6. COMPLEX NUMBER

1. INTRODUCTION

The number system can be briefly summarized as $N \subset W \subset I \subset Q \subset R \subset C$, where N, W, I, Q, R and C are the standard notations for the various subsets of the numbers belong to it.

N - Natural numbers = {1, 2, 3 .... n}

W - Whole numbers = {0, 1, 2, 3 .... n}

I - Integers = {....2, -1, 0, 1, 2 .....}

Q – Rational numbers = $\left\{\frac{1}{2}, \frac{3}{5} .....\right\}$

IR – Irrational numbers = $\left\{\sqrt{2}, \sqrt{3}, \pi\right\}$

C – Complex numbers

A complex number is generally represented by the letter "z". Every complex number z, can be written as, $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$.

x is called the real part of complex number, and

y is the imaginary part of complex number.

Note that the sign + does not indicate addition as normally understood, nor does the symbol "i" denote a number. These are parts of the scheme used to express numbers of a new class and they signify the pair of real numbers (x, y) to form a single complex number.

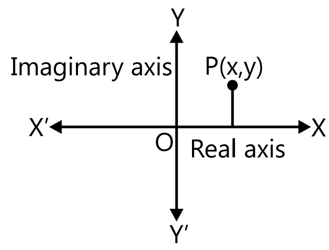

Figure 6.1: Representation of a complex number on a plane

Swiss-born mathematician Jean Robert Argand, after a systematic study on complex numbers, represented every complex number as a set of ordered pair (x, y) on a plane called complex plane.

All complex numbers lying on the real axis were called purely real and those lying on imaginary axis as purely imaginary.

Hence, the complex number $0 + 0i$ is purely real as well as purely imaginary but it is not imaginary.

Note

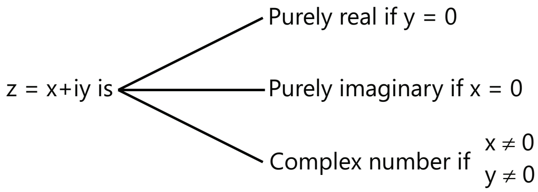

Figure 6.2: Classification of a complex number

(a) The symbol i combines itself with real number as per the rule of algebra together with

$i^2 = -1$; $i^3 = -i$; $i^4 = 1$; $i^{2014} = -1$; $i^{2015} = -i$ and so on.

In general, $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, $n \in I$ and $i^{4n} + i^{4n+1} + i^{4n+2} + i^{4n+3} = 0$

Hence, $1 + i^1 + i^2 + \ldots\ldots + i^{2014} + i^{2015} = 0$

(b) The imaginary part of every real number can be treated as zero. Hence, there is one-one mapping between the set of complex numbers and the set of points on the complex plane.

PLANCESS CONCEPTS

Complex number as an ordered pair: A complex number may also be defined as an ordered pair of real numbers and may be denoted by the symbol (a, b). For a complex number to be uniquely specified, we need two real numbers in a particular order.

Vaibhav Gupta (JEE 2009, AIR 54)

2. ALGEBRA OF COMPLEX NUMBERS

(a) **Addition:** $(a+ib)+(c+id) = (a+c)+i(b+d)$

(b) **Subtraction:** $(a+ib)-(c+id) = (a-c)+i(b-d)$

(c) **Multiplication:** $(a+ib)\,(c+id) = (ac-bd)+i\,(ad+bc)$

(d) **Reciprocal:** If at least one of a, b is non-zero, then the reciprocal of $a+ib$ is given by

$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)\,(a-ib)} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

(e) **Quotient:** If at least one of c, d is non-zero, then quotient of $a+ib$ and $c+id$ is given by

$$\frac{a+ib}{c+id} = \frac{(a+ib)\,(c-id)}{(c+id)\,(c-id)} = \frac{(ac+bd)+i\,(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

(f) Inequality in complex numbers is not discussed/defined. If a + ib > c + id is meaningful only if b = d = 0. However, equalities in complex numbers are meaningful. Two complex numbers z_1 and z_2 are said to be equal if $\text{Re}\,(z_1) = \text{Re}(z_2)$ and $\text{Im}\,(z_1) = \text{Im}(z_2)$. (Geometrically, the position of complex number z_1 on complex plane)

(g) In real number system if $p^2 + q^2 = 0$ implies, $p = 0 = q$. But if z_1 and z_2 are complex numbers then $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$. For e.g. $z_1 = i$ and $z_2 = 1$.

However if the product of two complex numbers is zero then at least one of them must be zero, same as in case of real numbers.

(h) In case x is real, then $|x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x < 0 \end{cases}$ but in case of complex number z, $|z|$ means the distance of the point z from the origin.

PLANCESS CONCEPTS

- The additive inverse of a complex number $z = a + ib$ is $-z$ (i.e. $-a - ib$).
- For every non-zero complex number z, the multiplicative inverse of z is $\frac{1}{z}$.
- $|z| \geq |Re(z)| \geq Re(z)$ and $|z| \geq |Im(z)| \geq Im(z)$.
- $\frac{z}{|z|}$ is always a uni-modular complex number if $z \neq 0$.

Vaibhav Krishnan (JEE 2009, AIR 22)

Illustration 1: Find the square root of 5 + 12i. **(JEE MAIN)**

Sol: $z = 5 + 12i$

Let the square root of the given complex number be a + ib. Use algebra to simplify and get the value of a and b.

Let its square root $= a + ib \quad \Rightarrow 5 + 12i = a^2 - b^2 + 2abi$

$\Rightarrow a^2 - b^2 = 5$ ... (i)

$\Rightarrow 2ab = 12$ ... (ii)

$\Rightarrow (a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2 \quad \Rightarrow (a^2 + b^2)^2 = 25 + 144 = 169 \quad \Rightarrow a^2 + b^2 = 13$ ... (iii)

(i) + (iii) $\Rightarrow 2a^2 = 18 \Rightarrow a^2 = 9 \Rightarrow a = \pm 3$

If $a = 3 \Rightarrow b = 2$ If $a = -3 \Rightarrow b = -2$

$\therefore$ Square root $= 3 + 2i, -3 - 2i$ $\therefore$ Combined form $\pm(3 + 2i)$

Illustration 2: If $z = (x, y) \in C$. Find z satisfying $z^2 \times (1 + i) = (-7 + 17i)$. **(JEE MAIN)**

Sol: Algebra of Complex Numbers.

$(x + iy)^2 (1 + i) = -7 + 17i$

$\Rightarrow (x^2 - y^2 + 2xyi)(1 + i) = -7 + 17i;$ $x^2 - y^2 + i(x^2 - y^2) + 2xyi - 2xy = -7 + 17i$

$\Rightarrow (x^2 - y^2 - 2xy) + i(x^2 - y^2 + 2xy) = -7 + 17i \Rightarrow x = 3, y = 2$ $\Rightarrow x = -3, y = -2$

$\Rightarrow z = -3 + i(-2) = -3 - 2i$

Illustration 3: If $x^2 + 2(1 + 2i)x - (11 + 2i) = 0$. Solve the equation. **(JEE ADVANCED)**

Sol: Use the quadratic formula to find the value of x.

$$\therefore x = \frac{-2(1 + 2i) \pm \sqrt{4 - 16 + 16i + 44 + 8i}}{2}$$

$\Rightarrow 2x = (-2)(1 + 2i) \pm \sqrt{32 + 24i}$

$\Rightarrow x = (-1)(1 + 2i) \pm \sqrt{8 + 6i} = -1 - 2i \pm (3 + i);$ $x = 2 - i, -4 - 3i$

Illustration 4: If $f(x)=x^4-4x^3+4x^2+8x+44$. Find $f(3+2i)$. **(JEE ADVANCED)**

Sol: Let x = 3+2i, and square it to form a quadratic equation. Then try to represent f(x) in terms of this quadratic.

$x=3+2i$

$\Rightarrow (x-3)^2=-4 \qquad \Rightarrow x^2-6x+13=0$

$x^4-4x^3+4x^2+8x+44=x^2(x^2-6x+13)+2x^3-9x^2+8x+44$

$\Rightarrow f(x)=x^2(x^2-6x+13)+2(x^3-6x^2+13x)+3(x^2-6x+13)+5 \qquad \Rightarrow f(x)=5$

3. IMPORTANT TERMS ASSOCIATED WITH COMPLEX NUMBER

Three important terms associated with complex number are conjugate, modulus and argument.

(a) Conjugate: If z = x + iy then its complex conjugate is obtained by changing the sign of its imaginary part and denoted by $\bar{z}$ i.e. $\bar{z}$ = x – iy (see Fig 6.3).

The conjugate satisfies following basic properties

(i) $z+\bar{z}=2\text{Re}(z)$

(ii) $z-\bar{z}=2i\,\text{Im}(z)$

(iii) $z\bar{z}=x^2+y^2$

(iv) If z lies in 1st quadrant then $\bar{z}$ lies in 4th quadrant and $-\bar{z}$ in the 2nd quadrant.

(v) If x + iy = f (a + ib) then x – iy = f (a – ib)

For e.g. If $(2+3i)^3=x+iy$ then $(2-3i)^3=x-iy$

and, $\sin(\alpha+i\beta)=x+iy \Rightarrow \sin(\alpha-i\beta)=x-iy$

(vi) $z+\bar{z}=0 \quad \Rightarrow z$ is purely imaginary

(vii) $z-\bar{z}=0 \quad \Rightarrow z$ is purely real

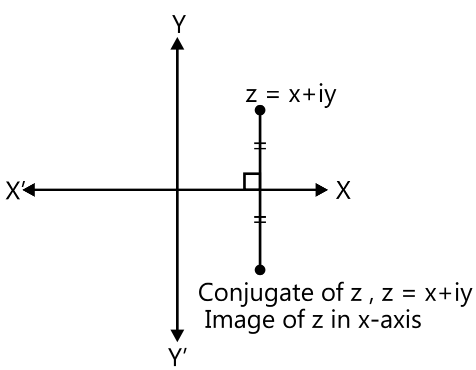

Figure 6.3: Conjugate of a complex number

(b) Modulus: If P denotes a complex number z = x + iy then, $OP=|z|=\sqrt{x^2+y^2}$. Geometrically, it is the distance of a complex number from the origin.

Hence, note that $|z|\ge 0$, $|i|=1$ i.e. $\left|\sqrt{-1}\right|=1$.

All complex number satisfying |z| = r lie on the circle having centre at origin and radius equal to 'r'.

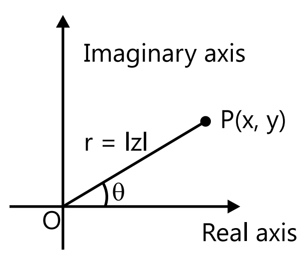

Figure 6.4: Modulus of a complex number

(c) Argument: If OP makes an angle θ (see Fig 6.4) with real axis in anticlockwise sense, then θ is called the argument of z. General values of argument of z are given by $2n\pi+\theta, n\in I$. Hence any two successive arguments differ by 2π.

Note: A complex number is completely defined by specifying both modulus and argument. However for the complex number 0 + 0i the argument is not defined and this is the only complex number which is completely defined by its modulus only.

(i) Amplitude (Principal value of argument): The unique value of θ such that $-\pi<\theta\le\pi$ is called principal value of argument. Unless otherwise stated, amp z refers to the principal value of argument.

(ii) Least positive argument: The value of θ such that $0<\theta\le 2\pi$ is called the least positive argument.

If $\phi=\tan^{-1}\left|\frac{y}{x}\right|$.

PLANCESS CONCEPTS

- If x > 0, y > 0 (i.e. z is in first quadrant), then arg z = $\theta = \tan^{-1}\left|\frac{y}{x}\right|$.
- If x < 0, y > 0 (i.e. z is in 2nd quadrant, then arg z = $\theta = \pi - \tan^{-1}\left|\frac{y}{x}\right|$
- If x < 0, y < 0 (i.e. z is in 3rd quadrant), then arg z = $\theta = -\pi + \tan^{-1}\left|\frac{y}{x}\right|$
- If x > 0, y < 0 (i.e. z is in 4th quadrant), then arg z = $\theta = -\tan^{-1}\left|\frac{y}{x}\right|$
- If y = 0 (i.e. z is on the X-axis), then arg (x + i0) = $\begin{cases} 0, & \text{if } x > 0 \\ \pi, & \text{if } x < 0 \end{cases}$
- If x = 0 (i.e. z is on the Y-axis), then arg (0 + iy) = $\begin{cases} \frac{\pi}{2}, & \text{if } y > 0 \\ \frac{3\pi}{2}, & \text{if } y < 0 \end{cases}$

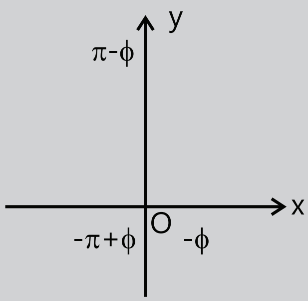

Figure 6.5

Shrikant Nagori (JEE 2009, AIR 30)

Illustration 5: For what real values of x and y, are $-3+ix^2y$ and x^2+y+4i complex conjugate to each other? **(JEE MAIN)**

Sol: As $-3+ix^2y$ and x^2+y+4i are complex conjugate of each other. Therefore $-3+ix^2y = \overline{x^2+y+4i}$.

$-3+ix^2y = x^2+y-4i$

Equating real and imaginary parts of the above question, we get

$-3 = x^2+y \Rightarrow y = -3-x^2$ ... (i)

and $x^2y = -4$ ... (ii)

Putting the value of y = $-3-x^2$ from (i) in (ii), we get

$x^2(-3-x^2) = -4 \quad \Rightarrow x^4+3x^2-4=0 \;\Rightarrow x^2 = \frac{-3\pm\sqrt{9+16}}{2} = \frac{-3\pm5}{2} = \frac{2}{2}, \frac{-8}{2} = 1, -4$

$\therefore\; x^2 = 1 \Rightarrow x = \pm1$

Putting value of x = ± 1 in (i), we get $\quad y = -3-(1)^2 = -3-1 = -4$

Hence, x = ± 1 and y = – 4.

Illustration 6: Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$. **(JEE MAIN)**

Sol: As | z | = $\sqrt{x^2+y^2}$, using algebra of complex number we will get the result.

Here, we have $\frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} - \frac{(1-i)(1-i)}{(1+i)(1-i)}$

$= \frac{1+i^2+2i}{1+1} - \frac{1+i^2-2i}{1+1} = \frac{1-1+2i}{2} - \frac{1-1-2i}{2} = \frac{2i}{2} - \frac{(-2i)}{2} = i + i = 2i, \;\therefore\; \Rightarrow \left|\frac{1+i}{1-i} - \frac{1-i}{1+i}\right| = |2i| = 2.$

Illustration 7: Find the locus of z if | z – 3 | = 3| z + 3|. **(JEE MAIN)**

Sol: Simply substituting z = x + iy and by using formula | z | = $\sqrt{x^2+y^2}$ we will get the result.

