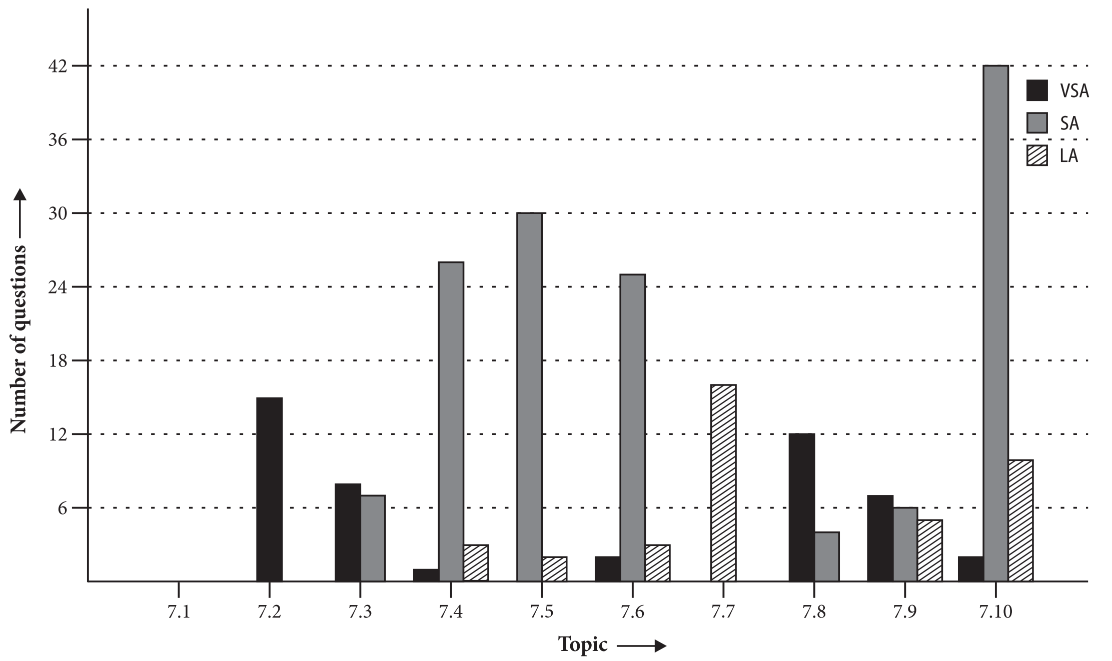
07 Integrals

Topicwise Analysis of Last 10 Years' CBSE Board Questions



- Maximum weightage is of *Some Properties of Definite Integrals*
- Maximum SA type questions were asked from *Some Properties of Definite Integrals*
- Maximum LA type questions were asked from *Definite Integral*
- No VBQ type questions were asked till now

QUICK RECAP

INDEFINITE INTEGRAL

- Integration is the inverse process of differentiation.

  i.e., $\frac{d}{dx}F(x) = f(x) \Rightarrow \int f(x)\,dx = F(x) + C,$

where C is the constant of integration.
Integrals are also known as antiderivatives.

Some Standard Integrals

- $\int dx = x + C$, where 'C' is the constant of integration
- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\log_e a} + C$, where $a > 0$
- $\int \frac{1}{x} dx = \log_e |x| + C$, where $x \neq 0$
- $\int \sin x\, dx = -\cos x + C$
- $\int \cos x\, dx = \sin x + C$
- $\int \sec^2 x\, dx = \tan x + C$
- $\int \text{cosec}^2 x\, dx = -\cot x + C$
- $\int \sec x \tan x\, dx = \sec x + C$
- $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + C$
- $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C = -\cos^{-1} x + C$, where $|x| < 1$
- $\int \frac{dx}{1+x^2} = \tan^{-1} x + C = -\cot^{-1} x + C$
- $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C = -\text{cosec}^{-1} x + C$, where $|x| > 1$
- $\int \tan x\, dx = \log|\sec x| + C = -\log|\cos x| + C$
- $\int \cot x\, dx = \log|\sin x| + C$
- $\int \sec x\, dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$
- $\int \text{cosec}\, x\, dx = \log|\text{cosec}\, x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

Properties of Indefinite Integral

(i) $\int f'(x) dx = f(x) + C$

(ii) $\int f(x) dx = \int g(x) dx + C$, f and g are indefinite integrals with the same derivative.

(iii) $\int [f(x) + g(x)] dx = \int f(x)\, dx + \int g(x)\, dx$

(iv) $\int k \cdot f(x)\, dx = k \int f(x)\, dx$, k being any real number.

METHODS OF INTEGRATION

Integration by Substitution

The given integral $\int f(x)\, dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$.

Integrals	Substitution
$\int f(ax+b) dx$	$ax + b = t$
$\int f(g(x)) g'(x) dx$	$g(x) = t$
$\int \frac{f'(x)}{f(x)} dx$	$f(x) = t$
$\int (f(x))^n f'(x) dx$	$f(x) = t$
$\int (px+q)\sqrt{cx+d}\, dx$ or $\int \frac{px+q}{\sqrt{cx+d}} dx$	$px + q = A(cx + d) + B$. Find A and B by equating coefficients of like powers of x on both sides.
$\int \frac{1}{(px+q)\sqrt{cx+d}} dx$ or $\int \frac{1}{(px^2+qx+r)\sqrt{cx+d}} dx$	$cx + d = t^2$
$\int \frac{1}{(px+q)\sqrt{cx^2+dx+e}} dx$	$px + q = \frac{1}{t}$
$\int \frac{1}{(px^2+q)\sqrt{cx^2+d}} dx$	$x = \frac{1}{t}$ and then $c + dt^2 = u^2$
$\int \frac{px+q}{ax^2+bx+c} dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$ or $\int (px+q)\sqrt{ax^2+bx+c}\, dx$	$(px + q) = A\frac{d}{dx}(ax^2+bx+c) + B$

- **Integration using Trigonometric Identities**
When the integrand consists of trigonometric functions, we use known identities to convert it into a form which can be easily integrated. Some of the identities useful for this purpose are given below :

(i) $2\sin^2\left(\frac{x}{2}\right)=(1-\cos x)$

(ii) $2\cos^2\left(\frac{x}{2}\right)=(1+\cos x)$

(iii) $2\sin x\cos y=\sin(x+y)+\sin(x-y)$

(iv) $2\cos x\sin y=\sin(x+y)-\sin(x-y)$

(v) $2\cos x\cos y=\cos(x+y)+\cos(x-y)$

(vi) $2\sin x\sin y=\cos(x-y)-\cos(x+y)$

- **Some Special Substitutions**

Expression	Substitution
$\sqrt{a^2-x^2}$	$x=a\sin\theta$ or $a\cos\theta$
$\sqrt{a^2+x^2}$ or (a^2+x^2)	$x=a\tan\theta$ or $a\cot\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$ or $a\,\text{cosec}\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x=a\cos2\theta$
$\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$	$x=a\sin^2\theta$ or $a\cos^2\theta$
$\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$	$x=a\tan^2\theta$ or $a\cot^2\theta$
$\sqrt{\frac{a-x}{x-b}}$ or $\sqrt{\frac{x-b}{a-x}}$ or $\sqrt{(a-x)(x-b)}$	$x=a\cos^2\theta+b\sin^2\theta$

- **Integrals of Some Particular Functions**

(i) $\int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$

(ii) $\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+C$

(iii) $\int\frac{1}{\sqrt{x^2-a^2}}dx=\log\left|x+\sqrt{x^2-a^2}\right|+C$

(iv) $\int\frac{1}{\sqrt{x^2+a^2}}dx=\log\left|x+\sqrt{x^2+a^2}\right|+C$

(v) $\int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$

(vi) $\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}+C$

Integration by Partial Fractions

- If $f(x)$ and $g(x)$ are two polynomials such that $\deg f(x)\ge\deg g(x)$, then we divide $f(x)$ by $g(x)$.

$\therefore\ \frac{f(x)}{g(x)}=\text{Quotient}+\frac{\text{Remainder}}{g(x)}$

- If $f(x)$ and $g(x)$ are two polynomials such that the degree of $f(x)$ is less than the degree of $g(x)$, then we can evaluate $\int\frac{f(x)}{g(x)}dx$ by decomposing $\frac{f(x)}{g(x)}$ into partial fraction.

Form of the Rational Function	Form of the Partial Fraction
$\frac{px+q}{(x-a)(x-b)}, a\ne b$	$\frac{A}{x-a}+\frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
$\frac{px+q}{(x-a)^3}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where x^2+bx+c can not be factorised further	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

Integration by Parts

If u and v are two differentiable functions of x, then

$$\int(uv)\,dx=\left[u\cdot\int v\,dx\right]-\int\left\{\frac{du}{dx}\cdot\int v\,dx\right\}dx.$$

In order to choose 1st function, we take the letter which comes first in the word ILATE.

I – Inverse Trigonometric Function
L – Logarithmic Function
A – Algebraic Function
T – Trigonometric Function
E – Exponential Function

- **Integral of the type**
$\int e^x (f(x) + f'(x))dx = e^x f(x) + C$

INTEGRALS OF SOME MORE TYPES

(i) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

(ii) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

(iii) $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$

DEFINITE INTEGRAL

- Let $F(x)$ be integral of $f(x)$, then for any two values of the independent variable x, say a and b, the difference $F(b) - F(a)$ is called the definite integral of $f(x)$ from a to b and is denoted by $\int_a^b f(x)dx$.
Here, $x = a$ is the lower limit and $x = b$ is the upper limit of the integral.

- **Definite Integral as a Limit of Sum**
Let $f(x)$ be a continuous real valued function defined on the closed interval $[a, b]$. Then
$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + \ldots + f(a+(n-1)h)],$$
where $h = \frac{b-a}{n} \to 0$ as $n \to \infty$

FUNDAMENTAL THEOREM OF CALCULUS

- **First Fundamental Theorem :** Let $f(x)$ be a continuous function in the closed interval $[a, b]$ and let $A(x)$ be the area function. Then $A'(x) = f(x)$, for all $x \in [a, b]$.

- **Second Fundamental Theorem :** Let $f(x)$ be a continuous function in the closed interval $[a, b]$ and $F(x)$ be an integral of $f(x)$, then
$\int_a^b f(x)dx = F(b) - F(a)$

EVALUATION OF DEFINITE INTEGRAL BY SUBSTITUTION

- When definite integral is to be found by substitution, change the lower and upper limits of integration. If substitution is $t = f(x)$ and lower limit of integration is a and upper limit is b, then new lower and upper limits will be $f(a)$ and $f(b)$ respectively.

SOME PROPERTIES OF DEFINITE INTEGRALS

(i) $\int_a^b f(x)dx = \int_a^b f(t)dt$

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

In Particular $\int_a^a f(x)dx = 0$

(iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$

(iv) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(v) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

(vi) $\int_{-a}^a f(x)dx = \begin{cases} 0, & \text{if } f(-x) = -f(x) \\ 2\int_0^a f(x)dx, & \text{if } f(-x) = f(x) \end{cases}$

(vii) $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

(viii) $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

Previous Years' CBSE Board Questions

7.2 Integration as an Inverse Process of Differentiation

VSA (1 mark)

1. Write the antiderivative of $\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)$. *(Delhi 2014)*

2. Evaluate : $\int \cos^{-1}(\sin x)dx$ *(Delhi 2014)*

3. Evaluate : $\int \frac{dx}{\sin^2 x \cos^2 x}$ *(Foreign 2014, Delhi 2014C)*

4. Find : $\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx$ *(Delhi 2014C)*

5. Evaluate : $\int (1-x)\sqrt{x}\,dx$ *(Delhi 2012)*

6. Write the value of $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}dx$ *(Delhi 2012C, 2011)*

7. Write the value of $\int \frac{2-3\sin x}{\cos^2 x}dx$ *(Delhi 2011)*

8. Write the value of $\int \sec x(\sec x+\tan x)dx$ *(Delhi 2011)*

9. Write the value of $\int (ax+b)^3 dx$ *(AI 2011)*

10. Evaluate : $\int \frac{x^3-x^2+x-1}{x-1}dx$ *(Delhi 2011C)*

11. Evaluate : $\int \frac{x^3-1}{x^2}dx$ *(Delhi 2010 C)*

12. Write the value of $\int \frac{1-\sin x}{\cos^2 x}dx$ *(AI 2010 C)*

13. Write the value of $\int 2^x dx$ *(AI 2010 C)*

7.3 Methods of Integration

VSA (1 mark)

14. Find : $\int \frac{\sin^6 x}{\cos^8 x}dx$ *(AI 2014C)*

15. Write the value of : $\int \frac{x+\cos 6x}{3x^2+\sin 6x}dx$ *(AI 2012C)*

16. Evaluate : $\int \frac{(\log x)^2}{x}dx$ *(AI 2011)*

17. Evaluate : $\int \frac{e^{\tan^{-1}x}}{1+x^2}\cdot dx$ *(AI 2011)*

18. Evaluate : $\int \frac{2\cos x}{3\sin^2 x}dx$ *(AI 2011C)*

19. Evaluate : $\int \frac{\log x}{x}dx$ *(Delhi 2010)*

20. Evaluate : $\int \sec^2(7-4x)dx$ *(AI 2010)*

21. Evaluate : $\int \cos 4x\cos 3x dx$. *(Delhi 2007)*

SA (4 marks)

22. Find $\int \frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}d\theta$. *(Delhi 2016, 2013C)*

23. Evaluate : $\int \frac{\sin(x-a)}{\sin(x+a)}dx$ *(Foreign 2015, Delhi 2013)*

24. Evaluate : $\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cdot\cos^2 x}dx$ *(Delhi 2014)*

25. Evaluate : $\int \frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}dx$ *(AI 2013)*

26. Evaluate : $\int \sin x\sin 2x\sin 3x\,dx$ *(Delhi 2012)*

7.4 Integrals of Some Particular Functions

VSA (1 mark)

27. Write the value of $\int \frac{dx}{x^2+16}$. *(Delhi 2011)*

SA (4 marks)

28. Find $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx.$ *(Delhi 2016)*

29. Find $\int \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$ *(AI 2015, 2014, 2013C, Delhi 2010C)*

30. Evaluate : $\int \frac{x+2}{2x^2 + 6x + 5} dx$ *(Delhi 2015C, AI 2007)*

31. Find $\int \frac{x+3}{\sqrt{5 - 4x - 2x^2}} dx$ *(AI 2015C)*

32. Evaluate : $\int \frac{x+2}{\sqrt{x^2 + 5x + 6}} dx$ *(AI 2014)*

33. Evaluate : $\int \frac{5x-2}{1 + 2x + 3x^2} dx$ *(Delhi 2014C, 2013)*

34. Evaluate : $\int \frac{x+2}{\sqrt{x^2 + 2x + 3}} dx$ *(AI 2013)*

35. Evaluate : $\int \frac{5x+3}{\sqrt{x^2 + 4x + 10}} dx$ *(AI 2012C, 2010, Delhi 2011)*

36. Evaluate : $\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx$ *(Delhi 2011C)*

37. Evaluate : $\int \frac{x^2 + 1}{x^4 + 1} dx$ *(Delhi 2011C, 2007)*

38. Evaluate : $\int \frac{x^2 + 4}{x^4 + 16} dx$ *(AI 2011C)*

39. Evaluate the following : $\int \frac{x+2}{\sqrt{(x-2)(x-3)}} dx$ *(AI 2010)*

40. Evaluate : $\int \frac{dx}{\sqrt{5 - 4x - 2x^2}}$ *(AI 2009)*

41. Evaluate : $\int \frac{2x+5}{\sqrt{7 - 6x - x^2}} dx$ *(Delhi 2009 C)*

42. Evaluate : $\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$ *(AI 2009 C)*

43. Evaluate : $\int \frac{x+3}{x^2 - 2x - 5} dx$ *(AI 2008 C)*

44. Evaluate : $\int \frac{1 - x^2}{1 + x^4} dx$ *(Delhi 2007)*

45. Evaluate : $\int \frac{x}{x^2 + x + 1} dx$ *(AI 2007)*

LA (6 marks)

46. Evaluate : $\int \frac{1}{\cos^4 x + \sin^4 x} dx$ *(AI 2014)*

47. Evaluate : $\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$ *(AI 2014)*

48. Evaluate : $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}} dx$ *(AI 2011)*

7.5 Integration by Partial Fractions

SA (4 marks)

49. Find : $\int \frac{x^2}{x^4 + x^2 - 2} dx$ *(AI 2016)*

50. Find : $\int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} dx$ *(Foreign 2016)*

51. Find : $\int \frac{dx}{\sin x + \sin 2x}$ *(Delhi 2015)*

52. Evaluate : $\int \frac{x^2 + x + 1}{(x^2+1)(x+2)} dx$ *(AI 2015, 2013C, 2009C)*

53. Evaluate : $\int \frac{x^2}{(x^2+4)(x^2+9)} dx$ *(Foreign 2015, Delhi 2013)*

54. Find : $\int \frac{x}{(x-1)^2 (x+2)} dx$ *(Delhi 2015C)*

55. Find $\int \frac{x}{(x^2+1)(x-1)} dx$ *(AI 2015C, Delhi 2013C)*

56. Evaluate : $\int \frac{x^2}{(x^2+1)(x^2+4)} dx$ *(Delhi 2014C, AI 2013C)*

57. Find : $\int \frac{x^3}{x^4+3x^2+2}dx$ *(AI 2014C)*

58. Evaluate : $\int \frac{2x^2+1}{x^2(x^2+4)}dx.$ *(Delhi 2013)*

59. Evaluate : $\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$ *(Delhi 2013)*

60. Evaluate : $\int \frac{dx}{x(x^5+3)}.$ *(AI 2013)*

61. Evaluate : $\int \frac{dx}{x(x^3+8)}.$ *(AI 2013)*

62. Evaluate : $\int \frac{dx}{x(x^3+1)}$ *(AI 2013)*

63. Evaluate : $\int \frac{3x+1}{(x+1)^2(x+3)}dx$ *(Delhi 2013C)*

64. Evaluate : $\int \frac{3x+5}{x^3-x^2-x+1}dx$ *(Delhi 2013C)*

65. Evaluate : $\int \frac{8}{(x+2)(x^2+4)}dx$ *(AI 2013C)*

66. Evaluate : $\int \frac{2}{(1-x)(1+x^2)}dx$ *(Delhi 2012)*

67. Evaluate : $\int \frac{2x\,dx}{(x^2+1)(x^2+3)}$ *(Delhi 2011)*

68. Evaluate : $\int \frac{1-x^2}{x(1-2x)}dx$ *(Delhi 2010)*

69. Evaluate : $\int \frac{dx}{(x^2+1)(x^2+2)}$ *(Delhi 2010 C)*

70. Evaluate : $\int \frac{dx}{\sin x-\sin 2x}$ *(AI 2010 C)*

71. Evaluate : $\int \frac{\sin x}{(1-\cos x)(2-\cos x)}dx$ *(Delhi 2007)*

72. Evaluate : $\int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx$ *(Delhi 2007)*

73. Evaluate : $\int \frac{2x+1}{(x+2)(x-3)}dx$ *(AI 2007)*

LA (6 marks)

74. Find $\int \frac{x^2+x+1}{(x+1)^2(x+2)}dx$ *(Delhi 2014C)*

75. Evaluate : $\int \frac{x^2+1}{(x-1)^2(x+3)}dx.$ *(AI 2012)*

7.6 Integration by Parts

VSA (1 mark)

76. Given $\int e^x(\tan x+1)\sec x\,dx = e^x f(x)+c$
Write $f(x)$ satisfying above. *(AI 2012)*

77. Evaluate : $\int x\log 2x\,dx.$ *(AI 2007)*

SA (4 marks)

78. Find : $\int (3x+1)\sqrt{4-3x-2x^2}\,dx$ *(AI 2016)*

79. Find : $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$ *(Foreign 2016, AI 2012)*

80. Find : $\int (2x+5)\sqrt{10-4x-3x^2}\,dx$ *(Foreign 2016)*

81. Find : $\int (x+3)\sqrt{3-4x-x^2}\,dx.$ *(Delhi 2015, 2014C)*

82. Integrate the following w.r.t. x. $\frac{x^2-3x+1}{\sqrt{1-x^2}}$ *(Delhi 2015)*

83. Evaluate : $\int (3-2x)\cdot\sqrt{2+x-x^2}\,dx$ *(AI 2015)*

84. Find : $\int \frac{\log x}{(x+1)^2}dx$ *(AI 2015)*

85. Evaluate : $\int e^{2x}\cdot\sin(3x+1)dx$ *(Foreign 2015)*

86. Find : $\int \frac{(x^2+1)e^x}{(x+1)^2}dx$ *(Delhi 2015C)*

87. Evaluate : $\int (x-3)\sqrt{x^2+3x-18}\,dx$ *(Delhi 2014)*

88. Evaluate : $\int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$ *(Foreign 2014)*

89. $\int (3x-2)\sqrt{x^2+x+1}\,dx$ *(Foreign 2014, AI 2014C)*

90. Evaluate : $\int e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$ *(Delhi 2013C)*

91. Evaluate : $\int \frac{\sqrt{1-\sin x}}{1+\cos x}\cdot e^{\frac{-x}{2}}\,dx$ *(AI 2013C)*

92. Evaluate : $\int \left(\frac{1+\sin x}{1+\cos x}\right)e^x\,dx$ *(Delhi 2012C)*

93. $\int \frac{x^2}{(x\sin x+\cos x)^2}dx$ *(Delhi 2012C)*

94. Evaluate : $\int e^x\left(\frac{\sin 4x-4}{1-\cos 4x}\right)dx$ *(Delhi 2010)*

95. Evaluate : $\int \left[\log(\log x)+\frac{1}{(\log x)^2}\right]dx$ *(Delhi 2010 C)*

96. Evaluate : $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right)dx$ *(AI 2010 C)*

97. Evaluate : $\int \frac{(x-4)e^x}{(x-2)^3}dx$ *(Delhi 2009)*

98. Evaluate : $\int x\cdot\log(x+1)\,dx$ *(Delhi 2008 C)*

99. Evaluate : $\int x\log 2x\,dx$ *(AI 2007)*

LA **(6 marks)**

100. Find : $\int \frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}dx$ *(AI 2014C, 2012C)*

101. Find : $\int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}\,dx,\ x\in[0,1]$ *(AI 2014C)*

7.7 Definite Integral

LA **(6 marks)**

102. Evaluate : $\int_1^3 (e^{2-3x}+x^2+1)\,dx$ as a limit of a sum. *(Delhi 2015)*

103. Evaluate : $\int_1^3 (3x^2+1)dx$ by the method of limit of sums. *(Delhi 2014C)*

104. Evaluate : $\int_1^3 (2x^2+5x)\,dx$ as limit of sums. *(Delhi 2012)*

105. Evaluate : $\int_2^5 (x^2+3)dx$ as limit of sums. *(Delhi 2012C)*

106. Evaluate : $\int_1^4 (x^2-x)dx$ as limit of sums. *(AI 2012C, Delhi 2010)*

107. Evaluate : $\int_0^2 (x^2-x)dx$ as limit of sums. *(Delhi 2011C)*

108. Evaluate : $\int_0^2 (3x^2-2)dx$ as limit of sums. *(AI 2011C)*

109. Evaluate : $\int_1^3 (3x^2+2x)dx$ as limit of sums. *(Delhi 2010)*

110. Evaluate : $\int_1^2 (x^2+5x)dx$ as limit of sums. *(Delhi 2010C)*

111. Evaluate : $\int_1^3 (2x^2+3)dx$ as limit of sums. *(Delhi 2010 C, 2009 C)*

112. Evaluate : $\int_2^5 (3x^2-5)dx$ as limit of sums. *(Delhi 2009 C)*

113. Evaluate : $\int_1^3 (x^2+x)dx$ as limit of sums. *(Delhi 2008 C)*

114. Evaluate : $\int_1^3 (x^2+5x)dx$ as limit of sums. *(Delhi 2008 C)*

115. Evaluate : $\int_0^2 (x^2+2x+1)dx$ as limit of sums. *(Delhi 2007)*

7.8 Fundamental Theorem of Calculus

VSA (1 mark)

