De Moiver’s Theorem

Q.1. Prove that $(1 + i\sqrt{3})^8 + (1 - i\sqrt{3})^8 = -2^8$ by using De Moiver’s Theorem.

Solution : 1

Let $r \cos \theta = 1$ and $r \sin \theta = \sqrt{3}$

Then $\tan \theta = \sqrt{3} \Rightarrow \theta = \pi/3$ and $r = \sqrt{(\sin^2 \theta + \cos^2 \theta)} = \sqrt{(1 + 3)} = 2$.

Therefore, L.H.S. $= (1+ i\sqrt{3})^8 + (1 - i\sqrt{3})^8$

$= [r(\cos \theta + i\sin \theta)]^8 + [r(\cos \theta - i\sin \theta)]^8$

$= 2^8[(\cos \pi/3 + i\sin\pi/3)8 + (\cos \pi/3 - i\sin \pi/3)8]$

$= 2^8[\cos 8/3\ i\sin 8\pi/3 + \cos 8\pi/3 - i\sin 8\pi/3]$

$= 2^8.2\cos 8\pi/3$

$= 2^8.2 \cos (3\pi - \pi/3)$

$= -2^8.2 \cos \pi/3$

$= -2^8.2 \times (1/2)$

$= -2^8$.

Q.2. Using De Moivre’s theorem, find the least value of $n \in N$ for which the expression : $(1+i)^n + (1 - i)^n$ is equal to $-2^{(n+2)/2}$.

Solution : 2

The given expression is :

$(1+i)^n + (1-i)^n$,

Let $1 + i = r(\cos \theta + i\sin \theta) \Rightarrow r \cos \theta = 1$; $r \sin \theta = 1$

Therefore, $\tan \theta = 1$ or $\theta = \pi/4$ and $r = \sqrt{2}$.

Therefore, $(! + i)^{\mathbf{n}} + (1 - i)^{\mathbf{n}} = (r \cos θ + i\, r \sin θ)^{\mathbf{n}} + (r \cos θ - i\sin θ)^{\mathbf{n}}$

$= rn(\cos π/4 + i \sin π/4)n + rn(\cos π/4 - i \sin π/4)^{\mathbf{n}}$

$= (\sqrt{2})^{\mathbf{n}} [(\cos π/4π + i\sin π/4)^{\mathbf{n}} + (\cos π/4 - i\sin π/4)^{\mathbf{n}}$

$= (\sqrt{2})^{\mathbf{n}}[\cos nπ/4 + i\sin nπ/4 + \cos nπ/4 - i\sin nπ/4]$

$= (\sqrt{2})^{\mathbf{n}} \times 2\cos nπ/4$

$= 2^{(n/2+2)} \cos nπ/4$

Or, $2^{(n/2+2)} \cos {}^{\mathbf{n}}π/4 = -\, 2^{(n/2+2)}$

Or, $\cos nπ/4 = -1 = \cos π$

Or, $nπ/4 = π$

Therefore, $n = 4$.

Q.3. Using De Moivre's Theorem find the value of : $(2 - 2i)^{1/3}$.

Solution : 3

We have, $(2 - 2i)^{\mathbf{1/3}}$.

Putting $2 = a \cos θ$ and $-2 = a \sin θ$ then $\tan θ = -1$, $θ = 135^o$.

$2 = a \cos 135^o = a (-1/\sqrt{2}) \Rightarrow a = -2\sqrt{2}$

Or, $(2 - 2i)1/3 = [- 2\sqrt{2} (\cos 135^o + i\sin 135^o)]1/3$

$= [-2\sqrt{2}(\cos 3π/4 + i\sin 3π/4)]1/3$

$= [(-\sqrt{2})3(\cos 3π/4 + i\sin 3π/4)]1/3$

$= (-\sqrt{2})3\times1/3[\cos \{(2nπ + 3π/4)\}/3 + i\sin \{(2nπ + 3π/4)/3\}$

where , $n = 0, 1, 2$.

When $n = 0$, $(-\sqrt{2})[1/\sqrt{2} + i/\sqrt{2}] = -(1 + i)$

When $n = 1$, $(-\sqrt{2})[\cos 11π/12 + i\sin 11π/12]$

When $n = 2$, $(-\sqrt{2})[\cos 19π/12 + i\sin 19π/12]$

Q.4. Using De Moivre's Theorem, find the least value of n for which $[(1 + \sqrt{3}i)/\sqrt{2}(1 + i)]n$ is purely imaginary.