Let z = x + iy

| x + iy – 3 | = 3 | x + iy + 3 | | (x – 3) + iy | = 3 | (x + 3) + iy |

$\sqrt{(x-3)^2+y^2} = 3\sqrt{(x+3)^2+y^2}$; $(x-3)^2+y^2 = 9(x+3)^2+9y^2$.

Illustration 8: If α and β are different complex numbers with $|\beta| = 1$, then find $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$. **(JEE ADVANCED)**

Sol: By using modulus and conjugate property, we can find out the value of $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$.

We have, $|\beta| = 1 \Rightarrow |\beta|^2 = 1 \Rightarrow \beta\bar{\beta} = 1$

Now, $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = \left|\frac{\beta-\alpha}{\beta\bar{\beta}-\bar{\alpha}\beta}\right| = \left|\frac{\beta-\alpha}{\beta(\bar{\beta}-\bar{\alpha})}\right| = \frac{|\beta-\alpha|}{|\beta||\overline{\beta-\alpha}|} = \frac{1}{|\beta|} = 1.$ $\{\text{as } |x+iy| = |\overline{x+iy}|\}$

Illustration 9: Find the number of non-zero integral solution of the equation $|1-i|^x = 2^x$. **(JEE ADVANCED)**

Sol: As | z | = $\sqrt{x^2+y^2}$, therefore by using this formula we can solve it.

We have, $|1-i|^x = 2^x$

$\Rightarrow \left[\sqrt{1^2+1^2}\right]^x = 2^x$ $\Rightarrow \left(\sqrt{2}\right)^x = 2^x$ $\Rightarrow 2^{\frac{x}{2}} = 2^x$ $\Rightarrow \frac{x}{2} = 0 \Rightarrow x = 0$.

∴ The number of non zero integral solution is zero.

Illustration 10: If $\frac{a+ib}{c+id}$ = p + iq. Prove that $\frac{a^2+b^2}{c^2+d^2} = p^2+q^2$. **(JEE MAIN)**

Sol: Simply by obtaining modulus of both side of $\frac{a+ib}{c+id}$ = p + iq .

We have, $\frac{a+ib}{c+id}$ = p + iq

$\left|\frac{a+ib}{c+id}\right| = \sqrt{\frac{a^2+b^2}{c^2+d^2}} \Rightarrow |p+iq| = \sqrt{p^2+q^2}$; $\left|\frac{a+ib}{c+id}\right| = |p+iq|$ $\Rightarrow \frac{a^2+b^2}{c^2+d^2} = p^2+q^2$.

Illustration 11: If $(x+iy)^{1/3}$ = a + ib. Prove that $\frac{x}{a}+\frac{y}{b} = 4(a^2-b^2)$. **(JEE ADVANCED)**

Sol: By using algebra of complex number. We have, $(x+iy)^{1/3}$ = a + ib

x + iy = $(a+ib)^3$ = $a^3+i^3b^3+3a^2ib+3a(ib)^2$ = $a^3-b^3i+3a^2bi-3ab^2$

x + iy = $(a^3-3ab^2)+(3a^2b-b^3)i$;   x = a^3-3ab^2 = a (a^2-3b^2);   y = $3a^2b-b^3$

$\frac{x}{a}+\frac{y}{b} = 4(a^2-b^2)$.

4. REPRESENTATION OF COMPLEX NUMBER

4.1 Graphical Representation

Every complex number x + iy can be represented in a plane as a point P (x, y). X-coordinate of point P represents the real part of the complex number and y-coordinate represents the imaginary part of the complex number. Complex number x + 0i (real number) is represented by a point (x, 0) lying on the x-axis. Therefore, x-axis is called the real axis. Similarly, a complex number 0 + iy (imaginary number) is represented by a point on y-axis. Therefore, y-axis is called the imaginary axis.

The plane on which a complex number is represented is called complex number plane or simply complex plane or Argand plane (see Fig 6.6). The figure represented by the complex numbers as points in a plane is known as Argand Diagram.

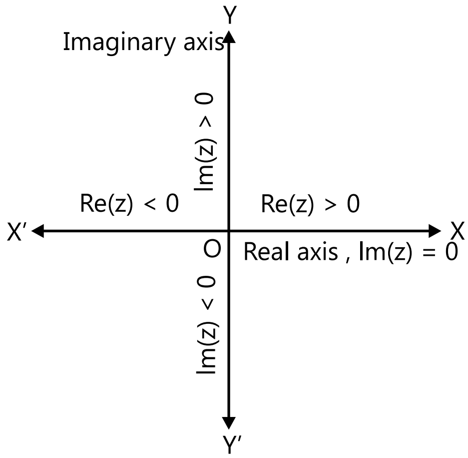

Figure 6.6: Graphical representation

4.2 Algebraic Form

If z = x + iy; then $|z| = \sqrt{x^2+y^2}$; $\bar{z} = x - iy$, and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Generally this form is useful in solving equations and in problems involving locus.

4.3 Polar Form

Figure 6.7 shows the components of a complex number along the x and y-axes respectively. Then

z = x + iy = $r(\cos\theta + i\sin\theta)$ = $r\,\text{cis}\,\theta$ where $|z| = r$; amp z = θ.

Aliter: z = x+iy

$$\Rightarrow z = \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\Rightarrow z = |z|\left(\cos\theta + i\sin\theta\right) = r\ \text{cis}\ \theta$$

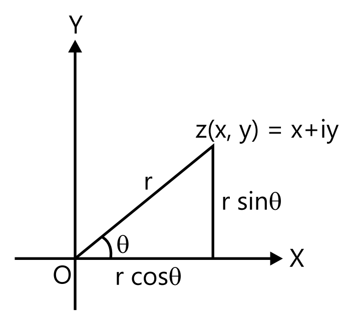

Figure 6.7: Polar form

Note: (a) $(\text{cis}\,\alpha)\,(\text{cis}\,\beta) = \text{cis}(\alpha+\beta)$

(b) $(\text{cis}\,\alpha)\,(\text{cis}(-\beta)) = \text{cis}(\alpha-\beta)$

(c) $\frac{1}{(\text{cis}\,\alpha)} = (\text{cis}\,\alpha)^{-1} = \text{cis}(-\alpha)$

PLANCESS CONCEPTS

The unique value of θ such that $-\pi < \theta \le \pi$ for which $x = r\cos\theta$ & $y = r\sin\theta$ is known as the principal value of the argument.

The general value of argument is $(2n\pi + \theta)$, where n is an integer and θ is the principal value of arg (z). While reducing a complex number to polar form, we always take the principal value.

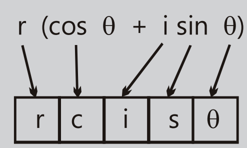

Figure 6.8

The complex number z = $r(\cos\theta + i\sin\theta)$ can also be written as $r\,\text{cis}\,\theta$.

Nitish Jhawar (JEE 2009, AIR 7)

4.4 Exponential Form

Euler's formula, named after the famous mathematician Leonhard Euler, states that for any real number x, $e^{ix} = \cos x + i\sin x$.

Hence, for any complex number $z = r(\cos\theta + i\sin\theta)$, $z = re^{i\theta}$ is the exponential representation.

Note: (a) cos x = $\frac{e^{ix} + e^{-ix}}{2}$ and sin x = $\frac{e^{ix} - e^{-ix}}{2i}$ are known as Eulers identities.

(b) cos ix = $\frac{e^{x} + e^{-x}}{2}$ = cos hx is always positive real $\forall\, x \in R$ and is > 1.

and, sin ix = i $\frac{e^{x} - e^{-x}}{2}$ = i sin hx is always purely imaginary.

4.5 Vector Representation

The knowledge of vectors can also be used to represent a complex number z = x + iy. The vector $\overrightarrow{OP}$, joining the origin O of the complex plane to the point P (x, y), is the vector representation of the complex number z=x+iy, (see Fig 6.9). The length of the vector $\overrightarrow{OP}$, that is, $|\overrightarrow{OP}|$ is the modulus of z. The angle between the positive real axis and the vector $\overrightarrow{OP}$, more exactly, the angle through which the positive real axis must be rotated to cause it to have the same direction as $\overrightarrow{OP}$ (considered positive if the rotation is counter-clockwise and negative otherwise) is the argument of the complex number z.

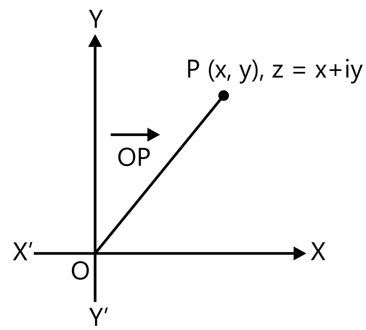

Figure 6.9 Vector representation

Illustration 12: Find locus represented by $\text{Re}\left(\frac{1}{x+iy}\right) < \frac{1}{2}$. **(JEE MAIN)**

Sol: Multiplying numerator and denominator by $x - iy$.

We have, $\text{Re}\left(\frac{1}{x+iy}\right) < \frac{1}{2}$ $\qquad$ $\text{Re}\left(\frac{x-iy}{x^2+y^2}\right) < \frac{1}{2}$

$\Rightarrow \frac{x}{x^2+y^2} < \frac{1}{2}$ $\qquad$ $\Rightarrow x^2 + y^2 - 2x > 0$

Locus is the exterior of the circle with centre (1, 0) and radius = 1.

Illustration 13: If $z = 1 + \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$. Find r and amp z. **(JEE MAIN)**

Sol: By using trigonometric formula we can reduce given equation in the form of $z = r(\cos\theta + i\sin\theta)$.

$z = 2\cos^2\frac{3\pi}{5} + 2i\sin\frac{3\pi}{5}\cos\frac{3\pi}{5} = 2\cos\frac{3\pi}{5}\left[\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right]$

$= -2\cos\frac{2\pi}{5}\left[-\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right] = 2\cos\frac{2\pi}{5}\left[\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right]$ Hence, $|z| = 2\cos\frac{2\pi}{5}$; amp z = $-\frac{2\pi}{5}$

Illustration 14: Show that the locus of the point P(ω)denoting the complex number $z+\frac{1}{z}$ on the complex plane is a standard ellipse where | z | = a, where a ≠ 0, 1. **(JEE ADVANCED)**

Sol: Here consider w = x + iy and z = $\alpha+i\beta$ and then solve this by using algebra of complex number.

Let w = $z+\frac{1}{z}$ where z = $\alpha+i\beta$, $\alpha^2+\beta^2=a^2$ (as | z | = a)

$$x+iy = \alpha+i\beta+\frac{1}{\alpha+i\beta} = \alpha+i\beta+\frac{\alpha-i\beta}{\alpha^2+\beta^2} = \left(\alpha+\frac{\alpha}{a^2}\right)+i\left(\beta-\frac{\beta}{a^2}\right) \quad \therefore x=\alpha\left(1+\frac{1}{a^2}\right);\ y=\beta\left(1-\frac{1}{a^2}\right)$$

$$\therefore \frac{x^2}{\left(1+\frac{1}{a^2}\right)^2}+\frac{y^2}{\left(1-\frac{1}{a^2}\right)^2} = \alpha^2+\beta^2=a^2; \qquad \therefore \frac{x^2}{\left(a+\frac{1}{a}\right)^2}+\frac{y^2}{\left(a-\frac{1}{a}\right)^2} = 1.$$

5. IMPORTANT PROPERTIES OF CONJUGATE, MODULUS AND ARGUMENT

For z, z_1 and $z_2 \in C$,

(a) Properties of Conjugate:

(i) $z+\bar{z} = 2\text{Re}(z)$

(ii) $z-\bar{z} = 2i\,\text{Im}(z)$

(iii) $(\bar{\bar{z}}) = z$

(iv) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

(v) $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$

(vi) $\overline{z_1\ z_2} = \bar{z}_1\cdot\bar{z}_2$

(vii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2};\ z_2 \neq 0$

(b) Properties of Modulus:

(i) $|z| \geq 0;\ |z| \geq \text{Re}(z);\ |z| \geq \text{Im}(z);\ |z| = |\bar{z}| = |-z|$

(ii) $z\bar{z} = |z|^2$; if $|z|$ = 1, then z = $\frac{1}{\bar{z}}$

(iii) $|z_1z_2| = |z_1|\cdot|z_2|$

(iv) $\left|\frac{z_1}{z_1}\right| = \frac{|z_1|}{|z_2|},\ z_2 \neq 0$

(v) $|z^n| = |z|^n$

(vi) $|z_1+z_2|^2+|z_1-z_2|^2 = 2\left[|z_1|^2+|z_2|^2\right]$

(vii) $\big||z_1|-|z_2|\big| \leq |z_1+z_2| \leq |z_1|+|z_2|$ [Triangle Inequality]

(c) Properties of Amplitude:

(i) $\text{amp}(z_1 \cdot z_2) = \text{amp } z_1 + \text{amp } z_2 + 2k\pi,\ k \in I$

(ii) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2 + 2k\pi,\ k \in I$

(iii) $\text{amp}(z^n) = n\,\text{amp}(z) + 2k\pi$, where the value of k should be such that RHS lies in $(-\pi, \pi]$

Based on the above information, we have the following

- $|\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2}\,|z|$
- $\big||z_1| - |z_2|\big| \le |z_1 - z_2| \le |z_1| + |z_2|$. Thus $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and $\big||z_1| - |z_2|\big|$ is the least possible value of $|z_1 + z_2|$.
- If $\left|z + \dfrac{1}{z}\right| = a$, the greatest and least values of |z| are respectively $\dfrac{a + \sqrt{a^2 + 4}}{2}$ and $\dfrac{-a + \sqrt{a^2 + 4}}{2}$.
- $|z_1 + \sqrt{z_1^2 - z_2^2}| + |z_2 - \sqrt{z_1^2 - z_2^2}| = |z_1 + z_2| + |z_1 - z_2|$
- If $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ and $\arg z_1 = \arg z_2$
- $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$ i.e. z_1 and z_2 are parallel.
- $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = 2n\pi$, where n is some integer.
- $|z_1 - z_2| = \big||z_1| - |z_2|\big| \Leftrightarrow \arg(z_1) - \arg(z_2) = 2n\pi$, where n is some integer.
- $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = (2n+1)\dfrac{\pi}{2}$, where n is some integer.
- If $|z_1| \le 1, |z_2| \le 1$, then $|z_1 + z_2|^2 \le (|z_1| - |z_2|)^2 + (\arg(z_1) - \arg(z_2))^2$, and
  $|z_1 + z_2|^2 \ge (|z_1| + |z_2|)^2 - (\arg(z_1) - \arg(z_2))^2$.

