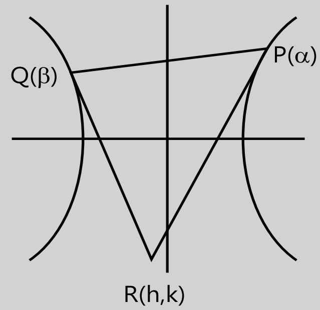


Figure 12.28

Nitish Jhawar (JEE 2009, AIR 7)

Illustration 23: If tangents to the parabola $y^2 = 4ax$ intersect the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B. **(JEE MAIN)**

Sol: The point of intersection of the tangents at A and B is nothing but the point for which AB is the chord of contact. Use this information to find the locus.

Let $P \equiv (h, k)$ be the point of intersection of tangent at A and B

$\therefore$ Equation of the chord of contact AB is $\dfrac{xh}{a^2} - \dfrac{yk}{b^2} = 1$ ....(i)

Which touches the parabola. Equation of the tangent to the parabola $y^2 = 4ax$

$y = mx - a/m \Rightarrow mx - y = -a/m$ ....(ii)

equation (i) and (ii) must be same

$\therefore \quad \dfrac{m}{((h/a^2))} = \dfrac{-1}{(-(k/b^2))} = \dfrac{-a/m}{1} \quad \Rightarrow \quad m = \dfrac{h}{k}\dfrac{b^2}{a^2}$ and $m = -\dfrac{ak}{b^2}$

$\therefore \quad \dfrac{hb^2}{ka^2} = -\dfrac{ak}{b^2} \Rightarrow$ locus of P is $y^2 = -\dfrac{b^4}{a^3}x$.

Illustration 24: A point P moves such that the chord of contact of a pair of tangents from P to $y^2 = 4x$ touches the rectangular hyperbola $x^2 - y^2 = 9$. If locus of 'P' is an ellipse, find e. **(JEE MAIN)**

Sol: Write the equation of the chord of contact to the parabola w.r.t. a point (h , k). Then solve this equation with the equation of the hyperbola.

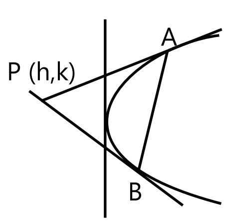


Figure 12.29

$yy_1 = 2a(x + x_1)$; $yk = 2(x + h) \quad \Rightarrow \quad y = \dfrac{2x}{k} + \dfrac{2h}{k}$; $\dfrac{4h^2}{k^2} = 9.\dfrac{4}{k^2} - 9$

$4h^2 = 36 - 9k^2$ $\quad \frac{x^2}{9}+\frac{y^2}{4}=1$ $\quad e^2 = 1 - \frac{4}{9}$ $\quad e = \frac{\sqrt{5}}{3}$

Illustration 25: Find the locus of the mid-point of focal chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$. **(JEE MAIN)**

Sol: Use the formula $T = S_1$ to get the equation of the chord and substitute the co-ordinates of the focus.

Let $P \equiv (h, k)$ be the mid-point

$\therefore$ Equation of the chord whose mid-point (h, k) is given $\frac{xh}{a^2}-\frac{yk}{b^2} - 1 = \frac{h^2}{a^2}-\frac{k^2}{b^2} - 1$ since it is a focal chord.

$\therefore$ It passes through the focus, either (ae, 0) or (–ae, 0)

$\therefore$ Locus is $\pm\frac{ex}{a} = \frac{x^2}{a^2}-\frac{y^2}{b^2}$

Illustration 26: Find the condition on 'a' and 'b' for which two distinct chords of the hyperbola $\frac{x^2}{2a^2}-\frac{y^2}{2b^2} = 1$ passing through (a, b) are bisected by the line x + y = b. **(JEE ADVANCED)**

Sol: Consider a point on the line x + y = b and then find a chord with this point as the mid-point. Then substitute the point in the equation of the chord to get the condition between 'a' and 'b'.

Let the line x + y = b bisect the chord at $P(\alpha, b - \alpha)$

$\therefore$ Equation of the chord whose mid-point is $P(\alpha, b - \alpha)$ is:

$$\frac{x\alpha}{2a^2}-\frac{y(b-\alpha)}{2b^2} = \frac{\alpha^2}{2a^2}-\frac{(b-\alpha)^2}{2b^2}$$

Since it passes through (a, b) $\therefore \frac{\alpha}{2a}-\frac{(b-\alpha)}{2b} = \frac{\alpha^2}{2a^2}-\frac{(b-\alpha)^2}{2b^2}$

$$\alpha^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+\alpha\left(\frac{1}{b}-\frac{1}{a}\right) = 0 \Rightarrow a = b$$

Illustration 27: Locus of the mid points of the focal chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ is another hyperbola whose eccentricity is e. **(JEE ADVANCED)**

Sol: Use the formula $T = S_1$ and proceed further.

$T = S_1$; $\frac{xh}{a^2}-\frac{yb}{b^2} = \frac{h^2}{a^2}-\frac{k^2}{b^2}$

It passes through focus $\Rightarrow$ $\frac{eh}{a} = \frac{h^2}{a^2}-\frac{k^2}{b^2}$

$$\Rightarrow \frac{x^2}{a^2}-\frac{ex}{a} = \frac{y^2}{b^2} \Rightarrow \frac{1}{a^2}[x^2 - eax] = \frac{y^2}{b^2}$$

$$\Rightarrow \frac{1}{a^2}\left[\left(x-\frac{ea}{2}\right)^2-\frac{e^2a^2}{4}\right] = \frac{y^2}{b^2} \Rightarrow \frac{\left(x-(ea/2)\right)^2}{a^2}-\frac{y^2}{b^2} = \frac{e^2}{4}$$

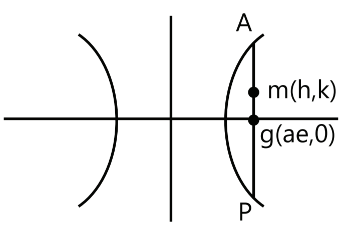


Figure 12.30

Hence the locus is a hyperbola of eccentricity e.

Illustration 28: Find the locus of the midpoint of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ which subtends a right angle at the origin. **(JEE ADVANCED)**

Sol: Use the formula $T = S_1$ and then homogenise the equation of the hyperbola using the equation of the chord to find the locus.

Let (h, k) be the mid-point of the chord of the hyperbola. Then its equation is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2} \quad \text{... (i)}$$

The equation of the lines joining the origin to the points of intersection of the hyperbola and the chord (i) is obtained by making a homogeneous hyperbola with the help of (i)

$$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{\left((hx/a^2) - (ky/b^2)\right)^2}{\left((h^2/a^2) - (k^2/b^2)\right)^2}$$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 x^2 - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 y^2 = \frac{h^2}{a^4}x^2 + \frac{k^2}{b^4}y^2 - \frac{2hk}{a^2b^2}xy \quad \text{... (ii)}$$

The lines represented by (ii) will be at right angles if the coefficient of x^2 + the coefficient of $y^2 = 0$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 - \frac{h^2}{a^4} - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 - \frac{k^2}{b^4} = 0 \quad \Rightarrow \quad \left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{h^2}{a^4} + \frac{k^2}{b^4}$$

hence, the locus of (h, k) is $\left(\frac{x^2}{a^2} - \frac{y^2}{b^4}\right)^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{x^2}{a^4} + \frac{y^2}{b^4}$

14. DIAMETER

The locus of the mid-points of a system of parallel chords of a hyperbola is called a diameter. The point where a diameter intersects the hyperbola is known as the vertex of the diameter.

14.1 Equation of Diameter

The equation of a diameter bisecting a system of parallel chords of slope m of the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = \frac{b^2}{a^2m}x$.

14.2 Conjugate Diameters

Two diameters of a hyperbola are said to be conjugate diameters if each bisects the chords parallel to the other.

Let $y = m_1x$ and $y = m_2x$ be conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Then, $y = m_2x$ bisects the system of chords parallel to $y = m_1x$. So, its equation is

$$y = \frac{b^2}{a^2m}x \quad \text{... (i)}$$

Clearly, (i) and $y = m_2x$ represent the same line. Therefore, $m_2 = \frac{b^2}{a^2m_1} \Rightarrow m_1m_2 = \frac{b^2}{a^2}$

Thus, $y = m_1x$ and $y = m_2x$ are conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $m_1m_2 = \frac{b^2}{a^2}$

CONCEPTS

- In a pair of conjugate diameters of a hyperbola, only one meets the hyperbola on a real point.
- Let P(a sec θ, b tan θ) be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 such that CP and CD are conjugate diameters of the hyperbola. Then, the coordinates of D are (a tan θ, b sec θ)
- If a pair of conjugate diameters meet the hyperbola and its conjugate in P and D respectively then $CP^2 - CD^2 = a^2 - b^2$.

Shivam Agarwal (JEE 2009, AIR 27)

15. POLE AND POLAR

Let $P(x_1, y_1)$ be any point inside the hyperbola. A chord through P intersects the hyperbola at A and B respectively. If tangents to the hyperbola at A and B meet at Q(h, k) then the locus of Q is called the polar of P with respect to the hyperbola and the point P is called the pole.

If $P(x_1, y_1)$ is any point outside the hyperbola and tangents are drawn, then the line passing through the contact points is polar of P and P is called the pole of the polar.

Note: If the pole lies outside the hyperbola then the polar passes through the hyperbola. If the pole lies inside the hyperbola then the polar lies completely outside the hyperbola. If pole the lies on the hyperbola then the polar becomes the same as the tangent.

Equation of polar: Equation of the polar of the point (x_1, y_1) with respect to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is given by $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$, i.e., T = 0

Coordinates of Pole: The pole of the line $lx+my+n=0$ with respect to hyperbola $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $P\left(\frac{-a^2l}{n}, \frac{b^2m}{n}\right)$.

Properties of pole and polar:

1. If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be conjugate points. Condition for conjugate points is $\frac{x_1x_2}{a^2}-\frac{y_1y_2}{b^2}=1$.

2. If the pole of line $l_1x+m_1y+n_1=0$ lies on another line $l_2x+m_2y+n_2=0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

3. Pole of a given line is the same as the point of intersection of the tangents at its extremities.

4. Polar of focus is its directrix.

16. ASYMPTOTES

An asymptote to a curve is a straight line, such that distance between the line and curve approaches zero as they tend to infinity.

In other words, the asymptote to a curve touches the curves at infinity i.e. asymptote to a curve is its tangent at infinity.

The equations of two asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 are y = ± $\frac{b}{a}$x or $\frac{x}{a}\pm\frac{y}{b}$ = 0

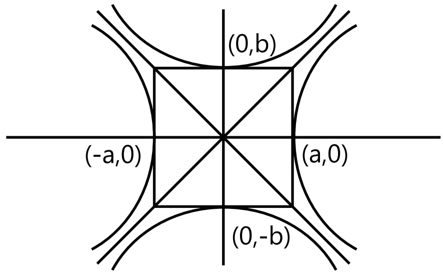


Figure 12.31

Combined equation of asymptote $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

Note: If the angle between the asymptotes of the hyperbola is θ, then its eccentricity is sec θ.

CONCEPTS

- The combined equation of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.
- When b = a, the asymptotes of the rectangular hyperbola $x^2 - y^2 = a^2$ are $y = \pm x$, which are at right angles.
- A hyperbola and its conjugate hyperbola have the same asymptotes.
- The equation of the pair of asymptotes differ from the hyperbola and the conjugate hyperbola by the same constant, i.e. Hyperbola – Asymptotes = Asymptotes – Conjugate hyperbola
- The asymptotes pass through the centre of the hyperbola.
- The bisectors of the angles between the asymptotes are the coordinates axes.
- The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.
- Asymptotes are the tangents to the hyperbola from the centre.
- The tangent at any point P on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with the centre C meets asymptotes at Q, R and cut off ΔCQR of constant area = ab.
- The parts of the tangent intercepted between the asymptote is bisected at the point of contact.
- If f(x, y) = 0 is an equation of the hyperbola then the centre of the hyperbola is the point of intersection of $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

Ravi Vooda (JEE 2009, AIR 71)

Illustration 29: Find the asymptotes of xy – 3y – 2x = 0. **(JEE MAIN)**

Sol: Proceed according to the definition of asymptotes.

Since the equation of a hyperbola and its asymptotes differ in constant terms only

$\therefore$ Pair of asymptotes is given by $xy - 3y - 2x + \lambda = 0$ ...(i)

where λ is any constant such that represents two straight lines

$\therefore$ $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow$ $0 + 2x - 3/2x - 1 + 1/2 - 0 - 0 - \lambda (1/2)^2 = 0$

$\therefore$ $\lambda = 6$

From (i) the asymptotes of given hyperbola are given by xy – 3y – 2x + 6 = 0 or (y – 2) (x – 3) = 0

$\therefore$ Asymptotes are x – 3 = 0 and y – 2 = 0

Illustration 30: Find the equation of that diameter which bisects the chord 7x + y – 20 = 0 of the hyperbola $\frac{x^2}{3} - \frac{y^2}{7} = 1$. **(JEE ADVANCED)**

Sol: Consider a diameter y = mx and solve it with the equation of the hyperbola to form a quadratic in x. Find the midpoint of the intersection of the chord and hyperbola. Use this point to find the slope of the diameter.

The centre of the hyperbola is (0, 0). Let the diameter be $y = mx$ ... (i)

The ends of the chord are found by solving

$$7x + y - 20 = 0 \quad \text{... (ii)}$$

and $$\frac{x^2}{3} - \frac{y^2}{7} = 1 \quad \text{... (iii)}$$

Solving (ii), (iii) we get $\frac{x^2}{3} - \frac{1}{7}(20 - 7x)^2 = 1$

or $7x^2 - 3(400 - 280x + 49x^2) = 21$ or $140x^2 - 840x + 1221 = 0$

Let the roots be x_1, x_2

Then $x_1 + x_2 = \frac{840}{140} = 6$ ... (iv)

If (x_1, y_1), (x_2, y_2) be ends then $7x_1 + y_1 - 20 = 0$, $7x_2 + y_2 - 20 = 0$

Adding, $7(x_1 + x_2) + (y_1 + y_2) - 40 = 0$

or $42 + y_1 + y_2 - 40 = 0$, using (iv) ; $\therefore\ y_1 + y_2 = -2$

$\therefore$ The middle point of the chord = $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{6}{2}, \frac{-2}{2}\right) = (3, -1)$

This lies on (i). So $-1 = 3m$; $\therefore\ m = -\frac{1}{3}$ $\therefore$ the equation of the diameter is $y = -\frac{1}{3}x$.

Illustration 31: The asymptotes of a hyperbola having centre at the point (1, 2) are parallel to the lines 2x + 3y = 0 and 3x + 2y = 0. If the hyperbola passes through the point (5, 3) show that its equation is (2x + 3y – 8) (3x + 2y + 7) = 154. **(JEE ADVANCED)**

Sol: With the information given, find out the equation of the asymptotes and then use the fact that the point (5, 3) lies on the hyperbola to find the equation of the hyperbola.

Let the asymptotes be $2x + 3y + \lambda = 0$ and $3x + 2y + \mu = 0$. Since the asymptote passes through (1, 2) then $\lambda = -8$ and $\mu = -7$

Thus the equation of the asymptotes are $2x + 3y - 8 = 0$ and $3x + 2y - 7 = 0$

Let the equation of the hyperbola be $(2x + 3y - 8)(3x + 2y - 7) + v = 0$ ... (i)

It passes through (5, 3), then $(10 + 9 - 8)(15 + 6 - 7) + v = 0$

$\Rightarrow\quad 11 \times 14 + v = 0$

$\therefore\quad v = -154$

putting the value of v in (i) we obtain $(2x + 3y - 8)(3x + 2y - 7) - 154 = 0$

which is the equation of the required hyperbola.

17. RECTANGULAR HYPERBOLA

A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola.

The equation of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by $y = \pm\frac{b}{a}x$.

The θ angle between these two asymptotes is given by

$$\tan\theta\left(\frac{(b/a)-(-(b/a))}{1+(b/a)(-(b/a))}\right) = \frac{2b/a}{1-b^2/a^2} = \frac{2ab}{a^2-b^2}$$

If the asymptotes are at right angles, then $\theta = \pi/2 \Rightarrow \tan\theta = \tan\pi/2 \Rightarrow \frac{2ab}{a^2-b^2} = \tan\frac{\pi}{2} \Rightarrow a = b$.

Thus, the transverse and conjugate axes of a rectangular hyperbola are equal and the equation of the hyperbola is $x^2 - y^2 = a^2$.

Remarks: Since the transverse and conjugate axis of a rectangular hyperbola are equal. So, its eccentricity e is given by

$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{2}$$

17.1 With Asymptotes as Coordinate Axes

Equation of the hyperbola referred to the transverse and conjugate axes along the axes of co-ordinates, the equation of the rectangular hyperbola is $x^2 - y^2 = a^2$ ....(i)

The asymptotes of (i) are y = x and y = –x. Each of these two asymptotes is inclined at an angle of 45° with the transverse axis. So, if we rotate the coordinate axes through an angle of –π/4 keeping the origin fixed, then the axes coincide with the asymptotes of the hyperbola and, we have

$x = X\cos(-\pi/4) - Y\sin(-\pi/4) = \frac{X+Y}{\sqrt{2}}$ and $y = X\sin(-\pi/4) + Y\cos(-\pi/4) = \frac{Y-X}{\sqrt{2}}$

Substituting the values of x and y in (i), we obtain the $\left(\frac{X+Y}{\sqrt{2}}\right)^2 - \left(\frac{Y-X}{\sqrt{2}}\right)^2 = a^2$

$$\Rightarrow XY = \frac{a^2}{2} \Rightarrow XY = c^2, \text{ where } c^2 = \frac{a^2}{2}$$

Thus, the equation of the hyperbola referred to its asymptotes as the coordinates axes is

$$xy = c^2, \text{ where } c^2 = \frac{a^2}{2}$$

Remark: The equation of a rectangular hyperbola having coordinate axes as its asymptotes is $xy = c^2$.

If the asymptotes of a rectangular hyperbola are $x = \alpha$, $y = \beta$, then its equation is

$(x-\alpha)(y-\beta) = c^2$ or $xy - ay - bx + \lambda = 0$; $(\lambda \le \alpha\beta)$

17.2 Tangent

Point Form

The equation of the tangent at (x_1, y_1) to the hyperbola $xy = c^2$ is $xy_1 + yx_1 = 2c^2$ or, $\frac{x}{x_1}+\frac{y}{y_1} = 2$.

Parametric Form

The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $\frac{x}{t} + yt = 2c$.

Note: Tangent at P $\left(ct_1, \frac{c}{t_1}\right)$ and Q $\left(ct_2, \frac{c}{t_2}\right)$ to the rectangular hyperbola $xy = c^2$ intersect at $\left(\frac{2ct_1t_2}{t_1+t_2}. \frac{2c}{t_1+t_2}\right)$

17.3 Normal

Point Form

The equation of the normal at (x_1, y_1) to the hyperbola $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$

Parametric Form

The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $xt - \frac{y}{t} = ct^2 - \frac{c}{t^2}$

Note:

(i) The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is a fourth degree equation in t. So, in general, at most four normals can be drawn from a point to the hyperbola $xy = c^2$.

(ii) The equation of the polar of any point $P(x_1, y_1)$ with respect to $xy = c^2$ is $xy_1 + yx_1 = 2c^2$.

(iii) The equation of the chord of the hyperbola $xy = c^2$ whose midpoint (x,y) is $xy_1 + yx_1 = 2x_1y_1$ or, T = S′. where T and S′ have their usual meanings.

(iv) The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the rectangular $xy = c^2$ is $xy_1 + yx_1 = 2c^2$.

Illustration 32: A, B, C are three points on the rectangular hyperbola $xy = c^2$, find

(i) The area of the triangle ABC

(ii) The area of the triangle formed by the tangents A, B and C **(JEE ADVANCED)**

Sol: Use parametric co-ordinates and the formula for the area to get the desired result.

Let co-ordinates of A, B and C on the hyperbola $xy = c^2$ be $\left(ct_1, \frac{c}{t_1}\right)$.$\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively

(i) Area of triangle ABC $= \frac{1}{2}\left(\begin{vmatrix} ct_1 & \frac{c}{t_1} \\ ct_2 & \frac{c}{t_2} \end{vmatrix} + \begin{vmatrix} ct_2 & \frac{c}{t_2} \\ ct_3 & \frac{c}{t_3} \end{vmatrix} + \begin{vmatrix} ct_3 & \frac{c}{t_3} \\ ct_1 & \frac{c}{t_1} \end{vmatrix}\right) = \frac{c^2}{2}\left|\frac{t_1}{t_2} - \frac{t_2}{t_1} + \frac{t_2}{t_3} - \frac{t_3}{t_2} + \frac{t_3}{t_1} - \frac{t_1}{t_3}\right|$

$$= \frac{c^2}{2t_1t_2t_3}\,|\,t_3^2t_3 - t_2^2t_3 + t_1t_2^2 - t_3^2t_1 + t_2t_3^2 - t_1^2t_2\,| = \frac{c^2}{2t_1t_2t_3}\,|\,(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\,|$$

(ii) Equation of tangents at A, B, C are $x + yt_1^2 - 2ct_1 = 0$, $x + yt_2^2 - 2ct_2 = 0$ and $x + yt_3^2 - 2ct_3 = 0$

$\therefore$ Required Area $= \frac{1}{2|C_1C_2C_3|}\begin{vmatrix} 1 & t_1^2 & -2ct_1 \\ 1 & t_2^2 & -2ct_2 \\ 1 & t_3^2 & -2ct_3 \end{vmatrix}^2$ ... (i)

where $C_1 = \begin{vmatrix} 1 & t_1^2 \\ 1 & t_3^2 \end{vmatrix}$, $C_2 = -\begin{vmatrix} 1 & t_1^2 \\ 1 & t_3^2 \end{vmatrix}$ and $C_3 = \begin{vmatrix} 1 & t_1^2 \\ 1 & t_3^2 \end{vmatrix}$

$\therefore$ $C_1 = t_3^2 - t_2^2$, $C_2 = t_1^2 - t_3^2$ and $C_3 = t_2^2 - t_1^2$

From (i) $= \frac{1}{2|(t_3^2 - t_2^2)(t_1^2 - t_3^2)(t_2^2 - t_1^2)|}4c^2(t_1 - t_2)^2(t_2 - t_3)^2(t_3 - t_1)^2 = 2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$

$\therefore$ Required area is, $2c^2\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$

PROBLEM SOLVING TACTICS

(a) In general convert the given hyperbola equation into the standard form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ and compare it with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Then solve using the properties of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. So, it is advised to remember the standard results.

(b) Most of the standard results of a hyperbola can be obtained from the results of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ just by changing the sign of b^2.