Solution : 4

$(1 + \sqrt{3}i)/\sqrt{2}(1 + i) = [(1 + \sqrt{3}i)(1 - i)]/[\sqrt{2}(1 + i)(1 - i)]$

$= [(1 + \sqrt{3}) + i(\sqrt{3} - 1)]/2\sqrt{2}$

Let $(\sqrt{3} + 1)/2\sqrt{2} + i(\sqrt{3} - 1)/2\sqrt{2} = (r \cos \theta) + i(r \sin \theta)$

Then, $r \cos \theta = (\sqrt{3} + 1)/2\sqrt{2}$ and $r \sin \theta = (\sqrt{3} - 1)/2\sqrt{2}$

Then, $r^{2} = (r \cos \theta)^{2} + (r \sin \theta)^{2}$

$= \{(\sqrt{3} + 1)/2\sqrt{2}\}^{2} + \{(\sqrt{3} - 1)/2\sqrt{2}\}^{2}$

$= (4 + 2\sqrt{3})/8 + (4 - 2\sqrt{3})/8 = 1$

Therefore, $r = \sqrt{1} = \pm 1$.

And $\tan \theta = r \sin \theta / r \cos \theta = [(\sqrt{3} - 1)/2\sqrt{2}]/[\sqrt{3} + 1)/2\sqrt{2}]$

$= (\sqrt{3} - 1)/(\sqrt{3} + 1)$

$= (\tan 60^{o} - \tan 45^{o})/(1 + \tan 60^{o} \times \tan 45^{o})$

$= \tan (60^{o} - 45^{o}) = \tan 15^{o}$

Therefore, $\theta = 15^{o} = \pi/12$.

Hence, $(1 + i\sqrt{3})/\sqrt{2}(1 + i) = \cos (\pi/12) + i\sin (\pi/12)$

Therefore, $[(1 + i\sqrt{3})/\sqrt{2}(1 + i)]n = [\cos (\pi/12) + i\sin (\pi/12)]n$

$= [\cos (n\pi/12) + i\sin (n\pi/12)]$

$\cos (n\pi/12)$ $i\sin (n\pi/12)$ will be purely imaginary if and only if $\cos (n\pi/12) = 0$

Or, $\cos (n\pi/12) = \cos \pi/2$. $\cos 3\pi/2$, $\cos 5\pi/2$, ………..

Or, $\cos (n\pi/12) = \cos (6\pi/12)$, $\cos (18\pi/12)$, $\cos (30\pi/12)$ ………….

Therefore, $n\pi/12 = 6\pi/12, 18\pi/12, 30\pi/12$, ………….

Or, n = 6, 18, 30, ……..

Therefore least value of n is 6.

Q.5. Using De Moivre's Theorem , find the least value of n for which the expression $\{(\sqrt{3} + 3i)/2\sqrt{3}\}^{2n-1}$ is purely real.

Solution : 5

We are given,

$\{(\sqrt{3} + 3i)/2\sqrt{3}\}2n - 1 = \{(1/2) + i(\sqrt{3}/2)\}^{\mathbf{2n-1}}$

$= (\cos \pi/3 + i\sin \pi/3)^{\mathbf{2n-1}}$

$= \cos (2n - 1)\pi/3 + i\sin (2n - 1)\pi/3$

The given expression is real, therefore its imaginary parts is zero. i.e.

$i\sin(2n - 1)\pi/3 = 0$, but $i \neq 0$.

Therefore, $\sin(2n - 1)\pi/3 = 0$

Therefore, value of $(2n - 1)\pi/3 = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots\ldots\ldots$

When $(2n - 1)\pi/3 = 0$, $2n - 1 = 0 \Rightarrow n = 1/2$.

This value is not acceptable as $n \in N$.

When $(2n - 1)\pi/3 = \pi$, $2n - 1 = 3 \Rightarrow n = 2 \in N$.

When $(2n - 1)\pi/3 = 2\pi$, $2n - 1 = 6 \Rightarrow n = 7/2$.

Which is again not acceptable as $n \in N$.