Illustration 15: If $z_1 = 3 + 5i$ and $z_2 = 2 - 3i$, then verify that $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ **(JEE MAIN)**

Sol: Simply by using properties of conjugate.

$$\frac{z_1}{z_2} = \frac{3+5i}{2-3i} = \frac{(3+5i)}{(2-3i)} \times \frac{(2+3i)}{(2+3i)} = \frac{6+9i+10i+15i^2}{4-9i^2} = \frac{6+19i+15(-1)}{4+9} = \frac{6+19i-15}{13} = \frac{-9+19i}{13} = \frac{-9}{13} + \frac{19}{13}i$$

$$\text{L.H.S.} = \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(-\frac{9}{13} + \frac{19}{13}i\right)} = -\frac{9}{13} - \frac{19}{13}i$$

$$\text{R.H.S.} = \frac{\overline{z_1}}{\overline{z_2}} = \frac{\overline{3+5i}}{\overline{2-3i}} = \frac{3-5i}{2+3i} = \frac{(3-5i)}{(2+3i)} \times \frac{(2-3i)}{(2-3i)}$$

$$= \frac{6-9i-10i+15i^2}{4-9i^2} = \frac{6-19i+15(-1)}{4+9} = \frac{6-19i-15}{13} = -\frac{9}{13} - \frac{19}{13}i \qquad \therefore \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

Illustration 16: If z be a non-zero complex number, then show that $\overline{(z^{-1})} = (\overline{z})^{-1}$. **(JEE MAIN)**

Sol: By considering z = a + ib and using properties of conjugate we can prove given equation.

Let z = a + ib Since, $z \neq 0$, we have $x^2 + y^2 > 0$

$$z^{-1} = \frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2} \Rightarrow \left(\overline{z^{-1}}\right) = \frac{a}{a^2+b^2} + \frac{ib}{a^2+b^2} \quad \text{... (i)}$$

$$\text{and } (\overline{z})^{-1} = \frac{1}{\overline{z}} = \frac{1}{\overline{a+ib}} = \frac{1}{a-ib} = \frac{1}{a-ib} \times \frac{a+ib}{a+ib} = \frac{a}{a^2+b^2} + i\frac{b}{a^2+b^2} \quad \text{... (ii)}$$

From (i) and (ii), we get $\overline{(z^{-1})} = (\overline{z})^{-1}$.

Illustration 17: If $\frac{(a+i)^2}{2a-i}$ = p + iq, then show that $p^2 + q^2 = \frac{(a^2+1)^2}{4a^2+1}$. **(JEE MAIN)**

Sol: Multiply given equation to its conjugate.

We have, $p + iq = \frac{(a+i)^2}{2a-i}$ ... (i)

Taking conjugate of both sides, we get $\overline{p+iq} = \overline{\left(\frac{(a+i)^2}{(2a-i)}\right)}$

$$\Rightarrow p - iq = \frac{\overline{(a+i)^2}}{\overline{(2a-i)}} \quad \left[\because \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}\right] \qquad \Rightarrow p - iq = \frac{(a-i)^2}{(2a+i)} \text{ ... (ii)} \left[\text{using } \overline{(z^2)} = \overline{z \cdot z} = \overline{z} \cdot \overline{z} = (\overline{z})^2\right]$$

Multiplying (i) and (ii), we get (p + iq) (p – iq) = $\left(\frac{(a+i)^2}{2a-i}\right)\left(\frac{(a-i)^2}{2a+i}\right)$

$$\Rightarrow p^2 - i^2q^2 = \frac{(a^2-i^2)^2}{4a^2-i^2} \qquad \Rightarrow p^2 + q^2 = \frac{(a^2+1)^2}{4a^2+1}.$$

Illustration 18: Let $z_1, z_2, z_3, \ldots\ldots z_n$ are the complex numbers such that $|z_1| = |z_2| = \ldots\ldots = |z_n| = 1$. If z = $\left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \frac{1}{z_k}\right)$ then prove that

(i) z is a real number (ii) $0 < z \le n^2$ **(JEE ADVANCED)**

Sol: Here $|z_1| = |z_2| = \ldots\ldots = |z_n| = 1$, therefore $z\overline{z} = 1 \Rightarrow z = \frac{1}{\overline{z}}$. Hence by substituting this to z = $\left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \frac{1}{z_k}\right)$, we can solve above problem.

Now, z = $(z_1 + z_2 + z_3 + \ldots\ldots + z_n)\left(\frac{1}{z_1} + \frac{1}{z_2} + \ldots\ldots + \frac{1}{z_n}\right)$

$= (z_1 + z_2 + z_3 + \ldots\ldots + z_n)\left(\overline{z}_1 + \overline{z}_2 + \ldots\ldots + \overline{z}_n\right) \quad = (z_1 + z_2 + z_3 + \ldots\ldots + z_n)\left(\overline{z_1 + z_2 + \ldots\ldots + z_n}\right)$

$= |z_1 + z_2 + z_3 + \ldots\ldots + z_n|^2$ which is real

$\le \left(|z_1| + |z_2| + |z_3| + \ldots\ldots + |z_n|\right)^2 = n^2 \qquad \therefore \qquad 0 < z \le n^2$.

Illustration 19: Let x_1, x_2 are the roots of the quadratic equation $x^2 + ax + b = 0$ where a, b are complex numbers and y_1, y_2 are the roots of the quadratic equation $y^2+|a|y+|b|=0$. If $|x_1|=|x_2|=1$, then prove that $|y_1|=|y_2|=1$.

(JEE ADVANCED)

Sol: Solve by using modulus properties of complex number.

Let $x^2+ax+b=0$ where x_1 and x_2 are complex numbers

$x_1+x_2 = -a$ ... (i)

and $x_1 x_2 = b$ ... (ii)

From (ii) $|x_1|\,|x_2| = |b| \Rightarrow |b|=1$ Also $|-a| = |x_1+x_2|$

$\therefore$ $|a| \le |x_1|+|x_2|$ or $|a| \le 2$

Now consider $y^2+|a|y+|b|=0$, $\left\langle \begin{matrix} y_1 \\ y_2 \end{matrix} \right.$ where y_1 and y_2 are complex numbers

$$y_{1,2} = \frac{-|a| \pm \sqrt{|a|^2-4|b|}}{2} = \frac{-|a| \pm \left(\sqrt{4-|a|^2}\right)i}{2} \qquad \therefore \qquad |y_{1,2}| = \frac{\sqrt{|a|^2+4-|a|^2}}{2} = 1$$

Hence, $|y_1|=|y_2|=1$.

6. TRIANGLE ON COMPLEX PLANE

In a ΔABC, the vertices A, B and C are represented by the complex numbers z_1, z_2 and z_3 respectively, then

(a) Centroid: The centroid 'G' is given by $\frac{z_1+z_2+z_3}{3}$. Refer to Fig 6.10.

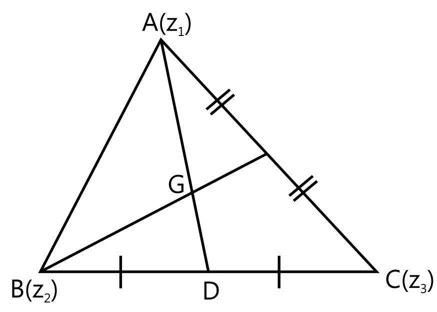

Figure 6.10: Centroid

(b) Incentre: The incentre 'I' is given by $\frac{az_1+bz_2+cz_3}{a+b+c}$. Refer to Fig 6.11.

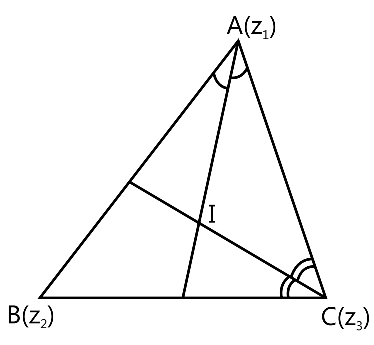

Figure 6.11: Incentre

(c) Orthocentre: The orhtocentre 'H' is given by $\dfrac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\sum \tan A}$.

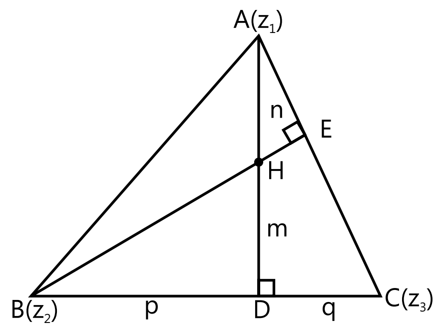

Figure 6.12: Orthocentre

Proof: From section formula, we have $z_D = \dfrac{p\,z_3 + q\,z_2}{a}$

In ΔABD and ΔACD, p = c cos B and q = b cos C. Refer to Fig 6.12.

Therefore, $z_D = \dfrac{b\cos C\, z_2 + c\cos B\, z_3}{a}$

Now, AE = c cos A; $n = AH = AE\,\text{cosec}\ C$ = c cos A cosec C

$\Rightarrow n = 2R\cos A$ [Using Sine Rule]

and m = c cos B cot C or, m = 2R cos B cos C [Using Sine Rule]

Hence, $z_H = \dfrac{mz_1 + nZ_D}{m+n}$.

$$= \frac{2R\cos B\cos C\, z_1 + 2R\cos A\left(\dfrac{b\cos C\, z_2 + c\cos B\, z_3}{a}\right)}{2R\,(\cos A + \cos B\cos C)}$$

$$= \frac{a\cos B\cos C\, z_1 + b\cos A\cos C\, z_2 + c\cos A\cos B\, z_3}{a(-\cos(B+C) + \cos B\cos C)}$$

$$= \frac{z_1(\sin A\cos B\cos C) + z_2(\sin B\cos C\cos A) + z_3(\sin C\cos A\cos B)}{\sin A\,(\sin B\sin C)}$$

$$\therefore\ Z_H = \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\sum \tan A} \quad \text{or} \quad \frac{z_1\tan A + z_2\tan B + z_3\tan C}{\prod \tan A}$$

[If A + B + C = π, then tan A + tan B + tan C = tan A tan B tan C]

(d) Circumcentre:

Let R be the circumradius and the complex number z_0 represent the circumcentre of the triangle as shown in Fig 6.11.

$\therefore |z_1 - z_0| = |z_2 - z_0| = |z_3 - z_0|$

Consider, $|z_1 - z_0|^2 = |z_2 - z_0|^2$

$(z_1 - z_0)(\bar{z}_1 - \bar{z}_0) = (z_2 - z_0)(\bar{z}_2 - \bar{z}_0)$

$$\bar{z}_1(z_1 - z_0) - \bar{z}_2(z_2 - z_0) = \bar{z}_0\left[(z_1 - z_0) - (z_2 - z_0)\right]$$

$$\bar{z}_1(z_1 - z_0) - \bar{z}_2(z_2 - z_0) = \bar{z}_0(z_1 - z_2) \qquad \text{... (i)}$$

Similarly 1st and 3rd gives

$$\bar{z}_1(z_1 - z_0) - \bar{z}_3(z_3 - z_0) = \bar{z}_0(z_1 - z_3) \qquad \text{... (ii)}$$

On dividing (i) by (ii), $\bar{z}_0$ gets eliminated and we obtain z_0.

Alternatively: From Fig 6.13, we have

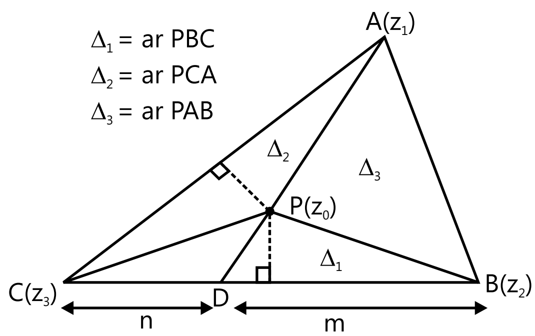

Figure 6.13: Circumcentre

$$\frac{BD}{DC} = \frac{m}{n} = \frac{\text{Ar. }\Delta ABD}{\text{Ar. }\Delta ADC} = \frac{\text{Ar. }\Delta PBD}{\text{Ar. }\Delta PDC}$$

$$\therefore \frac{m}{n} = \frac{\text{Ar. }\Delta ABD - \text{Ar. }\Delta PBD}{\text{Ar. }\Delta ADC - \text{Ar. }\Delta PDC} = \frac{\Delta_3}{\Delta_2}$$

$$\therefore \frac{m}{n} = \frac{\frac{R^2}{2}\sin 2C}{\frac{R^2}{2}\sin 2B} = \frac{\sin 2C}{\sin 2B}$$

Hence, $Z_D = \dfrac{\sin 2B\,(z_2) + \sin 2C\,(z_3)}{\sin 2B + \sin 2C}$

Now, $\dfrac{PA}{PD} = \dfrac{l}{k} = \dfrac{\Delta ABP}{\Delta PBD} = \dfrac{\Delta APC}{\Delta CPD} = \dfrac{\Delta ABP + \Delta APC}{\Delta PBD + \Delta CPD}$ $\therefore \dfrac{l}{k} = \dfrac{\Delta_3 + \Delta_2}{\Delta_1} = \dfrac{\sin 2C + \sin 2B}{\sin 2A}$

Hence, $z_0 = \dfrac{kz_1 + l z_D}{k + l} = \dfrac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\sum \sin 2A}$

PLANCESS CONCEPTS

- The area of the triangle whose vertices are z, iz and z + iz is $\frac{1}{2}|z|^2$.
- The area of the triangle with vertices z, ωz and z + ωz is $\frac{\sqrt{3}}{4}|z|^2$.
- If z_1, z_2, z_3 be the vertices of an equilateral triangle and z_0 be the circumcentre, then $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.
- If $z_1, z_2, z_3, \ldots\ldots z_n$ be the vertices of a regular polygon of n sides and z_0 be its centroid, then $z_1^2 + z_2^2 + \ldots\ldots + z_n^2 = nz_0^2$.
- If z_1, z_2, z_3 be the vertices of a triangle, then the triangle is equilateral if $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$ or $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ or $\dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} = 0$.
- If z_1, z_2, z_3 are the vertices of an isosceles triangle, right angled at z_2 then $z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$.
- If z_1, z_2, z_3 are the vertices of a right-angled isosceles triangle, then $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.
- If z_1, z_2, z_3 be the affixes of the vertices A, B, C respectively of a triangle ABC, then its orthocentere is $\dfrac{a(\sec A)z_1 + b(\sec B)z_2 + c(\sec C)z_3}{a\sec A + b\sec B + c\sec C}$.

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Illustration 20: If z_1, z_2, z_3 are the vertices of an isosceles triangle right angled at z_2 then prove that $z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$ **(JEE MAIN)**

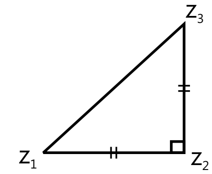

Figure 6.14

Sol: Here $(z_1 - z_2) = (z_3 - z_2)e^{\frac{i\pi}{2}}$. Hence by squaring both side we will get the result.

$\Rightarrow (z_1 - z_2)^2 = i^2(z_3 - z_2)^2$

$\Rightarrow z_3^2 + z_2^2 - 2z_3z_2 = -z_1^2 - z_2^2 + 2z_1z_2 \Rightarrow z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$.

