19.CONIC SECTION PEREBOLE

1. CONIC SECTIONS:

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

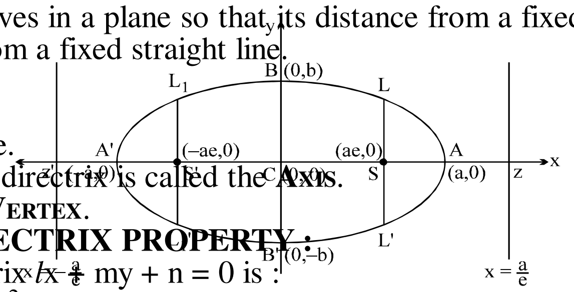
☞ The fixed point is called the **Focus.**

☞ The fixed straight line is called the **Directrix.**

☞ The constant ratio is called the **Eccentricity** denoted by e.

☞ The line passing through the focus & perpendicular to the directrix is called the **Axis.**

☞ A point of intersection of a conic with its axis is called a **Vertex.**



2. GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY :

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is :

$(l^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(lx + my + n)^2 \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

3. DISTINGUISHING BETWEEN THE CONIC :

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (I) : When The Focus Lies On The Directrix.

In this case $D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines if :www.MathsBySuhag.com , www.TekoClasses.com

$e > 1$ the lines will be real & distinct intersecting at S.

$e = 1$ the lines will coincident.

$e < 1$ the lines will be imaginary.

Case (II) : When The Focus Does Not Lie On Directrix.

a parabola	an ellipse	a hyperbola	rectangular hyperbola
$e = 1$; $D \neq 0$,	$0 < e < 1$; $D \neq 0$;	$e > 1$; $D \neq 0$;	$e > 1$; $D \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h^2 > ab$; $a + b = 0$

4. PARABOLA : DEFINITION :

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

Standard equation of a parabola is $y^2 = 4ax$. For this parabola :

(i) Vertex is (0, 0) (ii) focus is (a, 0) (iii) Axis is $y = 0$ (iv) Directrix is $x + a = 0$

FOCAL DISTANCE : The distance of a point on the parabola from the focus is called the **Focal Distance Of The Point.**

FOCAL CHORD :

A chord of the parabola, which passes through the focus is called a **Focal Chord.**

DOUBLE ORDINATE : A chord of the parabola perpendicular to the axis of the symmetry is called a **Double Ordinate.**

LATUS RECTUM :

A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the **Latus Rectum.** For $y^2 = 4ax$.

■Length of the latus rectum = 4a. ■ends of the latus rectum are L(a, 2a) & L' (a, – 2a).

Note that: **(i)** Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

(iii) Two parabolas are laid to be equal if they have the same latus rectum.

Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$

5. POSITION OF A POINT RELATIVE TO A PARABOLA :

The point $(x_1\ y_1)$ lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.

6. LINE & *A* PARABOLA :

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a \gtrless cm \Rightarrow$ condition of tangency is, $c = \frac{a}{m}$.

7. Length of the chord intercepted by the parabola on the line $y = mx + c$ is : $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a - mc)}$.

Note: length of the focal chord making an angle α with the x– axis is $4a\,\text{Cosec}^2\,\alpha$.

8. PARAMETRIC REPRESENTATION :

The simplest & the best form of representing the co–ordinates of a point on the parabola is $(at^2, 2at)$. The equations $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter. The equation of a chord joining t_1 & t_2 is $2x - (t_1 + t_2)\,y + 2\,at_1\,t_2 = 0$.

Note: If chord joining t_1, t_2 & t_3, t_4 pass a through point (c, 0) on axis, then $t_1t_2 = t_3t_4 = -c/a$.

9. TANGENTS TO THE PARABOLA $y^2 = 4ax$:

(i) $y\,y_1 = 2a\,(x + x_1)$ at the point (x_1, y_1) ; **(ii)** $y = mx + \frac{a}{m}$ $(m \neq 0)$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(iii) $t\,y = x + a\,t^2$ at $(at^2, 2at)$.

Note : Point of intersection of the tangents at the point t_1 & t_2 is $[\,at_1\,t_2,\ a(t_1 + t_2)\,]$.

10. NORMALS TO THE PARABOLA $y^2 = 4ax$:

(i) $y - y_1 = -\frac{y_1}{2a}\,(x - x_1)$ at (x_1, y_1) ; **(ii)** $y = mx - 2am - am^3$ at $(am^2, -2am)$

(iii) $y + tx = 2at + at^3$ at $(at^2, 2at)$.