116. Evaluate : $\int_0^3 \frac{dx}{9+x^2}$ *(Delhi 2014)*

117. Evaluate : $\int_0^{\pi/2} e^x(\sin x - \cos x)dx$ *(Delhi 2014)*

118. If $f(x) = \int_0^x t \sin t \, dt$, then write the value of $f'(x)$. *(AI 2014)*

119. If $\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$, find the value of a. *(AI 2014)*

120. Evaluate : $\int_0^{\pi/4} \tan x \, dx$ *(Foreign 2014)*

121. Evaluate : $\int_0^{\pi/4} \sin 2x \, dx$ *(Foreign 2014)*

122. Evaluate : $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ *(AI 2014C, 2011C)*

123. Evaluate : $\int_1^2 \frac{x^3-1}{x^2}dx$ *(AI 2014C)*

124. Evaluate : $\int_2^3 \frac{1}{x}dx$ *(Delhi 2012)*

125. Evaluate : $\int_0^2 \sqrt{4-x^2}\, dx$ *(AI 2012)*

126. Evaluate : $\int_0^1 \frac{dx}{1+x^2}$ *(Delhi 2011C)*

SA (4 marks)

127. Evaluate : $\int_0^{\pi/2} x^2 \sin x \, dx$ *(Delhi 2014C)*

128. Evaluate : $\int_0^{\frac{\pi}{2}} \frac{x+\sin x}{1+\cos x}dx$ *(AI 2011)*

129. Evaluate : $\int_0^1 \frac{x^4+1}{x^2+1}dx$ *(AI 2011C)*

130. Evaluate the following : $\int_1^2 \frac{5x^2}{x^2+4x+3}dx$ *(AI 2010)*

7.9 Evaluation of Definite Integrals by Substitution

VSA (1 mark)

131. Evaluate : $\int_2^4 \frac{x}{x^2+1}dx$ *(AI 2014)*

132. Evaluate : $\int_e^{e^2} \frac{dx}{x \log x}$. *(AI 2014)*

133. Evaluate : $\int_0^1 xe^{x^2} dx$ *(Foreign 2014)*

134. Evaluate : $\int_0^1 \frac{\tan^{-1} x}{1+x^2}dx$ *(AI 2014C)*

135. Write the value of $\int_0^1 \frac{e^x}{1+e^{2x}}dx$ *(Delhi 2012 C)*

136. Write the value of : $\int_0^1 \frac{2x}{1+x^2}dx$ *(AI 2012C, 2011C)*

SA (4 marks)

137. Evaluate $\int_0^{\pi} e^{2x} \cdot \sin\left(\frac{\pi}{4}+x\right)dx.$ *(Delhi 2016)*

138. Find : $\int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$ *(AI 2015)*

139. Evaluate : $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}dx$ *(AI 2014C, Delhi 2010)*

140. Evaluate : $\int_0^{\pi/4} \frac{\sin 2\theta}{\sin^4\theta + \cos^4\theta}d\theta$ *(AI 2013C)*

141. Evaluate : $\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}dx$ *(AI 2008)*

LA (6 marks)

142. Evaluate : $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$

(Foreign 2014, Delhi 2014C)

143. Prove that $\int_0^{\pi/4} (\sqrt{\tan x} + \sqrt{\cot x})\, dx = \sqrt{2} \cdot \frac{\pi}{2}$

(Delhi 2012)

144. Evaluate : $\int_{\pi/4}^{\pi/2} \cos 2x \log \sin x\, dx$ *(Delhi 2012C)*

145. Evaluate : $\int_0^{\pi/2} 2\sin x \cos x \tan^{-1}(\sin x) dx$

(Delhi 2011)

7.10 Some Properties of Definite Integrals

VSA (1 mark)

146. Write the value of the following integral :

$\int_{-\pi/2}^{\pi/2} \sin^5 x\, dx$ *(AI 2010)*

147. Evaluate : $\int_{-\pi/4}^{\pi/4} \sin^3 x\, dx$ *(Delhi 2010C)*

SA (4 marks)

148. Evaluate $\int_{-1}^{2} |x^3 - x| dx.$

(Delhi 2016, AI 2012, 2011)

149. Evaluate : $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ *(AI 2016)*

150. Evaluate : $\int_0^{3/2} |x \cos \pi x| dx$ *(AI 2016)*

151. Evaluate : $\int_0^{\pi} \frac{x}{1 + \sin\alpha \sin x} dx$ *(Foreign 2016)*

152. Evaluate : $\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx$ *(Delhi 2015)*

153. Evaluate : $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + e^x} dx$ *(Foreign 2015)*

154. Evaluate : $\int_0^{\pi/2} \frac{dx}{1 + \sqrt{\tan x}}$ *(Delhi 2015C)*

155. Evaluate : $\int_0^{\pi/4} \log(1 + \tan x) dx$

(AI 2015C, 2011, 2007, Delhi 2009)

156. Evaluate : $\int_0^{\pi} \frac{x \tan x}{\sec x \text{ cosec } x} dx$

(Delhi 2014, 2011C)

157. Evaluate : $\int_0^{\pi} \frac{4x \sin x}{1 + \cos^2 x} dx$ *(AI 2014)*

158. Show that :

$$\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$$

(AI 2014C)

159. Evaluate : $\int_0^4 (|x| + |x-2| + |x-4|) dx$

(Delhi 2013)

160. Evaluate : $\int_2^5 [|x-2| + |x-3| + |x-5|] dx$

(Delhi 2013)

161. Evaluate : $\int_1^3 [|x-1| + |x-2| + |x-3|]\, dx.$

(Delhi 2013)

162. Evaluate : $\int_0^{2\pi} \frac{1}{1 + e^{\sin x}} dx$ *(AI 2013)*

163. Evaluate : $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$

(AI 2013, 2012, Delhi 2011C, 2009C, 2008, 2007)

164. Evaluate : $\int_0^{\pi} \frac{x}{1 + \sin x} dx$

(AI 2012C, 2011C, Delhi 2010)

165. Evaluate : $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$ *(AI 2011)*

166. Using properties of definite integrals, evaluate the following :

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}dx$$ *(AI 2011, Delhi 2007)*

167. Evaluate : $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$ *(AI 2011C)*

168. Evaluate : $\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)dx$ *(Delhi 2009)*

169. Evaluate : $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx$ *(Delhi 2009)*

170. Evaluate : $\int_0^1 \cot^{-1}(1-x+x^2)dx$ *(Delhi 2008)*

171. Prove that : $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}dx = a\pi$ *(Delhi 2008)*

172. Evaluate : $\int_0^{\frac{\pi}{2}} \frac{x}{\sin x+\cos x}dx$ *(AI 2008)*

173. Evaluate : $\int_0^{\frac{\pi}{2}} \log\sin x dx$ *(AI 2008)*

174. Using properties of definite integrals, evaluate the following :

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$$ *(Delhi 2008C)*

175. Using properties of definite integrals prove the following :

$$\int_0^{\pi} \frac{x\tan x}{\sec x \operatorname{cosec} x}dx = \frac{\pi^2}{4}$$ *(Delhi 2007)*

LA (6 marks)

176. Evaluate: $\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx$ *(Delhi 2014, 2011, AI 2010C)*

177. Evaluate : $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$ *(Delhi 2014)*

178. Evaluate : $\int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}dx$ *(Foreign 2014, Delhi 2014C, 2010, AI 2008)*

179. Evaluate : $\int_0^{\pi} \frac{xdx}{a^2\cos^2 x+b^2\sin^2 x}$ *(Foreign 2014)*

180. Evaluate : $\int_1^4 [|x-1|+|x-2|+|x-4|]dx$ *(Delhi 2011C)*

Detailed Solutions

1. The antiderivative of $3\sqrt{x}+\frac{1}{\sqrt{x}}$

$=\int\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx=3\int x^{1/2}dx+\int x^{-1/2}dx$

$=3\cdot\frac{x^{3/2}}{3/2}+\frac{x^{1/2}}{1/2}+C=2x\sqrt{x}+2\sqrt{x}+C$

$=2\sqrt{x}(x+1)+C$

2. $\int\cos^{-1}(\sin x)dx=\int\cos^{-1}\left[\cos\left(\frac{\pi}{2}-x\right)\right]dx$

$=\int\left(\frac{\pi}{2}-x\right)dx=\frac{\pi}{2}x-\frac{x^2}{2}+C$

3. We have $\int\frac{dx}{\sin^2 x\cos^2 x}=\int\frac{(\sin^2 x+\cos^2 x)}{\sin^2 x\cos^2 x}dx$

$=\int(\sec^2 x+\text{cosec}^2 x)dx=\tan x-\cot x+C$

4. *Refer to answer 3.*

5. $\int(1-x)\sqrt{x}\,dx=\int(\sqrt{x}-x^{3/2})dx$

$=\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}+C$

6. $\int\frac{\sec^2 x}{\text{cosec}^2 x}dx=\int\frac{\sin^2 x}{\cos^2 x}dx=\int\tan^2 x\,dx$

$=\int(\sec^2 x-1)dx=\tan x-x+C$

7. $\int\frac{2-3\sin x}{\cos^2 x}dx=\int\left(\frac{2}{\cos^2 x}-\frac{3\sin x}{\cos^2 x}\right)dx$

$=\int(2\sec^2 x-3\sec x\tan x)dx=2\tan x-3\sec x+C$

8. $\int\sec x(\sec x+\tan x)dx$

$=\int(\sec^2 x+\sec x\tan x)dx=\tan x+\sec x+C$

9. $\int(ax+b)^3\,dx=\frac{(ax+b)^4}{4a}+C$

10. Let $I=\int\frac{x^3-x^2+x-1}{x-1}dx$

$=\int\frac{x^2(x-1)+1(x-1)}{x-1}dx=\int\frac{(x^2+1)(x-1)}{x-1}dx$

$=\int(x^2+1)dx=\frac{1}{3}x^3+x+C$

11. $\int\frac{x^3-1}{x^2}dx=\int\left(\frac{x^3}{x^2}-\frac{1}{x^2}\right)dx$

$=\int\left(x-\frac{1}{x^2}\right)dx=\frac{x^2}{2}+\frac{1}{x}+C$

12. *Refer to answer 7.*

13. $\int 2^x\,dx=\frac{2^x}{\log 2}+C$

14. Let $I=\int\frac{\sin^6 x}{\cos^8 x}dx=\int\frac{\sin^6 x}{\cos^6 x\cdot\cos^2 x}dx$

$=\int\tan^6 x\sec^2 x\,dx$

Put $\tan x=t\Rightarrow\sec^2 x\,dx=dt$

$\therefore\quad I=\int t^6dt=\frac{t^7}{7}+C=\frac{1}{7}\tan^7 x+C$

15. Let $I=\int\frac{x+\cos 6x}{3x^2+\sin 6x}dx$

Put $3x^2+\sin 6x=t$

$\Rightarrow\quad(6x+6\cos 6x)dx=dt$

$\Rightarrow\quad(x+\cos 6x)dx=\frac{1}{6}dt\quad\therefore\ I=\int\frac{\frac{1}{6}dt}{t}=\frac{1}{6}\log t+C$

$=\frac{1}{6}\log(3x^2+\sin 6x)+C$

16. Let $I=\int\frac{(\log x)^2}{x}dx$

Put $\log x=t\Rightarrow\frac{1}{x}dx=dt$

$\therefore\quad I=\int t^2dt=\frac{t^3}{3}+C=\frac{(\log x)^3}{3}+C$

17. Let $I=\int\frac{e^{\tan^{-1}x}}{1+x^2}\cdot dx$

Put $\tan^{-1}x=t\Rightarrow\frac{1}{1+x^2}\cdot dx=dt$

$\therefore\quad I=\int e^t\,dt=e^t+C=e^{\tan^{-1}x}+C$

18. Let $I=\int \frac{2\cos x}{3\sin^2 x}dx$

Put $\sin x = t \Rightarrow \cos x dx = dt$

$\therefore \quad I=\frac{2}{3}\int \frac{dt}{t^2}=\frac{2}{3}\int t^{-2}dt=\frac{2}{3}\cdot\frac{t^{-1}}{-1}+C=\frac{2}{-3\sin x}+C$

19. Let $I=\int \frac{\log x}{x}dx$

Put $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$\therefore \quad I=\int t\,dt=\frac{t^2}{2}+C=\frac{(\log x)^2}{2}+C.$

20. Let $I=\int \sec^2(7-4x)dx$

Put $7-4x = t \Rightarrow dx = \frac{-1}{4}dt$

$\therefore \quad I=\int \frac{\sec^2 t}{-4}dt \Rightarrow I=\frac{\tan t}{-4}+C=\frac{\tan(7-4x)}{-4}+C$

21. Let $I=\int \cos 4x\cos 3x\,dx$

$I=\frac{1}{2}\int 2\cos 4x\cos 3x\,dx$

$I=\frac{1}{2}\int[\cos(4x+3x)+\cos(4x-3x)]dx$

$I=\frac{1}{2}\int[\cos 7x+\cos x]dx$

$I=\frac{1}{2}\left[\frac{\sin 7x}{7}+\sin x\right]+C$

$I=\frac{1}{14}\sin 7x+\frac{1}{2}\sin x+C$

22. Let $I=\int \frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta}d\theta$

$=3\int \frac{\sin\theta\cos\theta}{4+\sin^2\theta-4\sin\theta}d\theta-2\int \frac{\cos\theta}{4+\sin^2\theta-4\sin\theta}d\theta$

$= 3I_1 - 2I_2$ (say)

Now, $I_1=\int \frac{\sin\theta\cos\theta}{4+\sin^2\theta-4\sin\theta}d\theta$

Put $\sin^2\theta = t \Rightarrow 2\sin\theta\cos\theta\, d\theta = dt$

$\therefore \quad I_1=\frac{1}{2}\int \frac{dt}{4+t-4\sqrt{t}}=\frac{1}{2}\int \frac{dt}{(\sqrt{t}-2)^2}$

Put $\sqrt{t}-2=u \Rightarrow \sqrt{t}=u+2$

$\Rightarrow \quad \frac{1}{2\sqrt{t}}dt=du \Rightarrow dt=2(u+2)du$

$\therefore \quad I_1=\int \frac{(u+2)}{u^2}du=\int \frac{du}{u}+2\int \frac{du}{u^2}$

$=\log u-\frac{2}{u}+C_1=\log(\sqrt{t}-2)-\frac{2}{\sqrt{t}-2}+C_1$

$=\log(\sin\theta-2)-\frac{2}{\sin\theta-2}+C_1$

Also, $I_2=\int \frac{\cos\theta}{4+\sin^2\theta-4\sin\theta}d\theta$

Put $\sin\theta = m \Rightarrow \cos\theta\, d\theta = dm$

$\therefore \quad I_2=\int \frac{dm}{4+m^2-4m}=\int \frac{dm}{(m-2)^2}$

$=\frac{-1}{m-2}+C_2=\frac{-1}{\sin\theta-2}+C_2$

$\therefore \quad I=3\log(\sin\theta-2)-\frac{6}{\sin\theta-2}+\frac{2}{\sin\theta-2}+C,$

where $C = 3C_1 - 2C_2$

$\Rightarrow \quad I=3\log(\sin\theta-2)-\frac{4}{\sin\theta-2}+C$

23. Let $I=\int \frac{\sin(x-a)}{\sin(x+a)}dx=\int \frac{\sin(x+a-2a)}{\sin(x+a)}dx$

$=\int\left(\frac{\sin(x+a)\cos 2a-\cos(x+a)\sin 2a}{\sin(x+a)}\right)dx$

$\Rightarrow \quad I=\cos 2a\int dx-\sin 2a\int \frac{\cos(x+a)}{\sin(x+a)}dx$

Put $\sin(x+a) = t \Rightarrow \cos(x+a)dx = dt$

$\Rightarrow \quad I=\cos 2a\int dx-\sin 2a\int \frac{dt}{t}$

$= x\cos 2a - \sin 2a \log|\sin(x+a)| + C$

24. $\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cdot\cos^2 x}dx$

$=\int \frac{\sin^6 x}{\sin^2 x\cdot\cos^2 x}dx+\int \frac{\cos^6 x}{\sin^2 x\cdot\cos^2 x}dx$

$=\int \frac{\sin^4 x}{\cos^2 x}dx+\int \frac{\cos^4 x}{\sin^2 x}dx$

$=\int \frac{\sin^2 x(1-\cos^2 x)}{\cos^2 x}dx+\int \frac{\cos^2 x(1-\sin^2 x)}{\sin^2 x}dx$

$=\int\tan^2x\,dx-\int\sin^2x\,dx+\int\cot^2x\,dx-\int\cos^2x\,dx$

$=\int(\sec^2x-1)dx+\int(\text{cosec}^2x-1)\,dx-\int\sin^2x\,dx-\int(1-\sin^2x)\,dx$

$=\tan x-x+(-\cot x)-x-x+C=\tan x-\cot x-3x+C$

25. Here $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha} dx$

$= \int \frac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{\cos x - \cos\alpha} dx$

$= 2\int \frac{\cos^2 x - \cos^2\alpha}{\cos x - \cos\alpha} dx$

$= 2\int \frac{(\cos x + \cos\alpha)\cdot(\cos x - \cos\alpha)}{\cos x - \cos\alpha} dx$

$= 2\int (\cos x + \cos\alpha) dx = 2[\sin x + x\cos\alpha] + C$

26. $\int \sin x \sin 2x \sin 3x\, dx$

$= \int \sin 3x \sin x \sin 2x\, dx$

$= \frac{1}{2}\int (\cos 2x - \cos 4x)\sin 2x\, dx$

$= \frac{1}{2}\int \sin 2x \cos 2x\, dx - \frac{1}{2}\int \cos 4x \sin 2x\, dx$

$= \frac{1}{4}\int \sin 4x\, dx - \frac{1}{4}\int (\sin 6x - \sin 2x)\, dx$

$= \frac{1}{4}\int \sin 4x\, dx - \frac{1}{4}\int \sin 6x\, dx + \frac{1}{4}\int \sin 2x\, dx$

$= \frac{1}{4}\left[\frac{-\cos 4x}{4} - \frac{(-\cos 6x)}{6} + \frac{(-\cos 2x)}{2}\right] + C$

$= \frac{1}{4}\left[\frac{\cos 6x}{6} - \frac{\cos 4x}{4} - \frac{\cos 2x}{2}\right] + C$

27. $\int \frac{dx}{x^2 + (4)^2} = \frac{1}{4}\tan^{-1}\frac{x}{4} + C$

28. Let $I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$

Put $x^{3/2} = t \Rightarrow \frac{3}{2}x^{1/2} dx = dt$

$\therefore \quad I = \frac{2}{3}\int \frac{dt}{\sqrt{a^3 - t^2}} = \frac{2}{3}\int \frac{dt}{\sqrt{(a^{3/2})^2 - t^2}}$

$= \frac{2}{3}\left[\sin^{-1}\left(\frac{t}{a^{3/2}}\right)\right] + C = \frac{2}{3}\left[\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)\right] + C$

$= \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + C$

29. Let $I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx$

$= \int \left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\right) dx = \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx$

$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{2\sin x \cos x}} dx = \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{\sin 2x + 1 - 1}} dx$

$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{1 - (1 - \sin 2x)}} dx = \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}} dx$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\, dx = dt$

$\therefore \quad I = \sqrt{2}\int \frac{dt}{\sqrt{1 - t^2}} = \sqrt{2}\sin^{-1} t + C$

$= \sqrt{2}\sin^{-1}(\sin x - \cos x) + C$

30. Let $I = \int \frac{x + 2}{2x^2 + 6x + 5} dx$

We write, $x + 2 = A\left[\frac{d}{dx}(2x^2 + 6x + 5)\right] + B$

$\Rightarrow \quad x + 2 = A(4x + 6) + B$

Equating coefficients of x and constant terms, we get

$4A = 1 \Rightarrow A = 1/4$ and $6A + B = 2 \Rightarrow B = 1/2$

$\therefore \quad I = \frac{1}{4}\int \frac{4x + 6}{2x^2 + 6x + 5} dx + \frac{1}{2}\int \frac{dx}{2x^2 + 6x + 5}$

$= \frac{1}{4}\log\left|2x^2 + 6x + 5\right| + \frac{1}{4}\int \frac{dx}{x^2 + 3x + \frac{5}{2}}$

$= \frac{1}{4}\log\left|2x^2 + 6x + 5\right| + \frac{1}{4}\int \frac{dx}{x^2 + 3x + \frac{9}{4} - \frac{9}{4} + \frac{5}{2}}$

$= \frac{1}{4}\log\left|2x^2 + 6x + 5\right| + \frac{1}{4}\int \frac{dx}{\left(x + \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$= \frac{1}{4}\log\left|2x^2 + 6x + 5\right| + \frac{1}{2}\tan^{-1}\left(\frac{x + \frac{3}{2}}{\frac{1}{2}}\right) + C$

$= \frac{1}{4}\log\left|2x^2 + 6x + 5\right| + \frac{1}{2}\tan^{-1}(2x + 3) + C$

31. Let $I = \int \frac{x + 3}{\sqrt{5 - 4x - 2x^2}} dx$

Let $x + 3 = A\frac{d}{dx}(5 - 4x - 2x^2) + B = A(-4 - 4x) + B$

On comparing the like coefficients, we get

$x + 3 = -\frac{1}{4}(-4 - 4x) + 2$

$$\Rightarrow \quad I = \int \frac{-\frac{1}{4}(-4-4x)+2}{\sqrt{5-4x-2x^2}}dx$$

$$= \frac{-1}{4}\int \frac{-4-4x}{\sqrt{5-4x-2x^2}}dx + 2\cdot\int \frac{1}{\sqrt{5-4x-2x^2}}dx$$

$$\Rightarrow \quad -\frac{1}{4}I_1 + 2I_2 \qquad \text{...(1)}$$

where $I_1 = \int \frac{-4-4x}{\sqrt{5-4x-2x^2}}dx$

Put $5 - 4x - 2x^2 = t \Rightarrow (-4 - 4x)dx = dt$

$$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}dt = 2\sqrt{t} + C_1$$

$$= 2\sqrt{5-4x-2x^2} + C_1 \qquad \text{...(2)}$$

and $I_2 = \int \frac{dx}{\sqrt{5-4x-2x^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{5}{2}-2x-x^2}}$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2-(x+1)^2}} = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{7/2}}\right) + C_2 \qquad \text{...(3)}$$