Illustration 21: A, B, C are the points representing the complex numbers z_1, z_2, z_3 respectively and the circumcentre of the triangle ABC lies at the origin. If the altitudes of the triangle through the opposite vertices meets the circumcircle at D, E, F respectively. Find the complex numbers corresponding to the points D, E, F in terms of z_1, z_2, z_3. **(JEE MAIN)**

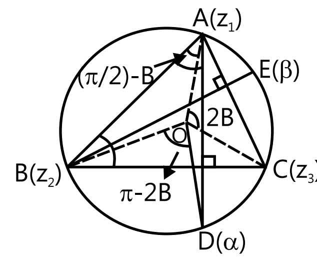

Figure 6.15

Sol: Here the $\angle BOD = \pi - 2B$, hence $\overrightarrow{OD} = \overrightarrow{OB}\, e^{i(\pi - 2B)}$.

From Fig 6.13, we have $\overrightarrow{OD} = \overrightarrow{OB}\, e^{i(\pi - 2B)}$;

$\alpha = z_2\, e^{i(\pi - 2B)} = -z_2\, e^{-i2B}$ ... (i)

also, $z_1 = z_3\, e^{i\,2B}$ ... (ii)

$\therefore\ \alpha z_1 = -z_2z_3 \qquad \Rightarrow \alpha = \dfrac{-z_2z_3}{z_1}$

Similarly, $\beta = \dfrac{-z_3z_1}{z_2}$ and $\gamma = \dfrac{-z_1z_2}{z_3}$.

Illustration 22: If z_r (r = 1, 2, ...,6) are the vertices of a regular hexagon then prove that $\sum_{r=1}^{6} z_r^2 = 6z_0^2$, where z_0 is the circumcentre of the regular hexagon. **(JEE MAIN)**

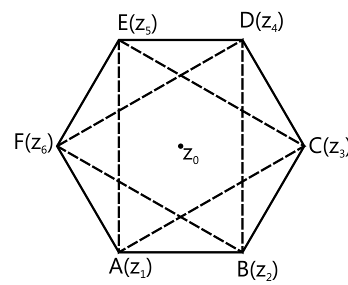

Figure 6.16

Sol: As we know If $z_1, z_2, z_3, \ldots\ldots. z_n$ be the vertices of a regular polygon of n sides and z_0 be its centroid, then $z_1^2 + z_2^2 + \ldots\ldots + z_n^2 = nz_0^2$.

Here by the Fig 6.14,

$3z_0^2 = z_1^2 + z_3^2 + z_5^2$

and, $3z_0^2 = z_2^2 + z_4^2 + z_6^2 \qquad \Rightarrow 6z_0^2 = \sum_{r=1}^{6} z_r^2$.

Illustration 23: If z_1, z_2, z_3 are the vertices of an equilateral triangle then prove that $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ and if z_0 is its circumcentre then $3z_0^2 = z_1^2 + z_2^2 + z_3^2$. **(JEE ADVANCED)**

Sol: By using triangle on complex plane we can prove

$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ and by using $z_0 = \dfrac{z_1 + z_2 + z_3}{3}$ we can prove $3z_0^2 = z_1^2 + z_2^2 + z_3^2$.

To Prove, $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

As seen in the Fig 6.17,

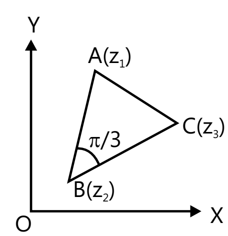

Figure 6.17

$\because \dfrac{z_1 - z_2}{z_2 - z_3} = \dfrac{(z_3 - z_2)\, e^{\frac{i\pi}{3}}}{(z_1 - z_3)\, e^{\frac{i\pi}{3}}} \Rightarrow (z_1 - z_2)(z_1 - z_3) = -(z_2 - z_3)^2$

$\Rightarrow z_1^2 - z_1z_3 - z_2z_1 + z_2z_3 + z_2^2 + z_3^2 - 2z_2z_3 = 0 \quad \therefore \sum z_1^2 = \sum z_1z_2$

Now if z_0 is the circumcentre of the Δ, then we need to prove $3z_0^2 = z_1^2 + z_2^2 + z_3^2$.

Since in an equilateral triangle, the circumcentre coincides with the centroid, we have $z_0 = \frac{z_1 + z_2 + z_3}{3}$

$\Rightarrow (z_1 + z_2 + z_3)^2 = (3z_0)^2$

$\Rightarrow \sum z_1^2 + 2\sum z_1 z_2 = 9z_0^2 \quad \therefore 3\sum z_1^2 = 9z_0^2$.

Illustration 24: Prove that the triangle whose vertices are the points z_1, z_2, z_3 on the Argand plane is an equilateral triangle if and only if $\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$. **(JEE ADVANCED)**

Sol: Consider ABC is the equilateral triangle with vertices z_1, z_2 and z_3 respectively.

Therefore $|z_2 - z_3| = |z_3 - z_1| = |z_1 - z_2|$.

Let ABC be a triangle such that the vertices A, B and C are z_1, z_2 and z_3 respectively.

Further, let $\alpha = z_2 - z_3$, $\beta = z_3 - z_1$ and $\gamma = z_1 - z_2$. Then $\alpha + \beta + \gamma = 0$ ... (i)

As shown in Fig 6.16, let ΔABC be an equilateral triangle. Then, BC = CA = AB

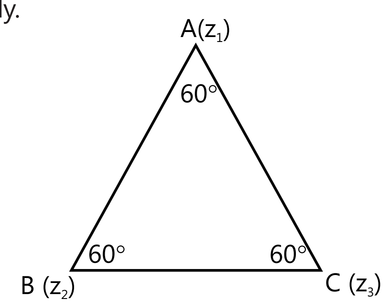

Figure 6.18

$\Rightarrow |z_2 - z_3| = |z_3 - z_1| = |z_1 - z_2| \quad \Rightarrow |\alpha| = |\beta| = |\gamma|$

$\Rightarrow |\alpha|^2 = |\beta|^2 = |\gamma|^2 = \lambda(\text{say})$

$\Rightarrow \alpha\bar{\alpha} = \beta\bar{\beta} = \gamma\bar{\gamma} = \lambda$

$\Rightarrow \bar{\alpha} = \frac{\lambda}{\alpha}, \bar{\beta} = \frac{\lambda}{\beta}, \bar{\gamma} = \frac{\lambda}{\gamma}$ ... (ii)

Now, $\alpha + \beta + \gamma = 0$ [from (i)]

$\Rightarrow \bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0 \qquad \Rightarrow \frac{\lambda}{\alpha} + \frac{\lambda}{\beta} + \frac{\lambda}{\gamma} = 0$ [Using (ii)]

$\Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0 \Rightarrow \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$ which is the required condition.

Conversely, let ABC be a triangle such that

$\Rightarrow \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$ i.e. $\Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0$

Thus, we have to prove that the triangle is equilateral. We have, $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 0$

$\Rightarrow \frac{1}{\alpha} = -\left(\frac{1}{\beta} + \frac{1}{\gamma}\right) \qquad \Rightarrow \frac{1}{\alpha} = -\left(\frac{\beta + \gamma}{\beta\gamma}\right) \qquad \Rightarrow \frac{1}{\alpha} = \frac{\alpha}{\beta\gamma} \qquad \Rightarrow \alpha^2 = \beta\gamma \qquad \Rightarrow |\alpha|^2 = |\beta\gamma|$

$\Rightarrow |\alpha|^2 = |\beta||\gamma| \qquad \Rightarrow |\alpha|^3 = |\alpha||\beta||\gamma|$

Similarly, $\Rightarrow |\beta|^3 = |\alpha||\beta||\gamma|$ and $|\gamma|^3 = |\alpha||\beta||\gamma|$

$\therefore \qquad |\alpha| = |\beta| = |\gamma|$

$\Rightarrow |z_2 - z_3| = |z_3 - z_1| = |z_1 - z_2| \qquad \Rightarrow$ BC = CA = AB

Hence, the given triangle is an equilateral triangle.

Illustration 25: Prove that the roots of the equation $\frac{1}{z-z_1}+\frac{1}{z-z_2}+\frac{1}{z-z_3}=0$ (where z_1, z_2, z_3 are pair wise distinct complex numbers) correspond to points on a complex plane, which lie inside a triangle with vertices z_1, z_2, z_3 excluding its boundaries. **(JEE ADVANCED)**

Sol: By using modulus and conjugate properties we can reduce given expression as $\frac{\bar{z}-\bar{z}_1}{|z-z_1|^2}+\frac{\bar{z}-\bar{z}_2}{|z-z_2|^2}+\frac{\bar{z}-\bar{z}_3}{|z-z_3|^2}$ = 0. Therefore by putting $|z-z_i|^2=\frac{1}{t_i}$, where i = 1, 2 and 3, we will get the result.

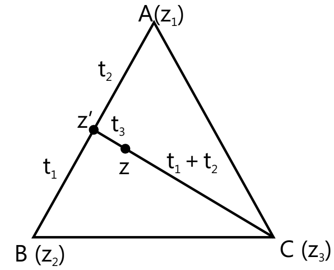

Figure 6.19

$t_1(\bar{z}-\bar{z}_1)+t_2(\bar{z}-\bar{z}_2)+t_3(\bar{z}-\bar{z}_3)=0$ where $|z-z_1|^2=\frac{1}{t_1}$ etc and $t_1, t_2, t_3 \in R^+$

$t_1(z-z_1)+t_2(z-z_2)+t_3(z-z_3)=0$

$(t_1+t_2+t_3)z=t_1z_1+t_2z_2+t_3z_3 \quad \Rightarrow z=\frac{t_1z_1+t_2z_2+t_3z_3}{t_1+t_2+t_3}$

$\Rightarrow z=\frac{t_1z_1+t_2z_2}{t_1+t_2}\cdot\frac{t_1+t_2}{t_1+t_2+t_3}+\frac{t_3z_3}{t_1+t_2+t_3}=\frac{t_1+t_2}{t_1+t_2+t_3}z'+\frac{t_3z_3}{t_1+t_2+t_3}$

$\Rightarrow z=\frac{(t_1+t_2)z'+t_3z_3}{t_1+t_2+t_3} \qquad \Rightarrow z$ lies inside the $\Delta z_1z_2z_3$

If $t_1=t_2=t_3$ $\qquad \Rightarrow$ z is the centroid of the triangle.

Also, it implies $|z-z_1|=|z-z_2|=|z-z_3|$ $\qquad \Rightarrow$ z is the circumcentre.

Illustration 26: Let z_1 and z_2 be roots of the equation $z^2+pz+q=0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB=\alpha\neq 0$ and OA = OB, where O is the origin, prove that $p^2=4q\cos^2\frac{\alpha}{2}$. **(JEE ADVANCED)**

Sol: Here $\overrightarrow{OB}=\overrightarrow{OA}e^{i\alpha}$. Therefore by using formula of sum and product of roots of quadratic equation we can prove this problem.

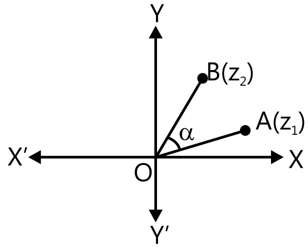

Figure 6.20

Since z_1 and z_2 are roots of the equation $z^2+pz+q=0$

$z_1+z_2=-p$ and $z_1z_2=q$ (1)

Since OA = OB. So $\overrightarrow{OB}$ is obtained by rotating $\overrightarrow{OA}$ in anticlockwise direction through angle α.

$\therefore\ \overrightarrow{OB}=\overrightarrow{OA}e^{i\alpha} \quad \Rightarrow z_2=z_1e^{i\alpha} \quad \Rightarrow \frac{z_2}{z_1}=e^{i\alpha}\Rightarrow\frac{z_2}{z_1}=\cos\alpha+i\sin\alpha$

$\Rightarrow \frac{z_2}{z_1}+1=1+\cos\alpha+i\sin\alpha\Rightarrow\frac{z_2+z_1}{z_1}=2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)=2\cos\frac{\alpha}{2}e^{\frac{i\alpha}{2}}$

$\Rightarrow \frac{z_2+z_1}{z_1}=2\cos\frac{\alpha}{2}e^{\frac{i\alpha}{2}}\Rightarrow\left(\frac{z_2+z_1}{z_1}\right)^2=4\cos^2\frac{\alpha}{2}e^{i\alpha}$

$\Rightarrow \left(\frac{z_2+z_1}{z_1}\right)^2=4\cos^2\frac{\alpha}{2}\frac{z_2}{z_1} \quad \Rightarrow (z_2+z_1)^2=4z_1z_2\cos^2\frac{\alpha}{2}$

$\Rightarrow (-p)^2=4q\cos^2\frac{\alpha}{2} \qquad \Rightarrow p^2=4q\cos^2\frac{\alpha}{2}$.

Illustration 27: On the Argand plane z_1, z_2 and z_3 are respectively the vertices of an isosceles triangle ABC with AC = BC and equal angles are θ. If z_4 is the incentre of the triangle then prove that $(z_2 - z_1)(z_3 - z_1) = (1 + \sec\theta)(z_4 - z_1)^2$ **(JEE ADVANCED)**

Sol: Here by using angle rotation formula we can solve this problem. From Fig 6.21, we have

$$\frac{z_2 - z_1}{|z_2 - z_1|} = \frac{z_4 - z_1}{|z_4 - z_1|}e^{i\theta/2} \quad \text{... (i) (clockwise)}$$

and $$\frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_4 - z_1}{|z_4 - z_1|}e^{i\theta/2} \quad \text{... (ii) (anticlockwise)}$$

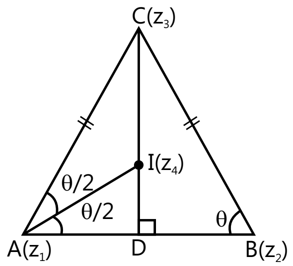

Figure 6.21

Multiplying (i) and (ii)

$$\frac{(z_2 - z_1)(z_3 - z_1)}{(z_4 - z_1)^2} = \frac{|(z_2 - z_1)||(z_3 - z_1)|}{|z_4 - z_1|^2} = \frac{AB|AC|}{(AI)^2} = \frac{2(AD)(AC)}{(AI)^2} = \frac{2(AD)^2}{(AI)^2}\cdot\frac{AC}{AD}$$

$$= 2\cos^2\frac{\theta}{2}\sec\theta = (1 + \cos\theta)\sec\theta.$$

7. REPRESENTATION OF DIFFERENT LOCI ON COMPLEX PLANE

(a) $|z - (1 + 2i)| = 3$ denotes a circle with centre (1, 2) and radius 3 (see Fig 6.22).

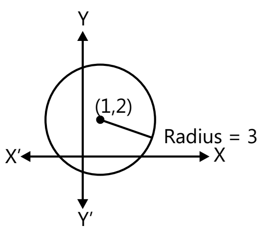

Figure 6.22: Circle on a complex plane

(b) $|z - 1| = |z - i|$ denotes the equation of the perpendicular bisector of join of (1, 0) and (0, 1) on the Argand plane (see Fig 6.24).