When $(2n - 1)\pi/3 = 3\pi$, $2n - 1 = 9 \Rightarrow n = 5 \in N$.

Therefore, the least value of n is 2.

Q.6. Using De Moivre's Theorem, find the value of $(1 + i\sqrt{3})^4 + (1 - i\sqrt{3})^4$.

Solution : 6

We are given, $(1 + i\sqrt{3})^{\mathbf{4}} + (1 - i\sqrt{3})^{\mathbf{4}}$

Let $r \cos \theta = 1$ and $r \sin \theta = \sqrt{3} \Rightarrow r \sin \theta / r \cos \theta = \sqrt{3}/1 \Rightarrow \tan \theta = \sqrt{3}$.

And $\theta = \pi/3$. also $r = \sqrt{\{(1)2 + (\sqrt{3})2\}} = \sqrt{(1 + 3)}\ \sqrt{4} = 2$.

Therefore, $(1 + i\sqrt{3}) = 2\ [1/2 + i\sqrt{3}/2] = 2\ [\cos \pi/3 + i\sin \pi/3]$

and $(1 - i\sqrt{3}) = 2[\cos \pi/3 - i\sin \pi/3]$

Hence, $(1 + i\sqrt{3})^{4} + (1 - i\sqrt{3})^{4} = (2)^{4}[\cos \pi/3 + i\sin \pi/3]^{4} + (2)^{4}[\cos \pi/3 - i\sin \pi/3]^{4}$

$= (2)^{4}[\cos^{4}\pi/3 + i\sin {}^{4}\pi/3 + \cos {}^{4}\pi/3 - i\sin {}^{4}\pi/3]$

$= (2)^{4}\times 2\cos {}^{4}\pi/3$

$=(2)5 \cos (\pi + \pi/3)$

$= (2)5\ (- \cos \pi/3)$

$= (2)5\ (- 1/2\ ) = -16$.

Q.7. Using De Moivre's Theorem , find the least value of n for which the expression $[(1 + i)/(1 - i)]^{n}$ is purely imaginary.

Solution : 7

We are given, $[(1 + i)/(1 - i)]^{n}$

Let $1 = r \cos \theta$ and $1 = r\sin \theta$

Then $r^{2}(\cos^{2}\theta + \sin {}^{2} \theta) = 1 + 1 = 2 \Rightarrow r^{2} = 2 \Rightarrow r = \sqrt{2}$

And $\tan \theta = 1 \Rightarrow \theta = \pi/4$.

Hence, $1 + i = r (\cos \theta + i\sin \theta)$ and $1 - i = r (\cos \theta - i \sin \theta)$

And $(1 + i)n/(1 - i)n = [rn(\cos \theta + i\sin \theta)n]/[rn(\cos \theta - i\sin \theta)^{n}]$

$= (\cos \theta\ i\sin \theta)n\ (\cos \theta - i\sin \theta)^{-n}$

$= (\cos n\theta + i\sin n\theta)(\cos n\theta + i\sin n\theta)$

$= (\cos n\theta + i\sin n\theta)^{2} = \cos 2n\theta + i\sin 2n\theta$

As the given expression is purely imaginary, $\cos 2n\theta = 0$

Or, $\cos (2n\times\pi/4) = 0$, $\cos \pi/2 = 0$

Or, $\cos n\pi/2 = 0$, $n = \pm1, \pm3, \pm5, \pm7$ ……….

If we take only positive values of n then least value of n is 1.

Q.8. If $z = (13 - 5i)/(4 - 9i)$, prove by using De Moivre's Theorem that $z^6 = -8i$.

Solution : 8

We have, $z = (13 - 15i)/(4 - 9i)$

$= \{(13 - 15i)(4 + 9i)\}/\{(4 - 9i)(4 + 9i)\}$

$= 97(1 + i)/97$

$= 1 + i$

$= \sqrt{2}[1/\sqrt{2} + 1/\sqrt{2}i]$

$= \sqrt{2}[\cos \pi/4 + i\sin \pi/4]$

Therefore, $z^{\mathbf{6}} = \{\sqrt{2}[\cos \pi/4 + i\sin \pi/4]\}^{\mathbf{6}}$

$= 8[\cos 6\pi/4 + i\sin 6\pi/4]$

$= 8[0 - i] = -8i.$