From (1), (2) and (3), we get

$$I = -\frac{1}{4}\cdot 2\sqrt{5-4x-2x^2} + 2\cdot\frac{1}{\sqrt{2}}\sin^{-1}\left[\sqrt{\frac{2}{7}}(x+1)\right] + C$$

where $C = C_1 + C_2$

$$\Rightarrow \quad I = -\frac{1}{2}\sqrt{5-4x-2x^2} + \sqrt{2}\sin^{-1}\left[\sqrt{\frac{2}{7}}(x+1)\right] + C$$

32. Let $I = \int \frac{x+2}{\sqrt{x^2+5x+6}}dx = \int \frac{\frac{1}{2}(2x+5)-\frac{1}{2}}{\sqrt{x^2+5x+6}}dx$

$$= \frac{1}{2}\int (x^2+5x+6)^{-1/2}(2x+5)dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+6}}$$

Put $x^2 + 5x + 6 = t \Rightarrow (2x + 5)\, dx = dt$

$$\Rightarrow \quad I = \frac{1}{2}\int t^{-1/2}dt - \frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$= \frac{1}{2}\frac{t^{1/2}}{\frac{1}{2}} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + C$$

$$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|x+\frac{5}{2}+\sqrt{x^2+5x+6}\right| + C$$

33. $I = \int \frac{5x-2}{1+2x+3x^2}dx$

Let $5x-2 = A\frac{d}{dx}(1+2x+3x^2) + B = A(2+6x)$

On comparing the like coefficients, we get

$$5x-2 = \frac{5}{6}(2+6x) - \frac{5}{3} - 2 = \frac{5}{6}\frac{d}{dx}(1+2x+3x^2) - \frac{11}{3}$$

$$\therefore \quad I = \int \frac{\frac{5}{6}\frac{d}{dx}(1+2x+3x^2) - \frac{11}{3}}{1+2x+3x^2}dx$$

$$= \frac{5}{6}\int \frac{\frac{d}{dx}(1+2x+3x^2)}{1+2x+3x^2}dx - \frac{11}{3}\int \frac{dx}{1+2x+3x^2}$$

$$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3\cdot 3}\int \frac{dx}{x^2+\frac{2}{3}x+\frac{1}{3}}$$

$$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2}$$

$$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{9}\cdot\frac{1}{\sqrt{2/3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\sqrt{2/3}}\right) + C$$

$$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

34. Let $I = \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \int \frac{(x+1)+1}{\sqrt{x^2+2x+3}}dx$

Put $x^2 + 2x + 3 = t$

$$\Rightarrow \quad (2x+2)dx = dt \Rightarrow (x+1)dx = \frac{1}{2}dt$$

$$\therefore \quad I = \frac{1}{2}\int t^{-1/2}dt + \int \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}dx$$

$$= \frac{1}{2}\frac{t^{1/2}}{(1/2)} + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

$$\left[\because \int \frac{1}{\sqrt{x^2+a^2}}dx = \log\left|x+\sqrt{x^2+a^2}\right| + C\right]$$

$$= \sqrt{x^2+2x+3} + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

35. Let $I = \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$

$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{dx}{\sqrt{x^2+4x+10}}$

$= I_1 + I_2$ (say) ...(1)

where $I_1 = \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$

Put $x^2 + 4x + 10 = t \Rightarrow (2x+4)dx = dt$

$\therefore \quad I_1 = \frac{5}{2}\int t^{-1/2}dt = \frac{5}{2}\cdot\frac{t^{1/2}}{(1/2)} = 5\sqrt{t}$

$= 5\sqrt{x^2+4x+10} + C_1$ ...(2)

and $I_2 = -7\int \frac{dx}{\sqrt{x^2+4x+10}}dx$

$= -7\int \frac{dx}{\sqrt{(x+2)^2+(\sqrt{6})^2}}dx$

$= -7\log|x+2+\sqrt{x^2+4x+10}| + C_2$ ...(3)

From (1), (2) and (3), we get

$I = 5\sqrt{x^2+4x+10} - 7\log|x+2+\sqrt{x^2+4x+10}| + C,$

where $C = C_1 + C_2$

36. Let $I = \int \frac{\sin x - \cos x}{\sqrt{\sin 2x}}dx = \int \frac{\sin x - \cos x}{\sqrt{1+\sin 2x - 1}}dx$

$= \int \frac{\sin x - \cos x}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x - 1}}dx$

$= \int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2 - 1}}\cdot dx$

Put $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)\, dx = dt$

$\therefore \quad I = \int \frac{-dt}{\sqrt{t^2-1}} = -\log|t + \sqrt{t^2-1}| + C$

(where $t = \sin x + \cos x$)

$= -\log|\sin x + \cos x + \sqrt{\sin 2x}| + C$

37. Let $I = \int \frac{x^2+1}{x^4+1}dx$

$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx = \int \frac{dt}{t^2+\left(\sqrt{2}\right)^2}$

$\left[\because x - \frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt\right]$

$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(x-\frac{1}{x}\right)}{\sqrt{2}} + C$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + C$

38. Let $I = \int \frac{x^2+4}{x^4+16}dx$

$\Rightarrow \quad I = \int \frac{1+\frac{4}{x^2}}{x^2+\frac{16}{x^2}}dx = \int \frac{1+\frac{4}{x^2}}{\left(x-\frac{4}{x}\right)^2+8}dx$

Put $x - \frac{4}{x} = t \Rightarrow \left(1+\frac{4}{x^2}\right)dx = dt$

$\therefore \quad I = \int \frac{dt}{t^2+8} = \int \frac{dt}{t^2+(2\sqrt{2})^2}$

$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t}{2\sqrt{2}}\right) + C = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-4}{2\sqrt{2}x}\right) + C$

39. Let $I = \int \frac{x+2}{\sqrt{(x-2)(x-3)}}dx = \int \frac{x+2}{\sqrt{x^2-5x+6}}dx$

We write,

$x+2 = A\left[\frac{d}{dx}\left(x^2-5x+6\right)\right] + B = A(2x-5) + B$

Equating coefficients of x and constant terms, we get

$2A = 1 \Rightarrow A = \frac{1}{2}$ and $-5A + B = 2 \Rightarrow B = \frac{9}{2}$

$\therefore \quad I = \frac{1}{2}\int \frac{2x-5}{\sqrt{x^2-5x+6}}dx + \frac{9}{2}\int \frac{dx}{\sqrt{x^2-5x+\frac{25}{4}-\frac{25}{4}+6}}$

$= \frac{1}{2}\cdot 2\cdot\sqrt{x^2-5x+6} + \frac{9}{2}\int \frac{dx}{\sqrt{\left(x-\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$

$= \sqrt{x^2-5x+6} + \frac{9}{2}\log\left|\left(x-\frac{5}{2}\right)+\sqrt{x^2-5x+6}\right| + C$

40. Let $I=\int\frac{dx}{\sqrt{5-4x-2x^2}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{5}{2}-2x-x^2}}$

$=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\frac{7}{2}-1-2x-x^2}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2-(x+1)^2}}$

$=\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{\frac{7}{2}}}\right)+C=\frac{1}{\sqrt{2}}\sin^{-1}\left[\sqrt{\frac{2}{7}}(x+1)\right]+C$

41. Let $I=\int\frac{2x+5}{\sqrt{7-6x-x^2}}dx$

We write,

$2x+5=A\left[\frac{d}{dx}(7-6x-x^2)\right]+B=A[-6-2x]+B$

Equating coefficients of x and constant terms, we get
$2=-2A\Rightarrow A=-1$ and $-6A+B=5\Rightarrow B=-1$

$\therefore\quad I=-\int\frac{(-6-2x)dx}{\sqrt{7-6x-x^2}}-\int\frac{dx}{\sqrt{7-6x-x^2}}$

$=-2\sqrt{7-6x-x^2}-\int\frac{dx}{\sqrt{7+9-9-6x-x^2}}$

$=-2\sqrt{7-6x-x^2}-\int\frac{dx}{\sqrt{4^2-(x+3)^2}}$

$=-2\sqrt{7-6x-x^2}-\sin^{-1}\left(\frac{x+3}{4}\right)+C$

42. *Refer to answer 36.*

43. Let $I=\int\frac{x+3}{x^2-2x-5}dx$

We write, $x+3=A\left[\frac{d}{dx}(x^2-2x-5)\right]+B$

$\Rightarrow\quad x+3=A(2x-2)+B$

Equating coefficients of x and constant terms, we get
$2A=1\Rightarrow A=1/2$ and $-2A+B=3\Rightarrow B=4$

$\therefore\quad I=\frac{1}{2}\int\frac{2x-2}{x^2-2x-5}dx+4\int\frac{dx}{x^2-2x-5}$

$=\frac{1}{2}\log\left|x^2-2x-5\right|+4\int\frac{dx}{x^2-2x+1-6}$

$=\frac{1}{2}\log\left|x^2-2x-5\right|+4\int\frac{dx}{(x-1)^2-(\sqrt{6})^2}$

$=\frac{1}{2}\log\left|x^2-2x-5\right|+\frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right|+C$

$=\frac{1}{2}\log\left|x^2-2x-5\right|+\frac{\sqrt{2}}{\sqrt{3}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right|+C$

44. *Refer to answer 37.*

45. Let $I=\int\frac{x}{x^2+x+1}dx$

We write, $x=A\left[\frac{d}{dx}\left(x^2+x+1\right)\right]+B=A[2x+1]+B$

Equating coefficients of x and constant terms, we get
$2A=1\Rightarrow A=1/2$ and $A+B=0\Rightarrow B=-1/2$

$\therefore\quad I=\frac{1}{2}\int\frac{2x+1}{x^2+x+1}dx-\frac{1}{2}\int\frac{dx}{x^2+x+1}$

$=\frac{1}{2}\log\left|x^2+x+1\right|-\frac{1}{2}\int\frac{dx}{x^2+x+\frac{1}{4}+\frac{3}{4}}$

$=\frac{1}{2}\log\left|x^2+x+1\right|-\frac{1}{2}\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$=\frac{1}{2}\log\left|x^2+x+1\right|-\frac{1}{2\cdot\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C$

$=\frac{1}{2}\log\left|x^2+x+1\right|-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$

46. Let $I=\int\frac{1}{\cos^4x+\sin^4x}dx=\int\frac{\sec^4x}{1+\tan^4x}dx$

$=\int\frac{(\tan^2x+1)\sec^2x}{1+\tan^4x}dx$

Put $\tan x=t\Rightarrow\sec^2x\,dx=dt$

$\therefore\quad I=\int\frac{t^2+1}{t^4+1}dt=\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt=\int\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt$

Put $t-\frac{1}{t}=y\Rightarrow\left(1+\frac{1}{t^2}\right)dt=dy$

$\therefore\quad I=\int\frac{dy}{y^2+(\sqrt{2})^2}=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right)+C$

$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x-\cot x}{\sqrt{2}}\right)+C$

47. Let $I=\int \frac{1}{\sin^4 x+\sin^2 x\cos^2 x+\cos^4 x}dx$

$=\int \frac{\sec^4 x\,dx}{\tan^4 x+\tan^2 x+1} = \int \frac{(1+\tan^2 x)\sec^2 x dx}{\tan^4 x+\tan^2 x+1}$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$I=\int \frac{1+t^2}{t^4+t^2+1}dt = \int \frac{1+\frac{1}{t^2}}{t^2+1+\frac{1}{t^2}}dt$

$=\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+3}dt$

Put $t-\frac{1}{t}=y \Rightarrow \left(1+\frac{1}{t^2}\right)dt = dy$

Thus, $I=\int \frac{dy}{y^2+\left(\sqrt{3}\right)^2} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{y}{\sqrt{3}}\right)+C$

$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{3}}+C=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x-\cot x}{\sqrt{3}}\right)+C$

48. Let $I=\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx=\int \frac{6x+7}{\sqrt{x^2-9x+20}}dx$

Let $6x+7=A\left[\frac{d}{dx}(x^2-9x+20)\right]+B$

$\therefore$ $6x + 7 = A[2x - 9] + B$

Equating the coefficients of like terms from both sides, we get

$2A = 6$ and $-9A + B = 7$

$\Rightarrow$ $A = 3$ and $-9(3) + B = 7 \Rightarrow B = 7 + 27 = 34$

$\therefore$ $I=\int \frac{3(2x-9)}{\sqrt{x^2-9x+20}}dx+\int \frac{34}{\sqrt{x^2-9x+20}}dx$

Put $x^2 - 9x + 20 = t$ in first integral

$\therefore$ $I=\int \frac{3}{\sqrt{t}}dt+34\int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2+20-\frac{81}{4}}}$

$=3\int t^{-1/2}dt+34\int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2-\frac{1}{4}}}$

$=3\frac{t^{1/2}}{1/2}+34\int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$

$=6\sqrt{t}+34\log\left|\left(x-\frac{9}{2}\right)+\sqrt{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+C$

$=6\sqrt{x^2-9x+20}+34\log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right|+C$

49. Let $I=\int \frac{x^2}{x^4+x^2-2}dx=\int \frac{x^2}{(x^2-1)(x^2+2)}dx$

Let $x^2 = z$

$\therefore$ $\frac{x^2}{(x^2-1)(x^2+2)}=\frac{z}{(z-1)(z+2)}$

Using partial fractions, we have

$\frac{z}{(z-1)(z+2)}=\frac{A}{z-1}+\frac{B}{z+2}$

$\Rightarrow$ $z = A(z + 2) + B(z - 1)$

When $z = 1$, we get $A=\frac{1}{3}$

and when $z = -2$, we get $B=\frac{2}{3}$

$\therefore$ $I=\int \frac{x^2}{(x^2-1)(x^2+2)}dx$

$=\int \frac{1/3}{(x^2-1)}dx+\int \frac{2/3}{(x^2+2)}dx$

$=\frac{1}{3}\int \frac{1}{x^2-1}dx+\frac{2}{3}\int \frac{1}{x^2+(\sqrt{2})^2}dx$

$=\frac{1}{3}\times\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+\frac{2}{3}\times\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$

$=\frac{1}{6}\log\left|\frac{x-1}{x+1}\right|+\frac{\sqrt{2}}{3}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$

50. Let $I=\int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx$

Let $x^2 = t$

$\therefore$ $\frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}=\frac{(t+1)(t+4)}{(t+3)(t-5)}$

$=\frac{t^2+5t+4}{(t+3)(t-5)}=1+\frac{7t+19}{(t+3)(t-5)}$

Let $\frac{7t+19}{(t+3)(t-5)} = \frac{A}{t+3} + \frac{B}{t-5}$

$\Rightarrow$ $7t + 19 = A(t - 5) + B(t + 3)$

Putting $t = 5$, we get $B = \frac{27}{4}$

Putting $t = -3$, we get $A = \frac{1}{4}$

$\therefore$ $\frac{t^2+5t+4}{(t+3)(t-5)} = 1 + \frac{1}{4(t+3)} + \frac{27}{4(t-5)}$

$$\Rightarrow \quad I = \int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}dx = \int dx + \frac{1}{4}\int \frac{1}{(x^2+3)}dx + \frac{27}{4}\int \frac{1}{(x^2-5)}dx$$

$$= x + \frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{27}{4}\times\frac{1}{2\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + C$$

$$= x + \frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{27}{8\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + C$$

51. Let $I = \int \frac{1}{\sin x + \sin 2x}dx$

$$= \int \frac{1}{\sin x + 2\sin x\cos x}dx = \int \frac{1}{\sin x(1+2\cos x)}dx$$

$$= \int \frac{\sin x}{\sin^2 x(1+2\cos x)}dx$$

Let $u = \cos x \Rightarrow du = -\sin x dx$

Also, $\sin^2 x = 1 - \cos^2 x = 1 - u^2$

$$\therefore \quad I = \int \frac{-1}{(1-u^2)(1+2u)}du$$

$$= \int \frac{-1}{(1+u)(1-u)(1+2u)}du$$

Using partial fractions, we have

$$\frac{-1}{(1+u)(1-u)(1+2u)} = \frac{A}{(1+u)} + \frac{B}{(1-u)} + \frac{C}{(1+2u)}$$

$\Rightarrow$ $-1 = A(1 - u)(1 + 2u) + B(1 + u)(1 + 2u) + C(1 + u)(1 - u)$

Put $u = 1$, we get $B = -1/6$

Put $u = -1$, we get $A = 1/2$

put $u = -\frac{1}{2}$, we get $C = \frac{-4}{3}$

So, $\frac{-1}{(1+u)(1-u)(1+2u)}$

$$= \frac{1}{2(1+u)} - \frac{1}{6(1-u)} - \frac{4}{3(1+2u)}$$

$$\Rightarrow \quad I = \int\left[\frac{1}{2(1+u)} - \frac{1}{6(1-u)} - \frac{4}{3(1+2u)}\right]du$$

$$= \frac{1}{2}\log(1+u) + \frac{1}{6}\log(1-u) - \frac{4}{3\times 2}\log(1+2u) + C$$

$$= \frac{1}{2}\log(1+\cos x) + \frac{1}{6}\log(1-\cos x) - \frac{2}{3}\log(1+2\cos x) + C$$

52. Let $I = \int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$ ...(1)

Let $\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2}$ ...(2)

$\Rightarrow$ $x^2 + x + 1 = (Ax + B)(x + 2) + C(x^2 + 1)$

Put $x = 0, 1, -2$ in it to get

$1 = 2B + C$; $3 = 3(A + B) + 2C$ and $3 = 5C$

$\Rightarrow$ $C = \frac{3}{5}, B = \frac{1}{5}$ and $A = \frac{2}{5}$

Hence, from (2)

$$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{\left(\frac{2}{5}x + \frac{1}{5}\right)}{x^2+1} + \frac{\frac{3}{5}}{x+2}$$

$$= \frac{1}{5}\cdot\frac{2x+1}{x^2+1} + \frac{3}{5}\cdot\frac{1}{x+2}$$

$$\therefore \quad I = \frac{1}{5}\int\frac{2x+1}{x^2+1}dx + \frac{3}{5}\int\frac{1}{x+2}dx$$

$$= \frac{1}{5}\left[\int\frac{2x}{x^2+1}dx + \int\frac{dx}{x^2+1}\right] + \frac{3}{5}\int\frac{dx}{x+2}$$

$$= \frac{1}{5}\left[\log|x^2+1| + \tan^{-1}x\right] + \frac{3}{5}\log|x+2| + C_1$$

53. Let $I = \int \frac{x^2}{(x^2+4)(x^2+9)}dx$

Put $x^2 = y$. Then $\frac{x^2}{(x^2+4)(x^2+9)} = \frac{y}{(y+4)(y+9)}$

Let $\frac{y}{(y+4)(y+9)} = \frac{A}{y+4} + \frac{B}{y+9}$ ...(1)

$\Rightarrow$ $y = A(y + 9) + B(y + 4)$ ...(2)

Putting $y = -4$ and $y = -9$ successively in (2), we get

$A = \frac{-4}{5}$ and $B = \frac{9}{5}$

Substituting the values of A and B in (1), we get

$$\frac{y}{(y+4)(y+9)} = \frac{-4/5}{(y+4)} + \frac{9/5}{(y+9)}$$

$\Rightarrow \quad \dfrac{x^2}{(x^2+4)(x^2+9)} = \dfrac{-4}{5(x^2+4)} + \dfrac{9}{5(x^2+9)}$

$\therefore \quad I = \int \dfrac{x^2}{(x^2+4)(x^2+9)} dx$

$= \dfrac{-4}{5}\int \dfrac{1}{(x^2+4)} dx + \dfrac{9}{5}\int \dfrac{1}{(x^2+9)} dx$

$= \dfrac{-4}{5} \times \dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right) + \dfrac{9}{5} \times \dfrac{1}{3}\tan^{-1}\left(\dfrac{x}{3}\right) + C$

$= \dfrac{-2}{5}\tan^{-1}\left(\dfrac{x}{2}\right) + \dfrac{3}{5}\tan^{-1}\left(\dfrac{x}{3}\right) + C$

54. Let $I = \int \dfrac{x}{(x-1)^2(x+2)} dx$. ...(1)

Let $\dfrac{x}{(x-1)^2(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2}$ ...(2)

$\Rightarrow \quad x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$ ...(3)

Comparing coeffs. of x^2, x and constants in (3), we get $0 = A + C$; $1 = A + B - 2C$; $0 = -2A + 2B + C$

Solving these, we get

$A = \dfrac{2}{9}$, $B = \dfrac{1}{3}$, $C = -\dfrac{2}{9}$

$\therefore \quad \dfrac{x}{(x-1)^2(x+2)} = \dfrac{2}{9}\cdot\dfrac{1}{x-1} + \dfrac{1}{3}\cdot\dfrac{1}{(x-1)^2} - \dfrac{2}{9}\cdot\dfrac{1}{x+2}$

$\therefore \quad I = \dfrac{2}{9}\int \dfrac{1}{x-1}dx + \dfrac{1}{3}\int \dfrac{1}{(x-1)^2}dx - \dfrac{2}{9}\int \dfrac{1}{x+2}dx + C_1$

$= \dfrac{2}{9}\log\left|\dfrac{x-1}{x+2}\right| - \dfrac{1}{3}\cdot\dfrac{1}{x-1} + C_1$

55. Let $I = \int \dfrac{x}{(x^2+1)(x-1)} dx$

Let $\dfrac{x}{(x^2+1)(x-1)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-1}$ ...(1)

$\Rightarrow \quad x = (Ax + B)(x - 1) + C(x^2 + 1)$ ...(2)

Comparing coefficients of x^2, x and constant terms, we get

$0 = A + C$; $1 = B - A$; $0 = -B + C$

Solving these, we get

$A = -\dfrac{1}{2}, C = \dfrac{1}{2}, B = \dfrac{1}{2}$

$\therefore$ From (1), we get

$\dfrac{x}{(x^2+1)(x-1)} = \dfrac{-\frac{1}{2}(x-1)}{x^2+1} + \dfrac{1}{2}\cdot\dfrac{1}{x-1}$

$= -\dfrac{1}{2}\cdot\dfrac{x}{x^2+1} + \dfrac{1}{2}\cdot\dfrac{1}{x^2+1} + \dfrac{1}{2}\cdot\dfrac{1}{x-1}$

$\therefore \quad I = -\dfrac{1}{4}\int \dfrac{2x}{x^2+1}dx + \dfrac{1}{2}\int \dfrac{dx}{x^2+1} + \dfrac{1}{2}\int \dfrac{dx}{x-1} + C$

$\Rightarrow \quad I = -\dfrac{1}{4}\log|x^2+1| + \dfrac{1}{2}\tan^{-1}x + \dfrac{1}{2}\log|x-1| + C$

56. *Refer to answer 53.*

57. $I = \int \dfrac{x^3}{x^4+3x^2+2} dx$

Put $x^2 = t \Rightarrow xdx = \dfrac{1}{2}dt$

$\therefore \quad I = \dfrac{1}{2}\int \dfrac{t}{t^2+3t+2}dt = \dfrac{1}{2}\int \dfrac{t}{(t+2)(t+1)}dt$