FORMULAE SHEET

HYPERBOLA

(a) Standard Hyperbola:

Imp. Terms \ **Hyperbola**	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	(0, 0)	(0, 0)
Length of transverse axis	2a	2b
Length of conjugate axis	2b	2a
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Eccentricity	$e = \sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$	$e = \sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$
Length of L.R.	$2b^2/a$	$2a^2/b$
Parametric co-ordinates	$(a \sec\phi, b \tan\phi)$ $0 \le \phi < 2\pi$	$(a \tan\phi, b \sec\phi)$ $0 \le \phi < 2\pi$
Focal radii	$SP = ex_1 - a$ $S¢P = ex_1 + a$	$SP = ey_1 - b$ $S¢P = ey_1 + b$
$\lvert S¢P - SP\rvert$	2a	2b
Tangents at the vertices	$x = -a, x = a$	$y = -b, y = b$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of the conjugate axis	$x = 0$	$y = 0$

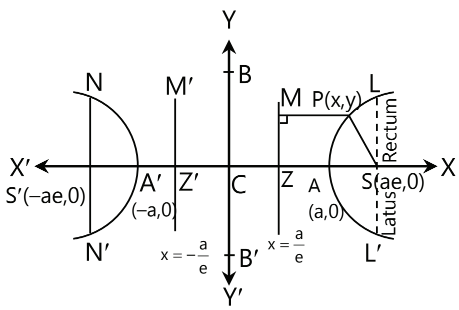


Figure 12.32: Hyperbola

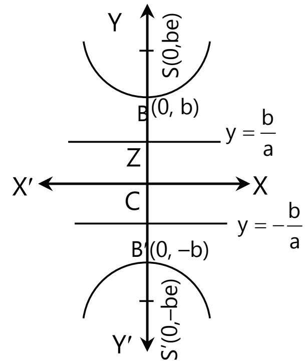


Figure 12.33: Conjugate Hyperbola

(b) Special form of hyperbola: If (h , k) is the centre of a hyperbola and its axes are parallel to the co-ordinate axes, then the equation of the hyperbola is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

(c) Parametric equations of a hyperbola: The equation x = a sec ϕ and y = b tan ϕ are known as the parametric equation of the standard hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

If $S=\frac{x^2}{a^2}-\frac{y^2}{b^2}$, then $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$; $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

(d) Position of a point and a line w.r.t. a hyperbola: n The point (x_1, y_1) lies inside, on or outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according to $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ being >, = or < zero.

The line y = mx + c intersects at 2 distinct points, 1 point or does not intersect with the hyperbola according as c^2 >, = or < $a^2m^2 - b^2$.

(e) Tangent:

(i) Point form: The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

(ii) Parametric form: The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at parametric coordinates (a sec ϕ, b tan ϕ) is $\frac{x}{a}\sec\phi-\frac{y}{b}\phi=1.$

(iii) Slope form: The equation of the tangents having slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are

$y=mx\pm\sqrt{a^2m^2-b^2}$ and the co-ordinates of points of contacts are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}},\pm\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

(f) Equation of a pair of tangents from an external point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $SS_1 = T^2$.

(g) Normal:

(i) Point form: The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2.$$

(ii) Parametric form: The equation of the normal at parametric coordinates (a sec θ, b tanθ) to the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\quad ax\cos\theta+by\cot\theta=a^2+b^2.$

(iii) Slope form: The equation of the normal having slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$y=mx\mp\frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$$

(iv) Condition for normality: y = mx + c is a normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if

$$c^2=\frac{m(a^2+b^2)^2}{(a^2-m^2b^2)}$$

(v) Points of contact: Co-ordinates of the points of contact are $\left(\pm\frac{a^2}{\sqrt{a^2-b^2m^2}},\mp\frac{mb^2}{\sqrt{a^2-b^2m^2}}\right)$.

(h) The equation of the director circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is given by $x^2+y^2=a^2-b^2$.

(i) Equation of the chord of contact of the tangents drawn from the external point (x_1, y_1) to the hyperbola is

given by $\quad \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$

(j) The equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ whose mid point is (x_1, y_1) is $T = S_1$.

(k) Equation of a chord joining points P(a sec f_1, b tan f_1) and Q (a sec f_2, b tan f_2) is

$$\frac{x}{a}\cos\left(\frac{\phi_1-\phi_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right)=\cos\left(\frac{\phi_1+\phi_2}{2}\right)$$

(l) Equation of the polar of the point (x_1, y_1) w.r.t. the hyperbola is given by T = 0.

The pole of the line lx + my + n = 0 w.r.t. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\left(-\frac{a^2\ell}{n},\frac{b^2m}{n}\right)$

(m) The equation of a diameter of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ corresponding to the chords of slope m is $y=\frac{b^2}{a^2m}x$

(n) The diameters $y = m_1x$ and $y = m_2x$ are conjugate if $m_1m_2=\frac{b^2}{a^2}$

(o) Asymptotes:

- Asymptote to a curve touches the curve at infinity.
- The equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=\pm\frac{b}{a}x$.

- The asymptote of a hyperbola passes through the centre of the hyperbola.
- * The combined equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$
- * The angle between the asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2\tan^{-1}\frac{a^2}{b^2}$ or $2\sec^{-1} e$.
- A hyperbola and its conjugate hyperbola have the same asymptotes.
- The bisector of the angles between the asymptotes are the coordinate axes.
- Equation of the hyperbola – Equation of the asymptotes = constant.

(p) Rectangular or Equilateral Hyperbola:

- A hyperbola for which a = b is said to be a rectangular hyperbola, its equation is $x^2 - y^2 = a^2$.
- $xy = c^2$ represents a rectangular hyperbola with asymptotes x = 0, y = 0.
- Eccentricity of a rectangular hyperbola is $\sqrt{2}$ and the angle between the asymptotes of a rectangular hyperbola is 90º.
- Parametric equation of the hyperbola $xy = c^2$ are $x = ct$, $y = \frac{c}{t}$, where t is a parameter.
- Equation of a chord joining t_1, t_2 on $xy = c^2$ is $x + y\,t_1t_2 = c(t_1 + t_2)$
- Equation of a tangent at (x_1, y_1) to $xy = c^2$ is $\frac{x}{x_1}+\frac{y}{y_1}=2$.
- Equation of a tangent at t is $x + yt^2 = 2ct$
- Equation of the normal at (x_1, y_1) to $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$.
- Equation of the normal at t on $xy = c^2$ is $xt^3 - yt - ct^4 + c = 0$.

  (i.e. Four normals can be drawn from a point to the hyperbola $xy = c^2$)
- If a triangle is inscribed in a rectangular hyperbola then its orthocentre lies on the hyperbola.
- Equation of chord of the hyperbola $xy = c^2$ whose middle point is given is $T = S_1$.
- Point of intersection of tangents at t_1 and t_2 to the hyperbola $xy = c^2$ is $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$

Solved Examples

JEE Main/Boards

Example 1: Find the equation of the hyperbola whose foci are (6, 4) and (–4, 4) and eccentricity is 2.

Sol: Calculate the value of 'a', by using the distance between the two foci and eccentricity. Then calculate the value of 'b'. Using these two values find the equation of the hyperbola.

Let S, S′ be the foci and C be the centre of the hyperbola S, S′ and C lie on the line y = 4. The co-ordinates of the centre are (1, 4).

The equation of the hyperbola is

$$\frac{(x-1)^2}{a^2}-\frac{(y-4)^2}{b^2}=1$$

The distance between the foci is 2ae = 10; ∴ $a = \frac{5}{2}$

$b^2 = a^2(e^2 - 1) = \frac{25}{4}(4-1) = \frac{75}{4}$

Hence the equation of the hyperbola is

$$\frac{(x-1)^2}{\frac{25}{4}}-\frac{(y-4)^2}{\frac{75}{4}}=1$$

Example 2: Obtain the equation of hyperbola whose asymptotes are the straight lines x + 2y + 3 = 0 & 3x + 4y + 5 = 0 and which passes through the point (1, –1)

Sol: Use the following formula:

Equation of hyperbola – Equation of asymptotes = constant.

The equation of the hyperbola, is

$(x + 2y + 3)(3x + 4y + 5) = k$, k being a constant.

This passes through the point (1, –1)

$\therefore (1 + 2(-1) + 3)(3(1) + 4(-1) + 5) = k$

$\Rightarrow k = 2 \times 4 = 8$

$\therefore$ The equation of the hyperbola is

$(x + 2y + 3)(3x + 4y + 5) = 8$

Example 3: If e and e′ are the eccentricities of two hyperbolas conjugate to each other,

show that $\frac{1}{e^2}+\frac{1}{e'^2} = 1$.

Sol: Start with the standard equation of two hyperbolas and eliminate 'a' and 'b'.

Let $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ and $\frac{y^2}{b^2}-\frac{x^2}{a^2} = 1$

be the two hyperbolas with eccentricities e and e′ respectively

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{1}{e^2} = \frac{a^2}{a^2+b^2}$

$a^2 = b^2(e¢^2 - 1) \Rightarrow \frac{1}{e'^2} = \frac{b^2}{a^2+b^2}$

$\therefore \frac{1}{e^2}+\frac{1}{e'^2} = \frac{a^2}{(a^2+b^2)}+\frac{b^2}{(a^2+b^2)} = 1$

Example 4: If any point P on the rectangular hyperbola $x^2 - y^2 = a^2$ is joined to its foci S, S′ show that SP.S¢P = CP^2, where C is the centre of the hyperbola.

Sol: The eccentricity of a rectangular hyperbola is $\sqrt{2}$. Consider a parametric point on the hyperbola and simplify the LHS.

Any point on the rectangular hyperbola $x^2 - y^2 = a^2$ is P(a secθ, a tanθ) ; eccentricity of a rectangular hyperbola is $\sqrt{2}$.

S is (ae, 0), S′ is (–ae, 0) and C is (0, 0)

$(SP)^2.(S'P)^2 = [(a\sec\theta - ae)^2 + a^2\tan^2 q] \times$

$[(a\sec\theta + ae)^2 + a^2\tan^2 q]$

$= a^4[(\sec^2\theta + \tan^2\theta + e^2)^2 - 4e^2\sec^2 q]$

$= a^4[(2\sec^2\theta - 1 + 2)^2 - 4.2\sec^2 q]$

$= a^4[(2\sec^2\theta + 1)^2 - 8\sec^2 q]$

$= a^4[(2\sec^2\theta - 1)^2]$

$\therefore SP.S'P = a^2(2\sec^2\theta - 1)$

$= a^2(\sec^2\theta + \tan^2\theta)$

$= CP^2$.

Example 5: Find the equation of the hyperbola conjugate to the hyperbola

$2x^2 + 3xy - 2y^2 - 5x + 5y + 2 = 0$

Sol: Use the formula:

Equation Hyperbola + Conjugate Hyperbola

= 2(Asympototes)

Let asymptotes be

$2x^2 + 3xy - 2y^2 - 5x + 5y + \lambda = 0$

The equation above represents a pair of lines if

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\therefore \quad \lambda = -5$

Equation Hyperbola + Conjugate Hyperbola

= 2(Asympototes)

$\therefore$ Conjugate Hyperbola

= 2(Asymptotes) – Hyperbola

$2x^2 + 3xy - 2y^2 - 5x + 5y - 8 = 0$

Example 6: If (5, 12) and (24, 7) are the foci of a hyperbola passing through the origin then the eccentricity of the hyperbola is

Sol: Use the definition of the hyperbola S¢P – SP = 2a.

Let S(5, 12) and S′(24, 7) be the two foci and P(0, 0) be a point on the conic then

$SP = \sqrt{25+144} = \sqrt{169} = 13;$

$S¢P = \sqrt{(24)^2+7^2} = \sqrt{625} = 25$

and $SS' = \sqrt{(24-5)^2+(7-12)^2} = \sqrt{19^2+5^2} = \sqrt{386}$

since the conic is a hyperbola, S¢P – SP = 2a, the length of transverse axis and SS′ = 2ae, e being the eccentricity.

$\Rightarrow \quad e = \frac{SS'}{S'P-SP} = \frac{\sqrt{386}}{12}$

Example 7: An equation of a tangent to the hyperbola. $16x^2 - 25y^2 - 96x + 100y - 356 = 0$ which makes an angle $\pi/4$ with the transverse axis is

Sol: Write the equation of the hyperbola in the standard form and compare to get the equation of the tangent.

Equation of the hyperbola can be written as

$X^2/5^2 - Y^2/4^2 = 1$ ....(i)

where X = x – 3 and Y = y – 2.

Equation of a tangent which makes an angle $\pi/4$, with the transverse axis X = 0 of (i) is

$$Y = \tan\frac{\pi}{4}X \pm \sqrt{25\tan^2\frac{\pi}{4} - 16}$$

$\Rightarrow \quad y - 2 = x - 3 \pm \sqrt{25 - 16}$

$\Rightarrow \quad y - 2 = x - 3 \pm 3$

$\Rightarrow \quad y = x + 2$ or $y = x - 4$.

Example 8: If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes of x and y in G and g respectively and C is the centre of the hyperbola, then prove that Gg=2PC.

Sol: In the equation of a normal, find the point of intersection with the axes and find the coordinates of G and g.

Let $P(x_1, y_1)$ be any point on the hyperbola $x^2 - y^2 = 4$ then equation of the normal at P is

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1)$$

$\Rightarrow \quad x_1y + y_1x = 2x_1y_1$.

Then coordinates of G are $(2x_1, 0)$ and of g are $(0, 2y_1)$ so that

$$PG = \sqrt{(2x_1 - x_1)^2 + y_1^2} = \sqrt{x_1^2 + y_1^2} = PC$$

$$Pg = \sqrt{x_1^2 + (2y_1 - y_1)^2} = \sqrt{x_1^2 + y_1^2} = PC$$

and

$$Gg = \sqrt{(2x_1)^2 + (2y_1)^2} = 2\sqrt{x_1^2 + y_1^2} = 2PC$$

Hence proved.

Example 9: The normal to the curve at P(x, y) meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is-

Sol: Similar to the previous question.

Equation of the normal at (x, y) is

$Y - y = -\frac{dx}{dy}(X - x)$ which meets the x-axis at G

$\left(0, x + y\frac{dy}{dx}\right)$, then $x + y\frac{dy}{dx} = \pm 2x$

$\Rightarrow \quad x + y\frac{dy}{dx} = 2x \Rightarrow \quad y\,dy = x\,dx$

$\Rightarrow \quad x^2 - y^2 = c$

or $\quad y\,dy = -3x\,dx$

$\Rightarrow \quad 3x^2 + y^2 = c$

Thus the curve is either a hyperbola or an ellipse.

Example 10: Find the centre, eccentricity, foci and directrices of the hyperbola

$16x^2 - 9y^2 + 32x + 36y - 164 = 0$

Sol: Represent the equation of the hyperbola in the standard form and compare.

Here,

$16x^2 + 32x + 16 - (9y^2 - 36y + 36) - 144 = 0$

or $\quad 16(x + 1)^2 - 9(y - 2)^2 = 144$

$$\therefore \quad \frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$$

Putting x + 1 = X and y – 2 = Y, the equation becomes

$$\frac{X^2}{9} - \frac{Y^2}{16} = 1$$

which is in the standard form.

$\because \quad b^2 = a^2(e^2 - 1)$, here $a^2 = 9$ & $b^2 = 16$

$\therefore \quad e^2 - 1 = \frac{16}{9} \Rightarrow \quad e^2 = \frac{25}{9}$, i.e., $e = \frac{5}{3}$

Now, centre $= (0, 0)_{X,Y} = (-1, 2)$

$$\text{foci} = (\pm ae, 0)_{X,Y} = \left(\pm 3.\frac{5}{3}, 0\right)_{X,Y} = (\pm 5, 0)_{x,Y}$$

$= (-1 \pm 5, 2) = (4, 2), (-6, 2)$

Directrices in X, Y coordinates have the equations

$$X \pm \frac{a}{e} = 0 \quad \text{or} \quad x + 1 \pm \frac{3}{5/3} = 0$$

i.e., $\quad x + 1 \pm \frac{9}{5} = 0$

$$\therefore \quad x = -\frac{14}{5} \text{ and } x = \frac{4}{5}$$

JEE Advanced/Boards

Example 1: S is the focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. M is the foot of the perpendicular drawn from S on a tangent to the hyperbola. Prove that the locus of M is $x^2 + y^2 = a^2$.

Sol: Use the definition of an auxiliary circle.

Let $M = (x_1, y_1)$ be any point on the locus.

Let the equation of the corresponding tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ be

$y = mx \pm \sqrt{a^2m^2-b^2}$

(the sign is chosen according to the position of M)

But $M(x_1, y_1)$ lies on it

$\therefore \quad y_1 = mx_1 \pm \sqrt{a^2m^2-b^2}$ ... (i)

Segment SM is perpendicular to the given tangent.

$\therefore$ Slope of segment SM is $-\frac{1}{m}$

and $S \equiv (ae, 0)$

$\therefore \quad$ Equation of SM is $(y - 0) = -\frac{1}{m}(x-ae)$

But $M(x_1, y_1)$ lies on it

$y_1 = -\frac{1}{m}(x_1-ae)$ ... (ii)

From (i), $(y_1 - mx_1) = \pm\sqrt{a^2m^2-b^2}$

From (ii), $(my_1 + x_1) = ae$

Squaring and adding we get the required locus of M

$y_1^2(1 + m^2) + x_1^2(1 + m^2) = a^2e^2 + a^2m^2 - a^2(e^2 - 1)$

$\therefore \quad x_1^2 + y_1^2 = a^2$

Note: This is the equation of the auxiliary circle

Example 2: PQ is the chord joining the points θ_1 and θ_2 on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If $\theta_1 - \theta_2 = 2\alpha$, where α is a constant, prove that PQ touches the hyperbola

$\frac{x^2\cos^2\alpha}{a^2}-\frac{y^2}{b^2} = 1$

Sol: Write the equation of the chord passing through the points q_1 and q_2. Represent this equation in the standard form of a tangent to a hyperbola and compare.

Equation of the chord PQ to the hyperbola is

$$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

$$\frac{x}{a}\cos\alpha-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

$$y = \frac{b}{a}\frac{\cos\alpha}{\sin\left((\theta_1+\theta_2)/2\right)}x-\frac{b\cos\left((\theta_1+\theta_2)/2\right)}{\sin\left((\theta_1+\theta_2)/2\right)} \quad \text{... (i)}$$

For line $y = mx + c$ to be a tangent to

$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ we have

$c^2 = a^2m^2 - b^2$

$$\frac{x^2\cos^2\alpha}{a^2}-\frac{y^2}{b^2} = 1 \quad \text{... (ii)}$$

If (i) is tangent to (ii), then, we must have

$$\left(\frac{b\cos\left((\theta_1+\theta_2)/2\right)}{\sin\left((\theta_1+\theta_2)/2\right)}\right)^2 = b^2\cot^2\left(\frac{\theta_1+\theta_2}{2}\right)$$

which is true.

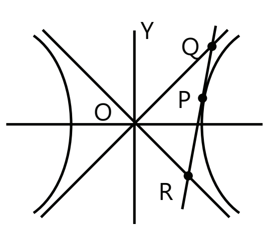
Example 3: Show that the portion of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ intercepted between the asymptotes is bisected at the point of contact. Also show that the area of the triangle formed by this tangent and the asymptotes is constant.

Sol: Calculate the point of intersection of the tangent and the asymptotes and then prove the statement.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1 \quad \text{... (i)}$$

equation of the tangent at $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2} = 1 \quad \text{... (ii)}$$



Equation of the asymptotes are

$$\frac{x}{a}-\frac{y}{b}=0 \quad \text{... (iii)}$$

and $$\frac{x}{a}+\frac{y}{b}=0 \quad \text{... (iv)}$$

If Q and R are the points of intersection of the tangent at P with the asymptotes, then solving the equation (ii) and (iii), we get

$$Q=\left(\frac{a}{(x_1/a)-(y_1/b)},\frac{b}{(x_1/a)-(y_1/b)}\right)$$

Solving the equation (ii) and (iv), we get

$$R=\left(\frac{a}{(x_1/a)+(y_1/b)},\frac{-b}{(x_1/a)+(y_1/b)}\right)$$

The midpoint of QR has coordinate (x_1, y_1) which is also the point of contact of the tangent.

Area of ΔOQR =

$$\frac{1}{2}\left|\left(\frac{a}{(x_1/a)-(y_1/b)}\right)\left(\frac{-b}{(x_1/a)+(y_1/b)}\right)-\left(\frac{b}{(x_1/a)-(y_1/b)}\right)\left(\frac{a}{(x_1/a)+(y_1/b)}\right)\right|$$

=ab sq. units

Example 4: Prove that if a rectangular hyperbola circumscribes a triangle it also passes through the orthocentre of the triangle.

Sol: Take three points on the hyperbola and find the coordinates of the orthocentre. Prove that the orthocentre satisfies the equation of the hyperbola.

Let the equation of the curve referred to its asymptotes be $xy = c^2$ ....(i)

Let the angular points of the triangle be P, Q and R and let their co-ordinates be

$P\equiv\left(ct_1,\frac{c}{t_1}\right)$, $Q\equiv\left(ct_2,\frac{c}{t_2}\right)$ and

$R\equiv\left(ct_3,\frac{c}{t_3}\right)$ respectively.

Equation of QR is $x + t_2t_3\, y = c\,(t_2 + t_3)$

The equation of altitude through P and perpendicular to QR is

$$y-\frac{c}{t_1}=t_2t_3(x-ct_1)$$

i.e. $$y + c\,t_1t_2t_3 = t_2t_3\left(x+\frac{c}{t_1t_2t_3}\right) \quad \text{... (ii)}$$

Similarly, the equation of altitude through Q perpendicular to RP is

$$y + ct_1t_2t_3 = t_3t_1\left(x+\frac{c}{t_1t_2t_3}\right) \quad \text{... (iii)}$$

Solving (ii) and (iii), we get

$\therefore$ Orthocentre = $\left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$

These co-ordinates satisfy (i)

Hence proved.

Example 5: Find the equation of the hyperbola, whose eccentricity is 5/4, whose focus is (a, 0) and whose directrix is 4x – 3y = a. Find also the coordinates of the centre and the equation to other directrix.

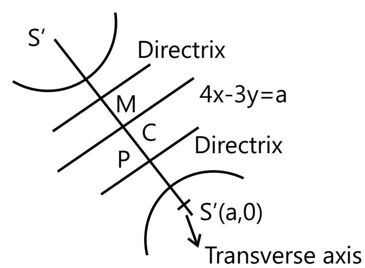
Sol: Use the basic definition of a hyperbola.

$$(x-a)^2+(y-0)^2=e^2\,\frac{(4x-3y-a)^2}{25}$$

$x^2 - 2ax + a^2 + y^2 =$

$$\frac{25}{16}(16x^2+9y^2+a^2-24xy-8ax+6ay)\times\frac{1}{25}$$

$$7y^2 + 24xy - 24ax - 6ay + 15a^2 = 0 \quad \text{... (i)}$$



Differentiating with respect to 'x'

$$24y - 24a = 0 \quad \text{... (ii)}$$

Differentiating with respect to 'y'

$$14y + 24x - 6a = 0 \quad \text{... (iii)}$$

Solving (ii) and (iii)

$C \equiv (-a/3, a)$

Transverse axis is

$3x + 4y = 3a$

'P' is the point of intersection of the transverse axis and the directrix:

$\therefore$ $P\equiv\left(\frac{13a}{25},\frac{9a}{25}\right)$ 'C' is mid point of MP

$\therefore \quad M = \left(\frac{-89a}{75}, \frac{41a}{25}\right)$

Equation of the other directrix $4x - 3y = \lambda$, passes through the 'M'

$\therefore \quad 12x - 9y + 29a = 0$

Example 6: Find the centre, eccentricity, foci, directrices and the length of the transverse and conjugate axes of the hyperbola, whose equation is $(x - 1)^2 - 2(y - 2)^2 + 6 = 0$.

Sol: Represent the equation of the hyperbola in the standard form and proceed.

The equation of the hyperbola can be re-written as $-\frac{(x-1)^2}{(\sqrt{6})^2} + \frac{(y-2)^2}{(\sqrt{3})^2} = 1$

$\Rightarrow \quad \frac{Y^2}{(\sqrt{3})^2} - \frac{X^2}{(\sqrt{6})^2} = 1$

Where $Y = (y - 2)$ and $X = (x - 1)$ ... (i)

$\therefore$ Centre: $X = 0, Y = 0$ i.e. $(1, 2)$

So $a = \sqrt{3}$ and $b = \sqrt{6}$

so transverse axis $= 2\sqrt{3}$,

and conjugate axis $= 2\sqrt{6}$

Also $b^2 = a^2(e^2 - 1)$

$\Rightarrow \quad 6 = 3(e^2 - 1)$ i.e. $e = \sqrt{3}$

In (X, Y) coordinates, foci are $(0, \pm ae)$

i.e. $(0, \pm 3)$

$\therefore \quad$ foci are $(1 + 0, 2 \pm 3)$

i.e. $\quad (1, 5)$ and $(1, -1)$

Equations of directrices $Y = \pm a/e$

$\therefore \quad$ Directrices are $y - 2 = \pm 1$

$\Rightarrow \quad y = 3, y = 1$.