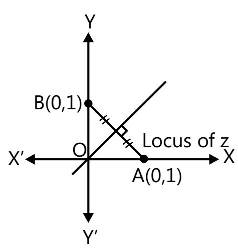

Figure 6.23: Perpendicular bisector complex plane

(c) $|z - 4i| + |z + 4i| = 10$ denotes an ellipse with foci at (0, 4) and (0, – 4); major axis 10; minor axis 6 with $e = \frac{4}{5}$ (see Fig 6.24).

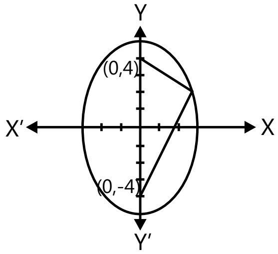

Figure 6.24: Ellipse on a complex plane

$$e^2 = 1 - \frac{36}{100} = \frac{64}{100} \Rightarrow \qquad e = \frac{4}{5} \left[\frac{x^2}{9} + \frac{y^2}{25} = 1\right]$$

(d) $|z - 1| + |z + 1| = 1$ denotes no locus. (Triangle inequality).

(e) $|z - 1| < 1$ denotes area inside a circle with centre (1, 0) and radius 1.

(f) $2 \le |z - 1| < 5$ denotes the region between the concentric circles of radii 5 and 2. Centred at (1, 0) including the inner boundary (see Fig 6.25).

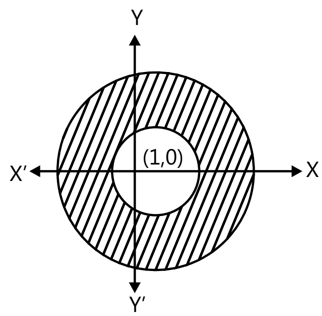

Figure 6.25: Circle disc on a complex plane

(g) $0 \le \arg z \le \frac{\pi}{4} (z \ne 0)$ where z is defined by positive real axis and the part of the line x = y in the first quadrant. It includes the boundary but not the origin. Refer to Fig 6.26.

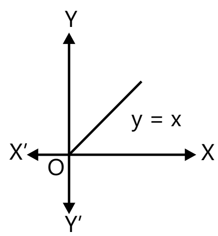

Figure 6.26

(h) $\text{Re}(z^2) > 0$ denotes the area between the lines x = y and x = – y which includes the x-axis.

Hint: $(x^2 - y^2) + 2xyi = 0 \quad \Rightarrow x^2 - y^2 > 0 \qquad \Rightarrow (x - y)(x + y) > 0$.

Illustration 28: Solve for z, if $z^2 + |z| = 0$. **(JEE MAIN)**

Sol: Consider z = $x + iy$ and solve this using algebra of complex number.

Let z = $x + iy \quad \Rightarrow (x + iy)^2 + \sqrt{x^2 + y^2} = 0 \qquad \Rightarrow \left(x^2 - y^2 + \sqrt{x^2 + y^2}\right) + (2ixy) = 0$

$\Rightarrow$ Either $x = 0$ or $y = 0$; $\quad x = 0 \Rightarrow -y^2 + |y| = 0 \qquad \Rightarrow y = 0, 1, -1 \quad \therefore z = 0, i, -i$

and, $y = 0 \Rightarrow x^2 + |x| = 0 \qquad \Rightarrow x = 0 \qquad \therefore z = 0$

Therefore, z = 0, z = i, z = – i.

Illustration 29: If the complex number z is to satisfy $|z| = 3, \left|z - \{a(1 + i) - i\}\right| \le 3$ and $|z + 2a - (a + 1)i| > 3$ simultaneously for at least one z then find all $a \in R$. **(JEE ADVANCED)**

Sol: Consider z = x + iy and solve these inequalities to get the result.

All z at a time lie on a circle | z | = 3 but inside and outside the circles | z – {a (1 + i) – i} | = 3 and | z + 2a – (a + 1) i | = 3, respectively.

Let z = x + iy then equation of circles are $x^2 + y^2 = 9$ ... (i)

$(x-a)^2 + (y-a+1)^2 = 9$ ... (ii)

and $(x+2a)^2 + (y-a-1)^2 = 9$ ... (iii)

Circles (i) and (ii) should cut or touch then distance between their centres $\leq$ sum of their radii.

$\Rightarrow \sqrt{(a-0)^2 + (a-1-0)^2} \leq 3+3 \Rightarrow a^2 + (a-1)^2 \leq 36$

$\Rightarrow 2a^2 - 2a - 35 \leq 0 \Rightarrow a^2 - a - \frac{35}{2} \leq 0$

$\Rightarrow \left(a - \frac{1}{2}\right)^2 \leq \frac{71}{4} \quad \therefore \frac{1-\sqrt{71}}{2} \leq a \leq \frac{1+\sqrt{71}}{2}$ ... (iv)

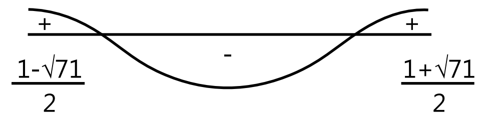

Figure 6.27

Again circles (i) and (iii) should not cut or touch then distance between their centres > sum of the radii

$\Rightarrow \sqrt{(-2a-0)^2 + (a+1-0)^2} > 3+3 \Rightarrow \sqrt{5a^2 + 2a + 1} > 6 \quad \Rightarrow 5a^2 + 2a + 1 > 36$

$\Rightarrow 5a^2 + 2a - 35 > 0 \qquad \Rightarrow a^2 + \frac{2a}{5} - 7 > 0$

Then $\left(a - \frac{-1-4\sqrt{11}}{5}\right)\left(a - \frac{-1+4\sqrt{11}}{5}\right) > 0$

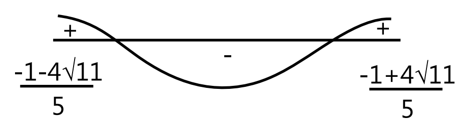

Figure 6.28

$\therefore \qquad a \in \left(-\infty, \frac{-1-4\sqrt{11}}{5}\right) \cup \left(\frac{-1+4\sqrt{11}}{5}, \infty\right)$ ... (v)

The common values of a satisfying (iv) and v are

$a \in \left(\frac{1-\sqrt{71}}{2}, \frac{-1-4\sqrt{11}}{5}\right) \cup \left(\frac{-1+4\sqrt{11}}{5}, \frac{1+\sqrt{71}}{2}\right)$

8. DEMOIVRE'S THEOREM

Statement: $(\cos n\theta + i\sin n\theta)$ is the value or one of the values of $(\cos\theta + i\sin\theta)^n$, $\forall\, n \in Q$. Value if n is an integer. One of the values if n is rational which is not integer, the theorem is very useful in determining the roots of any complex quantity.

Note: We use the theory of equations to find the continued product of the roots of a complex number.

PLANCESS CONCEPTS

The theorem is not directly applicable to $(\sin\theta + i\cos\theta)^n$, rather

$$(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^n = \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$$

8.1 Application

Cube root of unity

(a) The cube roots of unity are 1, $\frac{-1+i\sqrt{3}}{2}$, $\frac{-1-i\sqrt{3}}{2}$

[Note that $1 - i\sqrt{3} = -2$ and $1 + i\sqrt{3} = -2\omega^2$]

(b) If ω is one of the imaginary cube roots of unity then $1+\omega+\omega^2=0$.

In general $1 + \omega^r + \omega^{2r} = 0$; where r = 1, and not a multiple of 3.

(c) In polar form the cube roots of unity are: $\cos 0 + i\sin 0$; $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$; $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

(d) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.

[Note that the 3 cube roots of i lies on the vertices of an isosceles triangle]

(e) The following factorization should be remembered.

For a, b, c $\in R$ and ω being the cube root of unity,

(i) $a^3 - b^3 = (a-b)(a-\omega b)(a-\omega^2 b)$

(ii) $x^2 + x + 1 = (x-\omega)(x-\omega^2)$

(iii) $a^3 + b^3 = (a+b)(a+\omega b)(a+\omega^2 b)$

(iv) $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)$

nth roots of unity: If 1, $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_{n-1}$ are the n, n^{th} roots of unity then

(i) They are in G.P. with common ratio $e^{i\left(\frac{2\pi}{n}\right)} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

(ii) $1^p + \alpha_1^p + \alpha_2^p + \ldots\ldots + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n

$1^p + (\alpha_1)^p + (\alpha_2)^p + \ldots\ldots + (\alpha_{n-1})^p = n$ if p is an integral multiple of n.

(iii) $(1-\alpha_1)(1-\alpha_2)\ldots\ldots(1-\alpha_{n-1}) = n$.

Steps to determine nth roots of a complex number

(i) Represent the complex number whose roots are to be determined in polar form.

(ii) Add 2mπ to the argument.

(iii) Apply De Moivre's TheoremT

(iv) Put m = 0, 1, 2, 3, ... (n – 1) to get all the n^{th} roots.

Explanation: Let $z = 1^{\frac{1}{n}} = (\cos 0 + i\sin 0)^{\frac{1}{n}} = (\cos 2m\pi + i\sin 2m\pi)^{\frac{1}{n}} = \left(\cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}\right)$

Put m = 0, 1, 2, 3, ... (n – 1), we get

$1, \underbrace{\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}}_{\alpha}, \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}, \ldots\ldots, \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}$ (n, n^{th} roots in G.P.)

Now, $S = 1^p + \alpha^p + \alpha^{2p} + \alpha^{3p} + \ldots\ldots + \alpha^{(n-1)p} = \dfrac{1-(\alpha^p)^n}{1-\alpha^p} = \dfrac{1-(\alpha^n)^p}{1-\alpha^p}$

$$= \frac{1-(\alpha^n)^p}{1-\alpha^p} = \begin{cases} \dfrac{0}{\text{non zero}} = 0, & \text{if p is not an integral multiple of n} \\ \dfrac{0}{0} = \text{indeterminant}, & \text{if p is an integral multiple of n} \end{cases}$$

Again, if x is one of the n^{th} root of unity then $x^n - 1 = (x-1)\ (x-\alpha_1)(x-\alpha_2) \ldots\ldots (x-\alpha_{n-1})$

$1 + x + x^2 + \ldots\ldots + x^{n-1} = \dfrac{x^n - 1}{x-1} \equiv (x-\alpha_1)(x-\alpha_2) \ldots\ldots (x-\alpha_{n-1})$

Put x = 1, to get $(1-\alpha_1)(1-\alpha_2) \ldots\ldots (1-\alpha_{n-1}) = n$

Similarly put x = – 1, is to get other result.

PLANCESS CONCEPTS

Square roots of z = a + ib are $\pm\left[\sqrt{\dfrac{|z|+a}{2}} + i\sqrt{\dfrac{|z|-a}{2}}\right]$ for b > 0.

If $1, \alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_{n-1}$ are the n, n^{th} roots of unity then

$(1+\alpha_1)(1+\alpha_2) \ldots\ldots (1+\alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

$1 \cdot \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \ldots\ldots \cdot \alpha_{n-1}$ = 1 or – 1 according as n is odd or even.

$$(\omega-\alpha_1)(\omega-\alpha_2) \ldots\ldots (\omega-\alpha_{n-1}) = \begin{cases} 0, & \text{if } n = 3k \\ 1, & \text{if } n = 3k+1 \\ 1+\omega, & \text{if } n = 3k+2 \end{cases}.$$

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Illustration 30: If $x = a+b$, $y = a\omega + b\omega^2$ and $z = a\omega^2 + b\omega$, then prove that $x^3 + y^3 + z^3 = 3(a^3 + b^3)$ **(JEE MAIN)**

Sol: Here $x + y + z = 0$. Take cube on both side.

$x + y + z = 0 \quad \Rightarrow x^3 + y^3 + z^3 = 3xyz \therefore$ LHS = 3xyz

$= 3(a+b)(a\omega + b\omega^2)(a\omega^2 + b\omega) = 3(a+b)(a\omega + b\omega^2)(a\omega^2 + b\omega.\omega^3) = 3\omega^3(a+b)(a+b\omega)(a+b\omega^2) = 3(a^3 + b^3)$

Illustration 31: The value of expression $1(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) + \ldots + (n-1)(n-\omega)(n-\omega^2)$.

(JEE ADVANCED)

Sol: The given expression represent as $x^3 - 1 = (x-1)\ (x-\omega)(x-\omega^2)$. Therefore by putting x = 2, 3, 4 … n, we will get the result.

$x^3 - 1 = (x-1)\ (x-\omega)(x-\omega^2)$

Put x = 2 $\ 2^3 - 1 = 1 \cdot (2-\omega)(2-\omega)^2$ Put x = 3 $\ 3^3 - 1 = 2 \cdot (3-\omega)(3-\omega^2)$:

Put x = n $\ n^3 - 1 = (n-1)(n-\omega)(n-\omega^2)$

$\therefore \text{LHS} = (2^3 + 3^3 + \ldots\ldots + n^3) - (n-1) \quad = (1^3 + 2^3 + 3^3 + \ldots\ldots + n^3) - n = \left(\dfrac{n(n+1)}{2}\right)^2 - n$

9. SUMMATION OF SERIES USING COMPLEX NUMBER

(a) $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots\ldots + \cos n\theta = \dfrac{\sin\left(\dfrac{n\theta}{2}\right)}{\sin\left(\dfrac{\theta}{2}\right)} \cos\left(\dfrac{n+1}{2}\right)\theta$

(b) $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots\ldots + \sin n\theta = \dfrac{\sin\left(\dfrac{n\theta}{2}\right)}{\sin\left(\dfrac{\theta}{2}\right)} \sin\left(\dfrac{n+1}{2}\right)\theta$

Note: If $\theta = \dfrac{2\pi}{n}$, then the sum of the above series vanishes.

9.1 Complex Number and Binomial Coefficients

Try the following questions using the binomial expansion of $(1+x)^n$ and substituting the value of x according to the binomial coefficients in the respective question.

Find the value of the following

(i) $C_0 + C_4 + C_8 + \ldots\ldots$ (ii) $C_1 + C_5 + C_9 + \ldots\ldots$

(iii) $C_2 + C_6 + C_{10} + \ldots\ldots$ (iv) $C_3 + C_7 + C_{11} + \ldots\ldots$

(v) $C_0 + C_3 + C_6 + C_9 + \ldots\ldots$

Hint (v): In the expansion of $(1+x)^n$, put $x = 1, \omega, \text{and}\, \omega^2$ and add the three equations.

Illustration 32: If $1, \omega, \omega^2, \ldots\ldots, \omega^{n-1}$ are n^{th} roots of unity, then the value of $(5-\omega)(5-\omega^2)\ldots\ldots(5-\omega^{n-1})$ is equal to **(JEE MAIN)**

Sol: Here consider x = $(1)^{\frac{1}{n}}$, therefore $x^n - 1 = 0$ (has n roots i.e. 1, $\omega, \omega^2, \ldots\ldots, \omega^{n-1}$).