Let $\dfrac{t}{(t+2)(t+1)} = \dfrac{A}{(t+2)} + \dfrac{B}{(t+1)}$

$\Rightarrow \quad t = A(t + 1) + B(t + 2)$

Put $t = -1, 2$ in it, we get $A = 2$, $B = -1$

$\therefore \quad \dfrac{t}{(t+2)(t+1)} = \dfrac{2}{t+2} - \dfrac{1}{t+1}$

$\Rightarrow \quad I = \dfrac{1}{2}\int \left[\dfrac{2}{(t+2)} - \dfrac{1}{(t+1)}\right]dt$

$= \dfrac{1}{2}[2\log|t+2| - \log|t+1|] + C$

$= \dfrac{1}{2}[2\log|x^2+2| - \log|x^2+1|] + C$

58. Let $I = \int \dfrac{2x^2+1}{x^2(x^2+4)} dx$

Let $x^2 = y$, then $\dfrac{2x^2+1}{x^2(x^2+4)} = \dfrac{2y+1}{y(y+4)}$

$\Rightarrow \quad \dfrac{2y+1}{y(y+4)} = \dfrac{A}{y} + \dfrac{B}{y+4}$

$\Rightarrow \quad 2y + 1 = A(y + 4) + By$

Put $y = 0, -4$ in it to get

$A = \dfrac{1}{4}$ and $B = \dfrac{7}{4}$

$\therefore \quad \dfrac{2y+1}{y(y+4)} = \dfrac{1}{4}\cdot\dfrac{1}{y} + \dfrac{7}{4}\cdot\dfrac{1}{y+4}$

$\Rightarrow \quad \dfrac{2x^2+1}{x^2(x^2+4)} = \dfrac{1}{4}\cdot\dfrac{1}{x^2} + \dfrac{7}{4}\cdot\dfrac{1}{x^2+4}$

Integrating both sides w.r.t. x, we get

$$\int \frac{2x^2+1}{x^2(x^2+4)}dx = \frac{1}{4}\int x^{-2}dx + \frac{7}{4}\int \frac{dx}{x^2+2^2}$$

$$= \frac{1}{4}\cdot\frac{x^{-1}}{-1} + \frac{7}{4}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= -\frac{1}{4x} + \frac{7}{8}\tan^{-1}\left(\frac{x}{2}\right) + C$$

59. Let $I = \int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$

Put $x^2 = y$. Then,

$$\frac{x^2+1}{(x^2+4)(x^2+25)} = \frac{y+1}{(y+4)(y+25)}$$

Let $\frac{y+1}{(y+4)(y+25)} = \frac{A}{y+4} + \frac{B}{y+25}$ ...(1)

$\Rightarrow$ $y+1 = A(y+25) + B(y+4)$ ...(2)

Putting $y = -4$ and $y = -25$ successively in (2), we get $A = -\frac{1}{7}$ and $B = \frac{8}{7}$

Substituting the values of A and B in (1), we get

$$\frac{y+1}{(y+4)(y+25)} = \frac{-1/7}{(y+4)} + \frac{8/7}{(y+25)}$$

$$\Rightarrow \quad \frac{x^2+1}{(x^2+4)(x^2+25)} = \frac{-1}{7(x^2+4)} + \frac{8}{7(x^2+25)}$$

$$\therefore \quad I = \int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$$

$$= -\frac{1}{7}\int \frac{1}{x^2+2^2}dx + \frac{8}{7}\int \frac{1}{x^2+5^2}dx$$

$$= -\frac{1}{7}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{7}\times\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C$$

$$= -\frac{1}{14}\tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right) + C$$

60. Let $I = \int \frac{dx}{x(x^5+3)}$

Put $x^5 + 3 = t \Rightarrow 5x^4dx = dt$

$$\Rightarrow \quad dx = \frac{dt}{5x^4}$$

$$\therefore \quad I = \frac{1}{5}\int \frac{dt}{t(t-3)} = \frac{1}{5}\int\left[\frac{1}{3(t-3)} - \frac{1}{3t}\right]dt$$

$$= \frac{1}{15}[\log|t-3| - \log|t|] + C = \frac{1}{15}\log\left|\frac{t-3}{t}\right| + C$$

$$= \frac{1}{15}\log\left|\frac{x^5}{x^5+3}\right| + C$$

61. Let $I = \int \frac{dx}{x(x^3+8)}$

Put $x^3 + 8 = t \Rightarrow 3x^2dx = dt$

$$\Rightarrow \quad dx = \frac{dt}{3x^2} \quad \therefore \quad I = \frac{1}{3}\int \frac{dt}{t(t-8)}$$

$$= \frac{1}{3}\int \frac{1}{8}\left[\frac{1}{t-8} - \frac{1}{t}\right]dt = \frac{1}{24}[\log|t-8| - \log|t|] + C$$

$$= \frac{1}{24}\log\left|\frac{t-8}{t}\right| + C = \frac{1}{24}\log\left|\frac{x^3}{x^3+8}\right| + C$$

62. *Refer to answer 61.*

63. Let $I = \int \frac{3x+1}{(x+1)^2(x+3)}dx$

Let $\frac{3x+1}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$ ...(1)

$\Rightarrow$ $3x + 1 = A(x+1)(x+3) + B(x+3) + C(x+1)^2$ ...(2)

Put $x = -1, -3, 0$ in (2), we get

$B = -1; C = -2; A = 2$

$\therefore$ From (1),

$$\frac{3x+1}{(x+1)^2(x+3)} = \frac{2}{x+1} - \frac{1}{(x+1)^2} - \frac{2}{x+3}$$

Integrating both sides w.r.t. x, we get

$$I = \int \frac{2}{x+1}dx - \int \frac{1}{(x+1)^2}dx - \int \frac{2}{(x+3)}dx$$

$$= 2\log|x+1| + \frac{1}{x+1} - 2\log|x+3| + C$$

$$= 2\log\left|\frac{x+1}{x+3}\right| + \frac{1}{x+1} + C$$

64. Let $I = \int \frac{3x+5}{x^3-x^2-x+1}dx$

Here, $x^3 - x^2 - x + 1 = x^2(x-1) - 1(x-1)$

$= (x^2-1)(x-1) = (x-1)(x+1)(x-1) = (x-1)^2(x+1)$

Let $\frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{(x-1)^2(x+1)}$

$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$ ...(1)

$\Rightarrow\quad 3x + 5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$ ...(2)

Put $x = 1, -1, 0$ in (2) to get

$B = 4;\ C = \frac{1}{2};\ A = -\frac{1}{2}$

From (1),

$$\frac{3x+5}{x^3-x^2-x+1} = -\frac{1}{2}\cdot\frac{1}{x-1} + \frac{4}{(x-1)^2} + \frac{1}{2}\cdot\frac{1}{x+1}$$

Integrating, we get

$$I = -\frac{1}{2}\int\frac{dx}{x-1} + 4\int\frac{dx}{(x-1)^2} + \frac{1}{2}\int\frac{dx}{x+1}$$

$$= -\frac{1}{2}\log|x-1| - \frac{4}{x-1} + \frac{1}{2}\log|x+1| + C_1$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + C_1$$

65. Let $I = \int\frac{8}{(x+2)(x^2+4)}\,dx$

Let $\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$ ...(1)

$\Rightarrow\quad 8 = A(x^2+4) + (Bx+C)(x+2)$ ...(2)

Put $x = -2, 0, 1$ in (2), we get

$A = 1;\ C = 2;\ B = -1$

$\therefore$ From (1),

$$\frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{-x+2}{x^2+4}$$

Integrating, we get

$$I = \int\frac{8}{(x-2)(x^2+4)}dx = \int\frac{dx}{x+2} + \int\frac{-x+2}{x^2+4}dx$$

$$= \log(x+2) - \frac{1}{2}\int\frac{2x}{x^2+4}\,dx + 2\int\frac{dx}{x^2+2^2} + C_1$$

$$= \log(x+2) - \frac{1}{2}\log(x^2+4) + 2\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C_1$$

$$= \log\left(\frac{x+2}{\sqrt{x^2+4}}\right) + \tan^{-1}\left(\frac{x}{2}\right) + C_1$$

66. Let $I = \int\frac{2}{(1-x)(1+x^2)}dx$

Let $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}$

$\Rightarrow\quad 2 = A(1+x^2) + (Bx+C)(1-x)$

$\Rightarrow\quad 2 = (A-B)x^2 + (B-C)x + (A+C)$

Comparing coefficients of x^2, x and constant term, we get $A = B = C = 1$

Hence $\frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$

$$\therefore\quad I = \int\frac{-1}{x-1}\,dx + \frac{1}{2}\int\frac{2x}{1+x^2}dx + \int\frac{1}{1+x^2}\,dx$$

$$= -\log(x-1) + \frac{1}{2}\log(1+x^2) + \tan^{-1}x + C_1$$

67. Let $I = \int\frac{2x}{(x^2+1)(x^2+3)}\,dx$

Put $x^2 = y \Rightarrow 2xdx = dy \quad \therefore \quad I = \int\frac{dy}{(y+1)(y+3)}$

We write, $\frac{1}{(y+1)(y+3)} = \frac{A}{y+1} + \frac{B}{y+3}$

$\Rightarrow\quad 1 = A(y+3) + B(y+1)$ ...(1)

Putting $y = -1$ in (1), we get $1 = 2A \Rightarrow A = \frac{1}{2}$

Putting $y = -3$ in (1), we get $1 = -2B \Rightarrow B = -\frac{1}{2}$

$$\Rightarrow\quad \frac{1}{(y+1)(y+3)} = \frac{1}{2(y+1)} - \frac{1}{2(y+3)}$$

$$\therefore\quad I = \int\left[\frac{1}{2(y+1)} - \frac{1}{2(y+3)}\right]dy$$

$$= \frac{1}{2}\int\frac{dy}{y+1} - \frac{1}{2}\int\frac{dy}{y+3}$$

$$= \frac{1}{2}\log|y+1| - \frac{1}{2}\log|y+3| + C$$

$$= \frac{1}{2}\log\left|\frac{y+1}{y+3}\right| + C = \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$$

68. Since $\frac{1-x^2}{x(1-2x)} = \frac{1-x^2}{x-2x^2}$ is an improper fraction, therefore, we convert it into a proper fraction by long division method. We get

$$\frac{x^2-1}{2x^2-x} = \frac{1}{2} + \frac{\frac{x}{2}-1}{2x^2-x}$$

$$\Rightarrow\quad \int\frac{(-1+x^2)}{-x+2x^2}dx = \frac{1}{2}\int dx + \frac{1}{2}\int\frac{x-2}{2x^2-x}\,dx$$

Let $\frac{x-2}{2x^2-x} = \frac{x-2}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1}$

$\Rightarrow\quad x - 2 = A(2x-1) + Bx$ ...(1)

Putting $x = 0$ in (1), we get

$-2 = A(-1) \Rightarrow A = 2$

Putting $x = \frac{1}{2}$ in (1), we get

$$\frac{1}{2} - 2 = B\left(\frac{1}{2}\right) \Rightarrow 1 - 4 = B \Rightarrow B = -3$$

$$\therefore \quad \frac{x-2}{2x^2 - x} = \frac{2}{x} - \frac{3}{2x-1} = \frac{2}{x} + \frac{3}{1-2x}$$

$$\Rightarrow \quad I = \int \frac{1-x^2}{x(1-2x)} dx$$

$$= \frac{1}{2}\int (1) dx + \frac{1}{2}\int \left(\frac{2}{x} + \frac{3}{1-2x}\right) dx$$

$$= \frac{1}{2}x + \log|x| - \frac{3}{4}\log|1-2x| + C$$

69. Let $I = \int \frac{dx}{(x^2+1)(x^2+2)}$.

Put $x^2 = y$

Let $\frac{1}{(x^2+1)(x^2+2)} = \frac{1}{(y+1)(y+2)} = \frac{A}{(y+1)} + \frac{B}{(y+2)}$

$\Rightarrow \quad 1 = A(y + 2) + B(y + 1)$ ...(1)

Putting $y = -1$ in (1), we get $1 = A(-1 + 2) \Rightarrow A = 1$

Putting $y = -2$ in (1), we get $1 = B(-2 + 1) \Rightarrow B = -1$

$$\therefore \quad I = \int \frac{dx}{x^2+1} - \int \frac{dx}{x^2+2}$$

$$= \tan^{-1} x - \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

70. *Refer to answer 51.*

71. Let $I = \int \frac{\sin x}{(1-\cos x)(2-\cos x)} dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$\therefore \ I = \int \frac{-dt}{(1-t)(2-t)}$$

We write, $\frac{-1}{(1-t)(2-t)} = \frac{A}{1-t} + \frac{B}{2-t}$

$\Rightarrow \ -1 = A(2 - t) + B(1 - t)$ ...(1)

Putting $t = 1$ in (1), we get

$-1 = A(2 - 1) \Rightarrow A = -1$

Putting $t = 2$ in (1), we get

$-1 = B(1 - 2) \Rightarrow B = 1$

$$\therefore \quad I = \int \frac{-dt}{1-t} + \int \frac{dt}{2-t} = \log|1-t| - \log|2-t| + C$$

$$= \log\left|\frac{1-t}{2-t}\right| + C = \log\left|\frac{1-\cos x}{2-\cos x}\right| + C$$

72. *Refer to answer 71.*

73. Let $I = \int \frac{2x+1}{(x+2)(x-3)} dx$

Let $\frac{2x+1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$

$\Rightarrow \quad 2x + 1 = A(x - 3) + B(x + 2)$ ...(1)

Putting $x = -2$ in (1), we get

$-3 = A(-2 - 3) \Rightarrow A = \frac{3}{5}$

Putting $x = 3$ in (1), we get

$7 = B(3 + 2) \Rightarrow B = \frac{7}{5}$

$$\therefore \quad I = \frac{3}{5}\int \frac{1}{x+2} dx + \frac{7}{5}\int \frac{1}{x-3} dx$$

$$= \frac{3}{5}\log|x+2| + \frac{7}{5}\log|x-3| + C$$

74. Let $I = \int \frac{x^2+x+1}{(x+1)^2(x+2)} dx$

Let $\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$ ...(1)

$\Rightarrow \quad x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$

Put $x = -1, -2, 0$ in it, to get

$B = 1; C = 3; A = -2$

From (1), we get

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}$$

Integrating both sides, we get

$$I = \int \frac{x^2+x+1}{(x+1)^2(x+2)} dx$$

$$= -2\int \frac{1}{x+1} dx + \int \frac{1}{(x+1)^2} dx + 3\int \frac{1}{x+2} dx$$

$$= -2\log|x+1| - \frac{1}{x+2} + 3\log|x+2| + C_1$$

75. Let $I = \int \frac{x^2+1}{(x-1)^2(x+3)} dx$

Let $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$

$\Rightarrow \quad x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$ ...(1)

Put $x = 1$ in (1) we get $B = \frac{1}{2}$

Put $x = -3$ in (1) we get $C = \frac{5}{8}$

Put $x = 0$ in (1) we get $A = \frac{3}{8}$

$$\therefore \quad \frac{x^2+1}{(x-1)^2(x+3)} = \frac{3}{8}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{(x-1)^2} + \frac{5}{8}\cdot\frac{1}{x+3}$$

Integrating both sides, we get

$$I = \int \frac{x^2+1}{(x-1)^2(x+3)}\,dx = \frac{3}{8}\int\frac{dx}{(x-1)} + \frac{1}{2}\int\frac{dx}{(x-1)^2} + \frac{5}{8}\int\frac{dx}{x+3}$$

$$= \frac{3}{8}\log|x-1| - \frac{1}{2}\cdot\frac{1}{(x-1)} + \frac{5}{8}\log|x+3| + C_1$$

76. Given : $\int e^x(\tan x+1)\sec x\,dx = e^x f(x) + C$ ...(1)

$$\text{L.H.S.} = \int e^x(\tan x+1)\sec x\,dx$$

$$= \int e^x(\sec x + \sec x\tan x)\,dx$$

$$= \int e^x \sec x\,dx + \int e^x \sec x\tan x\,dx$$

(Integrating first integral by parts)

$$= e^x\sec x - \int e^x \sec x\tan x\,dx + \int e^x\sec x\tan x\,dx + C$$

$= e^x \sec x + C = e^x f(x) + C$ (by (1))

On comparing, we get $f(x) = \sec x$

77. $I = \int x\log 2x\,dx.$

$$I = \log 2x\int x\,dx - \int\left[\frac{d}{dx}\log 2x\int x\,dx\right]dx$$

$$= \log 2x\cdot\frac{x^2}{2} - \int\frac{1}{2x}\cdot 2\cdot\frac{x^2}{2}\,dx + C$$

$$= \frac{x^2}{2}\log 2x - \frac{1}{2}\cdot\int x\,dx + C$$

$$= \frac{x^2}{2}\log 2x - \frac{1}{2}\cdot\frac{x^2}{2} + C = \frac{x^2}{2}\log 2x - \frac{x^2}{4} + C$$

78. Let $I = \int (3x+1)\sqrt{4-3x-2x^2}\,dx$

Let $3x+1 = \lambda\frac{d}{dx}(4-3x-2x^2) + \mu$

$\Rightarrow \quad 3x + 1 = \lambda(-3 - 4x) + \mu$

$\Rightarrow \quad 3x + 1 = -3\lambda + \mu - 4\lambda x \Rightarrow 3 = -4\lambda,\ -3\lambda + \mu = 1$

$\Rightarrow \quad \lambda = \frac{-3}{4}, \mu = \frac{-5}{4}$

$$\therefore \quad I = \int(3x+1)\sqrt{4-3x-2x^2}\,dx$$

$$= \int\left[-\frac{3}{4}(-3-4x) - \frac{5}{4}\right]\sqrt{4-3x-2x^2}\,dx$$

$$= -\frac{3}{4}\int(-3-4x)\sqrt{4-3x-2x^2}\,dx - \frac{5}{4}\int\sqrt{4-3x-2x^2}\,dx$$

Let $4 - 3x - 2x^2 = t$ in the first integral

$\Rightarrow \quad (-3 - 4x)dx = dt$

$$\therefore \quad I = \frac{-3}{4}\int\sqrt{t}\,dt - \frac{5}{4}\int\sqrt{-2\left(x^2+\frac{3}{2}x-2\right)}\,dx$$

$$= \frac{-3}{4}\times\frac{2}{3}t^{3/2} + C_1 - \frac{5}{4}\int\sqrt{-2\left(x^2+\frac{3}{2}x-2+\frac{9}{16}-\frac{9}{16}\right)}\,dx$$

$$= \frac{-1}{2}(4-3x-2x^2)^{\frac{3}{2}} + C_1 - \frac{5}{4}\int\sqrt{-2\left[\left(x+\frac{3}{4}\right)^2 - \left(\frac{\sqrt{41}}{4}\right)^2\right]}\,dx$$

$$= -\frac{1}{2}(4-3x-2x^2)^{\frac{3}{2}} + C_1 - \frac{5\times\sqrt{2}}{4}\int\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x+\frac{3}{4}\right)^2}\,dx$$

$$= -\frac{1}{2}(4-3x-2x^2)^{\frac{3}{2}} + C_1 - \frac{5\sqrt{2}}{4}\left[\frac{1}{2}\left(x+\frac{3}{4}\right)\sqrt{\left(\frac{41}{16}\right) - \left(x+\frac{3}{4}\right)^2} + \frac{1}{2}\left(\frac{41}{16}\right)\sin^{-1}\left(\frac{x+\frac{3}{4}}{\frac{\sqrt{41}}{4}}\right) + C_2\right]$$

$$= \frac{-1}{2}(4-3x-2x^2)^{\frac{3}{2}} - \frac{5}{4\sqrt{2}}\left(x+\frac{3}{4}\right)\sqrt{\left(\frac{41}{16}\right) - \left(x+\frac{3}{4}\right)^2} - \frac{205}{64\sqrt{2}}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right) + C,$$

where $C = C_1 - \frac{5\sqrt{2}}{4}C_2$

79. Let $I = \int\frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

$$= \sin^{-1}x\int\frac{x}{\sqrt{1-x^2}}\,dx - \int\left[\frac{d}{dx}\left(\sin^{-1}x\right)\int\frac{x}{\sqrt{1-x^2}}\,dx\right]dx$$

(Applying integration by parts)

Firstly, let us evaluate the integral $\int \frac{x}{\sqrt{1-x^2}}dx$

Put $t = 1 - x^2$ and $dt = -2x\,dx$. So,

$$\int \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$$

$$\therefore \quad I = \sin^{-1} x\left(-\sqrt{1-x^2}\right) - \int \frac{1}{\sqrt{1-x^2}}\left(-\sqrt{1-x^2}\right)dx$$

$$= -\sqrt{1-x^2}\sin^{-1} x + \int dx = -\sqrt{1-x^2}\sin^{-1} x + x + C$$

80. Let $I = \int (2x+5)\sqrt{10-4x-3x^2}\,dx$

Let $2x+5 = \lambda\frac{d}{dx}(10-4x-3x^2)+\mu$

$\Rightarrow \quad 2x + 5 = \lambda(-4 - 6x) + \mu$

$\Rightarrow \quad 2x + 5 = -6\lambda x - 4\lambda + \mu$

$\Rightarrow \quad -6\lambda = 2, -4\lambda + \mu = 5$

$\Rightarrow \quad \lambda = -\frac{1}{3}, \mu = \frac{11}{3}$

$$\therefore \quad I = \int (2x+5)\sqrt{10-4x-3x^2}\,dx$$

$$= \int\left[-\frac{1}{3}(-4-6x)+\frac{11}{3}\right]\sqrt{10-4x-3x^2}\,dx$$

$$= \int -\frac{1}{3}\int(-4-6x)\sqrt{10-4x-3x^2}\,dx + \frac{11}{3}\int\sqrt{10-4x-3x^2}\,dx$$

Put $10 - 4x - 3x^2 = t$ in the first integral.