Example 7: Prove that the locus of a point whose chord of contact touches the circle described on the straight line joining the foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as the diameter is $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{(a^2 + b^2)}$.

Sol: Check if the line $T = 0$ is a tangent to the circle with two foci as the end points of the diameter.

Circle with foci $(ae, 0)$ and $(-ae, 0)$ as diameter is

$(x - ax)(x + ae) + (y - 0)(y - 0) = 0$

i.e. $x^2 + y^2 = a^2e^2 = a^2 + b^2$ ... (i)

$[\because a^2e^2 = a^2 + b^2]$

Let the chord of contact of $P(x_1, y_1)$ touch the circle (i).

Equation of the chord of contact of P is $[T = 0]$

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

i.e. $b^2x_1x - a^2y_1y - a^2b^2 = 0$ ... (ii)

This equation is tangent to the circle if

$\frac{a^2b^2}{\sqrt{(b^4x_1^2 + a^4y_1^2)}} = \pm\sqrt{(a^2 + b^2)}$

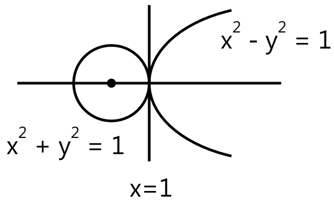
Hence locus of $P(x_1, y_1)$ is $(b^4x^2 + a^4y^2)(a^2 + b^2) = a^4b^4$.

Example 8: An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $P\left(\frac{1}{2}, 1\right)$. One of its directrices is the common tangent, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$, nearer to P. The equation of the ellipse in the standard form is.

Sol: he circle $x^2 + y^2 = 1$ is the auxiliary circle of the hyperbola $x^2 - y^2 = 1$ and they touch each other at the points $(\pm 1, 0)$. Use the definition of the ellipse to get the final equation.

The common tangent at these points are $x = \pm 1$.

Since $x = 1$ is near to the focus $P\left(\frac{1}{2}, 1\right)$, this is the directrix of the required ellipse.



Therefore, by definition, the equation of the ellipse is

$$\left(x - \frac{1}{2}\right)^2 + (y-1)^2 = \left(\frac{1}{2}\right)^2\left(\frac{x-1}{1}\right)^2$$

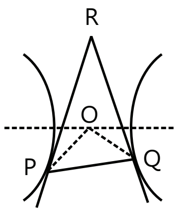
$$\Rightarrow \quad 9\left(x - \frac{1}{3}\right)^2 + 12(y - 1)^2 = 1.$$

Example 9: Prove that the angle subtended by any chord of a rectangular hyperbola at the centre is the supplement of the angle between the tangents at the ends of the chord.

Sol: Using the equation of chord, find the angle subtended at the centre and at the intersection of the tangents.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two ends of a chord of the rectangular hyperbola

$$x^2 - y^2 = 1 \quad .....(i)$$



Now, 'm' of OP $= \frac{y_1}{x_1}$

'm' of OQ $= \frac{y_2}{x_2}$

$$\therefore \quad \tan\theta = \frac{\frac{y_1}{x_1} - \frac{y_2}{x_2}}{1 + \frac{y_1}{x_1}.\frac{y_2}{x_2}} = \frac{x_2y_1 - x_1y_2}{x_1x_2 + y_1y_2},$$

Where $\angle POQ = \theta$,

The equations of tangents at P and Q are

$xx_1 - yy_1 = 1$ and $xx_2 - yy_2 = 1$.

Their slopes are $\frac{x_1}{y_1}$ and $\frac{x_2}{y_2}$.

$$\therefore \quad \tan\phi = \frac{\frac{x_1}{y_1} - \frac{x_2}{y_2}}{1 + \frac{x_1}{y_1}.\frac{x_2}{y_2}} = \frac{x_1y_2 - x_2y_1}{y_1y_2 + x_1x_2}$$

$\therefore$ $\tan\theta$ and $\tan\phi$ are equal in magnitude but opposite in sign

$\therefore \tan\theta = -\tan\phi = \tan(\pi - \phi)$

$\therefore \theta + \phi = \pi$. Hence, proved.

Example 10: If a chord of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, show that the locus of its middle point is $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$.

Sol: Apply the condition of tangency in the equation of the chord.

Let $M(\alpha, \beta)$ be the middle point of the chord PQ of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ....(i)$$

The equation of the chord is

$$\frac{x\alpha}{a^2} + \frac{y\beta}{b^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}$$

$$\Rightarrow \quad y = -\frac{xb^2\alpha}{a^2\beta} + \frac{b^2}{\beta}\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right)$$

This line is tangent to hyperbola if

$$c^2 = a^2m^2 - b^2$$

$$\Rightarrow \quad \left(\frac{b^2}{\beta}\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right)\right)^2 = a^2\left(\frac{b^2\alpha}{a^2\beta}\right)^2 - b^2$$

$$\Rightarrow \quad \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right)^2 = \frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2}$$

$\therefore$ The equation of the required locus of the middle point (α, β) is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

JEE Main/Boards

Exercise 1

Q.1 Find the centre, eccentricity and foci of the hyperbola $9x^2 - 16y^2 - 18x - 64y - 199 = 0$

Q.2 Find the equation to the tangent to the hyperbola $4x^2 - 3y^2 = 13$ at the point (2,1).

Q.3 Show that the line $21x + 5y = 116$ touches the hyperbola $7x^2 - 5y^2 = 232$ and find the co-ordinates of the point of contact.

Q.4 Find the locus of the middle points of the portion of the tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ included between the axes.

Q.5 A point P moves such that the tangents PT_1 and PT_2 from it to the hyperbola $4x^2 - 9y^2 = 36$ are mutually perpendicular. Find the equation of the locus of P.

Q.6 Find the equations of the two tangents to the hyperbola xy = 27 which are perpendicular to the straight line $4x - 3y = 7$.

Q.7 Find the equation of the hyperbola which has $3x - 4y + 7 = 0$ and $4x+3y+1 = 0$ for its asymptotes and which passes through the origin.

Q.8 Find the equation of chord of contact of tangents drawn from the point (–5, 2) to the hyperbola xy = 25.

Q.9 Find the eccentric angle of the point lying in fourth quadrant on the hyperbola $x^2 - y^2 = 4$ whose distance from the centre is 12 units.

Q.10 Find the acute angle between the asymptotes of $4x^2 - y^2 = 16$.

Q.11 If the tangent and normal to a rectangular hyperbola cut off intercepts a_1 and a_2 on one axis and b_1 and b_2 on the other axis, shows that $a_1a_2 + b_1b_2 = 0$.

Q.12 Show that the area of the triangle formed by the two asymptotes of the rectangular hyperbola $xy = c^2$ and the normal at (x_1, y_1) on the hyperbola is $\frac{1}{2}\left[\frac{x_1^2 - y_1^2}{c}\right]^2$.

Q.13 PN is the ordinate of any point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If Q divides AP in the ratio $a^2: b^2$, show that NQ is perpendicular to A¢P where A'A is the transverse axis of the hyperbola.

Q.14 A normal to the hyperbola $x^2 - 4y^2 = 4$ meets the x and y axes at A and B respectively. Find the locus of the point of intersection of the straight lines drawn through A and B perpendicular to the x and y axes respectively.

Q.15 In any hyperbola, prove that the tangent at any point bisects the angle between the focal distances of the point.

Q.16 If the normals at four points $P_i(x_i, y_i)$ i = 1, 2, 3, 4 on the rectangular hyperbola $xy = c^2$ meet at the point Q(h, k), prove that

(i) $x_1 + x_2 + x_3 + x_4 = h$ (ii) $y_1 + y_2 + y_3 + y_4 = k$

(iii) $x_1x_2x_3x_4 = y_1y_2y_3y_4 = -c^4$

Q.17 Find the locus of the points of intersection of two tangents to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if sum of their slopes is a constant λ.

Q.18 A variable tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse axis at Q and to the tangent at the vertex(a,0) at R. Show that the locus of the mid point of QR is $x(4y^2 + b^2) = ab^2$.

Q.19 A tangent to the parabola $x^2 = 4ay$ meets the hyperbola $xy = k^2$ in two points P and Q. Prove that the middle point of PQ lies on a parabola.

Q.20 Show that the locus of the middle points of the normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$.

Q.21 Given a hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ and circle $x^2 + y^2 = 9$. Find the locus of mid point of chord of contact drawn from a point on the hyperbola to the circle.

Q.22 A rectangular hyperbola whose centre is C, is cut by a circle of radius r in four points P, Q, R, S. Prove that $CP^2 + CQ^2 + CR^2 + CS^2 = 4r^2$.

Q.23 The normal at the three points P, Q, R on a rectangular hyperbola, intersect at a point S on the curve. Prove that the centre of the hyperbola is the centroid of the triangle PQR.

Q.24 A parallelogram is constructed with its sides parallel to the asymptotes of a hyperbola and one of its diagonals is a chord of the hyperbola, show that the other diagonal passes through the centre.

Q.25 If the straight line $y = mx + 2c\sqrt{-m}$ touches the hyperbola $xy = c^2$ then the co-ordinates of the point contact are (.................,)

Q.26 If the normal to the rectangular hyperbola $xy = c^2$ at the point 't' meets the curve again at 't_1' then t^3t_1 has the value equal to ....

Exercise 2

Single Correct Choice Type

Q.1 The line 5x + 12y = 9 touches the hyperbola $x^2 - 9y^2 = 9$ at the point-

(A) (–5, 4/3) (B) (5, –4/3)

(C) (3, –1/2) (D) None of these

Q.2 The length of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is-

(A) $\frac{2a^2}{b}$ (B) $\frac{2b^2}{a}$ (C) $\frac{b^2}{a}$ (D) $\frac{a^2}{b}$

Q.3 The area of the square whose sides are the directrixes of the hyperbola $x^2 - y^2 = a^2$ and its conjugate hyperbola, is-

(A) a^2 (B) $2a^2$ (C) $4a^2$ (D) $8a^2$

Q.4 The number of possible tangents which can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line 5x + 2y – 10 = 0 is -

(A) Zero (B) 1 (C) 2 (D) 4

Q.5 If m is a variable, the locus of the point of intersection of the lines $\frac{x}{3} - \frac{y}{2} = m$ and $\frac{x}{3} + \frac{y}{2} = \frac{1}{m}$ is a/an -

(A) Parabola (B) Ellipse

(C) Hyperbola (D) None of these

Q.6 The eccentricity of the hyperbola with its principal axes along the co-ordinate axes and which passes through (3, 0) and $(3\sqrt{2}, 2)$ is-

(A) $\frac{1}{3}$ (B) $\frac{\sqrt{13}}{3}$ (C) $\frac{\sqrt{5}}{3}$ (D) $\frac{2}{3}$

Q.7 The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is-

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) 1/2

Q.8 An ellipse and a hyperbola have the same centre origin, the same foci and the minor-axis of the one is the same as the conjugate axis of the other. If e_1, e_2 be their eccentricities respectively, then $\frac{1}{e_1^2} + \frac{1}{e_2^2} =$

(A) 1 (B) 2 (C) 4 (D) None of these

Q.9 Which of the following pair may represent the eccentricities of two conjugate hyperbola for all $\alpha \in (0, \pi/2)$?

(A) sin α, cos α (B) tan α, cot α

(C) sec α, cosec α (D) 1 + sinα, 1 + cosα

Q.10 The number of normals to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from an external point is-

(A) 2 (B) 4 (C) 6 (D) 5

Q.11 A rectangular hyperbola circumscribe a triangle ABC, then it will always pass through its-

(A) Orthocentre (B) Circum centre

(C) Centroid (D) Incentre

Q.12 If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again at t′ then-

(A) $t' = \frac{-1}{t^3}$ (B) $t' = \frac{1}{t}$

(C) $t' = \frac{1}{t^2}$ (D) $t¢^2 = \frac{-1}{t^2}$

Q.13 The centre of the hyperbola $9x^2 - 16y^2 - 36x + 96y - 252 = 0$ is-

(A) (2, 3) (B) (–2, –3) (C) (–2, 3) (D) (2, –3)

Q.14 The tangents from $(1, 2\sqrt{2})$ to the hyperbola $16x^2 - 25y^2 = 400$ include between them an angle equal to-

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Q.15 The number of points from where a pair of perpendicular tangents can be drawn to the hyperbola, $x^2\sec^2\alpha - y^2\text{cosec}^2\alpha = 1$, $\alpha \in (0, \pi/4)$ is-

(A) 0 (B) 1 (C) 2 (D) Infinite

Q.16 If hyperbola $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ passes through the focus of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then eccentricity of hyperbola is-

(A) $\sqrt{2}$ (B) $\frac{2}{\sqrt{3}}$ (C) $\sqrt{3}$ (D) None of these

Q.17 If the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (a > b) and $x^2 - y^2 = c^2$ cut at right angles then-

(A) $a^2 + b^2 = 2c^2$ (B) $b^2 - a^2 = 2c^2$

(C) $a^2 - b^2 = 2c^2$ (D) $a^2b^2 = 2c^2$

Q.18 Two conics $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $x^2 = -\frac{1}{b}y$ intersect if -

(A) $0 < b \le \frac{1}{2}$ (B) $0 < a < \frac{1}{2}$

(C) $a^2 < b^2$ (D) $a^2 > b^2$

Q.19 The locus of the mid points of the chords passing through a fixed point (α, β) of the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is-

(A) A circle with centre $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

(B) An ellipse with centre $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

(C) A hyperbola with centre $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

(D) Straight line through $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

Q.20 If the eccentricity of the hyperbola $x^2 - y^2\sec^2 a = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2 a + y^2 = 25$, then a value of α is-

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

Q.21 For all real values of m, the straight line $y = mx + \sqrt{9m^2 - 4}$ is a tangent to the curve-

(A) $9x^2 + 4y^2 = 36$ (B) $4x^2 + 9y^2 = 36$

(C) $9x^2 - 4y^2 = 36$ (D) $4x^2 - 9y^2 = 36$

Q.22 Locus of the middle points of the parallel chords with gradient m of the rectangular hyperbola $xy = c^2$ is-

(A) $y + mx = 0$ (B) $y - mx = 0$

(C) $my - mx = 0$ (D) $my + x = 0$

Q.23 The locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$ is-

(A) $3x - 4y = 4$ (B) $3y - 4x + 4 = 0$

(C) $4x - 4y = 3$ (D) $3x - 4y = 2$

Previous Years' Questions

Q.1 The equation $\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1$, $|r| < 1$ represents- ***(1981)***

(A) An ellipse (B) A hyperbola

(C) A circle (D) None of these

Q.2 Let $P(a \sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of the intersection of the normals at P and Q, then k is equal to- ***(1999)***

(A) $\frac{a^2+b^2}{a}$ (B) $-\left(\frac{a^2+b^2}{a}\right)$ (C) $\frac{a^2+b^2}{b}$ (D) $-\left(\frac{a^2+b^2}{b}\right)$

Q.3 If x = 9 is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is- ***(1999)***

(A) $9x^2 - 8y^2 + 18x - 9 = 0$

(B) $9x^2 - 8y^2 - 18x + 9 = 0$

(C) $9x^2 - 8y^2 - 18x - 9 = 0$

(D) $9x^2 - 8y^2 + 18x + 9 = 0$

Q.4 For hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$, which of the following remains constant with change in 'α' ? ***(2003)***

(A) Abscissa of vertices (B) Abscissa of foci

(C) Eccentricity (D) Directrix

Q.5 If the line $2x + \sqrt{6}\,y = 2$ touches the hyperbola $x^2 - 2y^2 = 4$, then the point of contact is- ***(2004)***

(A) $(-2, \sqrt{6})$ (B) $(-5, 2\sqrt{6})$

(C) $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ (D) $(4, -\sqrt{6})$

Q.6 If e_1 is the eccentricity of the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ and e_2 is the eccentricity of the hyperbola passing through the foci of the ellipse and $e_1e_2 = 1$, then equation of the hyperbola is- ***(2006)***

(A) $\frac{x^2}{9} - \frac{y^2}{16} = 1$ (B) $\frac{x^2}{16} - \frac{y^2}{9} = -1$

(C) $\frac{x^2}{9} - \frac{y^2}{25} = 1$ (D) None of these

Q.7 A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is- ***(2007)***

(A) $x^2\text{cosec}^2\theta - y^2\sec^2\theta = 1$ (B) $x^2\sec^2\theta - y^2\text{cosec}^2\theta = 1$

(C) $x^2\sin^2\theta - y^2\cos^2\theta = 1$ (D) $x^2\cos^2\theta - y^2\sin^2\theta = 1$

Q.8 Consider a branch of the hyperbola

$x^2 - 2y^2 - 2\sqrt{2}\,x - 4\sqrt{2}\,y - 6 = 0$

with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is- ***(2008)***

(A) $1-\sqrt{\frac{2}{3}}$ sq. unit (B) $\sqrt{\frac{3}{2}}-1$ sq. unit

(C) $1+\sqrt{\frac{2}{3}}$ sq. unit (D) $\sqrt{\frac{3}{2}}+1$ sq. unit

Q.9 Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$. If the normal at the point P intersect the x-axis at (9, 0), then the eccentricity of the hyperbola is- ***(2011)***

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

Q.10 The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is: ***(2016)***

(A) $\frac{4}{\sqrt{3}}$ (B) $\frac{2}{\sqrt{3}}$ (C) $\sqrt{3}$ (D) $\frac{4}{3}$

JEE Advanced/Boards

Exercise 1

Q.1 Find the equation to the hyperbola whose directrix is $2x + y = 1$ focus (1, 1) and eccentricity $\sqrt{3}$. Find also the length of its latus rectum.

Q.2 The hyperbola $x^2/a^2 - y^2/b^2 = 1$ passes through the point of inter-section of the lines. $7x + 13y - 87 = 0$ and $5x - 8y + 7 = 0$ and the latus rectum is $32\sqrt{3}/5$. Find 'a' & 'b'.

Q.3 For the hyperbola $x^2/100 - y^2/25 = 1$, prove that the

(i) eccentricity = $\sqrt{5}/2$

(ii) SA.S'A = 25, where S and S' are the foci and A is the vertex.

Q.4 Find the centre, the foci, the directrices, the length of the latus rectum, the length and the equations of the axes and the asymptotes of the hyperbola

$16x^2 - 9y^2 + 32x + 36y - 164 = 0$.

Q.5 If a rectangular hyperbola have the equation, $xy = c^2$, prove that the locus of the middle point of the chords of constant length 2d is $(x^2 + y^2)(xy - c^2) = d^2xy$.

Q.6 If m_1 and m_2 are the slopes of the tangents to the hyperbola $x^2/25 - y^2/16 = 1$ through the point (6, 2), find the value of (i) $m_1 + m_2$ and (ii) m_1m_2.

Q.7 Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Q.8 If θ_1 and θ_2 are the parameters of the extremities of a chord through (ae, 0) of a hyperbola $x^2/a^2 - y^2/b^2 = 1$, then show that

$$\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}+\frac{e-1}{e+1} = 0.$$

Q.9 If C is the centre of hyperbola $x^2/a^2 - y^2/b^2 = 1$, S, S' its foci and P a point on it. Prove that SP.S'P = $CP^2 - a^2 + b^2$.

Q.10 Tangents are drawn to the hyperbola $3x^2 - 2y^2 = 25$ from the point (0, 5/2). Find their equations.

Q.11 If the tangent at the point (h, k) to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ cuts the auxiliary circle in points whose ordinates are y_1 and y_2 then prove that $1/y_1 + 1/y_2 = 2/k$.

Q.12 Tangents are drawn from the point (α, β) to the hyperbola $3x^2 - 2y^2 = 6$ and are inclined at angles θ and ϕ to the x-axis. If $\tan\theta . \tan\phi = 2$, prove that $\beta^2 = 2\alpha^2 - 7$.

Q.13 Find the number of normal which can be drawn from an external point on the hyperbola $\frac{x^2}{a^2} \frac{-y^2}{b^2} = 1$.

Q.14 The perpendicular from the centre upon the normal on any point of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ meets at R. Find the locus of R.

Q.15 If the normal at a point P to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ meets the x-axis at G, show that SG = e. SP, S being the focus of the hyperbola.

Q.16 Show that the area of the triangle formed by the lines $x - y = 0$, $x + y = 0$ and any tangent to the hyperbola $x^2 - y^2 = a^2$ is a^2.

Q.17 Find the locus of the middle point of the chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$.

Q.18 The line $y = mx + 6$ is tangent to the hyperbola $\frac{x^2}{10^2} - \frac{y^2}{7^2} = 1$ at certain point. Find the value of m.

Q.19 A point P divides the focal length of the hyperbola $9x^2 - 16y^2 = 144$ in the ratio S¢P: SP = 2: 3 where S and S′ are the foci of the hyperbola. Through P a straight line is drawn at an angle of 135° to the axes OX. Find the points of intersection of the line with the asymptotes of the hyperbola.

Q.20 Find the equation of tangent to the hyperbola $x^2 - 2y^2 = 18$ which is perpendicular to the line $y = x$.

Q.21 If a chord joining the points $P(a \sec\theta, a \tan\theta)$ and $Q(a \sec\phi, a \tan\phi)$ on the hyperbola $x^2 - y^2 = a^2$ is a normal to it at P, then show that $\tan\phi = \tan\theta\,(4\sec^2\theta - 1)$.

Q.22 Find the equations of the tangents to the hyperbola $x^2 - 9y^2 = 9$ that are drawn from (3, 2). Find the area of the triangle that these tangents form with their chord of contact.

Q.23 Let 'p' be the perpendicular distance from the centre C of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ to the tangent drawn at point R on the hyperbola. If S and S′ are the two foci of the hyperbola, then show that

$(RS + RS')^2 = 4a^2\left(1 + \frac{b^2}{p^2}\right)$.