Note : Point of intersection of normals at t_1 & t_2 are, $a\,(t_1^2 + t_2^2 + t_1t_2 + 2)$; $-\,a\,t_1\,t_2\,(t_1 + t_2)$.

11. THREE VERY IMPORTANT RESULTS :

(a) If t_1 & t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1t_2 = -1$. Hence the co-ordinates at the extremities of a focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

(b) If the normals to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(c) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1\,t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point (–2a, 0).

General Note :

(i) Length of subtangent at any point P(x, y) on the parabola $y^2 = 4ax$ equals twice the abscissa of the point P. Note that the subtangent is bisected at the vertex.

(ii) Length of subnormal is constant for all points on the parabola & is equal to the semi latus rectum.

(iii) If a family of straight lines can be represented by an equation $\lambda^2P + \lambda Q + R = 0$ where λ is a parameter and P, Q, R are linear functions of x and y then the family of lines will be tangent to the curve $Q^2 = 4\,PR$.

12. The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the parabola $y^2 = 4ax$ is given by : $SS_1 = T^2$ where :

$S \equiv y^2 - 4ax$; $S_1 = y_1^2 - 4ax_1$; $T \equiv y\,y_1 - 2a(x + x_1)$.

13. DIRECTOR CIRCLE :

Locus of the point of intersection of the perpendicular tangents to the parabola $y^2 = 4ax$ is called the **Director Circle.** It's equation is $x + a = 0$ which is parabola's own directrix.

14. CHORD OF CONTACT : www.MathsBySuhag.com , www.TekoClasses.com

Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a\,(x + x_1)$.

Remember that the area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $(y_1^2 - 4ax_1)^{3/2} \div 2a$. Also note that the chord of contact exists only if the point P is not inside.

15. POLAR & POLE :

(i) Equation of the Polar of the point $P(x_1, y_1)$ w.r.t. the parabola $y^2 = 4ax$ is,

$y\,y_1 = 2a(x + x_1)$

(ii) The pole of the line $lx + my + n = 0$ w.r.t. the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, -\frac{2am}{l}\right)$.

Note:

(i) The polar of the focus of the parabola is the directrix.

(ii) When the point (x_1, y_1) lies without the parabola the equation to its polar is the same as the equation to the chord of contact of tangents drawn from (x_1, y_1) when (x_1, y_1) is on the parabola the polar is the same as the tangent at the point.

(iii) If the polar of a point P passes through the point Q, then the polar of Q goes through P.

(iv) Two straight lines are said to be conjugated to each other w.r.t. a parabola when the pole of one lies on the other.

(v) Polar of a given point P w.r.t. any Conic is the locus of the harmonic conjugate of P w.r.t. the two points is which any line through P cuts the conic.

16. CHORD WITH A GIVEN MIDDLE POINT :

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $y - y_1 = \frac{2a}{y_1}(x - x_1)$. This reduced to $T = S_1$

where $T \equiv y\,y_1 - 2a\,(x + x_1)$ & $S_1 \equiv y_1^2 - 4ax_1$.

17. DIAMETER :

The locus of the middle points of a system of parallel chords of a Parabola is called a **DIAMETER**. Equation to the diameter of a parabola is $y = 2a/m$, where m = slope of parallel chords.

Note:

(i) The tangent at the extremity of a diameter of a parabola is parallel to the system of chords it bisects.

(ii) The tangent at the ends of any chords of a parabola meet on the diameter which bisects the chord.

(iii) A line segment from a point P on the parabola and parallel to the system of parallel chords is called the ordinate to the diameter bisecting the system of parallel chords and the chords are called its double ordinate.

18. IMPORTANT HIGHLIGHTS :

(a) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.

(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the **focus.**www.MathsBySuhag.com , www.TekoClasses.com

(c) The tangents at the extremities of a focal chord intersect at right angles on the **directrix**, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangtent at the vertex.

(e) If the tangents at P and Q meet in T, then :■ TP and TQ subtend equal angles at the focus S. ■ $ST^2 = SP.\,SQ$ &■ The triangles SPT and STQ are similar.

(f) Tangents and Normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being $(-a, 0)$ & $(3a, 0)$.

(g) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola is ; $2a = \frac{2bc}{b+c}$ **i.e.** $\frac{1}{b} + \frac{1}{c} = \frac{1}{a}$.

(h) The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

(i) The orthocentre of any triangle formed by three tangents to a parabola $y^2 = 4ax$ lies on the directrix & has the co-ordinates $-a,\ a(t_1 + t_2 + t_3 + t_1t_2t_3)$.