$\therefore \quad (-4 - 6x)\,dx = dt$

$$\Rightarrow \quad I = -\frac{1}{3}\int\sqrt{t}\,dt + \frac{11}{3}\int\sqrt{-3\left(x^2+\frac{4}{3}x-\frac{10}{3}\right)}dx$$

$$\Rightarrow \quad I = -\frac{1}{3}\times\frac{2}{3}t^{\frac{3}{2}} + C_1 + \frac{11}{3}\int\sqrt{-3\left(x^2+\frac{4}{3}x-\frac{10}{3}+\frac{4}{9}-\frac{4}{9}\right)}dx$$

$$\Rightarrow \quad I = -\frac{2}{9}\left(10-4x-3x^2\right)^{3/2} + C_1 + \frac{11}{3}\int\sqrt{-3\left[\left(x+\frac{2}{3}\right)^2-\left(\frac{\sqrt{34}}{3}\right)^2\right]}dx$$

$$\Rightarrow \quad I = -\frac{2}{9}\left(10-4x-3x^2\right)^{3/2} + C_1 + \frac{11\times\sqrt{3}}{3}\int\sqrt{\left(\frac{\sqrt{34}}{3}\right)^2-\left(x+\frac{2}{3}\right)^2}\,dx$$

$$\Rightarrow \quad I = -\frac{2}{9}\left(10-4x-3x^2\right)^{3/2} + C_1 + \frac{11\sqrt{3}}{3}\left[\frac{1}{2}\left(x+\frac{2}{3}\right)\sqrt{\frac{34}{9}-\left(x+\frac{2}{3}\right)^2} + \frac{\frac{34}{9}}{2}\sin^{-1}\left(\frac{x+\frac{2}{3}}{\frac{\sqrt{34}}{3}}\right)+C_2\right]$$

$$\Rightarrow \quad I = -\frac{2}{9}\left(10-4x-3x^2\right)^{3/2} + \frac{11}{2\sqrt{3}}\left(x+\frac{2}{3}\right)\sqrt{\left(\frac{34}{9}\right)-\left(x+\frac{2}{3}\right)^2} + \frac{187}{9\sqrt{3}}\sin^{-1}\left(\frac{3x+2}{\sqrt{34}}\right)+C$$

where $C = C_1 + \frac{11}{\sqrt{3}}C_2$

81. Let $I = \int (x+3)\sqrt{3-4x-x^2}\,dx$

$$= \int (x+2+1)\sqrt{3-4x-x^2}\,dx$$

$$= \frac{-1}{2}\int -2(x+2)\sqrt{3-4x-x^2}\,dx + \int\sqrt{3-4x-x^2}\,dx$$

$= I_1 + I_2$ (say)

$$\therefore \quad I_1 = -\frac{1}{2}\int -2(x+2)\sqrt{3-4x-x^2}\,dx$$

Put $3 - 4x - x^2 = t \Rightarrow (-4 - 2x)dx = dt$

$$\Rightarrow \quad I_1 = -\frac{1}{2}\int\sqrt{t}\,dt = -\frac{1}{2}\times\frac{2}{3}(t)^{3/2}+C_1$$

$$= -\frac{1}{3}\times(3-4x-x^2)^{3/2}+C_1 \qquad ...(1)$$

and $I_2 = \int\sqrt{3-4x-x^2}\,dx$

$$= \int\sqrt{-(x^2+4x-3+4-4)}\,dx$$

$$= \int\sqrt{-((x+2)^2-7)}\,dx = \int\sqrt{7-(x+2)^2}\,dx$$

$$= \frac{(x+2)\sqrt{3-4x-x^2}}{2}+\frac{7}{2}\sin^{-1}\frac{(x+2)}{\sqrt{7}}+C_2 \qquad ...(2)$$

From (1) & (2), we get

$$I = -\frac{1}{3}(3-4x-x^2)^{3/2} + \frac{(x+2)\sqrt{3-4x-x^2}}{2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right)+C$$

where $C = C_1 + C_2$

82. Let $I=\int\frac{x^2-3x+1}{\sqrt{1-x^2}}dx$

$$=-\int\frac{(-x^2+3x-1)}{\sqrt{1-x^2}}dx=-\int\frac{(1-x^2)+3x-2}{\sqrt{1-x^2}}dx$$

i.e., $I=-\int\sqrt{1-x^2}\,dx+\int\frac{-3x+2}{\sqrt{1-x^2}}dx$

$$=-\int\sqrt{1-x^2}\,dx+\frac{3}{2}\int\frac{-2x}{\sqrt{1-x^2}}dx+2\int\frac{1}{\sqrt{1-x^2}}dx$$

$$=-\int\sqrt{1-x^2}\,dx+\frac{3\times 2}{2}\sqrt{1-x^2}+2\sin^{-1}x+C$$

$$=-\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]+3\sqrt{1-x^2}+2\sin^{-1}x+C$$

$$=-\frac{x}{2}\sqrt{1-x^2}+\frac{3}{2}\sin^{-1}x+3\sqrt{1-x^2}+C$$

83. Let $I=\int(3-2x)\sqrt{2+x-x^2}$

Here $\frac{d}{dx}(2+x-x^2)=1-2x$

We can write

$I=\int\sqrt{2+x-x^2}\,(1-2x+2)dx$

$$=\int\sqrt{2+x-x^2}\cdot(1-2x)dx+2\int\sqrt{2+x-x^2}\,dx$$

$= I_1 + 2I_2$ (say) ...(1)

where, $I_1=\int\sqrt{2+x+x^2}\,(1-2x)dx$

Take $2 + x - x^2 = t \Rightarrow (1 - 2x)\,dx = dt$

$\therefore\ I_1=\int t^{1/2}dt=\frac{2}{3}t^{3/2}+C_1=\frac{2}{3}(2+x-x^2)^{3/2}+C_1$ ...(2)

and $I_2=\int\sqrt{2+x-x^2}\,dx=\int\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$

$$=\frac{\left(x-\frac{1}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}{2}+\frac{(3/2)^2}{2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{3/2}\right)+C'$$

$=\frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{2+x-x^2}+\frac{9}{8}\sin^{-1}\left(\frac{2x-1}{3}\right)+C'$ ...(3)

From (1), (2) and (3), we get

$$I=\frac{2}{3}(2+x-x^2)^{3/2}+\left(x-\frac{1}{2}\right)\sqrt{2+x-x^2}+\frac{9}{4}\sin^{-1}\left(\frac{2x-1}{3}\right)+C$$

where $C = C_1 + 2C'$

84. Let $I=\int\frac{\log x}{(x+1)^2}dx=\int(x+1)^{-2}\cdot\log x\,dx$

Integrating by parts, taking log x as first function.

$$I=\frac{(x+1)^{-1}}{-1}\cdot\log x-\int\frac{(x+1)^{-1}}{-1}\cdot\frac{1}{x}\,dx$$

$$=\frac{-\log x}{x+1}+\int\frac{dx}{x(x+1)}=\frac{-\log x}{x+1}+\int\left[\frac{1}{x}-\frac{1}{x+1}\right]dx$$

$$=-\frac{\log x}{x+1}+\log x-\log(x+1)+C$$

$$=-\frac{\log x}{x+1}+\log\left(\frac{x}{x+1}\right)+C$$

85. Let $I=\int e^{2x}\sin(3x+1)dx$

$$=e^{2x}\int\sin(3x+1)dx-\int\left(\frac{d(e^{2x})}{dx}\cdot\int\sin(3x+1)dx\right)dx$$

$$=e^{2x}\frac{[-\cos(3x+1)]}{3}-\int 2e^{2x}\cdot\frac{[-\cos(3x+1)]}{3}dx$$

$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{3}\int e^{2x}\cos(3x+1)dx$$

$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{3}\left[e^{2x}\int\cos(3x+1)dx-\int\left(\frac{d}{dx}(e^{2x})\cdot\int\cos(3x+1)dx\right)dx\right]$$

$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)-\frac{4}{9}\int e^{2x}\sin(3x+1)dx$$

$$=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)-\frac{4}{9}I+C_1$$

$$\therefore\quad I+\frac{4}{9}I=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)+C_1$$

$$\Rightarrow\quad \frac{13I}{9}=\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)+C_1$$

$$\Rightarrow\quad I=\frac{9}{13}\left[\frac{-e^{2x}\cos(3x+1)}{3}+\frac{2}{9}e^{2x}\sin(3x+1)+C_1\right]$$

$$=\frac{9}{13}e^{2x}\left[\frac{2\sin(3x+1)-3e^{2x}\cos(3x+1)}{9}\right]+\frac{9}{13}C_1$$

$$=\frac{1}{13}e^{2x}[2\sin(3x+1)-3\cos(3x+1)]+C$$

86. Let $I=\int \frac{x^2+1}{(x+1)^2}e^x dx$

$=\int e^x\cdot\left[\frac{(x^2-1)+2}{(x+1)^2}\right]dx$

$=\int e^x\cdot\left[\frac{x^2-1}{(x+1)^2}+\frac{2}{(x+1)^2}\right]dx$

$=\int e^x\cdot\left[\frac{x-1}{x+1}+\frac{2}{(x+1)^2}\right]dx \quad ...(1)$

Put $f(x)=\frac{x-1}{x+1}$

$\Rightarrow \quad f'(x)=\frac{(x+1)\cdot 1-(x-1)\cdot 1}{(x+1)^2}=\frac{2}{(x+1)^2}$

From (1), we get

$I=\int e^x[f(x)+f'(x)]dx$

$=e^x f(x)+C=e^x.\left[\frac{x-1}{x+1}\right]+C$

87. $I=\int(x-3)\sqrt{x^2+3x-18}\ dx$

$=\int x\sqrt{x^2+3x-18}\ dx-3\int\sqrt{x^2+3x-18}\ dx$

$=\int\frac{1}{2}(2x+3-3)\sqrt{x^2+3x-18}\ dx$
$-3\int\sqrt{x^2+3x-18}\,dx$

$=\frac{1}{2}\int(2x+3)\sqrt{x^2+3x-18}\,dx-\frac{3}{2}\int\sqrt{x^2+3x-18}\ dx$
$-3\int\sqrt{x^2+3x-18}\ dx$

$=\frac{1}{2}\int(2x+3)\times\sqrt{x^2+3x-18}\,dx-\frac{9}{2}\int\sqrt{x^2+3x-18}\,dx$

$=\frac{1}{2}I_1-\frac{9}{2}I_2\text{(say)}$

For I_1, put $x^2 + 3x - 18 = z \Rightarrow (2x + 3)dx = dz$

$\therefore \quad I_1=\int\sqrt{z}\,dz=\frac{2}{3}z^{3/2}+C_1$

$\therefore \quad =\frac{2}{3}(x^2+3x-18)^{\frac{3}{2}}+C_1$

and $I_2=\int\sqrt{x^2+3x-18}\,dx$

$=\int\sqrt{x^2+3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2-18}\,dx$

$=\int\sqrt{\left(x+\frac{3}{2}\right)^2-\frac{81}{4}}\,dx=\int\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}\ dx$

$=\frac{x+\frac{3}{2}}{2}\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}$
$-\frac{81/4}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{\left(x+\frac{3}{2}\right)^2-\frac{81}{4}}\right|+C_2$

$=\frac{2x+3}{4}\sqrt{x^2+3x-18}$
$-\frac{81}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x-18}\right|+C_2$

The given integral is written as

$I=\frac{1}{3}(x^2+3x-18)^{3/2}-\frac{9}{8}(2x+3)\sqrt{x^2+3x-18}$
$+\frac{729}{16}\log\left|\frac{(2x+3)}{2}+\sqrt{x^2+3x-18}\right|+C,$

where $C=\frac{1}{2}C_1-\frac{9}{2}C_2$ is a constant

88. Let $I=\int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

Put $\cos^{-1}x = \theta \Rightarrow x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

$\Rightarrow \quad I=\int\frac{\cos\theta(\theta)}{\sqrt{1-\cos^2\theta}}(-\sin\theta)d\theta$

$\Rightarrow \quad I=-\int\theta\cos\theta d\theta$

$\Rightarrow \quad -I=\theta\int\cos\theta\,d\theta-\int\left(\frac{d}{d\theta}\theta\int\cos\theta d\theta\right)d\theta$

$\Rightarrow \quad -I=\theta\sin\theta-\int\sin\theta d\theta$

$\Rightarrow \quad -I=\theta\sin\theta+\cos\theta+C$

$\Rightarrow \quad I=-[\cos^{-1}x\sqrt{1-\cos^2\theta}+x]+C$

$\therefore \quad I=-\left[\sqrt{1-x^2}\cos^{-1}x+x\right]+C$

89. Let $I=\int(3x-2)\sqrt{x^2+x+1}\,dx$

Let $3x-2=A\frac{d}{dx}(x^2+x+1)+B$

$\Rightarrow \quad 3x - 2 = A(2x + 1) + B$

On equating the coefficients of like terms, we get $A = 3/2, B = -7/2$.

$\therefore \quad I=\int\left[\frac{3}{2}(2x+1)-\frac{7}{2}\right]\sqrt{x^2+x+1}\,dx$

$\Rightarrow \quad I=\frac{3}{2}\int(2x+1)\sqrt{x^2+x+1}\,dx-\frac{7}{2}\int\sqrt{x^2+x+1}\,dx$

Put $x^2+x+1=t$ in 1st integral $\Rightarrow (2x+1)dx=dt$

i.e., $I=\frac{3}{2}\int\sqrt{t}\,dt-\frac{7}{2}\int\sqrt{(x+(1/2))^2+(\sqrt{3}/2)^2}\,dx$

$$I=\frac{3}{2}\left(\frac{t^{3/2}}{3/2}\right)-\frac{7}{2}\left[\frac{x+\frac{1}{2}}{2}\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right.$$

$$\left.+\frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|\right]+C$$

$$\Rightarrow \quad I=(x^2+x+1)^{3/2}-\frac{7}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x+1}\right.$$

$$\left.+\frac{3}{8}\log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|\right]+C$$

90. Let $I=\int e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$

Put $2x=t \Rightarrow \quad dx=\frac{1}{2}dt$

$\therefore \quad I=\frac{1}{2}\int e^t\,\frac{1-\sin t}{1-\cos t}dt$

$$=\frac{1}{2}\int e^t\left(\frac{1-2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}}\right)dt$$

$$=\frac{1}{2}\int e^t\left(\frac{1}{2}\text{cosec}^2\frac{t}{2}-\cot\frac{t}{2}\right)dt$$

$$=\frac{1}{2}\int e^t\left(-\cot\frac{t}{2}+\frac{1}{2}\text{cosec}^2\frac{t}{2}\right)dt$$

[Compare with $\int e^x[f(x)+f'(x)]dx = e^x f(x)$

Take $f(t)=-\cot\frac{t}{2} \Rightarrow f'(t)=\frac{1}{2}\text{cosec}^2\frac{t}{2}$]

$\therefore \quad I=\frac{1}{2}e^t\left(-\cot\frac{t}{2}\right)+C=\frac{-1}{2}e^{2x}\cot x+C$

91. Let $I=\int\frac{\sqrt{1-\sin x}}{1+\cos x}\cdot e^{\frac{-x}{2}}\,dx$

Put $-\frac{x}{2}=t \Rightarrow x=-2t \Rightarrow dx=-2dt$

$\therefore \quad I=\int\frac{\sqrt{1-\sin(-2t)}}{1+\cos(-2t)}\cdot e^t(-2)dt$

$$=-2\int\frac{\sqrt{1+\sin 2t}}{1+\cos 2t}\cdot e^t\,dt$$

$$=-2\int\frac{\sqrt{\cos^2 t+\sin^2 t+2\sin t\cos t}}{2\cos^2 t}\cdot e^t\,dt$$

$$=-\int\frac{\cos t+\sin t}{\cos^2 t}\cdot e^t\,dt=-\int(\sec t+\sec t\tan t)e^t\,dt$$

$$=-\int e^t\sec t\,dt-\int e^t\sec t\tan t\,dt$$

$$=-\left[e^t\sec t-\int e^t\sec t\tan t\,dt\right]-\int e^t\sec t\tan t\,dt+C$$

$$=-e^t\sec t+C=-e^{-x/2}\cdot\sec\frac{x}{2}+C$$

92. Let $I=\int\left(\frac{1+\sin x}{1+\cos x}\right)e^x dx$

$$=\int\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\cdot e^x dx$$

$$=\int\left(\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right)e^x dx$$

$$=\int e^x\cdot\left(\tan\frac{x}{2}+\frac{1}{2}\sec^2\frac{x}{2}\right)dx$$

Compare it with

$\int e^x[f(x)+f'(x)]dx=e^x\cdot f(x)$

Here $f(x)=\tan\frac{x}{2} \Rightarrow f'(x)=\frac{1}{2}\sec^2\frac{x}{2}$

$\therefore \quad I=e^x\cdot\tan\frac{x}{2}+C$

93. Let $I=\int\frac{x^2}{(x\sin x+\cos x)^2}dx$

$$=\int\frac{x\cos x}{(x\sin x+\cos x)^2}\cdot x\sec x\,dx \qquad ...(1)$$

Let $I_1=\int\frac{x\cos x}{(x\sin x+\cos x)^2}dx$

Put $x\sin x+\cos x=t$

$\Rightarrow \quad (x\cos x+1\cdot\sin x-\sin x)dx=dt$

$\Rightarrow \quad x\cos x\,dx=dt$

$\therefore \quad I_1=\int\frac{dt}{t^2}=-\frac{1}{t}=\frac{-1}{x\sin x+\cos x}$

Now integrating (1) by parts taking $x\sec x$ as the first function and using I_1, we get

$$I = x\sec x \cdot \frac{-1}{x\sin x+\cos x} - \int (1\cdot \sec x + x\sec x\tan x)\cdot \frac{-dx}{x\sin x+\cos x}$$

$$= \frac{-x\sec x}{x\sin x+\cos x} + \int \sec x\left(1+\frac{x\sin x}{\cos x}\right)\frac{dx}{x\sin x+\cos x}$$

$$= \frac{-x\sec x}{x\sin x+\cos x} + \int \sec^2 x\,dx$$

$$= \frac{-x\sec x}{x\sin x+\cos x} + \tan x + C$$

94. Let $I = \int e^x\left(\frac{\sin 4x-4}{1-\cos 4x}\right)dx$

$\Rightarrow \quad I = \int e^x\left(\frac{\sin 4x}{1-\cos 4x} - \frac{4}{1-\cos 4x}\right)dx$

$\Rightarrow \quad I = \int e^x\left(\frac{2\sin 2x\cos 2x}{2\sin^2 2x} - \frac{4}{2\sin^2 2x}\right)dx$

$\Rightarrow \quad I = \int e^x\left(\cot 2x - 2\operatorname{cosec}^2 2x\right)dx$

Consider, $f(x) = \cot 2x \Rightarrow f'(x) = -2\operatorname{cosec}^2 2x$

$\therefore \quad I = \int e^x(\cot 2x - 2\operatorname{cosec}^2 2x)dx = e^x\cot 2x + C$

95. Let $I = \int\left(\log(\log x) + \frac{1}{(\log x)^2}\right)dx$

Put $\log x = t$, then $x = e^t \Rightarrow dx = e^t dt$

$\therefore\ I = \int\left(\log t + \frac{1}{t^2}\right)e^t dt = \int e^t\left(\log t - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2}\right)dt$

Consider, $f(t) = \log t - \frac{1}{t} \Rightarrow f'(t) = \frac{1}{t} + \frac{1}{t^2}$

Thus, the given integral is in the form

$\int e^t[(f(t) + f'(t))]dt$

$\therefore \quad I = \int e^t\left(\log t \frac{-1}{t} + \frac{1}{t} + \frac{1}{t^2}\right)dt = e^t\left(\log t - \frac{1}{t}\right) + C$

$\therefore \quad I = x\left(\log(\log x) - \frac{1}{\log x}\right) + C$

96. *Refer to answer 90.*

97. Let $I = \int \frac{(x-4)e^x}{(x-2)^3}dx$

$\therefore \quad I = \int e^x\left[\frac{x-2}{(x-2)^3} - \frac{2}{(x-2)^3}\right]dx$

$\Rightarrow \quad I = \int e^x\left[\frac{1}{(x-2)^2} - \frac{2}{(x-2)^3}\right]dx$

Consider, $f(x) = \frac{1}{(x-2)^2} \Rightarrow f'(x) = \frac{-2}{(x-2)^3}$

$\therefore \quad I = \int e^x\left[\frac{1}{(x-2)^2} - \frac{2}{(x-2)^3}\right]dx = \frac{e^x}{(x-2)^2} + C$

98. Let $I = \int x\log(x+1)dx$

Integrating by parts, we get

$$I = \log(x+1)\frac{x^2}{2} - \int \frac{1}{x+1}\cdot\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}dx$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\left[\int \frac{x^2-1}{x+1}dx + \int \frac{dx}{x+1}\right]$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\left[\int (x-1)dx + \int \frac{dx}{x+1}\right]$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\cdot\frac{x^2}{2} + \frac{1}{2}x - \frac{1}{2}\log|x+1| + C$$

$$= \frac{1}{2}(x^2-1)\log(x+1) - \frac{x^2}{4} + \frac{x}{2} + C$$

99. Let $I = \int x\log 2x\,dx$

$\therefore \quad I = \log 2x\int x\,dx - \int\left[\frac{d}{dx}\log 2x\int x\,dx\right]dx + C$

$$= \log 2x\cdot\frac{x^2}{2} - \int \frac{1}{2x}\cdot 2\cdot\frac{x^2}{2}dx + C$$

$$= \frac{x^2}{2}\log 2x - \frac{1}{2}\cdot\int x\,dx + C = \frac{x^2}{2}\log 2x - \frac{1}{2}\cdot\frac{x^2}{2} + C$$

$$= \frac{x^2}{2}\log 2x - \frac{x^2}{4} + C$$

100. Let $I = \int \frac{\sqrt{x^2+1}[\log(x^2+1) - 2\log x]}{x^4}dx$

$$= \int \frac{\sqrt{x^2+1}\,\log\left(\frac{x^2+1}{x^2}\right)}{x^4}dx$$

Put $\frac{x^2+1}{x^2} = t \Rightarrow x^2 = \frac{1}{t-1}$

$\Rightarrow\ 2x\,dx = \frac{-1}{(t-1)^2}dt \Rightarrow\ dx = -\frac{1}{2x}\cdot\frac{1}{(t-1)^2}\,dt$

$= -\frac{1}{2}\cdot\sqrt{t-1}\cdot\frac{1}{(t-1)^2}\,dt\ = -\frac{dt}{2(t-1)^{3/2}}$

Also $\sqrt{x^2+1} = \sqrt{\frac{1}{t-1}+1} = \sqrt{\frac{t}{t-1}}$

$\therefore\quad I = \int\sqrt{\frac{t}{t-1}}\cdot\log t\cdot\frac{1}{1/(t-1)^2}\times\frac{-dt}{2(t-1)^{3/2}}$

$= -\frac{1}{2}\int\sqrt{t}.\log t\,dt$

$= -\frac{1}{2}\left[\frac{t^{3/2}}{3/2}\cdot\log t - \int\frac{t^{3/2}}{3/2}\cdot\frac{1}{t}dt\right]+C$

$= -\frac{1}{3}\left[t^{3/2}\log t - \int t^{1/2}dt\right]+C$

$= \frac{-1}{3}\left[t^{3/2}\log t - \frac{2}{3}t^{3/2}\right]+C$

$= \frac{-1}{3}\left[\left(\frac{x^2+1}{x^2}\right)^{3/2}\log\left(\frac{x^2+1}{x^2}\right) - \frac{2}{3}\left(\frac{x^2+1}{x^2}\right)^{\frac{3}{2}}\right]+C$

101. Let $I = \int\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}\,dx, x\in[0,1]$

We know that $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x} = \frac{\pi}{2}$

$\Rightarrow\ \sin^{-1}\sqrt{x} = \frac{\pi}{2}-\cos^{-1}\sqrt{x}$

$\therefore\quad I = \int\frac{\frac{\pi}{2}-2\cos^{-1}\sqrt{x}}{\pi/2}\,dx$

$= \int 1\cdot dx - \frac{4}{\pi}\int 1\cdot\cos^{-1}\sqrt{x}\,dx$

$= x - \frac{4}{\pi}\left[x\cdot\cos^{-1}\sqrt{x} - \int x\cdot\frac{-1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}dx\right]+C$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$

$\therefore\ I = x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\cdot 2\sin\theta\cos\theta\,d\theta + C$

$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int\frac{\sin\theta}{\cos\theta}\cdot 2\sin\theta\cos\theta\,d\theta + C$

$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int(1-\cos 2\theta)\,d\theta + C$

$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]+C$

$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}[\theta - \sin\theta\cos\theta]+C$