Exercise 2

Single Correct Choice Type

Q.1 Locus of middle point of all chords of $\frac{x^2}{4} - \frac{y^2}{9}$ = 1. Which are at distance of '2' units from vertex of parabola $y^2 = -8ax$ is-

(A) $\left(\frac{x^2}{4} + \frac{y^2}{9}\right) = \frac{xy}{6}$ (B) $\left(\frac{x^2}{4} - \frac{y^2}{9}\right)^2 = 4\left(\frac{x^2}{16} + \frac{y^2}{81}\right)$

(C) $\left(\frac{x^2}{4} + \frac{y^2}{9}\right)^2 = \left(\frac{x^2}{9} + \frac{y^2}{4}\right)$ (D) None of these

Q.2 Tangents at any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cut the axes at A and B respectively. If the rectangle OAPB (where O is origin) is completed then locus of point P is given by-

(A) $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$ (B) $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$

(C) $\frac{a^2}{y^2} - \frac{b^2}{x^2} = 1$ (D) None of these

Q.3 The locus of the foot of the perpendicular from the centre of the hyperbola $xy = c^2$ on a variable tangent is-

(A) $(x^2 - y^2)2 = 4c^2xy$ (B) $(x^2 + y^2)2 = 2c^2xy$

(C) $(x^2 - y^2) = 4x^2xy$ (D) $(x^2 + y^2)^2 = 4c^2xy$

Q.4 The point of intersection of the curves whose parametric equation are $x = t^2 + 1$, $y = 2t$ and $x = 2s$, $y = 2/s$ is given by-

(A) (1, –3) (B) (2, 2) (C) (–2, 4) (D) (1, 2)

Q.5 P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at p meets the transverse axis at T. If O is the centre to the hyperbola, the OT.ON is equal to-

(A) e^2 (B) a^2 (C) b^2 (D) b^2/a^2

Q.6 The equation to the chord joining two point (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is-

(A) $\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$ (B) $\frac{x}{x_1 - x_2} + \frac{y}{y_1 - y_2} = 1$

(C) $\frac{x}{y_1 - y_2} + \frac{y}{x_1 - x_2} = 1$ (D) $\frac{x}{y_1 - y_2} + \frac{y}{x_1 - x_2} = 1$

Q.7 The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half the distance between the foci, is-

(A) $\frac{4}{3}$ (B) $\frac{4}{\sqrt{3}}$ (C) $\frac{2}{\sqrt{3}}$ (D) None of these

Q.8 The equation to the chord of the hyperbola $x^2 - y^2 = 9$ which is bisected at (5, –3) is-

(A) $5x + 3y = 9$ (B) $5x - 3y = 16$

(C) $5x + 3y = 16$ (D) $5x - 3y = 9$

Q.9 The differential equation $\frac{dx}{dy} = \frac{3y}{2x}$ represents a family of hyperbolas (except when it represents a pair of lines) with eccentricity-

(A) $\sqrt{\frac{3}{5}}$ (B) $\sqrt{\frac{5}{3}}$ (C) $\sqrt{\frac{2}{5}}$ (D) $\sqrt{\frac{5}{2}}$

Multiple Correct Choice Type

Q.10 Equation of a tangent passing through (2, 8) to the hyperbola $5x^2 - y^2 = 5$ is-

(A) $3x - y + 2 = 0$ (B) $3x + y - 14 = 0$

(C) $23x - 3y - 22 = 0$ (D) $3x - 23y + 178 = 0$

Q.11 The equation $16x^2 - 3y^2 - 32x + 12y - 44 = 0$ represent a hyperbola -

(A) The length of whose transverse axis is $4\sqrt{3}$

(B) The length of whose conjugate axis is 8

(C) Those centre is (1, 2)

(D) Those eccentricity is $\sqrt{\frac{19}{3}}$

Q.12 A common tangent to $9x^2 - 16y^2 = 144$ and $x^2 + y^2 = 9$ is-

(A) $y = \frac{3}{\sqrt{7}}x + \frac{15}{\sqrt{7}}$ (B) $y = 3\frac{\sqrt{2}}{7}x + \frac{15}{\sqrt{7}}$

(C) $y = 2\frac{\sqrt{3}}{7}x + 15\sqrt{7}$ (D) $y = 3\frac{\sqrt{2}}{7}x - \frac{15}{\sqrt{7}}$

Q.13 Which of the following equation in parametric form can represent a hyperbola, profile, where 't' is a parameter

(A) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$ & $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$

(B) $\frac{tx}{a} - \frac{y}{b} + t = 0$ & $\frac{x}{a} + \frac{ty}{b} - 1 = 0$

(C) $x = e^t + e^{-t}$ & $y = e^t - e^{-t}$

(D) $x^2 - 6 = 2\cot$ & $y^2 + 2 = 4\cos^2\frac{t}{2}$

Q.14 Circles are drawn on chords of the rectangular hyperbola $xy = a^2$ parallel to the line $y = x$ as diameters. All such circles pass through two fixed points whose co-ordinates are-

(A) (c, c) (B) (c, –c) (C) (–c, c) (D) (–c, –c)

Q.15 If the normal at (x_i, y_i) i = 1, 2, 3, 4 to its rectangular hyperbola $xy = 2$ meet at the point (3, 4), then-

(A) $x_1 + x_2 + x_3 + x_4 = 3$ (B) $y_1 + y_2 + y_3 + y_4 = 4$

(C) $x_1x_2x_3x_4 = -4$ (D) $y_1y_2y_3y_4 = -4$

Q.16 If (5, 12) and (24, 7) are the foci of a conic passing through the origin then the eccentricity of conic is-

(A) $\sqrt{386}/12$ (B) $\sqrt{386}/13$ (C) $\sqrt{386}/25$ (D) $\sqrt{386}/38$

Q.17 The value of m for which $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{49} = 1$ is-

(A) $\sqrt{\left(\frac{17}{20}\right)}$ (B) $-\sqrt{\left(\frac{17}{20}\right)}$ (C) $\sqrt{\left(\frac{20}{17}\right)}$ (D) $-\sqrt{\left(\frac{20}{17}\right)}$

Q.18 The equation $\frac{x^2}{12-k} + \frac{y^2}{k-8} = 1$ represents-

(A) A hyperbola if k < 8

(B) An ellipse if 8 < k < 12, k ≠ 10

(C) A hyperbola if 8 < k < 12

(D) Circle if k = 10

Q.19 Equations of a common tangent to the two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is-

(A) $y = x + \sqrt{a^2 - b^2}$ (B) $y = x - \sqrt{a^2 - b^2}$

(C) $y = -x + \sqrt{a^2 - b^2}$ (D) $y = -x - \sqrt{a^2 - b^2}$

Q.20 The equation of the tangent lines to the hyperbola $x^2 - 2y^2 = 18$ which are perpendicular the line y = x are-

(A) $y = -x + 7$ (B) $y = -x + 3$

(C) $y = -x - 4$ (D) $y = -x - 3$

Q.21 The co-ordinate of a focus of the hyperbola $9x^2 - 16y^2 + 18x + 32y - 151 = 0$ are-

(A) (–1, 1) (B) (6, 1) (C) (4, 1) (D) (–6, 1)

Q.22 If $(a \sec \theta, b \tan \theta)$ & $(a \sec \phi, b \tan \phi)$ are the ends of a focal chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\tan\frac{\theta}{2}\tan\frac{\varphi}{2}$ equal to-

(A) $\frac{e-1}{e+1}$ (B) $\frac{1-e}{1+e}$ (C) $\frac{e+1}{1-e}$ (D) $\frac{e+1}{e-1}$

Q.23 If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes in G and g and C is the centre of the hyperbola, then-

(A) PG = PC (B) Pg = PC (C) PG = Pg (D) Gg = PC

Previous Years' Questions

Q.1 If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_1, y_1)$, $R(x_3, y_3)$, $S(x_4, y_4)$, then- ***(1998)***

(A) $x_1 + x_2 + x_3 + x_4 = 0$ (B) $y_1 + y_2 + y_3 + y_4 = 0$

(C) $x_1x_2x_3x_4 = c^4$ (D) $y_1y_2y_3y_4 = c^4$

Q.2 An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal to that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then ***(2009)***

(A) Equation of ellipse is $x^2 + 2y^2 = 2$

(B) The foci of ellipse are (± 1, 0)

(C) Equation of ellipse is $x^2+2y^2 = 4$

(D) The foci of ellipse are $(\pm\sqrt{2}, 0)$

Q.3 Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then ***(2011)***

(A) The equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

(B) A focus of the hyperbola is (2, 0)

(C) The eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

(D) The equation of the hyperbola is $x^2 - 3y^2 = 3$

Q.4 For any real t, $x = \frac{e^t + e^{-t}}{2}$, $y = \frac{e^t - e^{-t}}{2}$ is a point on the hyperbola $x^2 - y^2 = 1$. Find the area bounded by this hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_2$. ***(1982)***

Q.5 Tangents are drawn from any point o the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to the circle $x^2 + y^2 = 9$. Find the locus of mid point of the chord of contact. ***(2005)***

Paragraph 6 to 7:

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B. ***(2010)***

Q.6 Equation of a common tangent with positive slope to the circle as well as to the hyperbola is-

(A) $2x - \sqrt{5}y - 20 = 0$ (B) $2x - \sqrt{5}y + 4 = 0$

(C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$

Q.7 Equation of the circle with AB as its diameter is-

(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$

(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$

Q.8 The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is …….. ***(2010)***

Q.9 Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is ***(2008)***

(A) $1 - \sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}} - 1$ (C) $1 + \sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{3}{2}} + 1$

Q.10 Match the conics in column I with the statements/ expressions in column II. ***(2009)***

Column I	Column II
(A) Circle	(p) The locus of the point (h, k) for which the line hx + ky = 1 touches the circle $x^2 + y^2 = 4$
(B) Parabola	(q) Points z in the complex plane satisfying $\left\|z+2\right\| - \left\|z-2\right\| = \pm 3$
(C) Ellipse	(r) Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right), y = \frac{2t}{1+t^2}$
(D) Hyperbola	(s) The eccentricity of the conic lies in the interval $1 \le x < \infty$
	(t) Points z in the complex plane satisfying $\text{Re}(z+1)^2 = \left\|z\right\|^2 + 1$

Q.11 The line 2x + y = 1 is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is ***(2010)***

Q.12 Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at (9, 0), then the eccentricity of the hyperbola is ***(2011)***

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

Q.13 Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then ***(2011)***

(A) The equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

(B) A focus of the hyperbola is (2, 0)

(C) The eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

(D) The equation of the hyperbola is $x^2 - 3y^2 = 3$

Q.14 Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, parallel to the straight line 2x −y = 1. The points of contact of the tangents on the hyperbola are ***(2012)***

(A) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (B) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(C) $\left(3\sqrt{3}, -2\sqrt{2}\right)$ (D) $\left(-3\sqrt{3}, 2\sqrt{2}\right)$

Questions

JEE Main/Boards

Exercise 1

Q.7 Q.12 Q.15 Q.21
Q.24 Q.25 Q.27

Exercise 2

Q.3 Q.11 Q.18 Q.19

Previous Years' Questions

Q.2 Q.6 Q.8

JEE Advanced/Boards

Exercise 1

Q.5 Q.11 Q.12 Q.15
Q.18 Q.22 Q.25

Exercise 2

Q.3 Q.6 Q.8 Q.11
Q.17 Q.23

Previous Years' Questions

Q.2 Q.3 Q.4 Q.8

Answer Key

JEE Main/Boards

Exercise 1

Q.1 C(1, 2), e = 5/4, (6, 2) and (–4, 2)

Q.2 $8x - 3y - 13 = 0$

Q.3 (6, –2)

Q.4 $a^2y^2 - b^2x^2 = 4x^2y^2$

Q.5 $x^2 + y^2 = 5$

Q.6 $3x + 4y \pm 36 = 0$

Q.7 $12x^2 - 7xy - 12y^2 + 31x + 17y = 0$

Q.8 $2x - 5y = 50$

Q.9 $7\frac{\pi}{4}$ rad.

Q.10 $\tan^{-1}\frac{4}{3}$

Q.14 $4x^2 - y^2 = 25$

Q.17 $\lambda(x^2 - a^2) = 2xy$

Q.21 $9x^2 - \frac{81y^2}{4} = (x^2 + y^2)^2$ is the required locus.

Q.25 $\frac{c}{\sqrt{-m}}$, $c\sqrt{-m}$

Q.26 –1

Exercise 2

Single Correct Choice Type

Q.1 B	**Q.2** A	**Q.3** B	**Q.4** A	**Q.5** C	**Q.6** B
Q.7 B	**Q.8** B	**Q.9** C	**Q.10** B	**Q.11** A	**Q.12** A
Q.13 A	**Q.14** D	**Q.15** D	**Q.16** C	**Q.17** C	**Q.18** B
Q.19 C	**Q.20** B	**Q.21** D	**Q.22** A	**Q.23** A	

Previous Years' Questions

Q.1 B	**Q.2** D	**Q.3** B	**Q.4** B	**Q.5** D	**Q.6** B
Q.7 A	**Q.8** B	**Q.9** B	**Q.10** B		

JEE Advanced/Boards

Exercise 1

Q.1 $7x^2 + 12xy - 2y^2 - 2x + 4y - 7 = 0$; $\frac{\sqrt{48}}{5}$

Q.2 $a^2 = 25/2$; $b^2 = 16$

Q.6 (i) 24/11 (ii) 20/11

Q.7 $x + y \pm 3\sqrt{3} = 0$

Q.10 $3x + 2y - 5 = 0$; $3x - 2y + 5 = 0$

Q.14 $(x^2+y^2)^2\ (a^2y^2-b^2y^2) = x^2y^2(a^2+b^2)^2$

Q.18 $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and $\frac{y^2}{16} - \frac{x^2}{9} = 1$

Q.19 (–4, 3) and $\left(-\frac{4}{7}, -\frac{3}{7}\right)$

Q.22 $y = \frac{5}{12}x + \frac{3}{4}$; $x - 3 = 0$; 8 sq. unit

Exercise 2

Single Correct Choice Type

Q.1 B **Q.2** A **Q.3** D **Q.4** B **Q.5** B **Q.6** A

Q.7 C **Q.8** C **Q.9** B

Multiple Correct Choice Type

Q.10 A,C **Q.11** B, C, D **Q.12** B, D **Q.13** A, C, D **Q.14** A, D **Q.15** A, B, C, D

Q.16 A, D **Q.17** A, B **Q.18** A, B, D **Q.19** A, B, C, D **Q.20** B, D **Q.21** C, D

Q.22 B, C **Q.23** A, B, C

Previous Years' Questions

Q.1 A, B, C, D **Q.2** A, B **Q.3** B, D **Q.4** t_1 **Q.5** $\frac{x^2}{9}-\frac{y^2}{4} = \frac{(x^2+y^2)^2}{81}$

Q.6 B **Q.7** A **Q.8** 2 **Q.9** B **Q.10** $A \to p$; $B \to s, t$; $C \to r$; $D \to q, s$

Q.11 2 **Q.12** B **Q.13** B, D **Q.14** A, B

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $9(x^2 – 2x + 1) – 16(y^2 – 4y + 4) – 199 – 9 + 64 = 0$

$9(x – 1)^2 – 16(y – 2)^2 = 144$

$\frac{(x-1)^2}{16}-\frac{(y-2)^2}{9} = 1$

so $a = \sqrt{16} = 4$ & $b = \sqrt{9} = 3$

so $e^2 = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{3^2}{4^2}} \Rightarrow e = \frac{5}{4}$

Now centre would be where

$x – 1 = 0$ and $y – 2 = 0$

$\Rightarrow$ $x = 1$ and $y = 2$

and focii distance = $ae = 4 \times \frac{5}{4}$ (in x-direction)

focii = (1 + 5, 2) and (1 – 5, 2)

(6, 2) and (–4, 2)

Sol 2: Tangent $\Rightarrow \frac{x.x_1}{a^2}-\frac{y.y_1}{b^2} = 1$

$\frac{2.x}{13/4}-\frac{1.y}{13/3} = 1 \Rightarrow 8x – 3y = 13$

Sol 3: We have $y = \left(-\frac{21}{5}\right)x+\left(\frac{116}{5}\right)$

[y = mx + c form]

Now, y = mx + c is tangent when

$a^2m^2 – b^2 = c^2$

So $\frac{232}{7}.\left(\frac{21}{5}\right)^2-\left(\frac{232}{5}\right)$

$= \frac{63\times 232}{25}-\frac{232\times 5}{25}\ \frac{(116)^2}{25} = \frac{(116)^2}{25}$

So LHS = RHS

Hence, the given line is tangent

Now tangent

$\Rightarrow \frac{x.x_1}{282/7}-\frac{y.y_1}{232/5} = 1$

Now, comparing with the given tangent

$\frac{21\times 232}{x_1\times 7} = \frac{5\times 232}{-5y_1} = \frac{116}{1}$

$\Rightarrow$ $x_1 = 6$ and $y_1 = –2$

Sol 4: Tangent = $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$

Now the tangent cuts the axes at (a cosθ, 0) and (0, b cot θ)

mid points ⇒ $\frac{a\cos\theta}{2}$ = h and k = $\frac{b\cot\theta}{2}$

$\Rightarrow \frac{a}{2h} = \sec\theta$ and $\frac{b}{2k} = \tan\theta$

$\Rightarrow \frac{a^2}{4h^2} - \frac{b^2}{4k^2} = 1 \Rightarrow \frac{a^2}{h^2} - \frac{b^2}{k^2} = 4$

Replacing h and k, we get locus as

$\frac{a^2}{h^2} - \frac{b^2}{k^2} = 4 \Rightarrow a^2y^2 - b^2x^2 = 4x^2y^2$

Sol 5: We have tangents

$\Rightarrow y = mx \pm \sqrt{a^2m^2 - b^2} \Rightarrow y = mx \pm \sqrt{9m^2 - 4}$

$\Rightarrow (y - mx)^2 = (\sqrt{9m^2 - 4})^2$

$y^2 + m^2x^2 - 4mxy = 9m^2 - 4$

$\Rightarrow (9 - x^2)m^2 + (4xy)m\ (4 + y^2) = 0$

Now h, k would satisfy this

$\Rightarrow (9 - h^2)m^2 + (4hk)m\ (4 + k^2) = 0$

So, $m_1m_2 = \frac{-(4+k^2)}{9-h^2} = 1$

$\Rightarrow 4 + k^2 = 9 - h^2 \Rightarrow h^2 + k^2 = 5$

Hence, the locus is $x^2 + y^2 = 5$

Sol 6: We have $m_1 = \frac{4}{3}$ (given line)

Given $m_1 . m_2 = -1 \Rightarrow m_2 = -\frac{3}{4}$

So $y = -\frac{3}{4}x + c$

⇒ Now putting this in the equation

$x.\left(-\frac{3}{4}x + c\right) = 27 \Rightarrow -\frac{3}{4}x^2 + cx = 27$

$\Rightarrow \frac{3}{4}x^2 - cx + 27 = 0$

has only one solution ⇒ D = 0

$\Rightarrow b^2 - 4ac = 0$

$c^2 - 4 \times \frac{3}{4} \times 27 = 0$

$\Rightarrow c = \pm 3 \times 3 = \pm 9$

$y = -\frac{3}{4}x + 9$ or $y = -\frac{3}{4}x - 9$

equation of asymptotes $\Rightarrow y = \pm\frac{b}{a}x$

Sol 7: Equation (3x – 4y + 7)(4x + 3y + 1) + c = 0

$\Rightarrow 12x^2 - 12y^2 - 7xy + 31x + 17y + (7 + c) = 0$

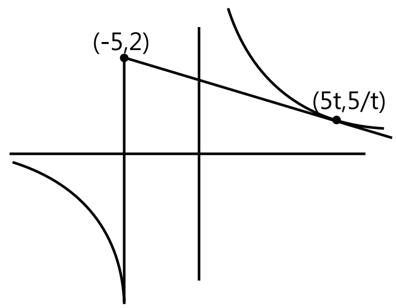
Now, it passes through origin

$\Rightarrow 7 + c = 0 \Rightarrow c = -7$

$\Rightarrow$ equation = $12x^2 - 12y^2 - 7xy + 31x + 17y = 0$

Sol 8: xy = 25 ⇒ parametric ⇒ 5t & $y = \frac{5}{t}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(-5)/t^2}{5} = \frac{-1}{t^2}$



Now slope = $\frac{-1}{t^2} = \frac{(5/t - 2)}{(5t + 5)}$

$\Rightarrow \frac{-1}{t} = \frac{(5-2t)}{5t(t+1)}$

$\Rightarrow -5(t + 1) = t(5 - 2t)$

$\Rightarrow 2t^2 - 10t - 5 = 0$

Now chord of contact

$\Rightarrow y = \frac{(5/t_1 - 5/t_2)x}{(5t_1 - 5t_2)} + c = \frac{-x}{t_1.t_2} + c$

Now, $\frac{5}{t_1} = \frac{-5t_1}{t_1.t_2} + c$

$\Rightarrow c = 5\left[\frac{1}{t_1} + \frac{1}{t_2}\right] \Rightarrow y = \frac{-x}{t_1 + t_2} + 5\left[\frac{t_1 + t_2}{t_1.t_2}\right]$

$\Rightarrow y = \frac{+x}{(+5/2)} + \frac{5.[5]}{-5/2}$

$\Rightarrow y = \frac{2x}{5} - 10 \Rightarrow 5y = 2x - 50$

Sol 9: $4.\sec^2\theta + 4\tan^2\theta = 12$

$\Rightarrow \sec^2\theta + \tan^2\theta = 3$

$\Rightarrow 2\tan^2\theta = 2$

$\Rightarrow \tan\theta = \pm 1$

$\Rightarrow \theta = \tan^{-1}(-1)$ [from 4th quadrant]

$\Rightarrow \theta = \dfrac{7\pi}{4}$

Sol 10: $\dfrac{x^2}{4} - \dfrac{y^2}{16} = 1$

asymptotes $\Rightarrow \dfrac{x}{2} - \dfrac{y}{4} = 0$ and $\dfrac{x}{2} + \dfrac{y}{4} = 0$

$\Rightarrow y = 2x$ and $y = -2x$

Now angle $\Rightarrow \tan^{-1}\dfrac{(m_1 - m_2)}{1 + m_1.m_2}$

$= \tan^{-1}\dfrac{[2-(-2)]}{1-4} = \tan^{-1}\dfrac{4}{3}$

Sol 11: Equation of hyperbola

$\Rightarrow ax\cos q_1 + by\cot q_1 = a^2 + b^2$

$[a\cos\theta, b\cot q]$

Equation of tangent

$\Rightarrow \dfrac{x}{a}\sec\theta_2 - \dfrac{y}{b}\tan\theta_2 = 1$

$[a\sec q_2, b\tan q_2]$

Intersection of tangents

$\Rightarrow (a\cos q_2, 0)$ and $(0, -b\cot q_2)$

Intersection of normal

$$\Rightarrow \left(\frac{\sec\theta_1(a^2+b^2)}{a}, 0\right) \text{ and } \left(0, \frac{(a^2+b^2)}{a}.\tan\theta_1\right)$$

Now, $a_1.a_2 + b_1.b_2 = \dfrac{a\cos\theta_2.\sec\theta_1.(a^2+b^2)}{a}$

$+ (-b).\cot q_2 \times \dfrac{(a^2+b^2)}{b}\tan q_1$

$= [\cos q_2.\sec q_1 - \cot q_2.\tan q_1](a^2 + b^2)$

$$= \left[\frac{\cos\theta_2}{\cos\theta_1} - \frac{\cos\theta_2.\sin\theta_1}{\cos\theta_1.\sin\theta_2}\right](a^2 + b^2]$$

Now if the point is same:

[i. e. , $q_1 = q_2$]

$(1 - 1)(a^2 + b^2) = 0$

Sol 12: The asymptotes are $x = 0, y = 0$

Let, $x = ct, y = \dfrac{c}{t}$,

tangent slope $= -\dfrac{1}{t^2}$

Now normal at x_1, y_1

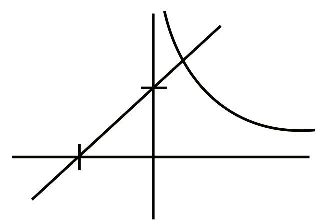
$\Rightarrow$ b has slope $= t^2$

so, $\dfrac{(y - c/t)}{(x - ct)} = t^2$

$$\Rightarrow \left(\frac{y - y_1}{x - x_1}\right) = \left(\frac{x_1}{c}\right)^2$$

$$\Rightarrow y - y_1 = \left(\frac{x_1^2}{c^2}\right).(x - x_1)$$

$$y = \frac{x_1^2}{c^2}.x + \left(y_1 - \frac{x_1^3}{c^2}\right)$$



Now putting

$$x = 0 \Rightarrow \boxed{y = y_1 - \frac{x_1^3}{c^2}}$$

and putting $y = 0$,

$$\frac{x_1^2}{c^2}.x = \left(\frac{x_1^3}{c^2} - y_1\right)$$

$$x = \frac{(x_1^3 - y_1c^2)}{x_1^2}$$

$$\boxed{x = x_1 - \frac{y_1}{x_1^2}.c^2}$$

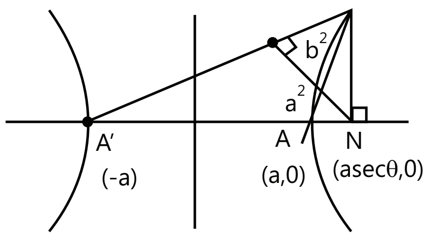
$$\text{Area} = \frac{1}{2}\left|\left(y_1 - \frac{x_1^3}{c^2}\right)\left(x_1 - \frac{y_1}{x_1^2}c^2\right)\right|$$

$$= \frac{1}{2}\left|\left[y_1.x_1 + x_1.y_1 - \frac{y_1^2.c^2}{x_1^2} - \frac{x_1^4}{c^2}\right]\right|$$

$$= \frac{1}{2}\left|\left[\frac{c^2x^2x_1.y_1}{c^2} - \frac{x_1^4}{c^2} - \frac{y_1^2.c^2}{(c^2/y_1)^2}\right]\right|$$

$$= \frac{1}{2}\left|\frac{x_1^4}{c^2} + \frac{y_1^4}{c^2} - 2(x_1.y_1)^2\right| = \frac{1}{2}\left|\frac{x_1^2 - y_1^2}{c}\right|^2$$

Sol 13:



$$Q = \left(\frac{ab^2 + a^3\sec\theta}{a^2+b^2}, a^2b\tan\theta\right)$$

slope of NQ

$$= \frac{a^2b\tan\theta}{\frac{ab^2+a^3\sec\theta}{a^2+b^2} - a\sec\theta} = \frac{a^2b\tan\theta - 0}{ab^2 - ab^2\sec\theta}$$

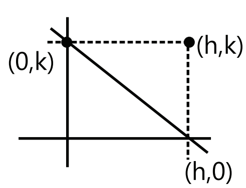
$$= \frac{a\tan\theta}{b - b\sec\theta} = \frac{a\tan\theta}{b(1-\sec\theta)}$$

slope of A$\Rightarrow$ P = $\frac{(b\tan\theta - 0)}{a(\sec\theta + 1)}$

$$\Rightarrow m_1.\, m_2 = \frac{b\tan\theta.a\tan\theta}{ab(1-\sec^2\theta)} = \frac{\tan^2\theta}{-\tan^2\theta} = -1$$

$\Rightarrow$ Hence proved.