(j) The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

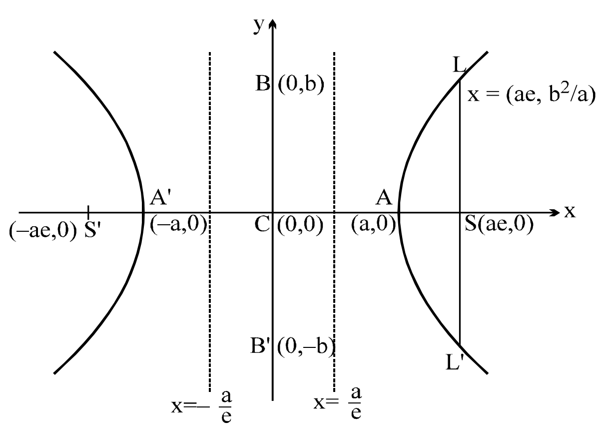
(k) If normal drawn to a parabola passes through a point P(h, k) then $k = mh - 2am - am^3$ i.e. $am^3 + m(2a - h) + k = 0$. Then gives $m_1 + m_2 + m_3 = 0$; $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$; $m_1\,m_2\,m_3 = -\frac{k}{a}$.

where m_1, m_2 & m_3 are the slopes of the three concurrent normals. Note that the algebraic sum of the:

■ slopes of the three concurrent normals is zero.

■ ordinates of the three conormal points on the parabola is zero.

■ Centroid of the Δ formed by three co-normal points lies on the x-axis.



(*l*) A circle circumscribing the triangle formed by three co–normal points passes through the vertex of the parabola and its equation is, $2(x^2 + y^2) - 2(h + 2a)x - ky = 0$

Suggested problems from S.L.Loney**: Exercise-25** (Q.5, 10, 13, 14, 18, 21), **Exercise-26 (Important)** (Q.4, 6, 7, 16, 17, 20, 22, 26, 27, 28, 34, 38), **Exercise-27** (Q.4, 7), **Exercise-28** (Q.2, 7, 11, 14, 17, 23), **Exercise-29** (Q.7, 8, 10, 19, 21, 24, 26, 27), **Exercise-30** (2, 3, 13, 18, 20, 21, 22, 25, 26, 30)

Note**:** Refer to the figure on Pg.175 if necessary.

ELLIPSE

Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Where a > b & $b^2 = a^2(1 - e^2) \Rightarrow a^2 - b^2 = a^2 e^2$. Where e = eccentricity (0 < e < 1). FOCI : $S \equiv (a\,e, 0)$ & $S' \equiv (-a\,e, 0)$.

EQUATIONS OF DIRECTRICES :

$x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

VERTICES :

$A' \equiv (-a, 0)$ & $A \equiv (a, 0)$.

MAJOR AXIS :

The line segment A′ A in which the foci S′ & S lie is of length 2a & is called the **major axis** (a > b) of the ellipse. Point of intersection of major axis with directrix is called **the foot of the directrix (z).**

MINOR AXIS :www.MathsBySuhag.com , www.TekoClasses.com

The y–axis intersects the ellipse in the points $B' \equiv (0, -b)$ & $B \equiv (0, b)$. The line segment B′B of length 2b (b < a) is called the **Minor Axis** of the ellipse.

PRINCIPAL AXIS :

The major & minor axis together are called **Principal Axis** of the ellipse.

CENTRE :

The point which bisects every chord of the conic drawn through it is called the **centre** of the conic. C ≡ (0, 0) the origin is the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

DIAMETER :

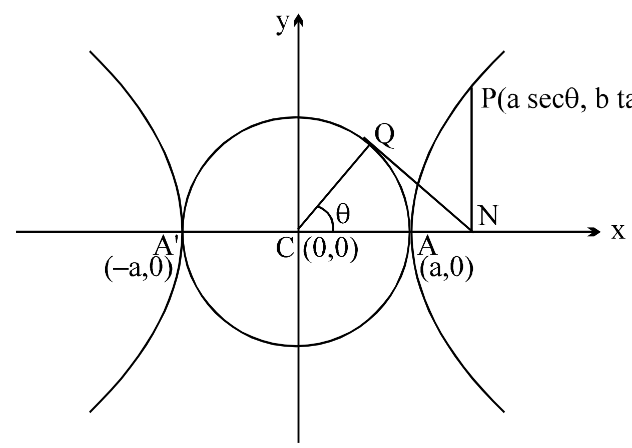
A chord of the conic which passes through the centre is called a **diameter** of the conic.