$\Rightarrow x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\ldots\ldots(x-\omega^{n-1})$ $\Rightarrow \dfrac{x^n-1}{x-1} = (x-\omega)(x-\omega^2)\ldots\ldots(x-\omega^{n-1})$

$\Rightarrow$ Putting x = 5 in both sides, we get $\therefore (5-\omega)(5-\omega^2)\ldots\ldots(5-\omega^{n-1}) = \dfrac{5^n-1}{4}$.

10. APPLICATION IN GEOMETRY

10.1 Distance Formula

Distance between $A(z_1)$ and $B(z_2)$ is given by AB = $|z_2 - z_1|$. Refer Fig 6.29.

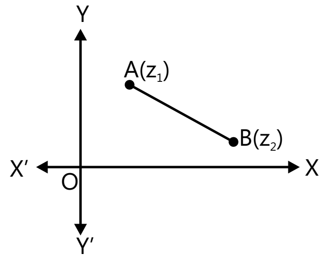


Figure 6.29

10.2 Section Formula

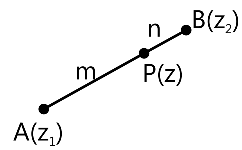


Figure 6.30

The point P(z) which divides the join of $A(z_1)$ and $B(z_2)$ in the ratio m: n is given by $z = \frac{mz_2 + nz_1}{m+n}$. Refer Fig 6.30.

10.3 Midpoint Formula

Mid-point M(z) of the segment AB is given by $z = \frac{1}{2}(z_1 + z_2)$.

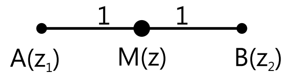


Figure 6.31 Mid point formula

10.4 Condition For Four Non-Collinear Points

Condition(s) for four non-collinear $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ to represent vertices of a

(a) Parallelogram: The diagonals AC and BD must bisect each other

$$\Leftrightarrow \quad \frac{1}{2}(z_1 + z_3) = \frac{1}{2}(z_2 + z_4)$$

$$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$$

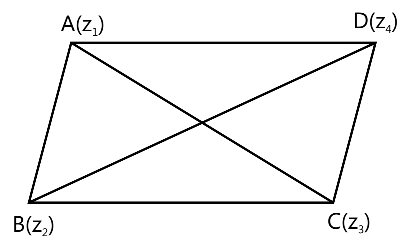


Figure 6.32

(b) Rhombus:

(i) The diagonals AC and BD bisect each other

$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$, and

(ii) A pair of two adjacent sides are equal, for instance AD = AB

$$\Leftrightarrow \quad |z_4 - z_1| = |z_2 - z_1|$$

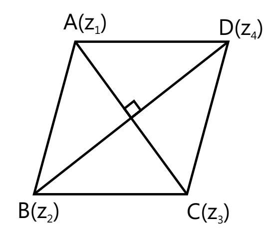


Figure 6.33

(c) Square:

(i) The diagonals AC and BD bisect each other

$$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$$

(ii) A pair of adjacent sides are equal; for instance, AD = AB

$$\Leftrightarrow \quad |z_4 - z_1| = |z_2 - z_1|$$

(iii) The two diagonals are equal, that is AC = BD

$$\Leftrightarrow \quad |z_3 - z_1| = |z_4 - z_2|$$

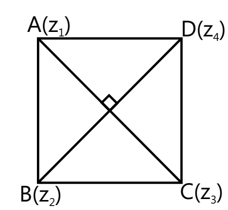


Figure 6.34

(d) Rectangle:

(i) The diagonals AC and BD bisect each other

$$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$$

(ii) The diagonals AC and BD are equal

$$\Leftrightarrow \quad |z_3 - z_1| = |z_4 - z_2|$$

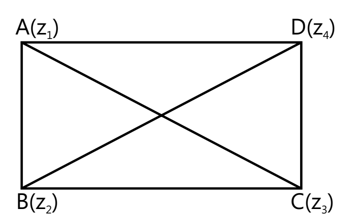


Figure 6.35

10.5 Triangle

In a triangle ABC, let the vertices A, B and C be represented by the complex numbers z_1, z_2, and z_3 respectively. Then

(a) Centroid: The centroid (G), is the point of intersection of medians of ΔABC. It is given by the formula

$$z = \frac{1}{3}(z_1 + z_2 + z_3)$$

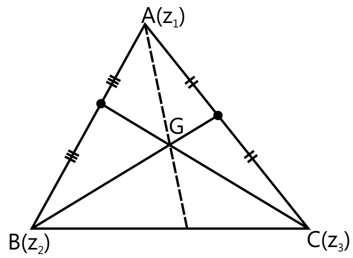


Figure 6.36 (a)

(b) Incentre: The incentre (I) of ΔABC is the point of intersection of internal angular bisectors of angles of ΔABC. It is given by the formula

$$z = \frac{az_1 + bz_2 + cz_3}{a + b + c},$$

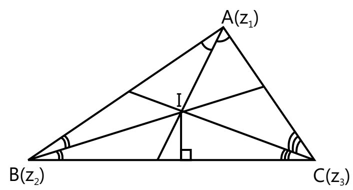


Figure 6.36 (b)

(c) Circumcentre: The circumcentre (S) of ΔABC is the point of intersection of perpendicular bisectors of sides of ΔABC. It is given by the formula

$$z = \frac{|z_1|^2 (z_2 - z_3) + |z_2|^2 (z_3 - z_1) + |z_3|^2 (z_1 - z_2)}{\overline{z}_1(z_2 - z_3) + \overline{z}_2(z_3 - z_1) + \overline{z}_3(z_1 - z_2)} = \frac{\begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix}}{\begin{vmatrix} \overline{z}_1 & z_1 & 1 \\ \overline{z}_2 & z_2 & 1 \\ \overline{z}_3 & z_3 & 1 \end{vmatrix}}$$

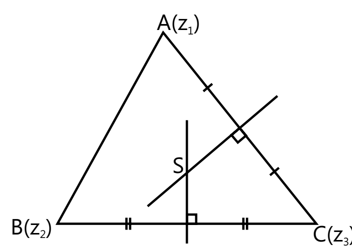


Figure 6.36 (c)

Also, $z = \dfrac{z_1(\sin 2A) + z_2(\sin 2B) + z_3(\sin 2C)}{\sin 2A + \sin 2B + \sin 2C}$

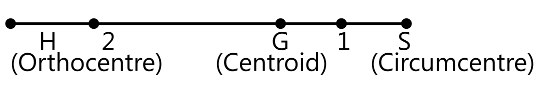


Figure 6.37

(d) Euler's Line: The orthocenter H, the centroid G and the circumcentre S of a triangle which is not equilateral lies on a straight line. In case of an equilateral triangle these points coincide.

G divides the join of H and S in the ratio 2 : 1 (see Fig 6.37).

Thus, $z_G = \frac{1}{3}(z_H + 2z_S)$

10.6 Area of a Triangle

Area of ΔABC with vertices $A(z_1), B(z_2)$ and $C(z_3)$ is given by

$$\Delta = \left| \frac{1}{4i}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} \right| = \left| \frac{1}{2}\text{Im}(\bar{z}_1 z_2 + \bar{z}_2 z_3 + \bar{z}_3 z_1) \right|$$

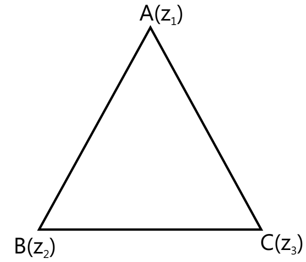


Figure 6.38

10.7 Conditions for Triangle to be Equilateral

The triangle ABC with vertices $A(z_1), B(z_2)$ and $C(z_3)$ is equilateral

iff $\dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} + \dfrac{1}{z_1 - z_2} = 0$

$\Leftrightarrow z_1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_3 z_1 + z_1 z_2 \quad \Leftrightarrow z_1\bar{z}_2 = z_2\bar{z}_3 = z_3\bar{z}_1 \Leftrightarrow z_1^2 = z_2 z_3$ and $z_2^2 = z_1 z_3$

$$\Leftrightarrow \begin{vmatrix} 1 & z_2 & z_3 \\ 1 & z_3 & z_1 \\ 1 & z_1 & z_2 \end{vmatrix} = 0 \quad \Leftrightarrow \frac{z_2 - z_1}{z_3 - z_2} = \frac{z_3 - z_2}{z_1 - z_2}$$

$\Leftrightarrow \dfrac{1}{z - z_1} + \dfrac{1}{z - z_2} + \dfrac{1}{z - z_3} = 0$ where $z = \dfrac{1}{3}(z_1 + z_2 + z_3)$.

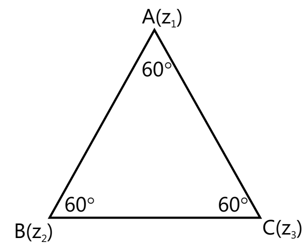


Figure 6.39

10.8 Equation of a Straight line

(a) Non-parametric form: An equation of a straight line joining the two points $A(z_1)$ and $B(z_2)$ is

$$\text{Arg}\left(\frac{z - z_1}{z_2 - z_1}\right) = 0 \quad \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

or $\dfrac{z - z_1}{z_2 - z_1} = \dfrac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$

or $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$

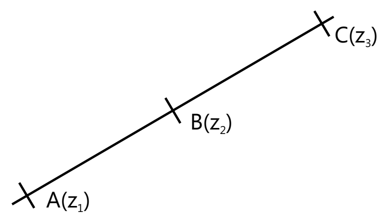


Figure 6.40

(b) Parametric form: An equation of the line segment between the points $A(z_1)$ and $B(z_2)$ is $z = tz_1 + (1 - t)z_2,\ t(0,1)$ where t is a real parameter.

(c) General equation of a straight line: The general equation of a straight line is $\bar{a}z + a\bar{z} + b = 0$ where, a is non-zero complex number and b is a real number.

10.9 Complex Slope of a Line

If $A(z_1)$ and $B(z_2)$ are two points in the complex plane, then complex slope of AB is defined to be $\mu = \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$

Two lines with complex slopes μ_1 and μ_2 are

(i) Parallel, if $\mu_1 = \mu_2$ (ii) Perpendicular, if $\mu_1 + \mu_2 = 0$

The complex slope of the line $\bar{a}z + a\bar{z} + b = 0$ is given by $\left(\dfrac{-a}{\bar{a}}\right)$.

10.10 Length of Perpendicular from a Point to a Line

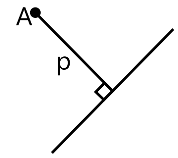


Figure 6.41

Length of perpendicular of point $A(\omega)$ from the line $\bar{a}z + a\bar{z} + b = 0$.

Where $a \in C - \{0\}$, and $b \in R$ is given by $p = \dfrac{|\bar{a}\omega + a\bar{\omega} + b|}{2|a|}$

10.11 Equation of Circle

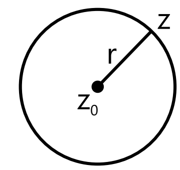


Figure 6.42

(a) An equation of the circle with centre z_0 and radius r is $|z - z_0| = r$ or

$z = z_0 + re^{i\theta}, 0 \le \theta < 2\pi$ (parametric form) or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + z_0\bar{z}_0 - r^2 = 0$

(b) General equation of a circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ ... (i)

Where a is a complex number and b is a real number such that $a\bar{a} - b \ge 0$. Centre of (i) is – a and its radius is $\sqrt{a\bar{a} - b}$

(c) Diameter form of a circle: An equation of the circle one of whose diameter is the segment joining $A(z_1)$ and $B(z_2)$ is $(z - z_1)(\bar{z} - \bar{z}_2) + (\bar{z} - \bar{z}_1)(z - z_2) = 0$

(d) An equation of the circle passing through two points $A(z_1)$ and $B(z_2)$

is $(z - z_1)(\bar{z} - \bar{z}_2) + (\bar{z} - \bar{z}_1)(z - z_2) + ik\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$ where k is a real parameter.

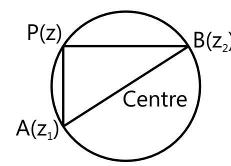


Figure 6.43

(e) Equation of a circle passing through three non-collinear points.

Let three non-collinear points be $A(z_1)$, $B(z_2)$ and $C(z_3)$ and P(z) be any point on the circle through A, B and C.

Then either $\angle ACB = \angle APB$ [when angles are in the same segment]

or, $\angle ACB + \angle APB = \pi$ [when angles are in the opposite segment] (see Fig 6.44).

$$\Rightarrow \arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) - \arg\left(\frac{z - z_2}{z - z_1}\right) = 0 \text{ or, } \arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) + \arg\left(\frac{z - z_1}{z - z_2}\right) = \pi$$

$$\Rightarrow \arg\left[\left(\frac{z_3 - z_2}{z_3 - z_1}\right)\left(\frac{z - z_1}{z - z_2}\right)\right] = 0$$

$$\text{or, } \arg\left[\left(\frac{z_3 - z_2}{z_3 - z_1}\right)\left(\frac{z - z_1}{z - z_2}\right)\right] = \pi$$

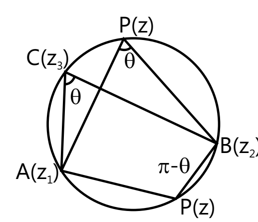


Figure 6.44

In any case, we get $\dfrac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}$ is purely real.

$$\Leftrightarrow \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)} = \frac{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}$$

(f) Condition for four points to be concyclic.

Four points z_1, z_2, z_3 and z_4 will lie on the same circle if and only if $\dfrac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)}$ is purely real.

$$\Leftrightarrow \frac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)} = \frac{(\bar{z}_4 - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}{(\bar{z}_4 - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}$$

PLANCESS CONCEPTS

Three points z_1, z_2 and z_3 are collinear if $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0.$

If three points $A(z_1), B(z_2)$ and $C(z_3)$ are collinear then slope of AB = slope of BC = slope of AC

$\Rightarrow \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = \dfrac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3} = \dfrac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}$

Akshat Kharaya (JEE 2009, AIR 235)

Illustration 33: If the imaginary part of $\dfrac{2z+1}{iz+1}$ is – 4, then the locus of the point representing z in the complex plane is

(a) A straight line (b) A parabola (c) A circle (d) An ellipse **(JEE MAIN)**

Sol: Put z = x + iy and then equate its imaginary part to – 4.

Let z = x + iy, then $\dfrac{2z+1}{iz+1} = \dfrac{2(x+iy)+1}{i(x+iy)+1} = \dfrac{(2x+1)+2iy}{(1-y)+ix} = \dfrac{[(2x+1)+2iy]\,[(1-y)-ix]}{(1-y)^2+x^2}$

As $\text{Im}\left(\dfrac{2z+1}{iz+1}\right) = -4$, we get $\dfrac{2y(1-y)-x(2x+1)}{x^2+(1-y)^2} = -4$

$\Rightarrow 2x^2+2y^2+x-2y = 4x^2+4(y^2-2y+1)$ $\Rightarrow 2x^2+2y^2-x-6y+4=0$. It represents a circle.

Illustration 34: The roots of $z^5 = (z-1)^5$ are represented in the argand plane by the points that are

(a) Collinear (b) Concyclic

(c) Vertices of a parallelogram (d) None of these **(JEE MAIN)**

Sol: Apply modulus on both the side of given expression.

