Factorisation of polynomials

- **Remainder Theorem**

  If $p(x)$ is a polynomial of degree greater than or equal to one and a is any real number then if $p(x)$ is divided by the linear polynomial $x - a$, the remainder is $p(a)$.

  Example:
  Find the remainder when $x^5 - x^2 + 5$ is divided by $x - 2$.
  Solution:
  $p(x) = x^5 - x^2 + 5$
  The zero of $x - 2$ is 2.
  $p(2) = 2^5 - 2^2 + 5 = 32 - 4 + 5 = 33$
  Therefore, by remainder theorem, the remainder is 33.

- **Factor Theorem**

If $p(x)$ is a polynomial of degree $n \geq 1$ and a is any real number, then

  - $x - a$ is a factor of $p(x)$, if $p(a) = 0$.
  - $p(a) = 0$, if $(x - a)$ is a factor of $p(x)$.

  Example:
  Determine whether $x + 3$ is a factor of $x^3 + 5x^2 + 5x - 3$.
  Solution:
  The zero of $x + 3$ is -3.
  Let $p(x) = x^3 + 5x^2 + 5x - 3$
  $p(-3) = (-3)^3 + 5(-3)^2 + 5(-3) - 3$
  $= -27 + 45 - 15 - 3$
  $= -45 + 45$
  $= 0$
  Therefore, by factor theorem, $x + 3$ is the factor of $p(x)$.

- Factorisation of quadratic polynomials of the form $ax^2 + bx + c$ can be done using Factor theorem and splitting the middle term.

Example 1:

Factorize $x^2 – 7x + 10$ using the factor theorem.

Solution:

Let $p(x) = x^2 - 7x + 10$

The constant term is 10 and its factors are ±1, ±2, ±5and ±10.

Let us check the value of the polynomial for each of these factors of 10.

$p(1) = 1^2 - 7 \cdot 1 + 10 = 1 - 7 + 10 = 4 \neq 0$

Hence, $x - 1$ is not a factor of $p(x)$.

$p(2) = 2^2 - 7 \cdot 2 + 10 = 4 - 14 + 10 = 0$

Hence, $x - 2$ is a factor of $p(x)$.

$p(5) = 5^2 - 7 \cdot 5 + 10 = 25 - 35 + 10 = 0$

Hence, $x - 5$ is a factor of $p(x)$.

We know that a quadratic polynomial can have a maximum of two factors. We have obtained the two factors of the given polynomial, which are $x – 2$ and $x – 5$.

Thus, we can write the given polynomial as:

$$p(x) = x^2 - 7x + 10 = (x - 2)\,(x - 5)$$

Example 2:

Factorize $2x^2 – 11x + 15$ by splitting the middle term.

Solution:

The given polynomial is $2x^2 - 11x + 15$.

Here, $a\,c = 2 \times 15 = 30$. The middle term is − 11. Therefore, we have to split − 11 into two numbers such that their product is 30 and their sum is –11. These numbers are − 5 and − 6 [As (–5) + (–6) = –11 and (–5) × (–6) = 30].

Thus, we have:

$$\begin{aligned} 2x^2 - 11x + 15 &= 2x^2 - 5x - 6x + 15 \\ &= x\,(2x - 5) - 3\,(2x - 5) \\ &= (2x - 5)\,(x - 3) \end{aligned}$$

Note: A quadratic polynomial can have a maximum of two factors.

- Factorisation of cubic polynomials of the form $ax^3 + bx^2 + cx + d$ can be done using factor theorem and hit and trial method.

A cubic polynomial can have a maximum of three linear factors. So, by knowing one of these factors, we can reduce it to a quadratic polynomial.

Thus, to factorize a cubic polynomial, we first find a factor by the hit and trial method or by using the factor theorem, and then reduce the cubic polynomial into a quadratic polynomial and it is then solved further.

Example:

Factorise $p(x) = x^3 - 7x + 6$

Solution:

The constant term is 6.

The factors of 6 are ±1, ±2, ±3 and ±6.

Let $x = 1$

$p\ (x = 1) = (1)^3 - 7\ (1) + 6$

$= 1 - 7 + 6$

$= 0$

Thus, $(x - 1)$ is a factor of $p(x)$.

Now we have to group the term of $p\ (x)$ such that we can take $(x - 1)$ as common.

Therefore, $p(x) = x^3 - 7x + 6$

$= x^3 - x^2 + x^2 - x - 6x + 6$

$= x^2\ (x - 1) + x\ (x - 1) - 6\ (x - 1)$

$= (x - 1)\ (x^2 + x - 6) \qquad \ldots (1)$

Now, we factorize $(x^2 + x - 6)$ by splitting its middle term.

$x^2 + x - 6 = x^2 + 3x - 2x - 6$

$= x\ (x + 3) -2\ (x + 3)$

$= (x - 2)\ (x + 3)$

From equation (1), we get

$p(x) = (x - 1)(x - 2)(x + 3)$

Hence, factors of polynomial $p(x)$ are $(x - 1)$, $(x - 2)$ and $(x + 3)$.