FOCAL CHORD : A chord which passes through a focus is called a **focal chord**.

DOUBLE ORDINATE :

A chord perpendicular to the major axis is called a **double ordinate.**

LATUS RECTUM : The focal chord perpendicular to the major axis is called the **latus rectum**. Length of latus rectum **(LL′)** = $\frac{2b^2}{a} = \frac{(minor\ axis)^2}{major\ axis} = 2a(1-e^2) =$ 2e (distance from focus to the corresponding directrix)



NOTE :

(i) The sum of the focal distances of any point on the ellipse is equal to the major Axis. Hence distance of focus from the extremity of a minor axis is equal to semi major axis. **i.e. BS = CA.**

(ii) If the equation of the ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ & nothing is mentioned, then the rule is to assume that a > b.

2. POSITION OF A POINT w.r.t. AN ELLIPSE :

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$ > < or = 0.

3. AUXILIARY CIRCLE / ECCENTRIC ANGLE :

A circle described on major axis as diameter is called the **auxiliary circle**.

Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P & Q are called as the **CORRESPONDING** on the ellipse ($0 \le \theta < 2\pi$). Note that $\frac{\ell(PN)}{\ell(QN)} = \frac{b}{a} = \frac{\textit{Semi minor axis}}{\textit{Semi major axis}}$ Hence " If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle".

4. PARAMETRIC REPRESENTATION :

The equations $x = a\cos\theta$ & $y = b\sin\theta$ together represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Where θ is a parameter. Note that if $P(\theta) \equiv (a\cos\theta,\ b\sin\theta)$ is on the ellipse then ; $Q(\theta) \equiv (a\cos\theta,\ a\sin\theta)$ is on the auxiliary circle.

5. LINE AND AN ELLIPSE :

The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two points real, coincident or imaginary according as c^2 is < = or > $a^2m^2 + b^2$. Hence $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

The equation to the chord of the ellipse joining two points with eccentric angles α& β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$

6. TANGENTS :

(i) $\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1$ is tangent to the ellipse at (x_1, y_1).

Note : The figure formed by the tangents at the extremities of latus rectum is rhoubus of area

(ii) $y = mx \pm \sqrt{a^2m^2 + b^2}$ is tangent to the ellipse for all values of m.
Note that there are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.

(iii) $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ is tangent to the ellipse at the point $(a\cos\theta,\ b\sin\theta)$.

(iv) The eccentric angles of point of contact of two parallel tangents differ by π. Conversely if the difference between the eccentric angles of two points is p then the tangents at these points are parallel.

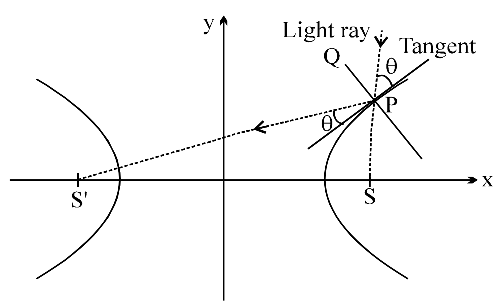
(v) Point of intersection of the tangents at the point α & β is $a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$.

7. NORMALS :www.MathsBySuhag.com , www.TekoClasses.com

(i) Equation of the normal at (x_1, y_1) is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2$.

(ii) Equation of the normal at the point (acos θ, bsin θ) is ; $ax \sec\theta - by \operatorname{cosec}\theta = (a^2 - b^2)$.

(iii) Equation of a normal in terms of its slope 'm' is $y = mx - \frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$.



8. DIRECTOR CIRCLE :

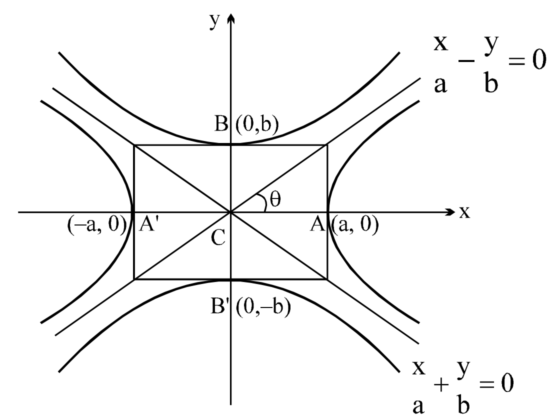
Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle**. The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axis.