Let z be a complex number satisfying $z^5 = (z-1)^5$.

$\Rightarrow |z^5| = |(z-1)^5|$ $\Rightarrow |z|^5 = |z-1|^5$ $\Rightarrow |z| = |z-1|$

Thus, z lies on the perpendicular bisector of the segment joining the origin and (1 + i0) i.e. z lies on Re(z) = $\dfrac{1}{2}$.

Illustration 35: Let z_1 and z_2 be two non-zero complex numbers such that $\dfrac{z_1}{z_2}+\dfrac{z_2}{z_1} = 1$, then the origin and points represented by z_1 and z_2

(a) Lie on straight line (b) Form a right triangle

(c) Form an equilateral triangle (d) None of these **(JEE ADVANCED)**

Sol: Here consider z = $\dfrac{z_1}{z_2}$ and z_1 and z_2 are represented by A and B respectively and O be the origin.

Let z = $\dfrac{z_1}{z_2}$, then z + $\dfrac{1}{z}$ = 1 $\Rightarrow z^2 - z + 1 = 0$

$\Rightarrow z = \dfrac{1 \pm \sqrt{3}i}{2}$ $\Rightarrow \dfrac{z_1}{z_2} = \dfrac{1 \pm \sqrt{3}i}{2}$

If z_1 and z_2 are represented by A and B respectively and O be the origin, then

$$\frac{OA}{OB} = \frac{|z_1|}{|z_2|} = \left|\frac{1 \pm \sqrt{3}i}{2}\right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1 \Rightarrow OA = OB$$

Also, $\frac{AB}{OB} = \frac{|z_2 - z_1|}{|z_2|} = \left|1 - \frac{z_1}{z_2}\right| = \left|1 - \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)\right| = \left|\frac{1}{2} \mp \frac{\sqrt{3}}{2}i\right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\Rightarrow AB = OB$ Thus, OA = OB = AB $\therefore$ ΔOAB is an equilateral triangle.

Illustration 36: If z_1, z_2, z_3 are the vertices of an isosceles triangle, right angled at the vertex z_2, then the value of $(z_1 - z_2)^2 + (z_2 - z_3)^2$ is

(a) -1 (b) 0 (c) $(z_1 - z_3)^2$ (d) None of these **(JEE ADVANCED)**

Sol: Here use distance and argument formula of complex number to solve this problem.

As ABC is an isosceles right angled triangle with right angle at B,

BA = BC and $\angle ABC = 90° \Rightarrow |z_1 - z_2| = |z_3 - z_2|$ and $\arg\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \frac{\pi}{2}$

$$\Rightarrow \frac{z_3 - z_2}{z_1 - z_2} = \frac{|z_3 - z_2|}{|z_1 - z_2|}\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right] = i$$

$\Rightarrow (z_3 - z_2)^2 = -(z_1 - z_2)^2$ $\Rightarrow (z_1 - z_2)^2 + (z_2 - z_3)^2 = 0$.

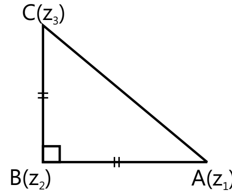


Figure 6.45

11. CONCEPTS OF ROTATION OF COMPLEX NUMBER

Let z be a non-zero complex number. We can write z in the polar form as follows:

z = r $(\cos\theta + i\sin\theta) = re^{i\theta}$ where r = | z | and arg (z) = θ (see Fig 6.46).

Consider a complex number $ze^{i\alpha}$.

$ze^{i\alpha} = (re^{i\theta})e^{i\alpha} = re^{i(\theta+\alpha)}$

Thus, $ze^{i\alpha}$ represents the complex number whose modulus is r and argument is $\theta + \alpha$.

Geometrically, $ze^{i\alpha}$ can be obtained by rotating the line segment joining O and P(z) through an angle α in the anticlockwise direction.

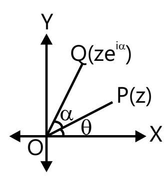


Figure 6.46

Corollary: If $A(z_1)$ and $B(z_2)$ are two complex number such that

$\angle AOB = \theta$, then $z_2 = \frac{|z_2|}{|z_1|}z_1e^{i\theta}$ (see Fig 6.47).

Let $z_1 = r_1e^{i\alpha}$ and $z_2 = r_2e^{i\beta}$ where $|z_1| = r_1, |z_2| = r_2$.

Then $\frac{z_2}{z_1} = \frac{r_2e^{i\beta}}{r_1e^{i\alpha}} = \frac{r_2}{r_1}e^{i(\beta-\alpha)}$

Thus, $\frac{z_2}{z_1} = \frac{r_2}{r_1}e^{i\theta}$ $(\because \beta - \alpha = \theta)$ $\Rightarrow z_2 = \frac{|z_2|}{|z_1|}z_1e^{i\theta}$

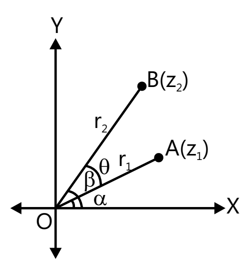


Figure 6.47

PLANCESS CONCEPTS

Multiplication of a complex number,z with i.

Let z = r $(\cos\theta + i\sin\theta)$ and i = $\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$, then iz = $r\left[\cos\left(\frac{\pi}{2}+\theta\right) + i\sin\left(\frac{\pi}{2}+\theta\right)\right]$.

Hence, iz can be obtained by rotating the vector z by right angle in the positive sense. And so on, to multiply a vector by – 1 is to turn it through two right angles.

Thus, multiplying a vector by $(\cos\theta + i\sin\theta)$ is to turn it through the angle θ in the positive sense.

Anvit Tawar (JEE 2009, AIR 9)

Illustration 37: Suppose $A(z_1)$, $B(z_2)$ and $C(z_3)$ are the vertices of an equilateral triangle inscribed in the circle | z | = 2. If $z_1 = 1 + \sqrt{3}i$, then z_2 and z_3 are respectively.

(a) $-2, 1-\sqrt{3}i$ (b) $-1+\sqrt{3}i, -2$

(c) $-2, -1+\sqrt{3}i$ (d) $-2, 2+\sqrt{3}i$ **(JEE ADVANCED)**

Sol: As we know x + iy = $re^{i\theta}$. Hence by using this formula we can obtain z_2 and z_3.

$z_1 = 1+\sqrt{3}i = 2e^{\frac{i\pi}{3}}$

Since, $\angle AOC = \frac{2\pi}{3}$ and $\angle BOC = \frac{2\pi}{3}$, $z_2 = z_1 e^{\frac{2\pi i}{3}}$ and $z_3 = z_2 e^{\frac{2\pi i}{3}}$

$\Rightarrow z_3 = 2e^{\pi i} = 2(\cos\pi + i\sin\pi) = -2$ and $z_3 = 2e^{\frac{5\pi i}{3}}$

$= 2\left[\cos\left(2\pi - \frac{\pi}{3}\right) + i\sin\left(2\pi - \frac{\pi}{3}\right)\right]$

$= 2\left[\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right] = 2\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] = 1-\sqrt{3}i.$

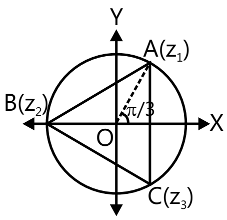


Figure 6.48

PROBLEM-SOLVING TACTICS

(a) On a complex plane, a complex number represents a point.

(b) In case of division and modulus of a complex number, the conjugates are very useful.

(c) For questions related to locus and for equations, use the algebraic form of the complex number.

(d) Polar form of a complex number is particularly useful in multiplication and division of complex numbers. It directly gives the modulus and the argument of the complex number.

(e) Translate unfamiliar statements by changing z into x+iy.

(f) Multiplying by $\cos\theta$ corresponds to rotation by angle θ about O in the positive sense.

(g) To put the complex number $\frac{a+ib}{c+id}$ in the form A + iB we should multiply the numerator and the denominator by the conjugate of the denominator.

(h) Care should be taken while calculating the argument of a complex number. If z = a + ib, then arg(z) is not always equal to $\tan^{-1}\left(\frac{b}{a}\right)$. To find the argument of a complex number, first determine the quadrant in which it lies, and then proceed to find the angle it makes with the positive x-axis.

For example, if z = – 1 – i, the formula $\tan^{-1}\left(\frac{b}{a}\right)$ gives the argument as $\frac{\pi}{4}$, while the actual argument is $\frac{-3\pi}{4}$.

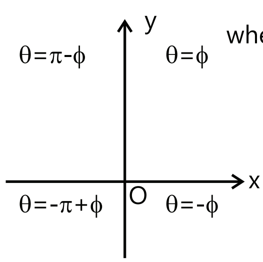
FORMULAE SHEET

(a) Complex number $z = x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$.

(b) If z = x + iy then its conjugate $\bar{z}$ = x – iy.

(c) Modulus of z, i.e. | z | = $\sqrt{x^2 + y^2}$

(d) Argument of z, i.e. $\theta = \begin{cases} \tan^{-1}\left|\frac{y}{x}\right| & x > 0, \quad y > 0 \\ \pi - \tan^{-1}\left|\frac{y}{x}\right| & x < 0, \quad y > 0 \\ -\pi + \tan^{-1}\left|\frac{y}{x}\right| & x < 0, \quad y < 0 \\ -\tan^{-1}\left|\frac{y}{x}\right| & x > 0, \quad y < 0 \end{cases}$



where, $\phi = \tan^{-1}\left|\frac{y}{x}\right|$

(e) If y=0, then argument of z, i.e. $\theta = \begin{cases} 0, & \text{if } x > 0 \\ \pi, & \text{if } x < 0 \end{cases}$

(f) If x=0, then argument of z, i.e. $\theta = \begin{cases} \frac{\pi}{2}, & \text{if } y > 0 \\ \frac{3\pi}{2}, & \text{if } y < 0 \end{cases}$

(g) In polar form $x = r\cos\theta$ and $y = r\sin\theta$, therefore $z = r(\cos\theta + i\sin\theta)$

(h) In exponential form complex number $z = re^{i\theta}$, where $e^{i\theta} = \cos\theta + i\sin\theta$.

(i) $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

(j) Important properties of conjugate

(i) $z + \bar{z} = 2\text{Re}(z)$ and $z - \bar{z} = 2\,\text{Im}(z)$

(ii) $z = \bar{z} \Leftrightarrow z$ is purely real

(iii) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary

(iv) $z\bar{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$

(v) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(vi) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(vii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(viii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ if $z_2 \neq 0$

(k) Important properties of modulus

If z is a complex number, then

(i) $|z| = 0 \Leftrightarrow z = 0$

(ii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iii) $-|z| \le \text{Re}(z) \le |z|$

(iv) $-|z| \le \text{Im}(z) \le |z|$

(v) $z\bar{z} = |z|^2$

If z_1, z_2 are two complex numbers, then

(i) $|z_1 z_2| = |z_1||z_2|$

(ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, if $z_2 \neq 0$

(iii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \bar{z}_1 z_2 + z_1 \bar{z}_2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1 \bar{z}_2)$

(iv) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - \bar{z}_1 z_2 - z_1 \bar{z}_2 = |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1 \bar{z}_2)$

(l) Important properties of argument

(i) $\arg(\bar{z}) = -\arg(z)$

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

In fact $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$

$$\text{where, } k = \begin{cases} 0, & \text{if } -\pi < \arg(z_1) + \arg(z_2) \le \pi \\ 1, & \text{if } -2\pi < \arg(z_1) + \arg(z_2) \le -\pi \\ -1, & \text{if } \pi < \arg(z_1) + \arg(z_2) \le 2\pi \end{cases}$$

(iii) $\arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2)$

(iv) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

(v) $|z_1 + z_2| = |z_1 - z_2| \quad \Leftrightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$

(vi) $|z_1 + z_2| = |z_1| + |z_2| \quad \Leftrightarrow \arg(z_1) = \arg(z_2)$

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

(vii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$

(viii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

(m) Triangle on complex plane

(i) Centroid (G), $z_G = \dfrac{z_1 + z_2 + z_3}{3}$

(ii) Incentre (I), $z_I = \dfrac{az_1 + bz_2 + cz_3}{a + b + c}$

(iii) Orthocentre (H), $z_H \dfrac{z_1\tan A + z_2\tan B + z_3\tan C}{\sum \tan A}$

(iv) Circumcentre (S), $z_S \dfrac{z_1(\sin 2A) + z_2(\sin 2B) + z_3(\sin 2C)}{\sin 2A + \sin 2B + \sin 2C}$

(n) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(o) $\sqrt{z} = \sqrt{x + iy} = \pm\left[\sqrt{\dfrac{|z| + x}{2}} + i\sqrt{\dfrac{|z| - x}{2}}\right]$ for $y > 0$

(p) Distance between $A(z_1)$ and $B(z_2)$ is given by $|z_2 - z_1|$

(q) Section formula: The point P (z) which divides the join of the segment AB in the ratio m : n is given by $z = \dfrac{mz_2 + nz_1}{m + n}$.

(r) Midpoint formula: $z = \dfrac{1}{2}(z_1 + z_2)$.

(s) Equation of a straight line

(i) Non-parametric form: $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$

(ii) Parametric form: $z = tz_1 + (1 - t)z_2$

(iii) General equation of straight line: $\bar{a}z + a\bar{z} + b = 0$

(t) Complex slope of a line, $\mu = \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$. Two lines with complex slopes μ_1 and μ_2 are

(i) Parallel, if $\mu_1 = \mu_2$

(ii) Perpendicular, if $\mu_1 + \mu_2 = 0$

(u) Equation of a circle: $|z - z_0| = r$

Solved Examples

JEE Main/Boards

Example 1: f z_1 and z_2 are 1 – i, – 2 + 4i respectively. Find $\text{Im}\left(\frac{z_1z_2}{\bar{z}_1}\right)$.

Sol: $\frac{z_1z_2}{\bar{z}_1} = \frac{(1-i)(-2+4i)}{1+i} = \frac{-2+2i+4i+4}{1+i}$

$= \frac{2+6i}{1+i}\times\frac{1-i}{1-i} = \frac{2+6i-2i+6}{2} = 4 + 2i$

$\therefore\ \text{Im}\left(\frac{z_1z_2}{\bar{z}_1}\right) = 2$.

Example 2: Find the square root of z = – 7 – 24i.

Sol: Consider $z_0 = x + iy$ be a square root then $z_0^{\,2} = -7-24i$.

$-7-24i = x^2 - y^2 + 2ixy$

Equating real and imaginary parts we get

$x^2 - y^2 = -7$ ... (i)

and 2xy = – 24 ... (ii)

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$= (-7)^2 + (-24)^2 = 625$

$\therefore\quad x^2 + y^2 = 25$ ... (iii)

Solving (i) and (iii), we get,

(x, y) = (3, – 4);(– 3, 4) by (ii)

$\therefore\quad z_0 = \pm(3-4i)$.