$= x - \frac{4}{\pi}x\cos^{-1}\sqrt{x} - \frac{2}{\pi}\left[\sin^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x}\right]+C$

102. We have, by definition

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + ... + f(a+(n-1)h)]$$

where, $nh = b - a$

Here, $a = 1$, $b = 3$, $f(x) = e^{2-3x} + x^2 + 1$ and $nh = 2$

$\therefore\ \int_1^3 (e^{2-3x}+x^2+1)\,dx = \lim_{h\to 0} h[\{e^{2-3}+1^2+1\}$

$+ \{e^{2-3(1+h)} + (1+h)^2 + 1\}$

$+ ...... + \{e^{2-3(1+(n-1)h)} + (1+(n-1)h)^2 + 1\}]$

$= \lim_{h\to 0} h[e^2e^{-3} + e^2e^{-3(1+h)} + e^2e^{-3(1+2h)} + ...$

$+ e^2e^{-3(1+(n-1)h)} + 1^2 + (1+h)^2 + (1+2h)^2$

$+ .... + (1+(n-1)h)^2 + n]$

$= \lim_{h\to 0} h\,[e^2e^{-3}(1 + e^{-3h} + e^{-6h} + ... + e^{-3(n-1)h}) +$

$n + h^2\times\{1^2 + 2^2 + ... + (n-1)^2\}$

$+ 2h(1 + 2 + 3 + ... + (n-1)) + n]$

$\lim_{h\to 0} h\left[\frac{e^{-1}(1-(e^{-3hn}))}{1-e^{-3h}} + h^2\times\frac{(n-1)n(2n-1)}{6} + 2h\times\frac{n(n-1)}{2} + 2n\right]$

$= \lim_{h\to 0}\left[he^{-1}\frac{(1-e^{-6})}{1-e^{-3h}} + \frac{(nh-h)nh(2nh-h)}{6} + nh\,(nh-h) + 2nh\right]$

$= \lim_{h\to 0}\left[\left[\frac{h(e^{-1}-e^{-7})}{(1-e^{-3h})} + \frac{(2-h)2(4-h)}{6} + 2(2-h) + 4\right]\right.$

$= \frac{(e^{-1}-e^{-7})}{\lim_{h\to 0}\left(\frac{1-\frac{1}{e^{3h}}}{h}\right)} + \frac{8}{3} + 4 + 4$

$= \frac{(e^{-1}-e^{-7})}{\lim_{h\to 0}\left(\frac{e^{3h}-1}{3h}\right)\times\frac{3h}{e^{3h}}\times\frac{1}{h}} + \frac{32}{3} = \frac{e^{-1}-e^{-7}}{3} + \frac{32}{3}$

103. We have, by definition

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+...+f(a+(n-1)h)]$$

where, $nh = b - a$

Here, $f(x) = 3x^2 + 1$, $a = 1$, $b = 3$ and $nh = 2$

$$\therefore \int_1^3 (3x^2+1)dx = \lim_{h\to 0} h[(3\cdot 1^2+1)+\{3(1+h)^2+1\}+...$$

$$....+3\{(1+(n-1)\cdot h)^2+1\}]$$

$$= \lim_{h\to 0} h[n+3\cdot n+6h\cdot\{1+2+3+....+(n-1)\}$$

$$+3h^2\{1^2+2^2+....+(n-1)^2\}]$$

$$= \lim_{h\to 0} h[4n+6h\cdot\frac{(n-1)n}{2}+3h^2\cdot\frac{1}{6}(n-1)n(2n-1)]$$

$$= \lim_{h\to 0}[4nh+3nh\ (nh-h)+\frac{1}{2}nh(nh-h)(2nh-h)]$$

$$= 4\times 2+3\times 2(2-0)+\frac{1}{2}\times 2\times(2-0)(2\times 2-0)$$

$$= 8+12+\frac{1}{2}\times 16 = 28$$

104. We have, by definition

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+$$

$$...+f(a+(n-1)h)], \text{ where } nh = b-a$$

Here, $a = 1$, $b = 3$, $f(x) = 2x^2 + 5x$ and $nh = 2$

$$\therefore \int_1^3 (2x^2+5x)dx = \lim_{h\to 0} h[\{2(1)^2+5\times 1\}+\{2(1+h)^2$$

$$+5(1+h)\}+\{2(1+2h)^2+5(1+2h)\}$$

$$+.....+\{2(1+(n-1)h)^2+5(1+(n-1)h)\}]$$

$$= \lim_{h\to 0} h[2\{1^2+(1+h)^2+(1+2h)^2$$

$$+...+(1+(n-1)h)^2\}+5\{1+(1+h)+(1+2h)$$

$$+...+(1+(n-1)h)\}]$$

$$= \lim_{h\to 0} h[2\{n+2h(1+2+3+...+(n-1))$$

$$+h^2(1^2+2^2+...+(n-1)^2)\}$$

$$+5\{n+h(1+2+....+(n-1))\}]$$

$$= \lim_{h\to 0} h[7n+9h(1+2+3+...+(n-1))$$

$$+2h^2(1^2+2^2+...+(n-1)^2)]$$

$$= \lim_{h\to 0} h\left\{7n+9h\times\frac{n(n-1)}{2}+2h^2\times\frac{n(n-1)(2n-1)}{6}\right\}$$

$$= \lim_{h\to 0}\left\{7nh+\frac{9}{2}nh\ (nh-h)+\frac{1}{3}nh\ (nh-h)(2nh-h)\right\}$$

$$= 7\times 2+\frac{9}{2}\times 2(2-0)+\frac{1}{3}\times 2(2-0)(2\times 2-0)$$

$$= 14+18+\frac{16}{3} = \frac{112}{3}$$

105. We have by definition,

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+....+f(a+(n-1)h)]$$

where $nh = b - a$

Here, $f(x) = x^2 + 3$, $a = 2$, $b = 5$ and $nh = 3$

$$\therefore \int_2^5 (x^2+3)dx = \lim_{h\to 0} h[(2^2+3)+\{(2+h)^2+3\}$$

$$+\{(2+2h)^2+3\}+.....+\{(2+(n-1)h)^2+3\}]$$

$$= \lim_{h\to 0} h[3n+2^2\cdot n+h^2(1^2+2^2+.....+(n-1)^2)$$

$$+4h(1+2+.....+(n-1))]$$

$$= \lim_{h\to 0}\left[7nh+h^3\cdot\frac{(n-1)\cdot n\cdot(2n-1)}{6}+4h^2\frac{(n-1)n}{2}\right]$$

$$= \lim_{h\to 0}[7nh+\frac{1}{6}(nh-h)nh(2nh-h)+2(nh-h)nh]$$

$$= 7\times 3+\frac{1}{6}(3-0)\times 3\times(2\times 3-0)+2(3-0)\times 3$$

$= 21 + 9 + 18 = 48$

106. We have, by definition

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+$$

$$....+f(a+(n-1)h)],$$

where $nh = b - a$

Here, $a = 1$, $b = 4$, $f(x) = x^2 - x$ and $nh = 3$

$$\therefore \int_1^4 (x^2-x)dx = \lim_{h\to 0} h[\{1^2-1\}+\{(1+h)^2-(1+h)\}+$$

$$...+\{(1+(n-1)h)^2-(1+(n-1)h)\}]$$

$$= \lim_{h\to 0} h[n+h^2(1^2+2^2+...+(n-1)^2)+2h(1+2$$

$$+...+(n-1))-n-h(1+2+...+(n-1))]$$

$$= \lim_{h\to 0} h[h^2\{1+2^2+3^2+...+(n-1)^2\}$$

$$+h\{1+2+3+...+(n-1)\}]$$

$$= \lim_{h\to 0} h\left[h^2\frac{(n-1)n(2n-1)}{6}+\frac{h(n-1)n}{2}\right]$$

$= \lim_{h\to 0}\left[\frac{(nh-h)nh(2nh-h)}{6}+\frac{(nh-h)nh}{2}\right]$

$= \lim_{h\to 0}\left[\frac{(3-h)3(6-h)}{6}+\frac{(3-h)3}{2}\right]$

$= \frac{3(3)(6)}{6}+\frac{3(3)}{2}=9+\frac{9}{2}=\frac{27}{2}$

107. *Refer to answer 106.*

108. We have, by definition

$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+...+f(a+(n-1)h)]$

where $nh = b - a$

Here $f(x) = 3x^2 - 2$, $a = 0$, $b = 2$ and $nh = b - a = 2$

$\Rightarrow \int_0^2 (3x^2-2)dx = \lim_{h\to 0} h[(0-2)+(3h^2-2)$
$+\{3(2h)^2-2\}+....+\{3((n-1)h)^2-2\}]$

$= \lim_{h\to 0} h[-2n+3h^2\{1^2+2^2+....+(n-1)^2\}]$

$= \lim_{h\to 0} h\left[-2n+3h^2\frac{(n-1)n(2n-1)}{6}\right]$

$= \lim_{h\to 0}\left[-2nh+\frac{1}{2}(nh-h)nh(2nh-h)\right]$

$=[-2\times 2+\frac{1}{2}(2-0)2(2\times 2-0)]=-4+8=4$

109. We have, by definition

$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)$
$+....+f(a+(n-1)h)]$

where $nh = b - a$

Here, $a = 1$, $b = 3$, $f(x) = 3x^2 + 2x$ and

$nh = b - a = 3 - 1 = 2$

$\therefore \int_1^3 (3x^2+2x)dx$

$= \lim_{h\to 0} h\ [\{3(1)^2 + 2(1)\} + \{3(1 + h)^2 + 2(1 + h)\} +$
$.... + \{3(1 + (n - 1)h)^2 + 2\ (1 + (n-1)h)\}$

$= \lim_{h\to 0} h\ [5 + 3(n - 1) + 3h^2\ (1^2 + 2^2 + ...(n - 1)^2)]$
$+6h(1+2+...+(n-1))+2\,(n-1)+2h\,(1+2...+(n-1))$

$= \lim_{h\to 0} h\left[\begin{array}{l}5n+3h^2(1+2^2+.....+(n-1)^2)\\ \quad +8h(1+2+.....+(n-1)\end{array}\right]$

$= \lim_{h\to 0} h\left[5n+3h^2\frac{n(n-1)(2n-1)}{6}+8h\frac{n(n-1)}{2}\right]$

$= \lim_{h\to 0}\left[5nh+\frac{nh(nh-h)(2nh-h)}{2}+4nh(nh-h)\right]$

$= \lim_{h\to 0}\left[5(2)+\frac{2(2-h)(2(2)-h)}{2}+4(2)(2-h)\right]$

$=10+\frac{2(2)(4)}{2}+4(2)(2)=10+8+16=34$

110. We have, by definition

$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)$
$+.....+f(a+(n-1)h)]$

where $nh = b - a$

Here, $a = 1$, $b = 2$, $f(x) = x^2 + 5x$ and $nh = 2 - 1 = 1$

$\therefore \int_1^2 (x^2+5x)dx$

$= \lim_{h\to 0} h[\{(1)^2 + 5(1)\} + \{(1 + h)^2 + 5(1 + h)\} +$
$.... + \{(1 + (n - h)^2 + 5(1 + (n - 1)h)\}]$

$= \lim_{h\to 0} h[6 + (n - 1) + h^2(1 + 2^2 + ...(n - 1)^2)$
$+2h(1+2\ ....\ (n-1))+5(n-1)+5h(1+2+....(n-1))]$

$= \lim_{h\to 0} h[6n+h^2(1+2^2+....+(n-1)^2)$
$+7h(1+2+.....+(n-1))]$

$= \lim_{h\to 0} h\left[6n+h^2\frac{n(n-1)(2n-1)}{6}+7h\frac{n(n-1)}{2}\right]$

$= \lim_{h\to 0}\left[6nh+\frac{nh(nh-h)(2nh-h)}{6}+\frac{7nh(nh-h)}{2}\right]$

$= \lim_{h\to 0}\left[6(1)+\frac{1(1-h)(2-h)}{6}+\frac{7(1)(1-h)}{2}\right]$

$=6+\frac{1\cdot 1\cdot 2}{6}+\frac{7\cdot 1\cdot 1}{2}=6+\frac{1}{3}+\frac{7}{2}=\frac{36+2+21}{6}=\frac{59}{6}$

111. We have, by definition,

$\int_a^b f(x)\ dx = \lim_{h\to 0} h\ [f(a) + f(a + h) + f(a + 2h) + ...$
$+ f(a + (n - 1)h)]$

where $nh = b - a$

Here $a = 1$, $b = 3$, $f(x) = 2x^2 + 3$ and $nh = 2$

$\therefore \int_1^3 (2x^2+3)dx = \lim_{h\to 0} h[\{2(1)^2+3\}+\{2(1+h)^2+3\}+$
$.... + \{2\ (1 + (n - 1)h)^2 + 3\}]$

$= \lim_{h\to 0} h\left[\begin{array}{l}5n+4h(1+2+...+(n-1))\\ \quad +2h^2(1^2+2^2+...+(n-1)^2)\end{array}\right]$

$$= \lim_{h\to 0}[5nh + \frac{4nh(nh-h)}{2} + \frac{2nh(nh-h)(2nh-h)}{6}]$$

$$= 5\times 2 + \frac{4\times 2\times 2}{2} + \frac{4\times 2\times 4}{6} = 10 + 8 + \frac{16}{3} = \frac{70}{3}$$

112. We have, by definition

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) .... + f(a+(n-1)h)]$$

where $nh = b - a$

Here $a = 2$, $b = 5$, $f(x) = 3x^2 - 5$ and $nh = 3$

$$\therefore \int_2^5 (3x^2-5)dx = \lim_{h\to 0} h[\{3(2)^2-5\} + \{3(2+h)^2-5\} + ... + \{3(2+(n-1)h)^2-5\}]$$

$$= \lim_{h\to 0} h[7n + 12h(1+2+....+(n-1)) + 3h^2(1^2+2^2+...+(n-1)^2)]$$

$$= \lim_{h\to 0}\left[7nh + \frac{12nh(nh-h)}{2} + \frac{3nh(nh-h)(2nh-h)}{6}\right]$$

$$= 7\times 3 + 6\times 3\times 3 + \frac{3\times 3\times 6}{2} = 21 + 54 + 27 = 102$$

113. We have, by definition

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)]$$

where $nh = b - a$

Here $a = 1$, $b = 3$, $f(x) = x^2 + x$ and $nh = 2$

$$\therefore \int_1^3 (x^2+x)dx$$

$$= \lim_{h\to 0} h[\{1^2+1\} + \{(1+h)^2 + (1+h)\} + ... + \{(1+(n-1)h)^2 + (1+(n-1)h)\}]$$

$$= \lim_{h\to 0} h[n + 2h(1+2+3+...(n-1)) + h^2(1+2^2+...(n-1)^2) + n + h(1+2+3+...(n-1))]$$

$$= \lim_{h\to 0} h[2n + 3h(1+2+3+...+(n-1)) + h^2(1^2+2^2+...+(n-1)^2)]$$

$$= \lim_{h\to 0}\left[2nh + \frac{3nh(nh-h)}{2} + \frac{nh(nh-h)(2nh-h)}{6}\right]$$

$$= 2\times 2 + \frac{3\times 2\times 2}{2} + \frac{2\times 2\times 4}{6} = 4 + 6 + \frac{8}{3} = \frac{38}{3}$$

114. *Refer to answer 110.*

115. We have, by definition

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)]$$

where $nh = b - a$

Here $a = 0$, $b = 2$, $f(x) = x^2 + 2x + 1$ and $nh = 2$

$$\therefore \int_0^2 (x^2+2x+1)dx$$

$$= \lim_{h\to 0} h[\{0^2+2(0)+1\} + \{h^2+2h+1\} + ... + \{((n-1)h)^2 + 2(n-1)h + 1\}]$$

$$= \lim_{h\to 0} h[n + h^2(1^2+2^2+...+(n-1)^2) + 2h(1+2+3+...+(n-1))]$$

$$= \lim_{h\to 0}\left[nh + \frac{nh(nh-h)(2nh-h)}{6} + \frac{2nh(nh-h)}{2}\right]$$

$$= 2 + \frac{8}{3} + 4 = \frac{26}{3}$$

116. $\int \frac{dx}{9+x^2} = \int \frac{dx}{x^2+3^2} = \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right] = F(x)$

$\therefore$ By second fundamental theorem of integral calculus, we have

$$\int_0^3 \frac{dx}{9+x^2} = F(3) - F(0) = \frac{1}{3}[\tan^{-1}1 - \tan^{-1}0]$$

$$= \frac{1}{3}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{12}$$

117. Let $I = \int_0^{\pi/2} e^x(\sin x - \cos x)dx$

$$= \int_0^{\pi/2} e^x \sin x\,dx - \int_0^{\pi/2} e^x \cos x\,dx$$

(Integrate II^nd integral by parts)

$$= \int_0^{\pi/2} e^x \sin x\,dx - \left[\{e^x \cos x\}_0^{\pi/2} - \int_0^{\pi/2} e^x(-\sin x)dx\right]$$

$$= \int_0^{\pi/2} e^x \sin x\,dx - [e^{\pi/2}\cdot 0 - e^0\cdot 1] - \int_0^{\pi/2} e^x \sin x\,dx = 1$$

118. By the first fundamental theorem of integral calculus

$$f'(x) = \frac{d}{dx}\int_0^x t\sin t\,dt = x\sin x.$$

119. Here, $\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$

$\Rightarrow \int_0^a \frac{1}{x^2+2^2}dx = \frac{\pi}{8} \Rightarrow \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^a = \frac{\pi}{8}$

$\Rightarrow \frac{1}{2}\tan^{-1}\frac{a}{2} = \frac{\pi}{8} \Rightarrow \tan^{-1}\frac{a}{2} = \frac{\pi}{4}$

$\Rightarrow \frac{a}{2} = \tan\frac{\pi}{4} = 1 \Rightarrow a = 2$

120. We have $\int_0^{\pi/4} \tan x\,dx = [\log|\sec x|]_0^{\pi/4}$

$= \log\left|\sec\frac{\pi}{4}\right| - \log|\sec 0| = \log|\sqrt{2}| - \log|1|$

$\therefore \int_0^{\pi/4} \tan x\,dx = \frac{1}{2}\log 2$

121. We have, $\int \sin 2x\,dx = -\frac{1}{2}[\cos 2x] = F(x)$

$\therefore$ By second fundamental theorem of integral calculus, we have $\int_0^{\pi/4} \sin 2x\,dx = F\left(\frac{\pi}{4}\right) - F(0)$

$\Rightarrow F(x) = -\frac{1}{2}[\cos(\pi/2) - \cos(0)] = \frac{1}{2}$

122. We have, $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = [\sin^{-1}x]_0^1$

$= \sin^{-1}1 - \sin^{-1}0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

123. Here, $\int_1^2 \frac{x^3-1}{x^2}dx = \int_1^2 (x - x^{-2})dx$

$= \left[\frac{x^2}{2} - \frac{x^{-1}}{-1}\right]_1^2 = \left[\frac{x^2}{2} + \frac{1}{x}\right]_1^2$

$= \left[\frac{4}{2} + \frac{1}{2}\right] - \left[\frac{1}{2} + 1\right] = \frac{5}{2} - \frac{3}{2} = 1$

124. $\int_2^3 \frac{1}{x}dx = [\log x]_2^3 = \log 3 - \log 2 = \log\left(\frac{3}{2}\right)$

125. $\int_0^2 \sqrt{4-x^2}\,dx = \int_0^2 \sqrt{2^2-x^2}\,dx$

$= \left[\frac{x}{2}\sqrt{2^2-x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2}\right]_0^2 = [2\sin^{-1}1 - 0] = \pi$

126. $\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1}x]_0^1$

$= \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

127. Let $I = \int_0^{\pi/2} x^2 \sin x\,dx$

Integrating by parts

$I = \left[x^2(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} 2x(-\cos x)dx$

$= -\frac{\pi^2}{4}.0 + 0 + 2\int_0^{\pi/2} x\cos x\,dx = 2\int_0^{\pi/2} x\cos x\,dx$

Again integrating by parts

$I = 2\left[[x\sin x]_0^{\pi/2} - \int_0^{\pi/2} 1\cdot\sin x\,dx\right]$

$= 2\left\{\frac{\pi}{2}\cdot 1 - 0 - [-\cos x]_0^{\pi/2}\right\} = 2\left[\frac{\pi}{2} + (0-1)\right] = \pi - 2$

128. Let $I = \int_0^{\frac{\pi}{2}} \frac{x+\sin x}{1+\cos x}dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x}dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x}dx$

$= \int_0^{\frac{\pi}{2}} \frac{x}{2\cos^2\frac{x}{2}}dx + \int_0^{\frac{\pi}{2}} \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} x\sec^2\frac{x}{2}dx + \int_0^{\frac{\pi}{2}} \tan\frac{x}{2}dx$

$= \frac{1}{2}\left[\left[2x\tan\frac{x}{2}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\left[\frac{d(x)}{dx}\int\sec^2\frac{x}{2}dx\right]dx\right] + \int_0^{\frac{\pi}{2}} \tan\frac{x}{2}dx$

$= \frac{2}{2}\left[\frac{\pi}{2}\right] - \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\tan\frac{x}{2}dx + \int_0^{\frac{\pi}{2}} \tan\frac{x}{2}dx = \frac{\pi}{2}$

129. Let $I=\int_0^1 \frac{x^4+1}{x^2+1}dx=\int_0^1 \frac{(x^4-1)+2}{x^2+1}dx$

Consider

$$\int \frac{(x^4-1)+2}{x^2+1}dx=\int\left[x^2-1+\frac{2}{x^2+1}\right]dx$$

$$=\left[\frac{x^3}{3}-x+2\tan^{-1}x\right]=F(x)$$

$\therefore$ By second fundamental theorem of calculus, we have, $I = F(1) - F(0)$

$$=\frac{1}{3}-1+2\tan^{-1}1-0=-\frac{2}{3}+\frac{\pi}{2}=\frac{3\pi-4}{6}$$

130. Let $I=\int_1^2 \frac{5x^2}{x^2+4x+3}dx$

$$=5\int_1^2\left(1+\frac{-4x-3}{x^2+4x+3}\right)dx$$

$$=5\int_1^2 dx-5\int_1^2\frac{4x+3}{x^2+4x+3}dx$$

$$=5[x]_1^2-5\int_1^2\frac{4x+3}{(x+3)(x+1)}dx$$

Let $\frac{4x+3}{(x+3)(x+1)}=\frac{A}{x+3}+\frac{B}{x+1}$

$\Rightarrow$ $4x + 3 = A(x + 1) + B(x + 3)$

Putting $x = -3$ and $x = -1$, we get $A=\frac{9}{2};\ B=-\frac{1}{2}$

$$\therefore\quad I=5(2-1)-5\int_1^2\left(\frac{9/2}{x+3}+\frac{-1/2}{x+1}\right)dx$$

$$=5-5\left[\frac{9}{2}\log|x+3|-\frac{1}{2}\log|x+1|\right]_1^2$$

$$=5-5\left[\left(\frac{9}{2}\log5-\frac{1}{2}\log3\right)-\left(\frac{9}{2}\log4-\frac{1}{2}\log2\right)\right]$$

$$=5-5\left[\frac{9}{2}(\log5-\log4)-\frac{1}{2}(\log3-\log2)\right]$$

$$=5-\frac{45}{2}\log\frac{5}{4}+\frac{5}{2}\log\frac{3}{2}$$

131. Let $I=\int_2^4\frac{x}{x^2+1}dx$

Put $x^2 + 1 = t \Rightarrow x\,dx=\frac{1}{2}dt$

Also $x = 2 \Rightarrow t = 5$ and $x = 4 \Rightarrow t = 17$

$$\therefore\quad I=\frac{1}{2}\int_5^{17}\frac{dt}{t}=\frac{1}{2}[\log t]_5^{17}=\frac{1}{2}[\log17-\log5]$$

$$=\frac{1}{2}\log\left(\frac{17}{5}\right).$$

132. Let $I=\int_e^{e^2}\frac{dx}{x\log x}$

Put $\log x = t \Rightarrow \frac{1}{x}dx=dt$

Also $x = e \Rightarrow t = \log e = 1$

and $x = e^2 \Rightarrow t = \log e^2 = 2\log e = 2$

$$\therefore\quad I=\int_1^2\frac{dt}{t}=[\log t]_1^2=\log2-\log1=\log2$$

133. Let $I=\int_0^1 xe^{x^2}dx$

Put $x^2=t\Rightarrow xdx=\frac{1}{2}dt.$

Also $x = 0 \Rightarrow t = 0$ and $x = 1 \Rightarrow t = 1.$

$$\therefore\quad I=\frac{1}{2}\int_0^1 e^t dt\Rightarrow I=\frac{1}{2}\left[e^t\right]_0^1=\frac{1}{2}(e-1)$$