Sol 14:



$ax\cos\theta + by\cot\theta = a^2 + b^2$

Which should be same as $\frac{x}{h} + \frac{y}{k} = 1$

$$\Rightarrow \frac{a\cos\theta}{1/h} = \frac{b\cot\theta}{1/k} = \frac{a^2+b^2}{1}$$

$$\Rightarrow h = \frac{a^2+b^2}{a\cos\theta},\ k = \frac{a^2+b^2}{b\cot\theta}$$

$$\Rightarrow \frac{ah}{a^2+b^2} = \sec\theta,\ \frac{bk}{a^2+b^2} = \tan\theta$$

$\Rightarrow \sec^2\theta - \tan^2\theta = 1$

$$\Rightarrow \frac{a^2h^2}{(a^2+b^2)^2} - \frac{b^2k^2}{(a^2+b^2)^2} = 1$$

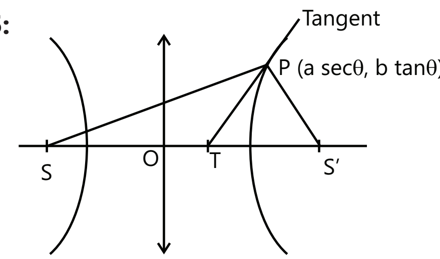
$$\Rightarrow \frac{a^2x^2}{(a^2+b^2)^2} - \frac{b^2y^2}{(a^2+b^2)^2} = 1$$

$\Rightarrow a^2x^2 - b^2y^2 = (a^2 + b^2)^2$

here, $a^2 = 4$, $b^2 = 1$

$\Rightarrow 4x^2 - y^2 = 25$

Sol 15:



Coordinates of $S' = (ae, 0)$

$S = (-ae, 0)$

$P = (a\sec\theta, b\tan\theta)$

Tangent at P cut the x axis at point T.

Eq. of target at P $= \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

Coordinates of $T = \left(\frac{a}{\sec\theta}, \frac{-b}{\tan\theta}\right)$

$T = (a\text{Cos}\theta, -b\cot\theta)$

$\Rightarrow ST = ae + a\cos\theta$

$\Rightarrow S'T = ae - a\cos\theta$

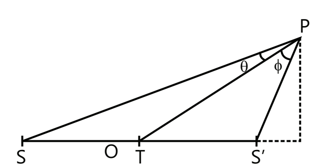
$$\Rightarrow \frac{ST}{ST} = \frac{ae + a\cos\theta}{ae - a\cos\theta} = \frac{e + \cos\theta}{e - \cos\theta}$$

Similarly on evaluating PS & Ps

$$\Rightarrow \frac{PS}{PS'} = \frac{PS}{PS'}$$

$\therefore$ Area of $\Delta PTS' = \frac{S'T \times -h}{2}$

Area of $\Delta PTS = \frac{S'T \times h}{2}$



Using fine rule:

Area of $\Delta PTS' = \frac{PS' \times PT\sin\phi}{2}$

Area of $\Delta PTS' = \frac{PS' \times PT\sin\theta}{2}$

$$\frac{\text{Area of } \Delta PTS'}{\text{Area of } \Delta PTS'} = \frac{PS\sin\theta}{PS'\text{Sin}\phi} = \frac{ST}{S'T}$$

For θ = d the conditions necessary are met & PT bisect the angle sps'

Sol 16: Let $x = ct$ and $y = \frac{c}{t}$

then, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1/t}{t} = \frac{-1}{t^2}$

so normal = t^2

thus, $\frac{(y - c/t)}{(x - ct)} = t^2 \Rightarrow y - \frac{c}{t} = t^2x - ct^3$

$\Rightarrow ty - c = t^3x - ct^4 \Rightarrow ct^4 - t^3x + ty - c = 0$

this satisfies h, k

thus, $ct^4 - ht^3 + kt - c = 0$

thus, $\sum_{i=1}^{h} t_i = \frac{h}{c} \Rightarrow \sum_{i=1}^{h} c.t_i = h$

$\Rightarrow x_1 + x_2 + x_3 + x_4 = h$

similarly we have $t_1.\ t_2.\ t_3.\ t_4 = -1$ and

$(t_1.\ t_2.\ t_3) + t_2.\ t_3.\ t_4 + t_3.\ t_4.\ t_1 + t_4.\ t_1.\ t_2 = \frac{-k}{e}$

dividing by $\prod_{i=1}^{h} t_i$ both sides

$\Rightarrow \frac{c}{t_1} + \frac{c}{t_2} + \frac{c}{t_3} + \frac{c}{t_4} = k \Rightarrow \boxed{\Sigma y_i = k}$

(iii) = –1

$\Rightarrow -c^4 = c^4.$

$-c^4 = x_1.\ x_2.\ x_3.\ x_4$

And = –1

$\Rightarrow -c^4 = \frac{c}{t_1}.\frac{c}{t_2}.\frac{c}{t_3}.\frac{c}{t_4} \qquad \Rightarrow y_1.\ y_2.\ y_3.\ y_4 = -c_1$

Sol 17: tangent $\Rightarrow \frac{a.\sec\theta.x}{a^2} - \frac{b.\tan.y}{b^2} = 1$

$$\boxed{\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1}$$

tangent $\Rightarrow y = mx \pm \sqrt{a^2m^2 - b^2}$

tangent passes through h, k

$k = mh + \sqrt{a^2m^2 - b^2}$

$(k - mh)^2 = a^2m^2 - b^2$

$k^2 + m^2h^2 - 2m.\ kh = a^2m^2 - b^2$

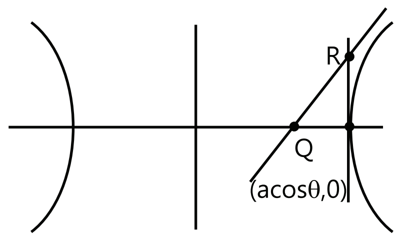
$(a^2 - h^2)m^2 + 2m.\ kh - (b^2 + k^2)$

Now $m_1 + m_2 = \lambda = \frac{-2kh}{a^2 - h^2}$

$\Rightarrow a^2 - h^2 = \frac{-2kh}{\lambda} \Rightarrow \boxed{a^2 - x^2 = \frac{-2xy}{\lambda}}$

Sol 18: $x = a\sec\theta$, $b = \tan\theta$

$\frac{x.\sec\theta}{a} - \frac{y}{b}.\tan\theta = 1$



Now coordinate of Q $\Rightarrow x = a\cos\theta$.

Coordinates of R

$\Rightarrow \boxed{\frac{(\sec\theta - 1)b}{\tan\theta} = y}$

Now

$2h = a + a\cos\theta = a(1 + \cos\theta)$. . (i)

and $2k = 0 + \frac{(\sec\theta - 1)b}{\tan\theta}$

$2k = \frac{(1 - \cos\theta)b}{\sin\theta}$

Now

$4k^2 + b^2 = \frac{(1 - \cos\theta)^2b^2 + b^2\sin^2\theta}{\sin^2\theta}$

$(4k^2 + b^2) = \frac{(2 - 2\cos\theta)^2b^2}{\sin^2\theta}$

$= \frac{2(1 - \cos\theta)b^2}{\sin^2\theta} = \frac{2.(b^2)}{(1 + \cos\theta)}$ from (i)

$\Rightarrow 4k^2 + b^2 = \frac{2b^2}{(2h/a)}$

$\Rightarrow ab^2 = h(4k^2 + b^2)$

Sol 19: Let $x = 2at$

$y = at^2$

then $\frac{dy}{dx} = \frac{2at}{2a} = t$

thus equation of tangent

$y - at^2 = t.(x - 2at)$

$y - at^2 = xt - 2at^2$

$\Rightarrow at^2 - xt + y = 0$

Now $x = \frac{k^2}{y}$ in the above eq^n

$\Rightarrow at^2 - \left(\frac{k^2}{y}\right)t + y = 0$

$\Rightarrow y^2 + yat^2 - k^2t = 0$

Now the$2k = -at^2$

and similarly, $y = \frac{k^2}{x}$ gives

$xat^2 - x^2t + \frac{k^2}{x} = 0$

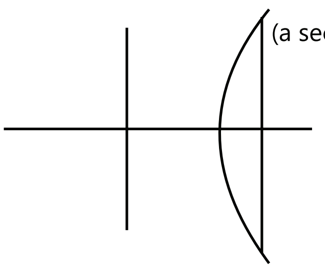
$x^2t - at^2x - k^2 = 0$

so $x_1 + x_2 = \frac{at^2}{t} = at \Rightarrow 2h = at$

so $\frac{4h^2}{a} = at^2 = -2k \Rightarrow \boxed{h^2 = \frac{-ak}{2}}$

Thus, it is a parabola.

Sol 20:



Now $\frac{dy}{dx} = \frac{a.\sec^2\theta}{a\tan\theta.\sec\theta}\ \frac{1}{\sin\theta}$

$\Rightarrow$ slope of normal $= -\sin\theta$

Now $hx - ky = -h^2 - k^2$ (chord of equation)

slope $= \frac{-h}{k} = \sin\theta$

$\cos\theta = \frac{\sqrt{k^2 - h^2}}{k}$ and $\tan\theta = \frac{-h}{\sqrt{k^2 - h^2}}$

so points $A\left(\frac{a.k}{\sqrt{k^2 - h^2}}, \frac{-ah}{\sqrt{k^2 - h^2}}\right)$

this satisfies the line

$\Rightarrow \frac{h.ak}{\sqrt{k^2 - h^2}} + \frac{ahk}{\sqrt{k^2 - h^2}} = (h^2 - k^2)$

$\Rightarrow 2ahk = (h^2 - k^2)\ (\sqrt{k^2 - h^2})$

$\Rightarrow 4a^2h^2k^2 = (k^2 - h^2)^3$

Sol 21: $(3\sec\theta, 2\tan\theta)$ = point on hyperbola

Now equation of the chord of contact is $hx + ky = h^2 + k^2$

and also$3\sec\theta\ x + 2\tan\theta.\ y = 9$

so $\frac{h}{3\sec\theta} = \frac{h^2 + k^2}{9} = \frac{k}{2\tan\theta}$

$\Rightarrow \sec\theta = \frac{3h}{h^2 + k^2}$ and $\tan\theta = \frac{9k}{2(h^2 + k^2)}$

$\Rightarrow \sec^2\theta - \tan^2\theta = 1$

$\Rightarrow \frac{9h^2}{(h^2 + k^2)^2} - \frac{81k^2}{4(h^2 + k^2)^2} = 1$

$\boxed{9h^2 - \frac{81k^2}{4} = (h^2 + k^2)^2}$

Sol 22: Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola then its conjugate hyperbola is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

If e_1 and e_2 are their eccentricities, then

$b^2 = a^2\left(e_1^2 - 1\right)$ and $a^2 = b^2\left(e_2^2 - 1\right)$

So $\frac{1}{e_1^2} = \frac{a^2}{\left(a^2 + b^2\right)}$ and $\frac{1}{e_2^2} = \frac{b^2}{\left(a^2 + b^2\right)}$

So $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1, \Rightarrow \frac{1}{e_1^2} = 1 - \frac{1}{e_2^2}$

Sol 23: Let the hyperbola be $xy = c^2$.

Let the x_1, y_1 be point where the other 3 normals intersect.

Now, equation of normal

$\Rightarrow \left(y - \frac{c}{t}\right) = t^2(x - ct)$

$\Rightarrow ty - c = t^3x - ct^4$

$\Rightarrow ct^4 - t^3x + ty - c = 0$

Thus, passes through

(x_1, y_1) or $(cx_1, c/t_1)$

So $ct^4 - t^3 . x_1 + ty_1 - c = 0$

Now $\Sigma t_i = \frac{x_1}{c}$ & product of roots $t_i = -1$

$\Rightarrow \Sigma x_i = x_1$ & $\Sigma t_1 . t_2 . t_3 = \frac{-y_1}{c}$

$\Rightarrow x_2 + x_3 + x_4 = 0 \quad \frac{\Sigma 1}{t_i} = \frac{y_1}{c}$

$\Downarrow$

$x_c = 0 \Rightarrow y_1+y_2+y_3+y_4=y_1$

$y_2+y_3+y_4 = 0$

$\Downarrow$

$y_c = 0$

Thus, the centroid of PQR is (0, 0)

Sol 24: Equations of normal at the points

on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$ are

$ax\cos\phi + by\cot\phi = a^2 + b^2$ ... (i)

and $ax\cos\phi + by\cot\phi = a^2 + b^2$

i.e. $ax\sin\theta + by\tan\theta = a^2 + b^2$ ... (ii)

$\because \quad \theta + \phi = \frac{\pi}{2}$

Solving (i) and (ii), $y = k = -\frac{(a^2 + b^2)}{b}$.

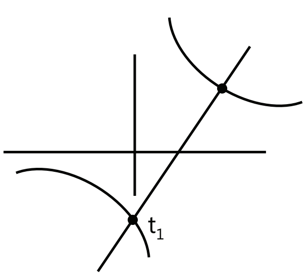
Sol 25: Tangent to the hyperbola $xy = c^2$ at $(ct, c/t)$ will be of the form $y = -\frac{1}{t^2}x + \frac{2c}{t}$

$y = mx + 2c\sqrt{-m} \Rightarrow t = \frac{1}{\sqrt{-m}}$

$\therefore$ Point is $\left(\frac{c}{\sqrt{-m}}, c\sqrt{-m}\right)$

Sol 26: $x = ct$ and $y = \frac{c}{t}$

Now $\frac{dy}{dx} = \frac{-1}{t^2}$



$\Rightarrow$ normal $= t^2$

Now slope $= t^2$

$= \frac{\frac{c}{t} - \frac{c}{t_1}}{c(t - t_1)} = \frac{-1}{t.t_1}$

$\Rightarrow t_1 . t^3 = -1$

Exercise 2

Single Correct Choice Type

Sol 1: (B) We have $x. x_1 - 9y. y_1 = 9$

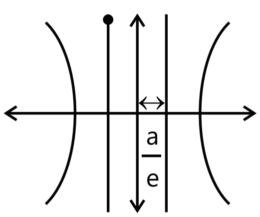
so $\frac{5}{x_1} = \frac{12}{-9y_1} = \frac{9}{9}$

$\Rightarrow y_1 = \frac{-4}{3}$ and $x_1 = 5$

Sol 2: (A) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

latus rectum $= \frac{2a^2}{b}$

Sol 3: (B)



area $= \frac{2a}{e} \times \frac{2a}{e} = \frac{4a^2}{e^2}$

for rectangular hyperbola $e = \sqrt{2}$

area $= 2a^2$

Sol 4: (A) $\frac{x^2}{9} - \frac{y^2}{4} = 1$

$y = \frac{2x}{5} + C$

$4\left(\frac{x^2 - 9}{9}\right) = \frac{4x^2}{25} + C^2 + \frac{4Cx}{5}$

$4x^2 \times \frac{16}{225} - \frac{4Cx}{5} - 4 - C^2 = 0$

$64x^2 - 180Cx - 180 - 45C^2 = 0$

$D = 0$

$180C^2 = 4 \times 64(-180 - 45C^2)$

$\Rightarrow C^2 = 64(-4 - C^2)$

$\Rightarrow C^2 < 0$ no possible tangent

Sol 5: (C) $\frac{x}{3}-\frac{y}{2} = m$

$\frac{x}{3}+\frac{y}{2} = \frac{1}{m}$

$\Rightarrow y^2 = \frac{1}{m^2} + m^2 - 2$

$\Rightarrow x = \frac{3}{2}\left(m+\frac{1}{m}\right)$

$\Rightarrow \frac{4x^2}{9} = m^2 + \frac{1}{m^2} + 2$

$\Rightarrow \frac{4x^2}{9} - y^2 = 4$

$\Rightarrow \frac{x^2}{9}-\frac{y^2}{4} = 1$

Sol 6: (B) $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ $\frac{9x^2}{a^2}-\frac{4}{b^2} = 1$

$\Rightarrow a^2 = 3^2 \Rightarrow \frac{4}{b^2} = 1$

$\Rightarrow a = 3 \Rightarrow b^2 = 4$

$\Rightarrow \boxed{\frac{x^2}{9}-\frac{y^2}{4}=1}$

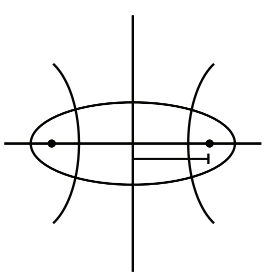
$\Rightarrow e = \sqrt{1+\frac{b^2}{a^2}} = 1 + \frac{4}{9} = \frac{\sqrt{13}}{3}$

Sol 7: (B) $(x - 2)^2 - (y - 2)^2 + 16 = 0$

$\Rightarrow \frac{(y-2)^2}{16}-\frac{(x-2)^2}{16} = 1$

$\Rightarrow e = \sqrt{1+\frac{16}{16}} = \sqrt{2}$

Sol 8: (B)



Ellipse Hyperbola

$\Rightarrow \frac{x^2}{a_1^2}+\frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{a_2^2}-\frac{y^2}{b^2} = 1$

Now $a_1e_1 = a_2e_2$

also $e_1^2 = 1 - \frac{b^2}{a_1^2}$ $b^2 = (e_2^2 - 1)a_2^2$

$\frac{1}{e_1^2}+\frac{1}{e_2^2} = \frac{1}{1-\frac{b^2}{a_1^2}}+\frac{1}{1+\frac{b^2}{a_2^2}}$

$= \frac{a_1^2}{a_1^2-b^2}+\frac{a_2^2}{a_2^2+b^2}$ ... (i)

Also we have,

$a_1e_1 = a_2e_2$

$\Rightarrow a_1^2 - b^2 = a_2^2 + b^2$ ... (ii)

$\Rightarrow a_1^2 + a_2^2 = 2(a_2^2 + b^2)$ ... (iii)

Now from (i)

$\Rightarrow \frac{1}{e_1^2}+\frac{1}{e_2^2} = \frac{a_1^2+a_2^2}{a_2^2+b^2} = \frac{2(a_2^2+b^2)}{a_2^2+b^2} = 2$

Sol 9: (C)

We have $e_1 = \sqrt{\frac{a^2+b^2}{a^2}}$ and $e_2 = \sqrt{\frac{b^2+a^2}{b^2}}$

$\Rightarrow \frac{1}{e_1^2}+\frac{1}{e_2^2} = 1$

Sol 10: (B) Equation of normal at any point x′, y′ of the curve $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ ... (i)

$\frac{a^2(x-x')}{x'}+\frac{b^2(y-y')}{y'} = 0$

$\Rightarrow \frac{a^2x}{x'}-a^2-b^2+\frac{b^2y}{y'} = 0$

$\frac{a^2x}{x'}+\frac{b^2y}{y'} = a^2 + b^2$ (h, k) satisfy this)

$\Rightarrow \frac{a^2h}{x_1}+\frac{b^2k}{y_1} = a^2 + b^2$

$\Rightarrow a^2h.\, y_1 + b^2k_1x_1 = (a^2 + b^2)(x_1, y_1)$ ... (iii)

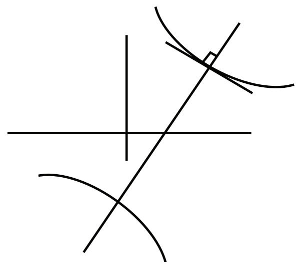
thus, (x_1, y_1) lies on curve (iii) and curve (i) these two points intersect at 4 points.

Sol 11: (A) A rectangular hyperbola circumscribing a triangle ABC always passes through the or the centre.