9. Chord of contact, pair of tangents, chord with a given middle point, pole & polar are to be interpreted as they are in parabola.

10. DIAMETER :

The locus of the middle points of a system of parallel chords with slope 'm' of an ellipse is a straight line passing through the centre of the ellipse, called its diameter and has the equation $y = -\frac{b^2}{a^2m}x$.

11. IMPORTANT HIGHLIGHTS : Refering to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



H – 1 If P be any point on the ellipse with S & S′ as its foci then $\ell(SP) + \ell(S'P) = 2a$.

H – 2 The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars Y,Y′ lie on its auxiliary circle.The tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is a similiar ellipse as that of the original one. Also the lines joining centre to the feet of the perpendicular Y and focus to the point of contact of tangent are parallel.

H – 3 If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then

(i) $PF \cdot PG = b^2$ **(ii)** $PF \cdot Pg = a^2$ **(iii)** $PG \cdot Pg = SP \cdot S'P$ **(iv)** $CG \cdot CT = CS^2$

(v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse. [where S and S′ are the focii of the ellipse and T is the point where tangent at P meet the major axis]

H – 4 The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice–versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.

H – 5 The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.www.MathsBySuhag.com , www.TekoClasses.com

H – 6 The circle on any focal distance as diameter touches the auxiliary circle.

H – 7 Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

H – 8 If the tangent at the point P of a standard ellipse meets the axis in T and t and CY is the perpendicular on it from the centre then,
(i) $Tt \cdot PY = a^2 - b^2$ and (ii) least value of Tt is $a + b$.

HYPERBOLA

The **HYPERBOLA** is a conic whose eccentricity is greater than unity. (e > 1).

1. STANDARD EQUATION & DEFINITION(S)

Standard equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Where $b^2 = a^2(e^2 - 1)$

or $a^2e^2 = a^2 + b^2$ i.e. $e^2 = 1 + \frac{b^2}{a^2}$

$= 1 + \left(\frac{C.A}{T.A}\right)^2$

FOCI :

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

EQUATIONS OF DIRECTRICES : $x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

VERTICES : $A \equiv (a, 0)$& $A' \equiv (-a, 0)$. ***l* (Latus rectum)** $= \frac{2b^2}{a} = \frac{(C.A.)^2}{T.A.} = 2a(e^2 - 1)$.

Note : l (L.R.) = 2e (distance from focus to the corresponding directrix)

Transverse Axis : The line segment A′A of length 2a in which the foci S′ & S both lie is called the **T.A. Of The Hyperbola.**

Conjugate Axis : The line segment B′B between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the **C.A. Of The Hyperbola.**

The T.A. & the C.A. of the hyperbola are together called the Principal axes of the hyperbola.

2. FOCAL PROPERTY :

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $\left||PS|-|PS'|\right| = 2a$. The distance SS' = focal length.

3. CONJUGATE HYPERBOLA :

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called **Conjugate Hyperbolas** of each other. eg. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ & $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are conjugate hyperbolas of each.

Note :(a) If e_1 & e_2 are the eccentrcities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

(b) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

(c) Two hyperbolas are said to be similiar if they have the same eccentricity.

4. RECTANGULAR OR EQUILATERAL HYPERBOLA :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an **Equilateral Hyperbola**. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and the length of its latus rectum is equal to its transverse or conjugate axis.

5. AUXILIARY CIRCLE : www.MathsBySuhag.com , www.TekoClasses.com

A circle drawn with centre C & T.A. as a diameter is called the **Auxiliary Circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Note from the figure that P & Q are called the **"Corresponding Points "** on the hyperbola & the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola. $(0 \le \theta < 2\pi)$.

Note : The equations $x = a \sec\theta$ & $y = b \tan\theta$ together represents the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where θ is a parameter. The parametric equations : $x = a \cosh\phi$, $y = b \sinh\phi$ also represents the same hyperbola.

General Note : Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.

6. POSITION OF A POINT 'P' w.r.t. A HYPERBOLA :

The quantity $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$ is positive, zero or negative according as the point (x_1, y_1) lies within, upon or without the curve.

7. LINE AND A HYPERBOLA : The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according as: $c^2 > = < a^2 m^2 - b^2$.

8. TANGENTS AND NORMALS : TANGENTS :

(a) Equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{x x_1}{a^2}-\frac{y y_1}{b^2}=1$.