134. Let $I=\int_0^1\frac{\tan^{-1}x}{1+x^2}dx$

Put $\tan^{-1}x=t\Rightarrow\frac{1}{1+x^2}dx=dt$

Also $x = 0 \Rightarrow t = 0$ and $x = 1 \Rightarrow t=\frac{\pi}{4}$

$$\therefore\quad I=\int_0^{\pi/4}t\,dt=\left[\frac{1}{2}t^2\right]_0^{\pi/4}=\frac{1}{2}\cdot\left[\left(\frac{\pi}{4}\right)^2-0\right]=\frac{\pi^2}{32}$$

135. Let $I=\int_0^1\frac{e^x}{1+e^{2x}}dx$

Put $e^x = t \Rightarrow e^x\,dx = dt$

Also $x = 0 \Rightarrow t = e^0 = 1$

and $x = 1 \Rightarrow t = e^1 = e$

$$\therefore\quad I=\int_1^e\frac{dt}{(1+t^2)}=[\tan^{-1}t]_1^e=\tan^{-1}e-\tan^{-1}1$$

$$=\tan^{-1}\left(\frac{e-1}{1+e}\right)$$

136. Let $I = \int_0^1 \frac{2x}{1+x^2} dx$

Put $1 + x^2 = t \Rightarrow 2xdx = dt$

Also $x = 0 \Rightarrow t = 1$ and $x = 1 \Rightarrow t = 2$

$\therefore \quad I = \int_1^2 \frac{dt}{t} = [\log t]_1^2$

$= \log 2 - \log 1 = \log 2 - 0 = \log 2.$

137. Let $I = \int_0^{\pi} e^{2x} \cdot \sin\left(\frac{\pi}{4} + x\right) dx$

Put $\frac{\pi}{4} + x = t \Rightarrow x = t - \frac{\pi}{4} \Rightarrow dx = dt$

When $x = 0$, $t = \frac{\pi}{4}$ and when $x = \pi$, $t = \frac{5\pi}{4}$

$$\therefore \quad I = \int_{\pi/4}^{5\pi/4} e^{2\left(t - \frac{\pi}{4}\right)} \sin t\, dt = e^{-\pi/2} \int_{\pi/4}^{5\pi/4} e^{2t} \sin t\, dt$$

$$= e^{-\pi/2}\left[\left(\sin t \frac{e^{2t}}{2}\right)_{\pi/4}^{5\pi/4} - \int_{\pi/4}^{5\pi/4} \cos t \frac{e^{2t}}{2} dt\right]$$

$$= e^{-\pi/2}\left[\frac{1}{2}\left(e^{5\pi/2} \sin\frac{5\pi}{4} - e^{\pi/2} \sin\frac{\pi}{4}\right)\right.$$

$$\left. - \left(\frac{e^{2t}}{4} \cos t\right)_{\pi/4}^{5\pi/4} - \int_{\pi/4}^{5\pi/4} \frac{e^{2t}}{4} \sin t\, dt\right]$$

$$= e^{-\pi/2}\left[\frac{1}{2}\left(\frac{-1}{\sqrt{2}} e^{5\pi/2} - \frac{1}{\sqrt{2}} e^{\pi/2}\right)\right.$$

$$\left. - \frac{1}{4}\left(-\frac{1}{\sqrt{2}} e^{5\pi/2} - \frac{1}{\sqrt{2}} e^{\pi/2}\right)\right] - \frac{I}{4}$$

$$\Rightarrow \quad I + \frac{1}{4} I = -\frac{1}{2\sqrt{2}}\left[e^{2\pi} + 1\right] + \frac{1}{4\sqrt{2}}[e^{2\pi} + 1]$$

$$\Rightarrow \quad \frac{5}{4} I = \frac{(e^{2\pi} + 1)}{2\sqrt{2}}\left[\frac{1}{2} - 1\right] = -\frac{1}{4\sqrt{2}}[e^{2\pi} + 1]$$

$$\Rightarrow \quad I = \frac{-1}{5\sqrt{2}}(1 + e^{2\pi})$$

138. Let $I = \int_0^{\pi/4} \frac{dx}{\cos^3 x \sqrt{2 \sin 2x}}$

$$= \int_0^{\pi/4} \frac{dx}{\cos^3 x \sqrt{2 \cdot 2 \sin x \cos x}}$$

$$= \frac{1}{2} \int \frac{dx}{\cos^{\left(3 + \frac{1}{2}\right)} x \cdot \sin^{\frac{1}{2}} x}$$

$$= \frac{1}{2} \int_0^{\pi/4} \frac{dx}{\cos^{\frac{7}{2}} x \cdot \tan^{\frac{1}{2}} x \cdot \cos^{\frac{1}{2}} x}$$

$$= \frac{1}{2} \int_0^{\pi/4} \frac{dx}{\cos^4 x \sqrt{\tan x}} = \frac{1}{2} \int_0^{\pi/4} \frac{\sec^4 x}{\sqrt{\tan x}} dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

Also $x = 0 \Rightarrow t = 0$ and $x = \frac{\pi}{4} \Rightarrow t = 1$

$$\therefore \quad I = \frac{1}{2} \int_0^1 \frac{(1 + t^2) dt}{\sqrt{t}} = \frac{1}{2} \int_0^1 t^{-\frac{1}{2}} + t^{\frac{3}{2}}\, dt$$

$$= \frac{1}{2}\left[\frac{t^{1/2}}{1/2} + \frac{t^{5/2}}{5/2}\right]_0^1 = 1 + \frac{1}{5} = \frac{6}{5}$$

139. Let $I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

$$= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1 - (1 - \sin 2x)}} dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}} dx$$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)dx = dt$

Also $x = \frac{\pi}{3} \Rightarrow t = \frac{\sqrt{3} - 1}{2} = \alpha$

and $x = \frac{\pi}{6} \Rightarrow t = \frac{1 - \sqrt{3}}{2} = -\alpha$

$$\therefore \quad I = \int_{-\alpha}^{\alpha} \frac{dt}{\sqrt{1 - t^2}}$$

$$= \left[\sin^{-1} t\right]_{-\alpha}^{\alpha} = 2 \sin^{-1} \alpha = 2 \sin^{-1}\left(\frac{\sqrt{3} - 1}{2}\right)$$

140. Let $I = \int_0^{\pi/4} \frac{\sin 2\theta}{\sin^4 \theta + \cos^4 \theta} d\theta$

$$= \int_0^{\pi/4} \frac{2 \sin \theta \cos \theta}{\sin^4 \theta + (1 - \sin^2 \theta)^2} d\theta$$

Put $\sin^2\theta = t \Rightarrow 2\sin \theta \cos \theta\, d\theta = dt$

Also $\theta = 0 \Rightarrow t = 0$ and $\theta = \frac{\pi}{4} \Rightarrow t = \frac{1}{2}$

$$\therefore \quad I = \int_0^{1/2} \frac{dt}{t^2 + (1-t)^2} = \int_0^{1/2} \frac{dt}{2t^2 - 2t + 1}$$

$$= \frac{1}{2}\int_0^{1/2} \frac{dt}{\left(t^2 - t + \frac{1}{2}\right)}$$

$$= \frac{1}{2}\int_0^{1/2} \frac{dt}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\cdot\frac{1}{\frac{1}{2}}\left[\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^{1/2}$$

$$= \tan^{-1}\left[\frac{2t-1}{1}\right]_0^{1/2}$$

$$= \tan^{-1} 0 - \tan^{-1}(-1) = \frac{\pi}{4}$$

141. Let $I = \int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

Put $x = a\tan^2 t \Rightarrow dx = a(2\tan t\sec^2 t)dt)$

When $x = 0$, $t = 0$ and when $x = a$, $t = \frac{\pi}{4}$

$$\therefore \quad I = \int_0^{\frac{\pi}{4}} \sin^{-1}\sqrt{\frac{a\tan^2 t}{a + a\tan^2 t}}\cdot 2a\tan t\sec^2 t\,dt$$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1}\sqrt{\frac{a\tan^2 t}{a\sec^2 t}}\cdot 2a\tan t\sec^2 t\,dt$$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin t)\cdot 2a\tan t\sec^2 t\,dt$$

$$= 2a\int_0^{\frac{\pi}{4}} t\tan t\sec^2 t\,dt$$

Integrating by parts, taking ($\tan t \sec^2 t$) as 2nd function.

$$\therefore \quad I = 2a\left[\left[t\frac{\tan^2 t}{2}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\frac{\tan^2 t}{2}dt\right]$$

$$= 2a\left[\left[\frac{\pi}{4}\times\frac{1}{2}\right] - \frac{1}{2}\int_0^{\frac{\pi}{4}}(\sec^2 t - 1)dt\right]$$

$$= 2a\left[\frac{\pi}{8} - \frac{1}{2}\left[\tan t - t\right]_0^{\frac{\pi}{4}}\right] = 2a\left[\frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right)\right]$$

$$= 2a\left[\frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8}\right] = 2a\left[\frac{\pi}{4} - \frac{1}{2}\right] \Rightarrow I = a\left(\frac{\pi}{2} - 1\right)$$

142. Let $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x}dx$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)dx = dt$

Also $x = 0 \Rightarrow t = -1$ and $x = \frac{\pi}{4} \Rightarrow t = 0$

$\Rightarrow \quad t^2 = (\sin x - \cos x)^2 = 1 - \sin 2x \Rightarrow \sin 2x = 1 - t^2$

$$\therefore \quad I = \int_{-1}^{0} \frac{dt}{(9 + 16(1-t^2))} = \int_{-1}^{0}\frac{dt}{25 - 16t^2}$$

$$= \frac{1}{16}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{4}\right)^2 - t^2} = \frac{1}{16}\cdot\frac{1}{2\cdot\frac{5}{4}}\left[\log\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right|_{-1}^{0}\right]$$

$$= \frac{1}{40}\left[\log 1 - \log\left(\frac{1/4}{9/4}\right)\right]$$

$$= \frac{1}{40}\left[0 - \log\left(\frac{1}{9}\right)\right] = \frac{1}{40}\log 9 = \frac{1}{20}\log 3$$

143. $\text{L.H.S.} = \int_0^{\pi/4} (\sqrt{\tan x} + \sqrt{\cot x})\,dx$

$$= \int_0^{\pi/4}\left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\right)dx = \int_0^{\pi/4}\frac{\sin x + \cos x}{\sqrt{\sin x\cos x}}dx$$

$$= \sqrt{2}\int_0^{\pi/4}\frac{(\sin x + \cos x)}{\sqrt{2\sin x\cos x}}dx$$

$$= \sqrt{2}\int_0^{\pi/4}\frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}}dx$$

Let $\sin x - \cos x = t$ then $(\cos x + \sin x)\,dx = dt$
Also, $x = 0 \Rightarrow t = -1$ and $x = \pi/4 \Rightarrow t = 0$.

$$\therefore \int_0^{\pi/4}(\sqrt{\tan x} + \sqrt{\cot x})\,dx = \sqrt{2}\int_{-1}^{0}\frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\left[\sin^{-1} t\right]_{-1}^{0} = \sqrt{2}\,[\sin^{-1} 0 - \sin^{-1}(-1)]$$

$$= \sqrt{2}\cdot\sin^{-1} 1 = \sqrt{2}\cdot\frac{\pi}{2} = \text{R.H.S.}$$

144. Let $I = \int_{\pi/4}^{\pi/2} \cos 2x \log \sin x\, dx$

Integrating by parts taking cos 2*x* as 2[nd] function

$$\therefore I = \left[\frac{\sin 2x}{2}\cdot \log \sin x\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} \frac{\sin 2x}{2}\cdot\frac{1}{\sin x}\cdot \cos x\, dx$$

$$= \frac{1}{2}\left[0 - 1\cdot \log \sin\frac{\pi}{4}\right] - \frac{1}{2}\int_{\pi/4}^{\pi/2} 2\sin x\cdot\cos x\cdot\frac{\cos x}{\sin x} dx$$

$$= -\frac{1}{2}\log\frac{1}{\sqrt{2}} - \int_{\pi/4}^{\pi/2} \cos^2 x\, dx$$

$$= -\frac{1}{2}(\log 1 - \log\sqrt{2}) - \int_{\pi/4}^{\pi/2} \frac{1+\cos 2x}{2}\, dx$$

$$= \frac{1}{4}\log 2 - \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{4}\log 2 - \frac{1}{2}\left[\frac{\pi}{2} + 0 - \left(\frac{\pi}{4} + \frac{1}{2}\right)\right]$$

$$= \frac{1}{4}\log 2 - \frac{\pi}{8} + \frac{1}{4}$$

145. Let $I = \int_0^{\pi/2} 2\sin x\cdot\cos x\cdot\tan^{-1}(\sin x) dx$

Put $\sin x = t \Rightarrow \cos x\cdot dx = dt$

Also, $x = 0 \Rightarrow t = 0$ and $x = \frac{\pi}{2} \Rightarrow t = 1$

$$\therefore \quad I = \int_0^1 2\sin x\cos x\tan^{-1}(\sin x) dx$$

$$= 2\int_0^1 t\times\tan^{-1} t\cdot dt$$

$$= 2\left[\frac{t^2}{2}\times\tan^{-1} t\right]_0^1 - 2\int_0^1 \frac{1}{1+t^2}\times\frac{t^2}{2}\cdot dt$$

$$= 2\times\frac{1}{2}\times\tan^{-1}\left(\tan\frac{\pi}{4}\right) - \int_0^1 \frac{1+t^2-1}{1+t^2}\cdot dt$$

$$= 1\times\frac{\pi}{4} - \int_0^1\left(1 - \frac{1}{1+t^2}\right)\cdot dt = \frac{\pi}{4} - \left[t - \tan^{-1} t\right]_0^1$$

$$= \frac{\pi}{4} - 1 + \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{2\pi}{4} - 1 = \left(\frac{\pi}{2} - 1\right)$$

146. Let $I = \int_{-\pi/2}^{\pi/2} \sin^5 x\, dx$

Let $f(x) = \sin^5 x \Rightarrow f(-x) = \sin^5(-x) = -\sin^5 x = -f(x)$

$$\therefore \quad I = 0 \quad \left[\because \int_{-a}^{a} f(x) dx = 0 \text{ if } f(-x) = -f(x)\right]$$

147. *Refer to answer 146.*

148. Let $I = \int_{-1}^{2} |x^3 - x| dx$

$$\text{Since, } |x^3 - x| = \begin{cases} x^3 - x, & x\in(-1,0)\cup(1,2) \\ -(x^3 - x), & x\in(0,1)\end{cases}$$

$$\therefore \quad I = \int_{-1}^{0}(x^3 - x)dx + \int_0^1 -(x^3 - x)dx + \int_1^2 (x^3 - x)dx$$

$$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{-x^4}{4} + \frac{x^2}{2}\right)_0^1 + \left(\frac{x^4}{4} - \frac{x^2}{2}\right)_1^2$$

$$= \left[0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] + \left[-\frac{1}{4} + \frac{1}{2} - 0\right] + \left[(4-2) - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{4} + \frac{1}{4} + 2 + \frac{1}{4} = \frac{11}{4}$$

149. Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx \quad \text{...(1)}$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx \quad \text{...(2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2}\left(\frac{\sin^2 x}{\sin x + \cos x} + \frac{\cos^2 x}{\sin x + \cos x}\right) dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{1}{\frac{2\tan(x/2)}{1+\tan^2(x/2)} + \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}} dx$$

$$\Rightarrow\ 2I=\int_0^{\pi/2}\frac{1+\tan^2(x/2)}{2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx$$

$$\Rightarrow\ 2I=\int_0^{\pi/2}\frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}dx$$

Put $\tan\frac{x}{2}=t \Rightarrow \sec^2\frac{x}{2}\cdot\frac{1}{2}dx=dt$

When $x=0 \Rightarrow t=0$ and when $x=\frac{\pi}{2}\Rightarrow t=1$

$$\therefore\ 2I=\int_0^1\frac{2dt}{2t+1-t^2}=2\int_0^1\frac{1}{(\sqrt{2})^2-(t-1)^2}dt$$

$$\Rightarrow\ 2I=2\times\frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|\right]_0^1$$

$$\Rightarrow\ 2I=\frac{1}{\sqrt{2}}\left\{\log\left(\frac{\sqrt{2}}{\sqrt{2}}\right)-\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}$$

$$\Rightarrow\ 2I=\frac{1}{\sqrt{2}}\left\{0-\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}=\frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$\Rightarrow\ 2I=\frac{1}{\sqrt{2}}\log\left\{\frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)(\sqrt{2}+1)}\right\}$$

$$\Rightarrow\ 2I=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)^2=\frac{2}{\sqrt{2}}\log(\sqrt{2}+1)$$

$$\Rightarrow\ I=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$$

150. When $0<x<\frac{1}{2} \Rightarrow 0<\pi x<\frac{\pi}{2} \Rightarrow \cos\pi x>0$

When $\frac{1}{2}<x<\frac{3}{2} \Rightarrow \frac{\pi}{2}<\pi x<\frac{3\pi}{2} \Rightarrow \cos\pi x<0$

$$\therefore\ |x\cos\pi x|=\begin{cases} x\cos\pi x, \text{ if } 0<x<\frac{1}{2}\\ -x\cos\pi x, \text{ if } \frac{1}{2}<x<\frac{3}{2}\end{cases}$$

Let $I=\int_0^{3/2}|x\cos\pi x|\,dx$

$$=\int_0^{1/2}x\cos\pi x\,dx+\int_{1/2}^{3/2}-(x\cos\pi x)dx$$

$$\therefore\ I=\left[\frac{x}{\pi}\sin\pi x+\frac{\cos\pi x}{\pi^2}\right]_0^{1/2}-\left[\frac{x}{\pi}\sin\pi x+\frac{\cos\pi x}{\pi^2}\right]_{1/2}^{3/2}$$

(Using by parts)

$$=\left[\frac{1}{\pi}\left(\frac{1}{2}-0\right)+\frac{1}{\pi^2}(0-1)\right]$$

$$-\left[\frac{1}{\pi}\left(\frac{3}{2}(-1)-\frac{1}{2}(1)\right)+\frac{1}{\pi^2}(0-0)\right]$$

$$=\left(\frac{1}{2\pi}-\frac{1}{\pi^2}\right)-\left(\frac{-2}{\pi}\right)=\left(\frac{5\pi-2}{2\pi^2}\right)$$

151. Let $I=\int_0^{\pi}\frac{x}{1+\sin\alpha\sin x}dx$

$$\Rightarrow\ I=\int_0^{\pi}\frac{\pi-x}{1+\sin\alpha\sin(\pi-x)}dx$$

$$\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$\Rightarrow\ I=\int_0^{\pi}\frac{\pi}{1+\sin\alpha\sin x}dx-\int_0^{\pi}\frac{x}{1+\sin\alpha\sin x}dx$$

$$\Rightarrow\ I=\int_0^{\pi}\frac{\pi}{1+\sin\alpha\sin x}dx-I$$

$$\Rightarrow\ 2I=\int_0^{\pi}\frac{\pi}{1+\sin\alpha\sin x}dx$$

$$\Rightarrow\ 2I=\pi\int_0^{\pi}\frac{1}{1+\sin\alpha\sin x}dx$$

$$\Rightarrow\ 2I=\pi\int_0^{\pi}\frac{1+\tan^2\frac{x}{2}}{\left(1+\tan^2\frac{x}{2}+\sin\alpha\times 2\tan\frac{x}{2}\right)}dx$$

$$\therefore\ I=\frac{\pi}{2}\int_0^{\pi}\frac{\sec^2\frac{x}{2}}{\left(1+\tan^2\frac{x}{2}+\sin\alpha\times 2\tan\frac{x}{2}\right)}dx$$

Let $\tan\frac{x}{2}=t \Rightarrow \sec^2\frac{x}{2}dx=2dt$

Also, when $x\to 0$, t tan0 = 0;

when $x\to\pi$, $t\to\tan\frac{\pi}{2}=\infty$

$$\therefore\ I=\frac{\pi}{2}\int_0^{\infty}\frac{2dt}{t^2+2t\sin\alpha+1}$$

$$\Rightarrow \quad I = \pi \int_0^{\infty} \frac{1}{(t+\sin\alpha)^2 + \cos^2\alpha} dt$$

$$\Rightarrow \quad I = \frac{\pi}{\cos\alpha}\left[\tan^{-1}\left(\frac{t+\sin\alpha}{\cos\alpha}\right)\right]_0^{\infty}$$

$$\Rightarrow \quad I = \frac{\pi}{\cos\alpha}\left[\tan^{-1}\infty - \tan^{-1}(\tan\alpha)\right]$$

$$\Rightarrow \quad I = \frac{\pi}{\cos\alpha}\left(\frac{\pi}{2} - \alpha\right)$$

152. Let $I = \int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 \, dx$

$$= \int_{-\pi}^{\pi} (\cos^2 ax + \sin^2 bx - 2\cos ax \sin bx) dx$$

$$= \int_{-\pi}^{\pi} \cos^2 ax \, dx + \int_{-\pi}^{\pi} \sin^2 bx \, dx - 2\int_{-\pi}^{\pi} \cos ax \sin bx \, dx$$

$$= 2\left[\int_0^{\pi} \cos^2 ax \, dx + \int_0^{\pi} \sin^2 bx \, dx\right]$$

$$\left[\because \int_{-a}^{a} f(x)dx = \begin{bmatrix} 0, \text{ if } f(x) \text{ is odd} \\ 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is even} \end{bmatrix}\right]$$

$$= 2\left[\int_0^{\pi}\left(\frac{1+\cos 2ax}{2}\right)dx + \int_0^{\pi}\left(\frac{1-\cos 2bx}{2}\right)dx\right]$$

$$= \int_0^{\pi}(1+\cos 2ax)dx + \int_0^{\pi}(1-\cos 2bx)dx$$

$$= |2x|_0^{\pi} + \frac{1}{2a}|\sin 2ax|_0^{\pi} - \frac{1}{2b}|\sin 2bx|_0^{\pi}$$

$$= 2\pi + \frac{\sin 2a\pi}{2a} - \frac{\sin 2b\pi}{2b}$$

153. Let $I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{e^{-x}+1} e^{-x} dx$

$$= \int_{-\pi/2}^{\pi/2} \frac{\cos x(e^{-x}+1-1)}{e^{-x}+1} dx$$

$$= \int_{-\pi/2}^{\pi/2} \cos x \, dx - \int_{-\pi/2}^{\pi/2} \frac{\cos x}{e^{-x}+1} dx$$