Sol 12: (A) We have $\frac{dy}{dx} = \frac{(c/t^2)}{c} = \frac{-1}{t^2}$

so normal slope = t^2

Now,



We have $t^2 = \frac{c/t - c/t'}{ct - ct'}$

$\Rightarrow t^2 = \frac{(t'-t)(-1)}{t.t'.(t-t')} \quad \Rightarrow t' = \frac{-1}{t^3}$

Sol 13: (A) $9(x^2 - 4x + 16) - 16(y^2 - 6y + 9)$

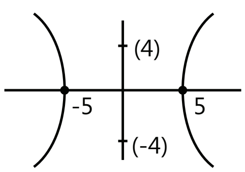
$-252 + 144 - 144 = 0$

$\Rightarrow 9(x-2)^2 - 16(y-3)^2 = 252$

$\Rightarrow$ Centre $\Rightarrow$ (2, 3)

Sol 14: (D) $\frac{x^2}{25} - \frac{y^2}{16} = 1$

tangents $\Rightarrow$



$y = mx + \sqrt{25m^2 - 16}$

$\Rightarrow (y - mx)^2 = 25m^2 - 16$

$\Rightarrow$ the point $(1, 2\sqrt{2})$ satisfy this

$\Rightarrow (1 - 2\sqrt{2}\,m)^2 = 25m^2 - 16$

$\Rightarrow 1 + 8m^2 - 4\sqrt{2}\,m = 25m^2 - 16$

$\Rightarrow 17m^2 + 4\sqrt{2}\,m - 17 = 0$

$\Rightarrow m_1 . m_2 = -1$

Sol 15: (D) $y = mx + \sqrt{a^2m^2 - b^2}$

$\Rightarrow y = mx + \sqrt{\cos^2\alpha . m^2 - \sin^2\alpha}$

$(k - mh)^2 = \cos^2\alpha . m^2 - \sin^2\alpha$

$\Rightarrow k^2 + m^2h^2 - 2mkh$

$= \cos^2\alpha . m^2 - \sin^2\alpha$

$\Rightarrow m^2(h^2 - \cos^2\alpha) - 2kh.\, m + (k^2 + \sin^2\alpha)$

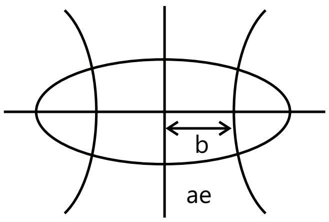
Now we have $m_1 . m_2 = -1$

$\frac{h^2 - \cos^2\alpha}{k^2 + \sin^2\alpha} = -1$

$\Rightarrow h^2 + k^2 = \cos^2\alpha - \sin^2\alpha$

$h^2 + k^2 = \cos 2\alpha$

Sol 16: (C) We have b = ae



$b = a\sqrt{1 - \frac{b^2}{a^2}}$

$\Rightarrow \boxed{\frac{b^2}{a^2} = \frac{1}{2}}$

$e_{hyperbola} = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1+2} = \sqrt{3}$

Sol 17: (C) Let any tangent of (x_1, y_1)

then $\frac{x.x_1}{a^2} + \frac{y.y_1}{b^2} = 1$ [1st tangent]

and $x.\, x_1 - y.\, y_1 = c^2$ [2nd tangent]

Now, $m_1 . m_2 = -1$

$\Rightarrow \left(\frac{-b^2}{a^2}\right)\left(\frac{x_1}{y_1}\right)\left(\frac{x_1}{y_1}\right) = -1$

$\Rightarrow \frac{-b^2}{a^2} . \left(\frac{x_1^2}{y_1^2}\right) = -1 \Rightarrow +b^2\left(\frac{x_1^2}{y_1^2}\right) = +a^2$ ... (i)

Now $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$ & $x_1^2 - y_1^2 = c^2$

$\Rightarrow \frac{y_1^2 + c^2}{a^2} + \frac{y_1^2}{b^2} = 1 \Rightarrow y_1^2\left[\frac{1}{a^2} + \frac{1}{b^2}\right] = \left[1 - \frac{c^2}{a^2}\right]$

$y_1^2 = \frac{[a^2 - c^2]}{a^2\left[\frac{1}{a^2} + \frac{1}{b^2}\right]} = \frac{b^2[a^2 - c^2]}{a^2 + b^2}$

And $x_1^2 = c^2 + \frac{[a^2 - c^2]b^2}{\frac{a^2[b^2 + a^2]}{a^2b^2}}$

$= \frac{b^2c^2 + a^2c^2 + a^2b^2 - c^2b^2}{b^2 + a^2} = \frac{a^2(b^2 + c^2)}{(a^2 + b^2)}$

so $b^2 . \frac{a^2(b^2+c^2)}{(a^2+b^2)} \times \frac{1}{\frac{b^2(a^2-c^2)}{a^2+b^2}} = a^2$

$\Rightarrow a^2 - b^2 = 2c^2$

Sol 18: (B) $\frac{y^2}{b^2} + \frac{y}{ba^2} + 1 = 0$

$a^2 - 4ac \Rightarrow \left(\frac{1}{ba^2}\right)^2 - \frac{4\times 1}{b^2} \geq 0$

$\Rightarrow \frac{1}{b^2a^4} - \frac{4}{b^2} \geq 0 [b^2 > 0] \Rightarrow \frac{1}{a^4} - \frac{4}{1} \geq 0$

$\Rightarrow \frac{1}{a^4} \geq 4 \Rightarrow \frac{1}{a^2} \geq 2 \Rightarrow \frac{1}{2} \geq a^2$

Sol 19: (C) $\frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$

$\frac{h\alpha}{a^2} - \frac{k\beta}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$

$\Rightarrow \left(\frac{h}{a} - \frac{\alpha}{2a}\right)^2 - \left(\frac{k}{b} - \frac{\beta}{2b}\right)^2 = \frac{\alpha^2}{4a^2} - \frac{\beta^2}{4b^2}$

$\Rightarrow$ Centre: $\frac{h}{a} = \frac{\alpha}{2a}$ and $k = \frac{\beta}{2}$

Sol 20: (B) $\frac{x^2}{5} - \frac{y^2}{5\cos^2\alpha} = 1$

so $e_1 = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \cos^2\alpha}$

$\frac{x^2}{25\cos^2\alpha} + \frac{y^2}{25} = 1$

$e_2 = \sqrt{1 - \cos^2\alpha}$

$1 + \cos^2\alpha = b. (1 - \cos^2\alpha)$

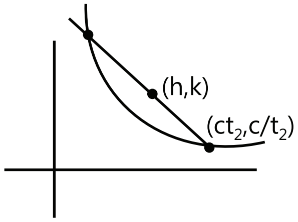
$\Rightarrow \cos^2\alpha = \frac{1}{2} \Rightarrow \cos\alpha = \frac{1}{\sqrt{2}}$

Sol 21: (D) $a^2 = 9$ and $b^2 = 4$

$\frac{x^2}{9} - \frac{y^2}{4} = 1$

$4x^2 - 9y^2 = 36$

Sol 22: (A) We have equation



Now, $2h = c(t_1 + t_2)$

and $2k = \frac{c}{t_1} + \frac{c}{t_2} = \frac{c.(t_1+t_2)}{t_1.t_2}$

$m = \frac{c/t_2 - c/t_1}{ct_2 - ct_1} = \frac{-1}{t_1t_2}$

$m = \frac{-c.(t_1+t_2)}{t_1.t_2c(t_1+t_2)}$

$m = \frac{-2k}{2h}$

$\Rightarrow k + mh = 0 \Rightarrow y + mx = 0$

Sol 23: (A) Let (h, k) be the midpoints of chords having slope 2

$\Rightarrow \tan\theta = 2 \quad \Rightarrow \sin\theta = \frac{2}{\sqrt{5}}$ and $\cos\theta = \frac{1}{\sqrt{5}}$

Let the two endpoints of the chord be a distance r from (h, k)

$\Rightarrow$ endpoints of the chord are

$(h + r\cos\theta, k + r\sin\theta)$ and $(h - r\cos\theta, k - r\sin\theta)$

$= \left(h + \frac{r}{\sqrt{5}}, k + \frac{2r}{\sqrt{5}}\right)$ and $= \left(h - \frac{r}{\sqrt{5}}, k - \frac{2r}{\sqrt{5}}\right)$

Plugging in the equation of the hyperbola

$3\left(h + \frac{r}{\sqrt{5}}\right)^2 - 2\left(k + \frac{2r}{\sqrt{5}}\right)^2 + 4\left(h + \frac{r}{\sqrt{5}}\right) - 6\left(k + \frac{2r}{\sqrt{5}}\right) = 0$ ...(i)

and

$3\left(h - \frac{r}{\sqrt{5}}\right)^2 - 2\left(k - \frac{2r}{\sqrt{5}}\right)^2 + 4\left(h - \frac{r}{\sqrt{5}}\right) - 6\left(k - \frac{2r}{\sqrt{5}}\right) = 0$ ...(ii)

Subtracting eqn. (ii) from (i),

$\frac{12\,hr}{\sqrt{5}} - \frac{8kr}{\sqrt{5}} + \frac{8r}{\sqrt{5}} - \frac{24r}{\sqrt{5}} = 0$

$\Rightarrow 3h - 2k - 4 = 0$

$\Rightarrow$ required locus is

$3x - 4y = 4.$

Previous Years' Questions

Sol 1: (B) Given equation is

$\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1$, where $|r| < 1$

$\Rightarrow$ $1 - r$ is (+ve) and $1 + r$ is (+ve)

$\therefore$ Given equation is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Hence, it represents a hyperbola when $|r| < 1$.

Sol 2: (D) Firstly we obtain the slope of normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec\theta, b\tan\theta)$

On differentiating w.r.t. x, we get

$\frac{2x}{a^2} - \frac{2y}{b^2} \times \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y}$

Slope, for normal at the point $(a\sec\theta, b\tan\theta)$ will be

$-\frac{a^2 b\tan\theta}{b^2 a\sec\theta} = -\frac{a}{b}\sin\theta$

$\therefore$ Equation of normal $(a\sec\theta, b\tan\theta)$ is

$y - b\tan\theta = -\frac{a}{b}\sin\theta(x - a\sec\theta)$

$\Rightarrow$ $(a\sin\theta)x + by = (a^2 + b^2)\tan\theta$

$\Rightarrow$ $ax + b\operatorname{cosec}\theta = (a^2 + b^2)\sec\theta$ ...(i)

Similarly, equation of normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec\theta, b\tan\theta)$ is

$ax + b\operatorname{cosec}\phi = (a^2 + b^2)\sec\theta$ ...(ii)

On subtracting eqs.(ii) from (i), we get

$b(\operatorname{cosec}\theta - \operatorname{cosec}\phi)y$

$= (a^2 + b^2)(\sec\theta - \sec\phi)$

$\Rightarrow y = \frac{a^2+b^2}{b}\frac{\sec\theta - \sec\varphi}{\operatorname{cosec}\theta - \operatorname{cosec}\varphi}$

But $\frac{\sec\theta - \sec\varphi}{\operatorname{cosec}\theta - \operatorname{cosec}\varphi}$

$= \frac{\sec\theta - \sec(\pi/2-\theta)}{\operatorname{cosec}\theta - \operatorname{cosec}(\pi/2-\theta)}$

$(\because \phi + \theta = \pi/2)$

$= \frac{\sec\theta - \operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sec\theta} = -1$

Thus, $y = -\frac{a^2+b^2}{b}$ i.e., $k = -\left(\frac{a^2+b^2}{b}\right)$

Sol 3: (B) Let (h, k) be point whose chord of contact with respect to hyperbola $x^2 - y^2 = 9$ is $x = 9$.

We know that, chord of contact of (h, k) with respect to hyperbola $x^2 - y^2 = 9$ is $T = 0$

$\Rightarrow$ $h.x + k(-y) - 9 = 0$

$\therefore hx - ky - 9 = 0$

But it is the equation of the line $x = 9$.

This is possible when $h = 1, k = 0$ (by comparing both equations).

Again equation of pair of tangents is $T^2 = SS_1$.

$\Rightarrow$ $(x - 9)^2 = (x^2 - y^2 - 9)(t^2 - 0^2 - 9)$

$\Rightarrow$ $x^2 - 18x + 81 = (x^2 - y^2 - 9)(-8)$

$\Rightarrow$ $x^2 - 18x + 81 = -8x^2 + 8y^2 + 72$

$\Rightarrow$ $9x^2 - 8y^2 - 18x + 9 = 0$

Sol 4: (B) Given equation of hyperbola is

$\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$

Here, $a^2 = \cos^2\alpha$ and $b^2 = \sin^2\alpha$

[We, comparing with standard equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$]

We know, foci = $(\pm ae, 0)$

where $ae = \sqrt{a^2+b^2} = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1$

$\Rightarrow$ foci = $(\pm 1, 0)$

whereas vertices are $(\pm\cos\alpha, 0)$

eccentricity, $ae = 1$ or $e = \frac{1}{\cos\alpha}$

Hence, foci remain constant with change in 'α'.

Sol 5: (D) The equation of tangent at (x_1, y_1) is $xx_1 - 2yy_1 = 4$, which is same as $2x + \sqrt{6}\,y = 2$

$\therefore \frac{x_1}{2} = \frac{-2y_1}{\sqrt{6}} = \frac{4}{2}$

$\Rightarrow$ $x_1 = 4$ and $y_1 = -\sqrt{6}$

Thus, the point of contact is $(4, -\sqrt{6})$

Sol 6: (B) The eccentricity of $\frac{x^2}{16} + \frac{y^2}{25} = 1$ is

$e_1 = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$

$\therefore e_2 = \frac{5}{3} (\because e_1e_2 = 1)$

$\Rightarrow$ Foci of ellipse $(0, \pm 3)$

$\Rightarrow$ Equation of hyperbola is

$\frac{x^2}{16} - \frac{y^2}{9} = -1$

Sol 7: (A) The given ellipse is

$\frac{x^2}{4} + \frac{y^2}{3} = 1 \Rightarrow a = 2, b = \sqrt{3}$

$\therefore 3 = 4(1 - e^2) \Rightarrow e = \frac{1}{2}$

$\therefore ae = 2 \times \frac{1}{2} = 1$

Hence, the eccentricity e_1, of the hyperbola is given by

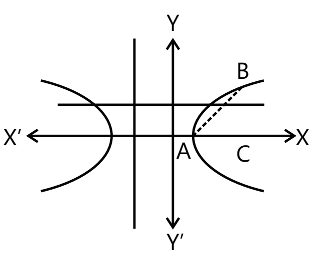
$1 = e_1 \sin\theta \Rightarrow e_1 = \text{cosec}\,\theta$

$\Rightarrow b^2 = \sin^2\theta(\text{cosec}^2\theta - 1) = \cos^2\theta$

Hence, equation of hyperbola is $\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1$

or $x^2\text{cosec}^2\theta - y^2\sec^2\theta = 1$

Sol 8: (B) Given equation can be rewritten as $\frac{(x-\sqrt{2})^2}{4} - \frac{(y+\sqrt{2})^2}{2} = 1$



For point A(x, y)

$e = \sqrt{1+\frac{2}{4}} = \sqrt{\frac{3}{2}}$

$x - \sqrt{2} = 2 \Rightarrow x = 2 + \sqrt{2}$

For point C(x, y)

$x - \sqrt{2} = ae = \sqrt{6}$

$x = \sqrt{6} + \sqrt{2}$

Now,

$AC = \sqrt{6} + \sqrt{2} - 2 - \sqrt{2} = \sqrt{6} - 2$

and$BC = \frac{b^2}{a} = \frac{2}{2} = 1$

Area of ΔABC

$= \frac{1}{2} \times (\sqrt{6} - 2) \times 1 = \sqrt{\frac{3}{2}} - 1$ sq. unit

Sol 9: (B) Equation of normal to hyperbola at (x_1, y_1) is

$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = (a^2 + b^2)$

$\therefore$ At $(6, 3) \Rightarrow \frac{a^2x}{6} + \frac{b^2y}{3} = (a^2 + b^2)$

It passes through (9, 0)

$\Rightarrow \frac{a^2.9}{6} = a^2 + b^2$

$\Rightarrow \frac{3a^2}{2} - a^2 = b^2 \Rightarrow \frac{a^2}{b^2} = 2$

$\therefore e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{1}{2}$

$\Rightarrow e = \sqrt{\frac{3}{2}}$

Sol 10: (B) $2b = \frac{1}{2}.(2ae) \Rightarrow b = \frac{ae}{2}$

$\Rightarrow a^2\left(e^2 - 1\right) = \frac{a^2e^2}{4} \Rightarrow 3e^2 = 4 \Rightarrow e = \frac{2}{\sqrt{3}}$

JEE Advanced/Boards

Exercise 1

Sol 1: $\frac{SP}{PM} = \sqrt{3}$

$\frac{\sqrt{(x-1)^2 + (y-1)^2}}{\frac{(2x+y-1)}{\sqrt{5}}} = \sqrt{3}$

Squaring

$5[(x - 1)^2 + (y - 1)^2] = 3(2x + y - 1)^2$

$\Rightarrow 7x^2 - 2y^2 + 12xy + 4y - 2x - 7 = 0$

Sol 2: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$7x + 13y = 87$

$5x - 8y = -7$

$\Rightarrow \frac{87-7x}{13} = \frac{5x+7}{8}$

$\Rightarrow 8.\ 87 - 7.\ 13 = 121\ x$

$\Rightarrow 121x = 605$

x = 5, y = 4

$\frac{2b^2}{a} = \frac{32\sqrt{2}}{5}$

$5b^2 = 16\sqrt{2}a$

$\frac{25}{a^2} - \frac{16}{b^2} = 1$

$\frac{25}{a^2} - \frac{16}{a\sqrt{2}} = 1$

$25\sqrt{2} - 5a = a^2\sqrt{2}$

$a^2\sqrt{2} + 5a - 25\sqrt{2} = 0$

$a = \frac{-5 \pm \sqrt{25+200}}{2\sqrt{5}} = \frac{5}{\sqrt{2}}$ or $\frac{-10}{\sqrt{2}}$

Now, $5b^2 = 16\sqrt{2}a$

$\Rightarrow a > 0 \Rightarrow a = \frac{5}{\sqrt{2}}$

Sol 3: $\frac{x^2}{100} - \frac{y^2}{25} = 1$

$e = \sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2}$

$S = (ae,0) = \left(\frac{\sqrt{5}}{2} \times 10, 0\right) = (5\sqrt{5}, 0)$

$S' = (-ae, 0) = (-5\sqrt{5}, 0)$

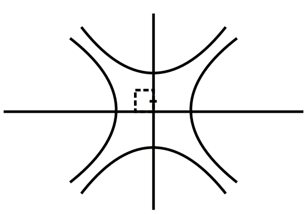
$A = (10, 0)$

$SA = (10 - 5\sqrt{5})$

$S'A = (10 + 5\sqrt{5})$

$SA. S'A = 100 - 75 = 25$

Sol 4:



$16x^2 - 9y^2 + 32x + 36y - 164 = 0$

$16x^2 + 32x - 9y^2 + 36y = 164$

$16(x + 1)^2 - 9(y - 2)^2$

$= 164 + 16 - 36 = 144$

$\frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$

Centre (–1, 2)

$e = \sqrt{1+\frac{16}{9}} = \frac{5}{3}$

foci = (–1 + ae, 2) = (4, 2)

= (–1 – ae, 2) = (–6, 2)

Directrix $x + 1 = \frac{9}{5} \Rightarrow x = \frac{4}{5}$

$x + 1 = \frac{-9}{5} \Rightarrow x = \frac{-14}{5}$

Latus rectum $= \frac{2b^2}{a} = \frac{2.16}{3} = \frac{32}{3}$

Length of major axis = 2×4 = 8

Length of minor axis = 2×3 = 6

Equation of axis is y = 2

Sol 5: $P_1(ct_1, c/t_1)$ $P_2(ct_2, c/t_2)$

$t_1 + t_2 = \frac{2h}{c}$

$\frac{1}{t_1} + \frac{1}{t_2} = \frac{2k}{c} \Rightarrow \frac{2h}{ct_1t_2} = \frac{2k}{c}$ $t_1t_2 = \frac{h}{k}$

$c^2(t_1 - t_2)^2 + c^2\left(\frac{1}{t_1} - \frac{1}{t_2}\right)^2 = 4d^2$

$(t_1 + t_2)^2 - 4t_1t_2 + \left(\frac{1}{t_1} + \frac{1}{t_2}\right)^2$

$-\frac{4}{t_1t_2} = \frac{4d^2}{c^2}$

$\left(\frac{2h}{c}\right)^2 + \left(\frac{2k}{c}\right)^2 - 4t_1t_2 - \frac{4}{t_1t_2} = \frac{4d^2}{c^2}$

$\frac{(2h)^2 + (2k)^2}{c^2} - 4\left(\frac{h}{k} + \frac{k}{h}\right) = \frac{4d^2}{c^2}$

$2\frac{(h^2+k^2)}{c^2} - 2\frac{(h^2+k^2)}{kh} = \frac{2d^2}{c^2}$

$(h^2 + k^2)hk - c^2(h^2 + k^2) = d^2kh$

$(h^2 + k^2)(hk - c^2) = d^2kh$

Hence proved.

Sol 6: $y - 2 = m(x - 6)$

$y = mx + 2 - 6m$

$\frac{x^2}{25} - 1 = \frac{(mx + 2 - 6m)^2}{16}$

$16(x^2 - 25) = 25(m^2x^2 + 4 + 36m^2 + 4mx - 24m - 12m^2x)$

$x^2(16 - 25m^2) + x(-100m + 300m^2) - 400 - 100 - 900 m^2 + 600 m = 0$

$(300m^2 - 100m)^2 = 4(16 - 25m^2)(-900m^2 + 600m - 500)$

$100(3m^2 - m)^2 = 4(16 - 25m^2)(-9m^2 + 6m - 5)$

$25(9m^4 + m^2 - 6m^3) = -144m^2 + 96m - 80 + 225m^4 - 150m^3 + 125m^2$

$25m^2 = -19m^2 + 96m - 80$

$44m^2 - 96m + 80 = 0$

$11m^2 - 24m + 20 = 0$

$m_1 + m_2 = \frac{24}{11}$

$m_1m_2 = \frac{20}{11}$

Sol 7: $y = -x + c$

$x^2 - 4(c - x)^2 = 36$

$x^2 - 4(c^2 + x^2 - 2cx) = 36$

$3x^2 - 8cx + 4c^2 + 36 = 0$

$\Rightarrow x + y = \pm 3\sqrt{3}$

$64c^2 = 12(4c^2 + 36)$

$16c^2 = 12(4c^2 + 36)$

$4c^2 = 3c^2 + 27$

$c^2 = 2 \Rightarrow c = \pm 3\sqrt{3}$

Sol 8: Equation of chord

$\frac{x}{a}\cos\left(\frac{\theta_1 - \theta_2}{2}\right) - \frac{y}{b}\left(\sin\left(\frac{\theta_1 + \theta_2}{2}\right)\right) = \cos\left(\frac{\theta_1 + \theta_2}{2}\right)$

If it pass through (ae, 0)

$e = \frac{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}$

$\frac{1}{e} = \frac{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}$

Using componendo rule we get

$\frac{1 - e}{1 + e} = \tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right)$

Sol 9: $e = (0, 0)$

$S(ae, 0)$

$S'(-ae, 0)$

$P = (a\sec\theta, b\tan\theta)$

SP. S$\Rightarrow$ P =

$\sqrt{((a\sec\theta - ae)^2 + b^2\tan^2\theta)((a\sec\theta + ae)^2 + b^2\tan^2\theta)}$

$= \sqrt{(a^2\sec^2\theta + a^2e^2 + b^2\tan^2\theta)^2 - (2a^2e\sec\theta)^2}$

$= a^2\sec^2\theta + b^2\tan^2\theta - (a^2 + b^2)$

$= CP^2 - (a^2 + b^2)$

Sol 10: $y - \frac{5}{2} = mx$

$y = mx + \frac{5}{2}$

$3x^2 - 25 = 2\left(m^2x^2 + \frac{25}{4} + 5mx\right)$

$x^2(3 - 2m)^2 - 10mx - \frac{75}{2} = 0$

$100m^2 = 4(3 - 2m^2)\left(-\frac{75}{2}\right)$

$50m^2 = 150m^2 - 225$

$100m^2 = 225$

$m^2 = \frac{9}{4}; \quad m = \pm\frac{3}{2}$

$2y = 3x + 5$ or $2y + 3x = 5$

Sol 11: $\frac{y - k}{x - h} = \frac{b^2h}{a^2k}$

$\Rightarrow x^2 + y^2 = a^2$

$\left[h + \frac{a^2k}{b^2h}(y - k)\right]^2 + y^2 = a^2$

$h^2 + \frac{a^4k^2}{b^4h^2}(y^2 + k^2 - 2ky) + \frac{2a^2k}{b^2}(y - k) + y^2 = a^2$

$$y^2\left[\frac{a^4k^2}{b^4h^2}+1\right] + y\left[\frac{-2k^3a^4}{b^4h^2}+\frac{2a^2k}{b^2}\right] + h^2 - a^2$$

$$+\frac{a^4k^4}{b^4h^2}-\frac{2a^2k^2}{b^2} = 0$$

$$\frac{y_1+y_2}{y_1y_2} = \frac{\frac{2a^2k}{b^2}-\frac{2k^3a^4}{h^2b^4}}{h^2-a^2+\frac{a^4k^4}{b^4h^2}-\frac{2a^2k^2}{b^2}}$$

$$= \frac{2a^2h^2kb^2-2k^3a^4}{h^4b^4-a^2h^2b^4+a^4k^4-2a^2k^2b^2h^2} = \frac{2a^2ka^2b^2}{k^2a^4b^2} = \frac{2}{k}$$

Sol 12: $\frac{x^2}{2}-\frac{y^2}{3} = 1;\quad y-\beta = m(x-\alpha)$

$$\frac{x^2-2}{\alpha} = \frac{1}{3}(mx - m\alpha + \beta)^2$$

$$3x^2-6 = 2(m^2x^2+m^2\alpha^2+\beta^2-2m^2\alpha x-2m\alpha+2mx\beta)$$

$$x^2(3-2m^2)+2x(2m^2\alpha-2m\beta)-6$$

$$-2m^2\alpha^2-2\beta^2+4m\alpha\beta = 0$$

$$(4m^2\alpha-4m\beta)^2 = 4(3-2m)^2(4m\alpha\beta-2m^2\alpha^2-2\beta^2-6)$$

$$2m^2(m\alpha-\beta)^2 = (3-2m^2)(2m\alpha\beta-m^2\alpha^2-\beta^2-3)$$

$$2m^4\alpha^2+2m^2\beta^2-4m^3\alpha\beta = -4m^3\alpha\beta+2m^4\alpha^2+$$

$$6m\alpha\beta-3m^2\alpha^2-3\beta^2+9+2m^2\beta^2+6m^2$$

$$m^2(3\alpha^2-6)-6m\alpha\beta+3\beta^2+9 = 0$$

$$\frac{3\beta^2+9}{3\alpha^2-6} = 2 \Rightarrow \beta^2+3 = 2\alpha^2-4$$

$$\beta^2 = 2\alpha^2-7$$

Sol 13: Equation of any normal to the hyperbola is

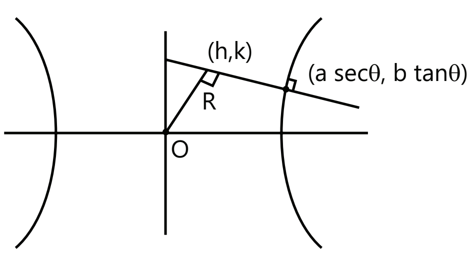
$$y = mx - \frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$$

$$\Rightarrow (a^2-b^2m^2)(y-mx)^2 = m^2(a^2+b^2)^2$$

If it passes through the point (x_1, y_1), then $(a^2-b^2m^2)(y_1-mx_1)^2 = m^2(a^2+b^2)^2$

It is a 4 degree equation in m, so it gives 4 values of m. corresponding to these 4 values, four normal can be drawn from the point (x_1, y_1).