Note: In general two tangents can be drawn from an external point $(x_1\, y_1)$ to the hyperbola and they are $y - y_1 = m_1(x - x_1)$ & $y - y_1 = m_2(x - x_2)$, where m_1 & m_2 are roots of the equation $(x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 + b^2 = 0$. If $D < 0$, then no tangent can be drawn from $(x_1\, y_1)$ to the hyperbola.

(b) Equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a \sec\theta, b \tan\theta)$ is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.

Note : Point of intersection of the tangents at θ_1 & θ_2 is $x = a\frac{\cos\frac{\theta_1-\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}}$, $y = b\frac{\sin\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}}$

(c) $y = mx \pm\sqrt{a^2m^2-b^2}$ can be taken as the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. **Note that there are two parallel tangents having the same slope m.**

(d) Equation of a chord joining α & β is $\frac{x}{a}\cos\frac{\alpha-\beta}{2}-\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}$

NORMALS:

(a) The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(x_1, y_1)$ on it is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2-b^2 = a^2e^2$.

(b) The equation of the normal at the point P (a secθ, b tanθ) on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2 = a^2e^2$.

(c) Equation to the chord of contact, polar, chord with a given middle point, pair of tangents from an external point is to be interpreted as in ellipse.www.MathsBySuhag.com , www.TekoClasses.com

9. DIRECTOR CIRCLE :

The locus of the intersection of tangents which are at right angles is known as the **Director Circle** of the hyperbola. The equation to the director circle is : $x^2 + y^2 = a^2 - b^2$.

If $b^2 < a^2$ this circle is real; if $b^2 = a^2$ the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve. If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no tangents at right angle can be drawn to the curve.

10. HIGHLIGHTS ON TANGENT AND NORMAL :

H–1 Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of the feet of these perpendiculars is $b^2 \cdot$ (semi C ·A)^2

H–2 The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

H–3 The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as **"An incoming light ray "** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.

Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}=1\,(a > k > b > 0)$ Xare confocal and therefore orthogonal.

H–4 The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.

11. ASYMPTOTES :

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the Hyperbola.

To find the asymptote of the hyperbola :

Let y = mx + c is the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
Solving these two we get the quadratic as
$(b^2 - a^2m^2)\, x^2 - 2a^2 mcx - a^2 (b^2 + c^2) = 0$ ....(1)
In order that y = mx + c be an asymptote, both roots of equation (1) must approach infinity, the conditions for which are :

coeff of $x^2 = 0$ & coeff of x = 0. $\Rightarrow$ $b^2 - a^2m^2 = 0$ or $m = \pm\frac{b}{a}$ &

$a^2 mc = 0 \Rightarrow c = 0$. $\therefore$ equations of asymptote are $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$.

combined equation to the asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

PARTICULAR CASE :

When b = a the asymptotes of the rectangular hyperbola. $x^2 - y^2 = a^2$ are, $y = \pm\, x$ which are at right angles.

Note : **(i)** Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola.

(ii) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

(iii) A hyperbola and its conjugate have the same asymptote.

(iv) The equation of the pair of asymptotes differ the hyperbola & the conjugate hyperbola by the same constant only.

(v) The asymptotes pass through the centre of the hyperbola & the bisectors of the angles between the asymptotes are the axes of the hyperbola.www.MathsBySuhag.com , www.TekoClasses.com

(vi) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(vii) Asymptotes are the tangent to the hyperbola from the centre.

(viii) A simple method to find the coordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as:
Let f (x, y) = 0 represents a hyperbola.
Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ gives the centre of the hyperbola.

12. HIGHLIGHTS ON ASYMPTOTES:

H–1 If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.

H–2 Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.

H–3 The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C, meets the asymptotes in Q and R and cuts off a Δ CQR of constant area equal to ab from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing the Δ CQR in case of a rectangular hyperbola is the hyperbola itself & for a standard hyperbola the locus would be the curve, $4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.

H–4 If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ then $e = \sec\theta$.

13.RECTANGULAR HYPERBOLA : Rectangular hyperbola referred to its asymptotesas axis of coordinates.**(a)**Eq. is $xy = c^2$ with parametric representation $x = ct$, $y = c/t$, $t \in R - \{0\}$.

(b) Eq.of a chord joining the points (t_1) & (t_2) is $x + t_1t_2y = c(t_1 + t_2)$ with slope $m = -\frac{1}{t_1t_2}$.

(c) Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at P (t) is $\frac{x}{t} + ty = 2c$.

(d) Equation of normal : $y - \frac{c}{t} = t^2(x - ct)$

(e) Chord with a given middle point as (h, k) is kx + hy = 2hk.