Now, put $x = -z$ in 2^{nd} integral,

$\therefore \quad dx = -dz$

Also, $x = \frac{-\pi}{2} \Rightarrow z = \frac{\pi}{2}$ and $x = \frac{\pi}{2} \Rightarrow z = \frac{-\pi}{2}$

$$\therefore \quad I = \int_{-\pi/2}^{\pi/2} \cos x \, dx + \int_{\pi/2}^{-\pi/2} \frac{\cos z}{e^z+1} dz$$

$$I = [\sin x]_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} \frac{\cos x}{e^x+1} dx$$

$$I = \left[\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right] - I \Rightarrow 2I = 2 \Rightarrow I = 1$$

154. Let $I = \int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}} = \int_0^{\pi/2} \frac{dx}{1+\sqrt{\frac{\sin x}{\cos x}}}$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx \qquad \text{...(1)}$$

By the property, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, we get

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}} dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2}\left[\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} + \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right]dx$$

$$= \int_0^{\pi/2} 1 \cdot dx = [x]_0^{\pi/2} = \frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi}{4}$$

155. Let $I = \int_0^{\pi/4} \log(1+\tan x) dx$

By using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, we get

$$I = \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$$

$$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan x}{1+\tan x}\right]dx = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} [\log 2 - \log(1+\tan x)]dx$$

$$= \log 2 \int_0^{\pi/4} 1\cdot dx - \int_0^{\pi/4} \log(1+\tan x)dx = \log 2 [x]_0^{\pi/4} - I$$

$$\Rightarrow \quad 2I = \log 2\left[\frac{\pi}{4} - 0\right] \Rightarrow I = \frac{\pi}{8}\log 2$$

156. Let $I = \int_0^{\pi} \frac{x\tan x}{\sec x \operatorname{cosec} x} dx$ ...(1)

Using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, we get

$$I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x)\operatorname{cosec}(\pi - x)}dx = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x \operatorname{cosec} x}$$ ...(2)

Adding (1) and (2), we get

$$2I = \int_0^{\pi}\left[\frac{x\tan x}{\sec x\operatorname{cosec} x} + \frac{(\pi - x)\tan x}{\sec x \operatorname{cosec} x}\right]dx$$

$$= \pi\int_0^{\pi} \frac{\tan x}{\sec x \operatorname{cosec} x}dx = \pi\int_0^{\pi} \frac{\sin x/\cos x}{\frac{1}{\cos x}\cdot\frac{1}{\sin x}}dx$$

$$= \pi\int_0^{\pi} \sin^2 x\, dx = \pi\int_0^{\pi} \frac{1-\cos 2x}{2}dx$$

$$= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{\pi^2}{2} \quad \Rightarrow \quad I = \frac{\pi^2}{4}$$

157. Let $I = \int_0^{\pi} \frac{4x\sin x}{1+\cos^2 x}dx$ ... (1)

$$I = \int_0^{\pi} \frac{4(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$$

$$\left[\text{By } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{4(\pi - x)\sin x}{1+\cos^2 x}dx$$ ... (2)

Adding (1) and (2), we get

Hence, $2I = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$

Put $\cos x = t \Rightarrow \sin x\, dx = -dt$

Also $x = 0 \Rightarrow t = 1$ and $x = \pi \Rightarrow t = -1$

$$\therefore \quad I = 2\pi\int_1^{-1} \frac{-dt}{1+t^2} = 2\pi\int_{-1}^{1} \frac{dt}{t^2+1^2} = 2\pi\left[\tan^{-1} t\right]_{-1}^{1}$$

$$= 2\pi\left[\tan^{-1}1 - \tan^{-1}(-1)\right] = 2\pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi^2$$

158. *Refer to answer 149.*

159. Let $I = \int_0^4 (|x| + |x-2| + |x-4|)dx$

$$= \int_0^2 [x - (x-2) - (x-4)]dx + \int_2^4 [x + (x-2) - (x-4)]dx$$

$$= \int_0^2 (-x+6)dx + \int_2^4 (x+2)dx$$

$$= \left[-\frac{x^2}{2} + 6x\right]_0^2 + \left[\frac{x^2}{2} + 2x\right]_2^4$$

= (–2 + 12 – 0) + [8 + 8 – (2 + 4)] = 10 + 10 = 20

160. Let $I = \int_2^5 [|x-2| + |x-3| + |x-5|]dx$

$$= \int_2^3 [(x-2) - (x-3) - (x-5)]dx$$

$$+ \int_3^5 [(x-2) + (x-3) - (x-5)]dx$$

$$= \int_2^3 (-x+6)dx + \int_3^5 (x)dx = \left[-\frac{x^2}{2} + 6x\right]_2^3 + \left[\frac{x^2}{2}\right]_3^5$$

$$= \left[\left(-\frac{9}{2} + 18\right) - (-2+12)\right] + \frac{1}{2}(25-9)$$

$$= -\frac{9}{2} + 8 + 8 = \frac{23}{2}$$

161. We have, $I = \int_1^3 [|x-1| + |x-2| + |x-3|]\,dx$

$$= \int_1^2 [|x-1| + |x-2| + |x-3|]dx +$$

$$+ \int_2^3 [|x-1| + |x-2| + |x-3|]dx$$

$$= \int_1^2 (x-1) - (x-2) - (x-3)dx + \int_2^3 (x-1) + (x-2) - (x-3)dx$$

$$=\int_1^2(-x+4)dx+\int_2^3 x\,dx=\left[-\frac{x^2}{2}+4x\right]_1^2+\left[\frac{x^2}{2}\right]_2^3$$

$$=\left[\left(-\frac{2^2}{2}+4(2)\right)-\left(-\frac{1}{2}+4\right)\right]+\left[\frac{3^2}{2}-\frac{2^2}{2}\right]$$

$$=\left[(-2+8)-\left(\frac{7}{2}\right)\right]+\left[\frac{9}{2}-\frac{4}{2}\right]=\frac{5}{2}+\frac{5}{2}=5$$

162. Let $I=\int_0^{2\pi}\frac{1}{1+e^{\sin x}}dx$

$$=\int_0^{\pi}\frac{dx}{1+e^{\sin x}}+\int_{\pi}^{2\pi}\frac{dx}{1+e^{\sin x}}=\int_0^{\pi}\frac{dx}{1+e^{\sin x}}+I_1 \quad ...(1)$$

where $I_1=\int_{\pi}^{2\pi}\frac{dx}{1+e^{\sin x}}$

$$=\int_0^{\pi}\frac{dy}{1+e^{-\sin y}} \quad (\text{Put}: x=\pi+y)$$

$$=\int_0^{\pi}\frac{e^{\sin y}\,dy}{e^{\sin y}+1}=\int_0^{\pi}\frac{e^{\sin x}}{1+e^{\sin x}}\,dx$$

$\therefore$ From (1),

$$I=\int_0^{\pi}\frac{dx}{1+e^{\sin x}}+\int_0^{\pi}\frac{e^{\sin x}}{1+e^{\sin x}}\,dx$$

$$\Rightarrow \quad I=\int_0^{\pi}\frac{1+e^{\sin x}}{1+e^{\sin x}}\,dx=\int_0^{\pi}1\cdot dx=[x]_0^{\pi}=\pi$$

163. *Refer to answer 157.*

164. Let $I=\int_0^{\pi}\frac{x}{1+\sin x}dx \quad ...(1)$

Using $\int_0^a f(x)dx=\int_0^a f(a-x)dx$, we get

$$I=\int_0^{\pi}\frac{\pi-x}{1+\sin(\pi-x)}dx=\int_0^{\pi}\frac{\pi-x}{1+\sin x}dx \quad ...(2)$$

Adding (1) and (2), we get

$$2I=\int_0^{\pi}\frac{\pi}{1+\sin x}dx=\pi\int_0^{\pi}\frac{dx}{1+\sin x}$$

$$=\pi\int_0^{\pi}\frac{1}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}dx=\pi\int_0^{\pi}\frac{1-\sin x}{\cos^2 x}dx$$

$$=\pi\int_0^{\pi}(\sec^2 x-\tan x\sec x)dx=\pi[\tan x-\sec x]_0^{\pi}$$

$= \pi[0 - \sec\pi - (0 - \sec 0)] = \pi[1 + 1] = 2\pi \Rightarrow I = \pi$

165. Let $I=\int_0^1\log\left(\frac{1}{x}-1\right)dx \quad ... (1)$

$$\Rightarrow \quad I=\int_0^1\log\left(\frac{1}{1-x}-1\right)dx$$

$$\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I=\int_0^1\log\left(\frac{1-1+x}{1-x}\right)dx$$

$$\Rightarrow \quad I=\int_0^1\log\left(\frac{x}{1-x}\right)dx \quad ... (2)$$

Adding (1) and (2), we get

$$2I=\int_0^1\left[\log\left(\frac{1-x}{x}\right)+\log\left(\frac{x}{1-x}\right)\right]dx$$

$$=\int_0^1\log 1\cdot dx=0 \quad \Rightarrow I=0$$

166. Let $I=\int_{\pi/6}^{\pi/3}\frac{dx}{(1+\sqrt{\tan x})}$

$$\Rightarrow \quad I=\int_{\pi/6}^{\pi/3}\frac{dx}{\left(1+\sqrt{\frac{\sin x}{\cos x}}\right)}=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \quad ...(1)$$

$$\Rightarrow \quad I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}dx$$

$$\left[\because \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$$

$$\Rightarrow \quad I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}dx$$

$$\Rightarrow \quad I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx=\int_{\pi/6}^{\pi/3}dx$$

$$\Rightarrow \quad 2I=[x]_{\pi/6}^{\pi/3}=\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{\pi}{6} \Rightarrow 2I=\frac{\pi}{6}$$

$$\Rightarrow \quad I=\frac{\pi}{12}$$

167. $I=\int_0^1\frac{\log(1+x)}{1+x^2}dx$

Put $x=\tan\theta \Rightarrow dx=\sec^2\theta d\theta$

Also, $x=0\Rightarrow\theta=0$ and $x=1\Rightarrow\theta=\frac{\pi}{4}$

$$\therefore \quad I=\int_0^{\pi/4}\frac{\log(1+\tan\theta)}{1+\tan^2\theta}\cdot\sec^2\theta d\theta$$

$$=\int_0^{\pi/4}\log(1+\tan\theta)d\theta$$

Refer to answer 155.

168. Let $I=\int_0^{\pi/2}(2\log\sin x-\log\sin 2x)dx$

$$=\int_0^{\pi/2}[2\log\sin x-\log(2\sin x\cos x)]dx$$

$$=\int_0^{\pi/2}[2\log\sin x-\log 2-\log\sin x-\log\cos x]dx$$

$$=\int_0^{\pi/2}\log\sin x\,dx-(\log 2)\int_0^{\pi/2}(1)dx-\int_0^{\pi/2}\log\cos\left(\frac{\pi}{2}-x\right)dx$$

$$\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$=\int_0^{\pi/2}\log\sin x\,dx-(\log 2)[x]_0^{\pi/2}-\int_0^{\pi/2}\log\sin x\,dx$$

$$=-(\log 2)\left(\frac{\pi}{2}-0\right)=-\frac{\pi}{2}\log 2$$

$$=\frac{\pi}{2}\log(2)^{-1}=\frac{\pi}{2}\log\left(\frac{1}{2}\right)$$

169. Let $I=\int_0^{\pi}\frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx \quad \ldots(1)$

$$\Rightarrow \quad I=\int_0^{\pi}\frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)}+e^{-\cos(\pi-x)}}dx$$

$$\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I=\int_0^{\pi}\frac{e^{-\cos x}}{e^{-\cos x}+e^{\cos x}}dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I=\int_0^{\pi}\frac{e^{\cos x}+e^{-\cos x}}{e^{\cos x}+e^{-\cos x}}dx=\int_0^{\pi}dx$$

$$\Rightarrow \quad 2I=[x]_0^{\pi}=\pi \Rightarrow I=\frac{\pi}{2}$$

170. Let $I=\int_0^1\cot^{-1}(1-x+x^2)dx$

$$=\int_0^1\tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx=\int_0^1\tan^{-1}\left[\frac{x+(1-x)}{1-x(1-x)}\right]dx$$

$$=\int_0^1[\tan^{-1}x+\tan^{-1}(1-x)]dx$$

$$=\int_0^1\tan^{-1}x\,dx+\int_0^1\tan^{-1}(1-x)dx$$

$$=\int_0^1\tan^{-1}x\,dx+\int_0^1\tan^{-1}(1-(1-x))dx$$

$$\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$=\int_0^1\tan^{-1}x\,dx+\int_0^1\tan^{-1}x\,dx=2\int_0^1\tan^{-1}x\,dx$$

Integrating by parts, we get

$$=2\left[\left[x\tan^{-1}x\right]_0^1-\frac{1}{2}\int_0^1\frac{2x}{1+x^2}dx\right]$$

$= 2\left[\left[\tan^{-1}1 - 0\right] - \frac{1}{2}\left[\log|1+x^2|\right]_0^1\right]$

$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right] = \frac{\pi}{2} - \log 2$

171. Let $I = \int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}dx$

$= \int_{-a}^{a}\sqrt{\frac{(a-x)(a-x)}{(a+x)(a-x)}}dx = \int_{-a}^{a}\frac{a-x}{\sqrt{a^2-x^2}}dx$

$= \int_{-a}^{a}\frac{a}{\sqrt{a^2-x^2}}dx - \int_{-a}^{a}\frac{xdx}{\sqrt{a^2-x^2}} = a\left[\sin^{-1}\frac{x}{a}\right]_{-a}^{a} - 0$

$\left[\because \frac{x}{\sqrt{a^2-x^2}} \text{ is an odd function}\right]$

$= a\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = a\pi$

172. Let $I = \int_0^{\pi/2}\frac{x}{\sin x + \cos x}dx$ ...(1)

$\Rightarrow \quad I = \int_0^{\pi/2}\frac{\left(\frac{\pi}{2} - x\right)dx}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}$

$\Rightarrow \quad I = \int_0^{\pi/2}\frac{\left(\frac{\pi}{2} - x\right)}{\sin x + \cos x}dx$ ...(2)

Adding (1) and (2), we get

$2I = \frac{\pi}{2}\int_0^{\pi/2}\frac{dx}{\sin x + \cos x} = \frac{\pi}{2\sqrt{2}}\int_0^{\pi/2}\frac{dx}{\frac{\sin x}{\sqrt{2}} + \frac{\cos x}{\sqrt{2}}}$

$= \frac{\pi}{2\sqrt{2}}\int_0^{\pi/2}\frac{dx}{\sin\left(x + \frac{\pi}{4}\right)} = \frac{\pi}{2\sqrt{2}}\int_0^{\pi/2}\operatorname{cosec}\left(x + \frac{\pi}{4}\right)dx$

$\Rightarrow \quad I = \frac{\pi}{4\sqrt{2}}\left[\log\left|\operatorname{cosec}\left(x + \frac{\pi}{4}\right) - \cot\left(x + \frac{\pi}{4}\right)\right|\right]_0^{\pi/2}$

$= \frac{\pi}{2\sqrt{2}}\log\left(\sqrt{2} + 1\right)$

173. Let $I = \int_0^{\pi/2}\log\sin x\, dx$ ...(1)

$\Rightarrow \quad I = \int_0^{\frac{\pi}{2}}\log\sin\left(\frac{\pi}{2} - x\right)dx$

$\Rightarrow \quad I = \int_0^{\frac{\pi}{2}}\log\cos x dx$ ...(2)

Adding (1) and (2), we get

$2I = \int_0^{\frac{\pi}{2}}\log\sin x dx + \int_0^{\frac{\pi}{2}}\log\cos x dx$

$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}}(\log\sin x + \log\cos x)dx$

$\Rightarrow \quad 2I = \int_0^{\pi/2}\log(\sin x\cos x)dx$

$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}}\log\frac{\sin 2x}{2}dx$

$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}}[\log\sin 2x - \log 2]dx$

$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}}\log\sin 2x dx - \int_0^{\frac{\pi}{2}}\log 2 dx$

$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}}\log\sin 2x dx - \log 2[x]_0^{\frac{\pi}{2}}$

$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{2}}\log\sin 2x dx - \frac{\pi}{2}\log 2$ ... (3)

Let $I_1 = \int_0^{\frac{\pi}{2}}\log\sin 2x dx$, put $2x = t \Rightarrow 2dx = dt$

When $x = 0 \Rightarrow t = 0$ and when $x = \frac{\pi}{2} \Rightarrow t = \pi$

$\therefore \quad I_1 = \frac{1}{2}\int_0^{\pi}\log\sin t dt$

$\Rightarrow \quad I_1 = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}}\log\sin t + \int_0^{\frac{\pi}{2}}\log\sin(\pi - t)dt\right]$

$$\Rightarrow \quad I_1 = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \log\sin t + \int_0^{\frac{\pi}{2}} \log\sin t\,dt\right]$$

$$\Rightarrow \quad I_1 = \frac{1}{2}\times 2\int_0^{\frac{\pi}{2}} \log\sin t\,dt = \int_0^{\frac{\pi}{2}} \log\sin t\,dt$$

Since $I = \int_0^{\frac{\pi}{2}} \log\sin x\,dx$, then

$$I_1 = I$$

From (3), we get

$$2I = I - \frac{\pi}{2}\log 2 \Rightarrow I = \frac{-\pi}{2}\log 2$$

174. Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$ ...(1)

$$\Rightarrow \quad I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{a-(a-x)}}dx$$

$$= \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx \qquad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx = \int_0^a (1)dx = [x]_0^a$$

$$= a - 0 = a$$

$$\Rightarrow \quad I = \frac{a}{2}$$

175. *Refer to answer 156.*

176. Let $I = \int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}dx$ ...(1)

$$\therefore \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)\sin x\cos x\,dx}{\cos^4 x+\sin^4 x} \qquad ...(2)$$

Adding (1) and (2), we get

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x\cos x\,dx}{\sin^4 x+\cos^4 x}$$

Dividing numerator and denominator by $\cos^4 x$, we get

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\tan x\sec^2 x\,dx}{1+\tan^4 x}.$$

Put $\tan^2 x = t \Rightarrow 2\tan x\sec^2 x\,dx = dt$

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = \infty$

$$\therefore \quad I = \frac{\pi}{8}\int_0^{\infty} \frac{dt}{1+t^2} = \frac{\pi}{8}\left[\tan^{-1} t\right]_0^{\infty} = \frac{\pi}{8}\times\frac{\pi}{2} = \frac{\pi^2}{16}$$

177. Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

Refer to answer 166.

178. Let $I = \int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}dx$ ...(1)

Also, $I = \int_0^{\pi} \frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}dx$

$$= \int_0^{\pi} \frac{(\pi-x)(-\tan x)}{(-\sec x)+(-\tan x)}dx = \int_0^{\pi} \frac{(\pi-x)\tan x}{\sec x+\tan x}dx \qquad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{(x+\pi-x)\tan x}{\sec x+\tan x} = \pi\int_0^{\pi} \frac{\tan x}{\sec x+\tan x}dx$$

$$= \pi\int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}dx = \pi\int_0^{\pi} \frac{\sin x}{1+\sin x}dx$$

$$= \pi\int_0^{\pi} \frac{(1+\sin x)-1}{1+\sin x}dx = \pi\int_0^{\pi}\left(1-\frac{1}{1+\sin x}\right)dx$$

$$= \pi\int_0^{\pi}(1)dx - \pi\int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}dx = \pi[x]_0^{\pi} - \pi\int_0^{\pi} \frac{1-\sin x}{\cos^2 x}dx$$

$$= \pi(\pi-0) - \pi\int_0^{\pi}(\sec^2 x - \sec x\tan x)dx$$

$$= \pi^2 - \pi[\tan x - \sec x]_0^{\pi}$$

$$= \pi^2 - \pi[(\tan\pi - \tan 0) - (\sec\pi - \sec 0)]$$

$$= \pi^2 - \pi(0-0) + \pi(-1-1) = \pi^2 - 2\pi = \pi(\pi-2)$$

$$\therefore \quad I = \frac{\pi}{2}(\pi-2)$$

179. Let $I = \int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$

$$\left[\text{Using} \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)dx}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)} \quad ...(1)$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)dx}{a^2\cos^2 x + b^2\sin^2 x} \quad ...(2)$$

Adding (1) and (2), we get

$$I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Let $f(x) = \frac{1}{a^2\cos^2 x + b^2\sin^2 x}$

$$\Rightarrow \quad f(\pi - x) = \frac{1}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}$$

$$\Rightarrow \quad f(\pi - x) = \frac{1}{a^2\cos^2 x + b^2\sin^2 x} = f(x)$$

$$\left[\therefore \text{ using} \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(2a - x) = f(x)\right]$$

$$\therefore \quad I = \frac{\pi}{2}\left(2\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}\right)$$

$$\Rightarrow \quad I = \pi\int_0^{\pi/2} \frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$.

Also when $x = 0 \Rightarrow t = \tan 0 = 0$.

And when $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{2} = \infty$

$$\therefore \quad I = \int_0^{\infty} \frac{dt}{a^2 + b^2t^2} \Rightarrow I = \frac{\pi}{b^2}\int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$$

$$\Rightarrow \quad I = \frac{\pi}{b^2}\left[\frac{b}{a}\tan^{-1}\left(\frac{bt}{a}\right)\right]_0^{\infty}$$

$$= \quad I = \frac{\pi}{ab}\left[\tan^{-1}\infty - \tan^{-1}0\right] = \frac{\pi^2}{2ab}$$

180. Here $|x - 1| + |x - 2| + |x - 4|$

$$= \begin{cases} (x-1) - (x-2) - (x-4), & \text{when } 1 \le x < 2 \\ (x-1) + (x-2) - (x-4), & \text{when } 2 \le x < 4 \end{cases}$$

$$= \begin{cases} -x + 5, & \text{when } 1 \le x < 2 \\ x + 1, & \text{when } 2 \le x < 4 \end{cases}$$

$$\therefore \quad \int_1^4 \{|x-1| + |x-2| + |x-4|\}dx$$

$$= \int_1^2 \{|x-1| + |x-2| + |x-4|\}dx + \int_2^4 \{|x-1| + |x-2| + |x-4|\}dx$$

$$= \int_1^2 (-x+5)dx + \int_2^4 (x+1)dx$$

$$= \left[-\frac{x^2}{2} + 5x\right]_1^2 + \left[\frac{x^2}{2} + x\right]_2^4$$

$$= (-2+10) - \left(-\frac{1}{2} + 5\right) + (8+4) - (2+2) = \frac{23}{2}$$