Sol 14:



Slope of normal = $\frac{-h}{k}$

[slope of OR = $\frac{k}{h}$]

that has equation:

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

$$\Rightarrow \text{slope} = -\frac{a^2}{b^2}\times\frac{y_1}{x_1} = \frac{-a^2}{b^2}\times\frac{b-\tan\theta}{a\sec\theta} = \frac{-a}{b}\sin\theta$$

$$\text{so } +\frac{h}{k} = +\frac{a}{b}\sin\theta \Rightarrow \frac{bh}{ak} = \sin\theta$$

$$\Rightarrow \cos\theta = \frac{\sqrt{a^2k^2-b^2h^2}}{ax}$$

$$\text{and } \tan\theta = \frac{bh}{\sqrt{a^2k^2-b^2h^2}}$$

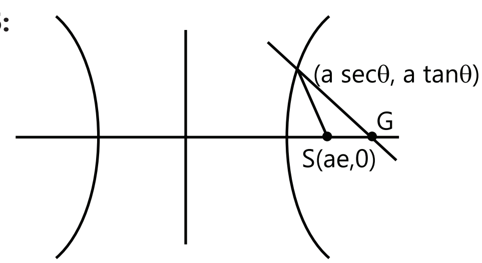
Putting

$$x_1 = b - \frac{bh}{\sqrt{a^2k^2-b^2h^2}},\ y_1 = a\sec\theta = \frac{a^2x}{\sqrt{a^2k^2-b^2h^2}}$$

in equation $\frac{a^2x}{x_1}+\frac{b^2y}{y_1} = a^2+b^2$ and simplifying, we get

locus as $(x^2+y^2)^2(a^2y^2-b^2y^2) = x^2y^2(a^2+b^2)^2$

Sol 15:



Normal: $\frac{a^2x}{x_1}+\frac{b^2y}{y_1} = a^2 + b^2$

$\Rightarrow$ ax. cosθ + by. cotθ = $a^2 + b^2$

Now for coordinates of G $\Rightarrow$ put y = 0 in above equation

$$\Rightarrow x = \frac{(a^2+b^2)}{a}.\sec\theta$$

$$\text{also } e^2 = 1 + \frac{b^2}{a^2} \Rightarrow e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{\frac{a^2+b^2}{a}}$$

Now

$$SG^2 = \left[\frac{(a^2+b^2)\sec\theta}{a}-\sqrt{a^2+b^2}\right]^2$$

and $SP^2 = (\sqrt{a^2+b^2} - a\sec\theta)^2 + (b\tan\theta)^2$

$SP^2 = a^2 + b^2 + a^2\sec^2\theta - 2a.\ \sqrt{a^2+b^2}\ \sec\theta + b^2\tan^2\theta$

$\Rightarrow\ e^2SP^2 = \dfrac{(a^2+b^2)}{a^2}[(a^2 + b^2) + a^2 \sec^2\theta -$

$2a\sqrt{a^2+b^2}\ \sec\theta + b^2\tan^2 q]$

$$= \left[\frac{(a^2+b^2)^2}{a^2} + (a^2+b^2)\sec^2\theta - \right.$$

$$\left. \frac{2\sqrt{a^2+b^2}\sec\theta}{a} + \frac{b^2(a^2+b^2)}{a^2}\tan 2\theta\right]$$

$$= \left[(a^2+b^2) + (a^2+b^2)\times\frac{b^2}{a^2}\right]\sec^2\theta$$

$$\left. + \ \frac{(a^2+b^2)^2}{a^2} - \frac{b^2(a^2+b^2)}{a^2}\ \frac{-2\sqrt{a^2+b^2}\sec\theta}{a}\right]$$

$$= \left[\frac{(a^2+b^2)\sec^2\theta}{a^2} + (a^2+b^2) - \frac{2\sqrt{a^2+b^2}\sec\theta}{a}\right]$$

$$e^2SP^2 = \left[\frac{(a^2+b^2)\sec\theta}{a} - \sqrt{a^2+b^2}\right]^2$$

$e^2SP^2 = SG^2 \Rightarrow\ eSP = SG$

Sol 16: Equation of any tangent to $x^2 - y^2 = a^2$ or $\dfrac{x^2}{a^2}\dfrac{-y^2}{a^2} = 1$ is

$\dfrac{x}{a}\tan\theta = \perp$ or $x\sec\theta - y\tan\theta = a$ ... (i)

Equation of other two sides of the triangle are

$x - y = 0$ ...(ii)

$x + y = 0$ ... (iii)

Solving (ii) and (iii), (iii) and (i), (i) and (ii) in pairs, the co-ordinates of the vertices of the triangle are (0, 0); $\left(\dfrac{a}{\sec\theta+\tan\theta}, \dfrac{a}{\sec\theta+\tan\theta}\right)$ and $\left(\dfrac{a}{\sec\theta-\tan\theta}, \dfrac{a}{\sec\theta-\tan\theta}\right)$

$\therefore$ Area of triangle =

$$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & \dfrac{a}{\sec\theta+\tan\theta} & -\dfrac{a}{\sec\theta+\tan\theta} \\ 1 & \dfrac{a}{\sec\theta-\tan\theta} & \dfrac{a}{\sec\theta+\tan\theta} \end{vmatrix}$$

$$= \frac{1}{2}(2a^2) = a^2$$

Sol 17: Let $P(x_1, y_1)$ be the middle point of the chord of the hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$

$\therefore$ Equation of the chord is $T = s_1$

$\Rightarrow 3xx_1 - 2yy_1 + 2(x + x_1) - 3(y + y_1)$

$\Rightarrow 3x_1^2 - 2y_1^2 + 4x_1 - 6y_1$

$\Rightarrow (3x_1 + 2)x - (2y_1 + 3)y + 2x_1 - 3y_1$

$\Rightarrow 3x_1^2 - 2y_1^2 + 4x_1 - 6y_1$

If this chord is parallel to line y = 2x, then

$m_1 = m_2 \Rightarrow\ -\dfrac{3x+2}{-(2y, 3)} = 2$

$\Rightarrow 3x_1 - 4y_1 = 4$

Hence, the locus of the middle point (x_1, y_1) is $3x - 4y = 4$

Sol 18: Eq. of Hyperbola $= \dfrac{x^2}{100} - \dfrac{y^2}{49} = 1$

Eqn. of tangent $= y = mx \pm \sqrt{a^2m^2 - 49}$ ...(i)

$\Rightarrow y = mx I\sqrt{100m^2 - 49}$

Given that $y = mx + 6$ ....(ii)

Equating (i) and (ii)

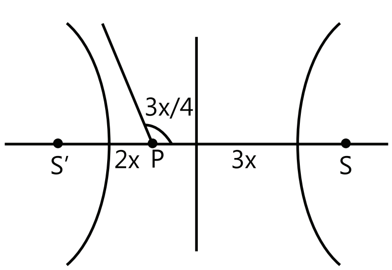
$\Rightarrow \sqrt{100m^2 - 49} = 6$

$\Rightarrow 100m^2 49 = 36$

$\Rightarrow 100m^2 = 85/100 = \dfrac{17}{20}$

$\Rightarrow m = \sqrt{17/20}$

Sol 19:



Now, a = 4, b = 3

$\Rightarrow e^2 = 1 + \frac{9}{16} = \frac{25}{16} \Rightarrow e = \frac{5}{4}$

so coordinates of S = (5, 0) and

S′ = (–5, 0)

so P = $\left(\frac{3\times(-5)+2\times 5}{3+2}, \frac{0\times 3+2\times 0}{3+2}\right)$ = (–1, 0)

Now slope of line through P ⇒ –1

⇒ y = –x + C

⇒ 0 = 1 + c ⇒ c = –1

so line through P = y = – x – 1

Now asymptotes ⇒ $\left(\frac{x}{4}-\frac{y}{3}\right) = 0$

and $\left(\frac{x}{4}+\frac{y}{3}\right) = 0$

Point of intersection ⇒

$\frac{x}{4}+\frac{(x+1)}{3} = 0 \quad \frac{x}{4}-\frac{(x+1)}{3} = 0$

7x + 4 = 0x = –4

$x = \frac{-4}{7}$ y = 3

and $y = \frac{-3}{7}$

$\left(\frac{-4}{7}, \frac{-3}{7}\right)$ and (–4, 3)

Sol 20: Eq. of Hyperbola; $x^2 - 2y^2 = 18 \Rightarrow \frac{x^2}{18} - \frac{y^2}{9} = 1$

Eq. of tangent $y = mx \pm \sqrt{m^2a^2 - b^2}$

$\Rightarrow y = mx \pm \sqrt{m^2.18 - 9}$

∴ this is perpendicular to y = x

⇒ the value of m =-1

$\Rightarrow y = -1\,x \pm \sqrt{18-9}$

$y = -x \pm \sqrt{9}$

$y = -x \pm 3$

Sol 21: The chord joining the points P(a sec θ, a tan θ) and given by $x\cos\frac{\theta-\phi}{2} - y\sin\frac{\theta-\phi}{2}$

$= a\cos\frac{\theta-\phi}{2}$ ... (i)

And normal to the hyperbola at P(a sec θ, a tan θ) is given by

$\frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a$ ... (ii)

Note that equation (i) and (ii) are the same lines comparing these lines, we get

$$\frac{\cos\frac{\theta-\phi}{2}}{\frac{1}{\sec\theta}} = \frac{-\sin\frac{\theta-\phi}{2}}{\frac{1}{\tan\theta}} = \frac{a\cos\frac{\theta-\phi}{2}}{\frac{1}{2a}}$$

Solving above and simplifying, we get

$\tan\phi = \tan\theta\,(4\sec^2\theta - 1)$

Sol 22: $\frac{x^2}{9} - y^2 = 1$

now, line: $y = mx + \sqrt{9m^2 - 1}$

$\Rightarrow 2 = 3m + \sqrt{9m^2-1}$

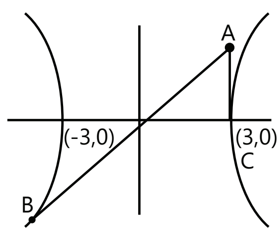
$\Rightarrow (2 - 3m)^2 = \sqrt{9m^2-1}$

$\Rightarrow 9m^2 - 6m \times 2 + 4 = 9m^2 - 1$

(one m = ∞)

S = 12 m

$\Rightarrow m = \frac{5}{12}$



so one tangent ⇒ x = 3

and one is $y = \frac{5}{12}x + \frac{3}{4}$

12y = 5x + 9

Now tangent at B

$\frac{x.x_1}{9} - y.\,y_1 = 1$

⇒ same –5x + 12y = 9

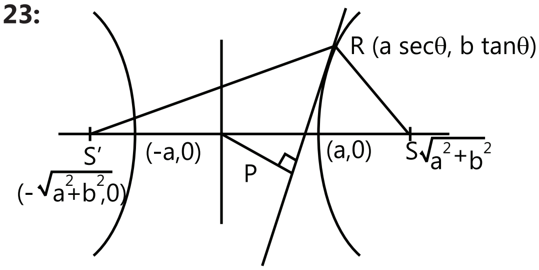
$\Rightarrow \frac{-5}{x_1/9} = \frac{12}{-y_1} = \frac{9}{1} \Rightarrow y_1 = \frac{-4}{3}$

$x_1 = -5$

so $\Delta = \frac{1}{2}$ × AC × height

$= \frac{1}{2} \times 2 \times [(3-(-5))] = 8$ sq. unit

Sol 23:



We have

$(S\Rightarrow R - SR)^2 = S\Rightarrow R^2 + SR^2 - 2S \Rightarrow R.\ RS = (2a)^2$

$\Rightarrow (S\Rightarrow R + SR)^2 = (S\Rightarrow R - SR)^2 + 4S\Rightarrow R.\ SR$

$= 4a^2 + 4.\ S\Rightarrow R \times SR$ ...(i)

Now, tangent

$\Rightarrow \frac{x.\sec\theta}{a} - \frac{y.\tan\theta}{b} = 1$

$\Rightarrow \frac{1}{\sqrt{\frac{\sec^2\theta}{a^2} + \frac{\tan^2\theta}{b^2}}}$

$\Rightarrow P^2 = \frac{a^2b^2}{b^2\sec^2\theta + a^2\tan^2\theta}$ ...(ii)

$SR^2 = \sqrt{(a\sec\theta - \sqrt{a^2+b^2})^2 + b^2\tan^2\theta}$

$SR^2 = a^2\sec^2\theta + a^2 + b^2 + b^2\tan^2\theta - 2a\sec\theta.\ \sqrt{a^2+b^2}$

$= (a^2 + b^2)\sec^2\theta + a^2 - 2a\sec\theta.\ \sqrt{a^2+b^2}$

$SR = (\sqrt{a^2+b^2}\ .\ \sec\theta - a)$

similarly $S\Rightarrow R = (\sqrt{a^2+b^2}\ .\ \sec\theta - a)$

$SR.\ S\Rightarrow R = (a^2 + b^2)\sec^2\theta - a^2$

$= b^2\sec^2\theta + a^2\tan^2\theta$

$SR.\ S\Rightarrow R = \frac{a^2b^2}{P^2}$ [(from (ii)]

putting in (i)

$(S\Rightarrow R + SR)^2 = 4a^2 + \frac{4a^2b^2}{P^2} = 4a^2\left(1 + \frac{b^2}{P^2}\right)$

Exercise 2

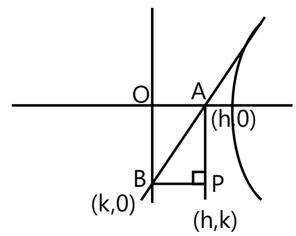
Single Correct Choice Type

Sol 1: (B) $\frac{hx}{4} - \frac{ky}{9} = \frac{h^2}{4} - \frac{k^2}{9}$

Now $d = \frac{\frac{h^2}{4} - \frac{k^2}{9}}{\sqrt{\frac{h^2}{16} + \frac{k^2}{81}}} = 2$

Sol 2: (A) Equation of tangent,

$\frac{x.\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$



so for h $\Rightarrow \frac{h.\sec\theta}{a} = 1$

$\Rightarrow h = a\cos\theta$

and $h' = -b\cot\theta$

$\Rightarrow \left(\frac{a}{h}\right)^2 - \left(\frac{-b}{k}\right)^2 = 1$

$$\boxed{\frac{a^2}{h^2} - \frac{b^2}{k^2} = 1}$$

Sol 3: (D) $\frac{h-0}{1} = \frac{k-0}{t^2} = \frac{-(-2ct)}{1+t^2}$

$h = \frac{2ct}{1+t^4}$, $k = \frac{2ct^3}{1+t^4}$

$\frac{k}{h} = t^2$

$k = \frac{2c\left(\frac{k}{h}\right)^{3/2}}{1+\frac{k^2}{h^2}}$

$k^2 = \frac{4c^2\left(\frac{k}{h}\right)^3}{\left(1+\frac{k^2}{h^2}\right)^2}$

$\frac{k^2(h^2+k^2)^2}{h^4} = \frac{4c^2k^3}{h^3}$

$(x^2 + y^2)^2 = 4c^2\ xy$

Sol 4: (B) We have

$2s = t^2 + 1$ and $2t = 2/s$

$\Rightarrow t = 1/s$

$\Rightarrow 2s = \frac{1}{s^2} + 1$

$\Rightarrow 2s^3 = 1 + s^3$

$\Rightarrow$ $2s^3 - s - 1 = 0$

$\Rightarrow$ $(s - 1)(2s^2 + s + 1) = 0$

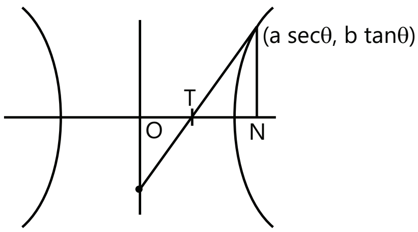
$\Rightarrow$ $s = 1$

$y = \frac{2}{s} = 2$

$x = 2s = 2$

$\Rightarrow$ (2, 2)

Sol 5: (B)



We hae NP = a sec θ and tangent slope:

$$\frac{dy}{dx} = \frac{b.\sec^2\theta}{a.\sec\theta.\tan\theta} = \frac{b}{a\sin\theta}$$

so $\frac{x.\sec\theta}{a} - \frac{y.b\tan\theta}{b} = 1$

so at y = 0

x = a cosθ

so OT = a cosθ

so OT × ON = a cosθ. a secθ = a^2

Sol 6: (A) We have, slope = $\frac{c/t_1 - c/t_2}{ct_1 - ct_2}$

$$= \frac{(t_2 - t_1)}{t_1.t_2(t_1 - t_2)} = \frac{-1}{t_1.t_2}$$

so $y = \frac{-x}{t_1.t_2} + N$

$\Rightarrow$ $y = \frac{-x}{t_1.t_2} + N$

this satisfies,

$$\frac{c}{t_1} = \frac{-c}{t_2} + N$$

$\Rightarrow$ $N = c\left[\frac{1}{t_1} + \frac{1}{t_2}\right]$

Now, $y = \frac{-x}{t_1.t_2} + c.\left[\frac{t_1 + t_2}{t_1.t_2}\right]$

$$\frac{y(t_1t_2)}{c(t_1 + t_2)} - \frac{x}{c(t_1 + t_2)} = 1$$

Now $c(t_1 + t_2) = x_1 + x_2$

and $\frac{c(t_1 + t_2)}{t_1.t_2} = y_1 + y_2$

$\Rightarrow$ $\frac{y}{y_1 + y_2} + \frac{x}{x_2 + x_1} = 1$

Sol 7: (C) We have 2b = ae

$\Rightarrow$ $\frac{b}{a} = \frac{e}{2}$

So $e^2 = 1 + \frac{e^2}{4}$

$\Rightarrow$ $e^2 = \frac{4}{3}$

$\Rightarrow$ $e = \frac{2}{\sqrt{3}}$

Sol 8: (C) $(5)x - (-3)y = (5)^2 - (-3)^2$

$5x + 3y = 16$

Sol 9: (B) We have,

$$2\int x.dx = 3\int y.dy$$

$\Rightarrow$ $x^2 = \frac{3y^2}{2} + c$

$\Rightarrow$ $x^2 - \frac{3y^2}{2} = c$

$\Rightarrow$ $e^2 = 1 + \frac{2/3}{1} = 1 + \frac{2}{3} = \sqrt{\frac{5}{3}}$

Multiple Correct Choice Type

Sol 10: (A, C) $\frac{x^2}{1} - \frac{y^2}{5} = 1$

tangent $\Rightarrow$ $y = mx \pm \sqrt{1m^2 - 5}$

$\Rightarrow$ $(8 - 2m)^2 = m^2 - 5$

$\Rightarrow$ $4m^2 + 64 - 32m = m^2 - 5$

$\Rightarrow$ $3m^2 - 32m + 69 = 0$

$\Rightarrow$ $3m^2 - 23m - 9m + 69 = 0$

$\Rightarrow$ $m(3m - 23) - 3(3m - 23) = 0$

$\Rightarrow$ m = 3 or m = $\frac{23}{3}$

Now y = 3x + 2(A)

or3y = $\frac{23x}{3} \pm \frac{\sqrt{(23)^2 - 45}}{3}$

$\Rightarrow$ 3y = 23x ± 22

Sol 11: (B, C, D) $16(x^2 - 2x) - 3(y - 4y) = 44$

$16(x-1)^2 - 3(y-2)^2 = 44 + 16 - 12$

$$\Rightarrow \frac{(x-1)^2}{3} - \frac{(y-2)^2}{16} = 1$$

Conjugate = 2b = 2 × 4 = 8

Centre = (1, 2)

and $e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{16}{9} \Rightarrow e = \sqrt{\frac{12}{3}}$

Sol 12: (B, D) $\frac{x^2}{16} - \frac{y^2}{9} = 0$

Now tangent

$1 \Rightarrow y = mx \pm \sqrt{a^2m^2 - b^2}$

$y = mx \pm \sqrt{16m^2 - 9}$

tangent $2 \Rightarrow y = mx \pm 3\sqrt{m^2 + 1}$

so$16m^2 - 9 = 9(m^2 + 1)$

$\Rightarrow 7m^2 = 18$

$\Rightarrow m = \pm 3\sqrt{\frac{2}{7}}$

soy = $3\sqrt{\frac{2}{7}}x \pm 3\sqrt{\frac{18}{7} + 1}$

$y = 3\sqrt{\frac{2}{7}}x \pm \frac{16}{\sqrt{7}}$

Sol 13: (A, C, D)

(A) $\left(\frac{2x}{a}\right)^2 - \left(\frac{2y}{b}\right)^2 = 4$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(C) $x^2 - y^2 = 4$

(D) $x^2 - 6 = 2\cos t$

and $y^2 + 2 = 2\left(\sin^2\frac{t}{2} - 1\right) + 2$

$y^2 = 2\cos t$

$\Rightarrow x^2 - y^2 = 6$

(B)t = $\frac{b}{y}\left(1 - \frac{x}{a}\right)$

so $\frac{x}{a}.\left(\frac{b}{y}\right).\left(1 - \frac{x}{a}\right) - \frac{b}{y}\left(1 - \frac{x}{a}\right) = 0$

$$\frac{bx.(a-x)}{a^2y} - \frac{y}{b} + \frac{b(a-x)}{ay} = 0$$

x ba^2y

$b^2x.(a - x) - a^2y^2 + ab^2(a - x) = 0$

$ab^2x - b^2x^2 - a^2y^2 + a^2b^2 - ab^2x = 0$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Sol 14: (A, D) We have equation of circle

$(x - x_1)(x - x_2) + (y_0 - y_1)(y - y_2) = 0$

Now,

$x_1 = ct_1$&$y_1 = c/t_1$

$x_2 = ct_2$&$y_2 = c/t_2$

so slope $\Rightarrow \frac{c/t_2 - c/t_1}{ct_2 - ct_1} = \frac{-1}{t_1.t_2}$

Now, slope = 1

$\Rightarrow -t_1.\frac{1}{t_2}$

$\Rightarrow t_1 = \frac{-1}{t_2}$

putting this above

$(x - ct_1)(x - ct_2) + (y - \frac{c}{t_1})(y - \frac{c}{t_2}) = 0$

$(x - ct_1)(x + \frac{c}{t_1}) + (y - \frac{c}{t_1})(y + ct_1) = 0$

$x^2 - c^2 + 2c\left[\frac{1}{t_1} - t_1\right]x + y^2$

$-c^2 + cy.\left[t_1 - \frac{1}{t_1}\right]$

$(x^2 + y^2 - 2c^2) + c[x - y]\left[\frac{1}{t_1} - t_1\right] = 0$

Now when $x = y$ & $x^2 + y^2 = 2c^2$

this is satisfied for

$x = c$ & $y = c$

$x = -c$ & $y = -c$

Sol 15: (A, B, C, D) $x = \sqrt{2}\,t$ and $y = \sqrt{2}/t$

Now slope of normal = t^2

so $\left(y - \frac{c}{t}\right) = t^2(x - ct)$

$ty - c = t^3x - ct^4$

$\Rightarrow ct^4 - t^3x + ty - c = 0$

passes through (3,4)

Now $ct^4 - 3t^3 + 4t - c = 0$

thus, $St_i = \frac{3}{c}$

$\Rightarrow$ c. Sti = 3

$\Rightarrow$ Sxi = 3(A)

Also $\pi t_i = -1$

$St_1.\ t_2.\ t_3 = \frac{-4}{c}$

$\Rightarrow \frac{\Sigma t_1.t_2.t_3}{\pi t_i} = +\frac{4}{c} \times \frac{1}{(-1)}$

$\Rightarrow c.\ \Sigma\frac{1}{t_i} = 4$

$\Rightarrow Sy_i = 4$(B)

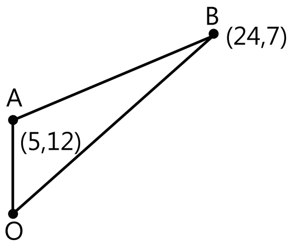
Now $\pi t_i = -1$

$\Rightarrow \pi(ct_i) = -c^4 = -(\sqrt{2})^4 = -4$

and $\frac{1}{\pi t_i} = -1$

$\Rightarrow \pi\left(\frac{c}{t_i}\right) = -c^4 = -4$(C) & (D)

Sol 16: (A, D)



Now AO + BO = 2a

$\sqrt{5^2 + 12^2} + \sqrt{24^2 + 7^2} = 13 + 25 = 38$

So2ae = $\sqrt{19^2 + 5^2}$

38e = $\sqrt{386}$

$\Rightarrow$ e = (0)(if ellipse)

BO – AO = 2a(hyperbola)

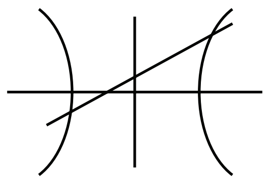
$\Rightarrow \sqrt{24^2 + 72} - \sqrt{52 + 122} = 2a$

$\Rightarrow 25 - 13 = 2a$

$\Rightarrow 12 = 2a$

So2ae = $\sqrt{386}$ $\Rightarrow$ e = $\boxed{e = \sqrt{386}/12}$

Sol 17: (A, B)



Now,

$6 = \sqrt{100m^2 - 49}$

$\Rightarrow 36 + 49 = 100\ m^2$

$\Rightarrow \pm\sqrt{\frac{85}{100}} = m \Rightarrow m = \pm\sqrt{\frac{17}{20}}$

Sol 18: (A, B, D)

k < 8 and k > 12 hyperbola (A)

8 < k < 12 ellipse and

if k = 10 circle

Sol 19: (A, B, C, D) $y = mx + \sqrt{a^2bm^2 - b^2}$

and $y = mx + \sqrt{a^2 - b^2m^2}$

so $\sqrt{a^2m^2 - b^2} = \sqrt{a^2 - b^2m^2}$

$\Rightarrow a^2m^2 - b^2 = a^2 - b^2m^2$

$\Rightarrow a^2(m^2 - 1) = (m^2 - 1)(-b^2)$

$\Rightarrow m = \pm 1$

So, $y = x \pm \sqrt{a^2 - b^2}$ or $y = -x \pm \sqrt{a^2 - b^2}$

Sol 20: (B, D) $\frac{x^2}{18} - \frac{y^2}{9} = 1$

Now m = –1

So $y = mx \pm \sqrt{a^2m^2 - b^2}$

$\Rightarrow y = -x \pm \sqrt{18(+1) - 9}$

$\Rightarrow y = -x \pm 3$

$\Rightarrow y = -x \pm 3$

$\Rightarrow x + y = 3$ and $x + y = -3$

Sol 21: (C, D) $9(x^2 + 2y) - 16(y^2 - 2y) = 151$

$9(x + 1)^2 - 16(y - 1)^2 = 151 + 9 - 16$

$\frac{(x+1)^2}{16} - \frac{(y-1)^2}{9} = 1$

Now $e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{9}{16} = \frac{25}{16} \Rightarrow e = \frac{5}{4}$

So distance from centre

$ae = 4 \times \frac{5}{4} = 5$

$\Rightarrow (-1 + 5, 1)$ and $(-1-5, 1)$

(4, 1) and (–6, 1)

Sol 22: (B, C) Equation of chord connecting the points $(a \sec\theta, b \sec\theta)$ and $(a \tan\phi, b \tan\phi)$ is

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right) \quad \text{... (i)}$$

If it passes through (ae, 0); we, have

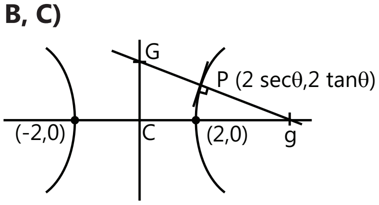
$$e \cos\left(\frac{\theta-\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$$

$$\Rightarrow \quad e = \frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)} = \frac{1 - \tan\frac{\theta}{2}.\tan\frac{\phi}{2}}{1 + \tan\frac{\theta}{2}.\tan\frac{\phi}{2}}$$

$$\Rightarrow \quad \tan\frac{\theta}{2}.\tan\frac{\phi}{2} = \frac{1-e}{1+e}$$

Similarly if (i) passes through (-ae, 0), tan.tan = $\frac{1+e}{1-e}$

Sol 23: (A, B, C)



Normal: $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

$\Rightarrow g \Rightarrow x_1 = x = \frac{x_1.(a^2 + b^2)}{a^2}$

$\boxed{x = \frac{\sec\theta.(a^2 + b^2)}{a}} = 4 \sec\theta$

and $G \Rightarrow y = \frac{\tan\theta(a^2 - b^2)}{b} \Rightarrow 4 \tan\theta$

$PC = 2\sqrt{\sec^2\theta + \tan^2\theta}$

$Og = \sqrt{\frac{\sec^2\theta.(a^2 + b^2)^2}{a^2} + \frac{\tan^2\theta(a^2 + b^2)}{b^2}}$

$= (a^2 + b^2)\sqrt{\frac{\sec^2\theta}{a^2} + \frac{\tan^2\theta}{b^2}} = \frac{8}{2}\sqrt{\sec^2\theta + \tan^2\theta}$

$PG = \sqrt{(2\sec\theta)^2 + (2\tan\theta)^2} = 2.\sqrt{\sec^2\theta + \tan^2\theta}$

$Pg = \sqrt{(4\sec\theta - 2\sec\theta)^2 + (2\tan\theta)^2}$

$= 2\sqrt{\sec^2\theta + \tan^2\theta}$

Previous Years' Questions

Sol 1: (A, B, C, D) It is given that

$x^2 + y^2 = +a^2$ ... (i)

and $xy = c^2$ ... (ii)

We obtain $x^2 = c^4/x^2 = a^2$

$\Rightarrow x^4 - a^2x^2 + c^4 = 0$ ... (iii)

Now, x_1, x_2, x_3, x_4 will be roots of Eq. (iii)

Therefore, $Sx_2 = x_1+x_2+2x_3+x_4 = 0$

and product of the roots $x_1x_2x_3x_4=c^4$

Similarly, $y_1 + y_2 + y_3 + y_4 = 0$ and $y_3y_2y_1y_4 = c^4$

Hence, all options are correct.

Sol 2: (A, B) Given, $2x^2 - 2y^2 = 1$

$$\Rightarrow \frac{x^2}{\left(\frac{1}{2}\right)} - \frac{y^2}{\left(\frac{1}{2}\right)} = 1 \quad \text{... (i)}$$

Eccentricity of hyperbola = $\sqrt{2}$ So eccentricity of ellipse

$= 1/\sqrt{2}$

Let equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ (a > b)

$\therefore \frac{1}{\sqrt{2}} = \sqrt{1-\frac{b^2}{a^2}}$

$\Rightarrow \frac{b^2}{a^2} = \frac{1}{2} \Rightarrow a^2 = 2b^2$

$\therefore x^2 + 2y^2 = 2b^2$ ... (ii)

Let ellipse and hyperbola intersect as

$A\left(\frac{1}{\sqrt{2}}\sec\theta, \frac{1}{\sqrt{2}}\tan\theta\right)$

On differentiating Eq. (i),

$4x - 4y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

$\therefore$ At (6, 3) $\Rightarrow \frac{a^2x}{6}+\frac{b^2y}{3} = (a^2+b^2)$

It passes through (9, 0)

$\Rightarrow \frac{a^2.9}{6} = a^2 + b^2$

$\Rightarrow \frac{3a^2}{2}-a^2 = b^2 \Rightarrow \frac{a^2}{b^2} = 2$

$\therefore e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{1}{2} \Rightarrow \sqrt{\frac{3}{2}}$

$\left.\frac{dy}{dx}\right|_{at\ A} = \frac{\sec\theta}{\tan\theta} = \text{cosec}\theta$

and differentiating Eq. (ii)

$2x + 4y\frac{dy}{dx} = 0$

$\left.\frac{dy}{dx}\right|_{at\ A} = -\frac{x}{2y} = -\frac{1}{2}\text{cosec}\theta$

Since, ellipse and hyperbola are orthogonal

$\therefore -\frac{1}{2}\text{cosec}^2\theta = -1$

$\Rightarrow \text{cosec}^2\theta = 2 \Rightarrow \theta = \pm\frac{\pi}{4}$

$\therefore A\left(1, \frac{1}{\sqrt{2}}\right)$ or $\left(1, -\frac{1}{\sqrt{2}}\right)$

$\therefore$ From Eq. (i),

$1 + 2\left(\frac{1}{\sqrt{2}}\right)^2 = 2b^2$

$\Rightarrow b^2 = 1$

Equation of ellipse is $x^2 + 2y^2 = 2$

Coordinate of foci

$(\pm ae, 0) = \left(\pm\sqrt{2}.\frac{1}{\sqrt{2}}, 0\right) = (\pm 1, 0)$

Hence, option (A) and (B) are correct.

If major axis is along y-axis, then

$\frac{1}{\sqrt{2}} = \sqrt{1-\frac{a^2}{b^2}}$

$\Rightarrow b^2 = 2a^2$

$\therefore 2x^2 + y^2 = 2a^2$

$\Rightarrow y' = -\frac{2x}{y}$

$\Rightarrow y'_{\left(\frac{1}{\sqrt{2}}\sec\theta, \frac{1}{\sqrt{2}}\tan\theta\right)} = \frac{-2}{\sin\theta}$

As ellipse and hyperbola are orthogonal

$\therefore \frac{-2}{\sin\theta}.\text{cosec}\theta = -1$

$\Rightarrow \text{cosec}^2\theta = 1 \Rightarrow \theta = \pm\frac{\pi}{4}$

$\therefore 2x^2 + y^2 = 2a^2$

$\Rightarrow 2 + \frac{1}{2} = 2a^2$

$\Rightarrow a^2 = \frac{5}{4}$

$\Rightarrow 2x^2 + y^2 = \frac{5}{2}$, corresponding foci are (0, ± 1)

Sol 3: (B, D) Here, equation of ellipse $\frac{x^2}{4}+\frac{y^2}{1} = 1$

$\Rightarrow e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore e = \frac{\sqrt{3}}{2}$ and focus (± ae, 0)

$\Rightarrow (\pm\sqrt{3}, 0)$

For hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$

$e_1^2 = 1 + \frac{b^2}{a^2}$

where, $e_1^2 = \frac{1}{e^2} = \frac{4}{3}$

$\Rightarrow 1 + \frac{b^2}{a^2} = \frac{4}{3}$

$\therefore \frac{b^2}{a^2} = \frac{1}{3}$ ... (i)

and hyperbola passes through ($\pm\sqrt{3}$,0)

$\Rightarrow \frac{3}{a^2} = 1$

$\Rightarrow a^2 = 3$ ... (ii)

From Eqs.(i) and (ii), we get

$b^2 = 1$ ... (iii)

$\therefore$ Equation of hyperbola is $\frac{x^2}{3} - \frac{y^2}{1} = 1$

Focus is ($\pm ae_1$, 0)

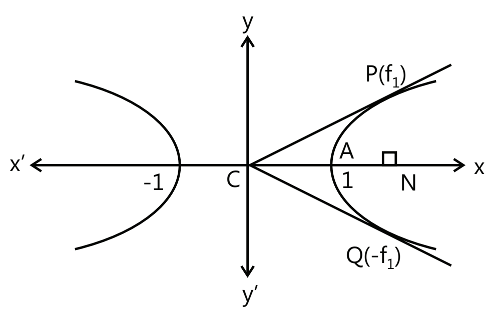
$\Rightarrow \left(\pm\sqrt{3}.\frac{2}{\sqrt{3}}, 0\right) \Rightarrow (\pm 2, 0)$

$\therefore$ (B) and (D) are correct answers.

Sol 4: Let P = $\left(\frac{e^{t_1} + e^{-t_1}}{2}, \frac{e^{t_1} - e^{-t_1}}{2}\right)$

and Q = $\left(\frac{e^{-t_1} + e^{t_1}}{2}, \frac{e^{-t_1} - e^{t}}{2}\right)$

We have to find the area of the region bounded by the curve $x^2 - y^2 = 1$ & the lines joining the centre x = 0, y = 0 to the points (t_1) and ($-t_1$)



Required area

$$= 2\left[\text{area of } \Delta PCN = \int_1^{\frac{e^{t_1}+e^{-t_1}}{2}} y dy\right]$$

$$= 2\left[\frac{1}{2}\left(\frac{e^{t_1} + e^{-t_1}}{2}\right)\left(\frac{e^{t_1} - e^{-t_1}}{2}\right) - \int_1^{t_1} y\frac{dx}{dy}.dt\right]$$

$$= 2\left[\frac{e^{2t_1} - e^{-2t_1}}{8} - \int_0^{t_1}\left(\frac{e^t - e^{-t}}{2}\right)^2.dt\right]$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - \frac{1}{2}\int_0^{t_1}(e^{2t} + e^{-2t} - 2)dt$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - \frac{1}{2}\left[\frac{e^{2t}}{2} - \frac{e^{-2t}}{2} - 2t\right]_0^{t_1}$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - \frac{1}{4}(e^{2t_1} - e^{-2t_1} - 4t_1) = t_1$$

Sol 5: Let any point on the hyperbola is (3secθ, 2tanθ)

$\therefore$ Chord of contact of the circle $x^2+y^2 = 9$ with respect to the point ($3\sec^t\theta$, 2tanθ) is,

$(3\sec\theta)x + (2\tan\theta)y = 9$ ... (i)

Let (x_1, y_1) be the mid point of the chord of contact

$\Rightarrow$ Equation of chord in mid point form is

$xx_1 + yy_1 = x_1^2 + y_1^2$ ... (ii)

Since, Eqs. (i) and (ii) are identically equal

$$\therefore \frac{3\sec\theta}{x_1} = \frac{2\tan\theta}{y_1} = \frac{9}{x_1^2 + y_1^2}$$

$$\Rightarrow \sec\theta = \frac{9x_1}{3(x_1^2 + y_1^2)}$$

$$\text{and } \tan\theta = \frac{9y_1}{2(x_1^2 + y_1^2)}$$

Thus, eliminating 'θ' from above equation, we get

$$\frac{81x_1^2}{9(x_1^2 + y_1^2)^2} - \frac{81y_1^2}{4(x_1^2 + y_1^2)^2} = 1$$

($\because \sec^2\theta - \tan^2\theta = 1$)

$\therefore$ Required locus is $\frac{x^2}{9} - \frac{y^2}{4} = \frac{(x^2 + y^2)^2}{81}$

Sol 6: (B) Equation of tangents to hyperbola having slope m is

$y = mx + \sqrt{9m^2 - 4}$ ... (i)

Equation of tangent to circle is

$y = m(x - 4) + \sqrt{16m^2 + 16}$ ... (ii)

Eqs. (i) and (ii) will be identical for m = $\frac{2}{\sqrt{5}}$ satisfy.

$\therefore$ Equation of common tangent is $2x - \sqrt{5}y + 4 = 0$.

Sol 7: (A) The equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{4} = 1$

and that of circle is $x^2 + y^2 - 8x = 0$

For their points of intersection $\frac{x^2}{9}+\frac{x^2-8x}{4} = 1$

$\Rightarrow 4x^2 + 9x^2 - 72x = 36$

$\Rightarrow 13x^2 - 72x - 36 = 0$

$\Rightarrow 13x^2 - 78x + 6x - 36 = 0$

$\Rightarrow 13x(x-6) = 6(x-6) = 0$

$\Rightarrow x = 6, x = -\frac{13}{6}$

$x = -\frac{13}{6}$ not acceptable

Now, for $x = 6$, $y = \pm 2\sqrt{3}$

Required equation is $(x-6)^2 + (y+2\sqrt{3})(y-2\sqrt{3})=0$

$\Rightarrow x^2 - 12x + y^2 + 24 = 0$

$\Rightarrow x^2 + y^2 - 12x + 24 = 0$

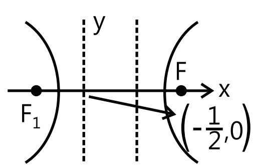
Sol 8: On substituting $\left(\frac{a}{e},0\right)$ in $y = -2x + 1$,

we get $0 = -\frac{2a}{e} + 1$

$\Rightarrow \frac{a}{e} = \frac{1}{2}$

Also, $y = -2x + 1$ is tangent to hyperbola

$\therefore 1 = 4a^2 - b^2$



$\Rightarrow \frac{1}{5} = 4 - (e^2 - 1)$

$\Rightarrow \frac{4}{e^2} = 5 - e^2$

$\Rightarrow e^4 - 5e^2 + 4 = 0$

$\Rightarrow (e^2 - 4)(e^2 - 1) = 0$

$\Rightarrow e = 2, e = 1$

$e = 1$ gives the conic as parabola. But conic is given as hyperbola, hence $e = 2$.

Sol 9: (B) Hyperbola is $\frac{\left(x-\sqrt{2}\right)^2}{4}-\frac{\left(y+\sqrt{2}\right)^2}{2}=1$

$a = 2, b = \sqrt{2}$

$e = \sqrt{\frac{3}{2}}$

Area $= \frac{1}{2}a(e-1)\times\frac{b^2}{a} = \frac{1}{2}\frac{\left(\sqrt{3}-\sqrt{2}\right)\times 2}{\sqrt{2}} = \frac{\left(\sqrt{3}-\sqrt{2}\right)}{\sqrt{2}}$

$\Rightarrow$ Area $= \left(\sqrt{\frac{3}{2}}-1\right)$.

Sol 10: A→p; B→s, t; C→ r; D→ q, s

(p) $\frac{1}{k^2} = 4\left(1+\frac{h^2}{k^2}\right)$

$\Rightarrow 1 = 4\left(k^2+h^2\right)$

$\therefore h^2+k^2 = \left(\frac{1}{2}\right)^2$ which is a circle.

(q) If $\left|z-z_1\right|-\left|z-z_2\right| = k$ where $k < \left|z_1-z_2\right|$ the locus is a hyperbola.

(r) Let $t = \tan\alpha$

$\Rightarrow x = \sqrt{3}\cos 2\alpha$ and $\sin 2\alpha = y$

or $\cos 2\alpha = \frac{x}{\sqrt{3}}$ and $\sin 2\alpha = y$

$\therefore \frac{x^2}{3}+y^2 = \sin^2 2\alpha+\cos^2 2\alpha = 1$ which is an ellipse.

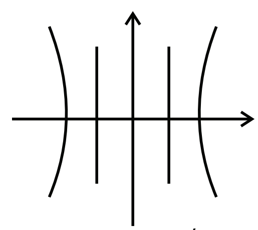
(s) If eccentricity is $[1,\infty)$, then the conic can be a parabola (if e = 1) and a hyperbola if $e\in(1,\infty)$.

(t) Let $z = x+iy;\ x,y\in R$

$\Rightarrow (x+1)^2-y^2 = x^2+y^2+1$

$\Rightarrow y^2 = x$; which is a parabola.

Sol 11: $y = -2x+1$



x = a/e

$0 = -\frac{2a}{e}+1$

$\Rightarrow \frac{a}{e} = \frac{1}{2}$

$e = 2a$

$c^2 = a^2m^2 - b^2$

$\Rightarrow 1 = 4a^2 - b^2$

$\Rightarrow 1 + b^2 - 4a^2 = 0$

$e^2 = 1+\frac{b^2}{a^2} = 1+\frac{\left(4a^2-1\right)}{a^2}$

$e^2 = 1+4-\frac{1}{a^2}$

$e^2 = 5-\frac{1}{e^2}$

$\Rightarrow e^4 - 5e^2 + 4 = 0$

$\Rightarrow \left(e^2-1\right)\left(e^2-4\right) = 0$

$e^2 - 1 \neq 0 \;\; e = 2$

Sol 12: (B) Equation of normal is

$(y-3) = \frac{-a^2}{2b^2}(x-6) \Rightarrow \frac{a^2}{2b^2} = 1 \Rightarrow e = \sqrt{\frac{3}{2}}$.

Sol 13: (B, D) Ellipse is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

$1^2 = 2^2(1-e^2) \Rightarrow e = \frac{\sqrt{3}}{2}$

$\therefore$ Eccentricity of the hyperbola is

$\frac{2}{\sqrt{3}} \Rightarrow b^2 = a^2\left(\frac{4}{3} - 1\right) \Rightarrow 3b^2 = a^2$

Foci of the ellipse are $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$.

Hyperbola passes through $(\sqrt{3}, 0)$

$\frac{3}{a^2} = 1 \Rightarrow a^2 = 3$ and $b^2 = 1$

$\therefore$ Equation of hyperbola is $x^2 - 3y^2 = 3$

Focus of hyperbola is (ae, 0)

$(ae, 0) \equiv \left(\sqrt{3} \times \frac{2}{\sqrt{3}}, 0\right) \equiv (2, 0)$

Sol 14: (A, B) Slope of tangent = 2

The tangents are $y = 2x \pm \sqrt{9 \times 4 - 4}$

i.e., $2x - y = \pm 4\sqrt{2}$

$\Rightarrow \frac{x}{2\sqrt{2}} - \frac{y}{4\sqrt{2}} = 1$ and $-\frac{x}{2\sqrt{2}} - \frac{y}{4\sqrt{2}} = 1$

Comparing it with $\frac{xx_1}{9} - \frac{yy_1}{4} = 1$

We get point of contact as $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

and $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.