Example 3: If n is a positive integer and ω be an imaginary cube root of unity, prove that

$1 + \omega^n + \omega^{2n} \begin{cases} 3, \text{ when n is a mulitple of 3} \\ 0, \text{ when n is not a mulitple of 3} \end{cases}$

Sol: Case I: n = 3m; m $\in$ I

$\therefore 1+\omega^n+\omega^{2n} = 1+\omega^{3m}+\omega^{6m}$

= 1 + 1 + 1 [$\because \omega^3 = 1$] = 3

Case II: n = 3m + 1 or 3m + 2; m $\in$ I

(a) Let n = 3m + 1

$\therefore\ \text{L.H.S} = 1+\omega^{3m+1}+\omega^{6m+2} = 1 + \omega+\omega^2 = 0$

(b) Let n = 3m + 2

$1+\omega^{3m+2}+\omega^{6m+4} = 1+\omega^2+\omega^4 = 1+\omega^2+\omega = 0$.

Example 4: Show that $\left|\frac{z-3}{z+3}\right| = 2$ represents a circle.

Sol: Consider z = x + iy and then by taking modulus we will get the result.

Let z = x + iy

$\therefore \left|\frac{z-3}{z+3}\right| = 2 \Rightarrow \left|\frac{x-3+iy}{x+3+iy}\right| = 2$

$\therefore |x-3+iy|^2 = 2^2\,|x+3+iy|^2$

or $(x-3)^2 + y^2 = 4\left((x+3)^2+y^2\right)$

$\Rightarrow 3x^2+3y^2+30x+27 = 0$

which represents a circle.

Example 5: If $|z_1| = |z_2| = \ldots\ldots = |z_n| = 1$

prove that $|z_1+z_2+\ldots\ldots+z_n| = \left|\frac{1}{z_1}+\frac{1}{z_2}+\ldots\ldots+\frac{1}{z_n}\right|$

Sol: $|z_j| = 1 \quad \Rightarrow z_j\bar{z}_j = 1\ \forall\ j = 1, \ldots\ldots, n$

($\because \quad z\bar{z} = |z^2|$)

L.H.S.

$|z_1+z_2+\ldots\ldots+z_n| = \left|\frac{1}{\bar{z}_1}+\frac{1}{\bar{z}_2}+\ldots\ldots+\frac{1}{\bar{z}_n}\right| =$

$\left|\overline{\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots\ldots+\frac{1}{z_n}}\right|$

$= \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots\ldots+\frac{1}{z_n}\right|$ = R.H.S.

Example 6: If $|z_1+z_2| = |z_1-z_2|$, prove that

$\arg z_1 - \arg z_2$ = odd multiple of $\frac{\pi}{2}$.

Sol: As we know $|z| = z.\bar{z}$. Apply this formula and consider $z = r(\cos\theta + i\sin\theta)$.

$|z_1 + z_2|^2 = |z_1 - z_2|^2$

$\Rightarrow (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$ or

$z_1\bar{z}_1 + z_2\bar{z}_2 + z_2\bar{z}_1 + z_1\bar{z}_2 = z_1\bar{z}_1 + z_2\bar{z}_2 - z_2\bar{z}_1 - z_1\bar{z}_2$

or $\quad 2(z_2\bar{z}_1 + z_1\bar{z}_2) = 0$; $\operatorname{Re}(z_1\bar{z}_2) = 0$

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$; then $z_1\bar{z}_2 = r_1r_2\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$

$\therefore \cos(\theta_1 - \theta_2) = 0 \left(\text{as } \operatorname{Re}(z_1\bar{z}_2) = 0\right)$

$\theta_1 - \theta_2 =$ odd multiple of $\dfrac{\pi}{2}$.

Example 7: If | z – 1 | < 3, prove that $|iz + 3 - 5i| < 8$.

Sol: Here we have to reduce iz + 3 – 5i as the sum of two complex numbers containing z – 1. because we have to use

| z – 1 | < 3.

| iz + 3 – 5i | = | iz – i + 3 – 4i |

= | 3 – 4i + i (z – 1) | $\le |3 - 4i| + |i(z-1)|$

(by triangle inequality) < 5 + 1 · 3 = 8

Example 8: If $(1 + x)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots + a_nx^n$, then show that

(a) $a_0 - a_2 + a_4 + \ldots\ldots = 2^{\frac{n}{2}}\cos\dfrac{n\pi}{4}$

(b) $a_1 - a_3 + a_5 + \ldots = 2^{\frac{n}{2}}\sin\dfrac{n\pi}{4}$

Sol: Simply put x = i in the given expansion and then by using formula

$z = r(\cos\theta + i\sin\theta)$ and $(\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$, we can solve this problem.

Put x = i in the given expansion

$(1 + i)^n = a_0 + a_1i + a_2i^2 + \ldots + a_ni^n$.

$$\left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n$$

$= (a_0 - a_2 + a_4 - \ldots) + i\,(a_1 - a_3 + a_5 - \ldots.)$

$$2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

$= (a_0 - a_2 + a_4 + \ldots\ldots.) + i(a_1 - a_3 + a_5 + \ldots..)$

Equating real and imaginary parts.

$2^{\frac{n}{2}}\cos\dfrac{n\pi}{4} = a_0 - a_2 + a_4 + \ldots..$

$2^{\frac{n}{2}}\sin\dfrac{n\pi}{4} = a_1 - a_3 + a_5 + \ldots..$

Example 9: Solve the equation $z^{n-1} = \bar{z} : n \in N$

Sol: Apply modulus on both side.

$Z^{n-1} = \bar{z}$; $\quad |z|^{n-1} = |\bar{z}| = |z|$

$\therefore$ |z| = 0 or |z| = 1 If |z| = 0 then z = 0,

Let |z| = 1; then, $z^n = z\bar{z} = 1$

$\therefore\ z = \cos\dfrac{2m\pi}{n} + i\sin\dfrac{2m\pi}{n}$: m = 0, 1, ..., n – 1

Example 10: If z = x + iy and $\omega = \dfrac{1 - iz}{z - i}$ with $|\omega| = 1$, show that, z lies on the real axis.

Sol: Substitute value of ω in $|\omega| = 1$.

$|\omega| = \left|\dfrac{1 - iz}{z - i}\right| = 1 \Rightarrow |1 - iz| = |z - i|$

or, |1 – ix + y| = |x + i(y – 1)|

or, $(1 + y)^2 + x^2 = x^2 + (y - 1)^2$ or, 4y = 0

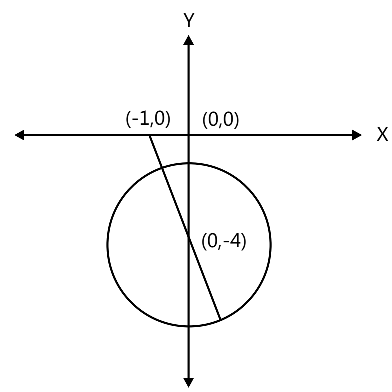
Hence z lies on the real axis.

Example 11: If a complex number z lies in the interior or on the boundary of a circle of radius as 3 and centre at (0, – 4) then greatest and least value of |z + 1| are-

(A) $3 + \sqrt{17}, \sqrt{17} - 3$ (B) 6, 1

(C) $\sqrt{17}$, 1 (D) 3, 1

Sol: Greatest and least value of |z + 1| means maximum and minimum distance of circle from the point (– 1, 0). In circle greatest and least distance of it from any point is along the normal.

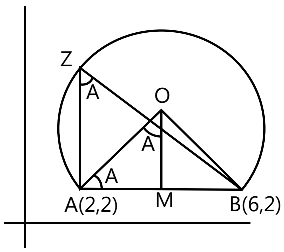


$\therefore$ Greatest distance = $3+\sqrt{1^2+4^2} = 3+\sqrt{17}$

Least distance = $\sqrt{1^2+4^2}-3 = \sqrt{17}-3$

Example 12: Find the equation of the circle for which $\arg\left(\dfrac{z-6-2i}{z-2-2i}\right)=\pi/4$.

Sol: $\arg\left(\dfrac{z-6-2i}{z-2-2i}\right)=\pi/4$ represent a major arc of circle of which Line joining (6, 2)) and (2, 2) is a chord that subtends an angle $\dfrac{\pi}{4}$ at circumference.



Clearly AB is parallel to real (x) axis, M is mid-point,

$M \equiv (4, 2)$, OM = AM = 2

$\therefore$ O = (4, 4) and $OA^2 = OM^2 + AM^2 = 2\sqrt{2}$ Equation of required circle is

$|z-4-4i|=2\sqrt{2}$

Example 13: If $|z| \geq 3$, prove that the least value of $\left|z+\dfrac{1}{z}\right|$ is $\dfrac{8}{3}$.

Sol: Here $\left|z+\dfrac{1}{z}\right| \geq |z|-\dfrac{1}{|z|}$.

Now $|z| \geq 3$

$\therefore \dfrac{1}{|z|}\leq\dfrac{1}{3}$ or $-\dfrac{1}{|z|}\geq-\dfrac{1}{3}$ ... (i)

Adding the two like inequalities

$|z|-\dfrac{1}{|z|}\geq 3-\dfrac{1}{3}=\dfrac{8}{3}$ ... (ii)

Hence from (i) and (ii), we get $\left|z+\dfrac{1}{z}\right|\geq\dfrac{8}{3}$

$\therefore$ Least value is $\dfrac{8}{3}$

Example 14: If z_1, z_2, z_3 are non-zero complex numbers such that $z_1 + z_2 + z_3 = 0$ and $z_1^{-1}+z_2^{-1}+z_3^{-1}=0$ then prove that the given points are the vertices of an equilateral triangle. Also show that $|z_1| = |z_2| = |z_3|$.

Sol: Use algebra to solve this problem.

Given $z_1 + z_2 + z_3 = 0$, and from 2nd relation $z_2z_3 + z_3z_1 + z_1z_2 = 0$

$\therefore z_2z_3 = -z_1(z_2 + z_3) = -z_1(-z_1) = z_1^2$

$\therefore z_1^3 = z_1z_2z_3 = z_2^3 = z_3^3$

$\therefore |z_1|^3 = |z_2|^3 = |z_3|^3$

Above shows that distance of origin from A, B, C is same.

Origin is circumcentre, but $z_1 + z_2 + z_3 = 0$

implies that centroid is also at the origin so that the triangle must be equilateral.

JEE Advanced/Boards

Example 1: For constant $c \geq 1$, find all complex numbers z satisfying the equation $z + c|z + 1| + i = 0$

Sol: Solve this by putting z = x + iy.

Let z = x + iy.

The equation $z + c|z + 1| + i = 0$ becomes

$x + iy + c\sqrt{(x+1)^2+y^2}+i=0$

or $x + c\sqrt{(x+1)^2+y^2}+i(y+1)=0$

Equating real and imaginary parts, we get

$y + 1 = 0 \Rightarrow y = -1$ ... (i)

and $x + c\sqrt{(x+1)^2+y^2}=0 : x < 0$ ...(ii)

Solving (i) and (ii), we get

$x + c\sqrt{(x+1)^2+1}=0$ or $x^2 = c^2[(x + 1)^2 + 1]$

or $(c^2 - 1)x^2 + 2c^2x + 2c^2 = 0$

If c =1, then x = – 1. Let c > 1 ; then,

$$x=\frac{-2c^2\pm\sqrt{4c^4-8c^2(c^2-1)}}{2(c^2-1)}=\frac{-c^2\pm c\sqrt{2-c^2}}{c^2-1}$$

As x is real and c > 1, we have: $1 < c \leq \sqrt{2}$

(Thus, for $c > \sqrt{2}$, there is no solution). Since both values of x satisfy (ii), both values are admissible.

Example 2: Find the sixth roots of z = 64i.

Sol: Here i = $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$ and sixth root of z i.e. $z_r = z^{1/6}$.

$$z = 64\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) \qquad \therefore z_r = z^{1/6}$$

$$= 2\left[\cos\frac{2r\pi+\frac{\pi}{2}}{6}+i\sin\frac{2r\pi+\frac{\pi}{2}}{6}\right]$$

Where r = 0, 1, 2, 3, 4, 5

The roots $z_0, z_1, z_2, z_3, z_4, z_5$ are given by

$$z_0 = 2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$

$$z_1 = 2\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$$

$$z_2 = 2\left(\cos\frac{9\pi}{12}+i\sin\frac{9\pi}{12}\right)$$

$$z_3 = 2\left(\cos\frac{13\pi}{12}+i\sin\frac{13\pi}{12}\right) = -2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$

$$z_4 = 2\left(\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}\right) = -2\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$$

$$z_5 = 2\left(\cos\frac{21\pi}{12}+i\sin\frac{21\pi}{12}\right) = -2\left(\cos\frac{9\pi}{12}+i\sin\frac{9\pi}{12}\right)$$

Example 3: Locate the region in the Argand plane for the complex number z satisfying

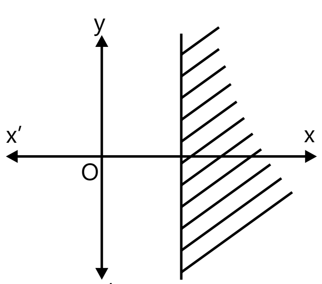
(a) $|z-4| < |z-2|$ (b) $\frac{\pi}{6} \le \arg z \le \frac{\pi}{4}$

Sol: Consider z = x + iy and solve by using properties of modulus and argument.

(a) Let z = x + iy

$|x + iy - 4| < |x + iy - 2|$

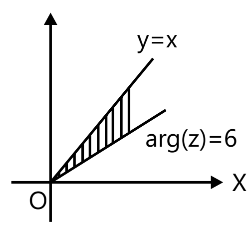
$(x-4)^2 + y^2 < (x-2)^2 + y^2$ or $-4x + 12 < 0$



Re(z) > 3 (see the Figure above)

(b) Let z = x + iy, then, x > 0 and y > 0

$$\arg z = \tan^{-1}\frac{y}{x} \quad \tan\frac{\pi}{6} \le \frac{y}{x} \le \tan\frac{\pi}{4}$$



$$\frac{1}{\sqrt{3}} \le \frac{y}{x} \le 1; \quad x \le \sqrt{3}y \text{ and } y \le x$$

Hence the given inequality represents the region bounded by the rays y = x and $y = \frac{1}{\sqrt{3}}x$ except the origin.

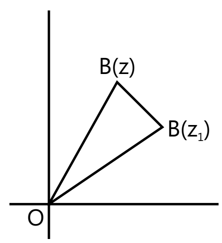
Example 4: If $z_1^2 + z_2^2 - 2z_1z_2\cos\theta = 0$, show that the points z_1, z_2 and the origin, in the argand plane, are the vertices of an isosceles triangle.

Sol: By using formula of roots of quadratic equation we can solve it.

$z_1^2 + z_2^2 - 2z_1z_2\cos\theta = 0$

$$\Rightarrow \left(\frac{z_1}{z_2}\right)^2 - 2\left(\frac{z_1}{z_2}\right)\cos\theta + 1 = 0$$

$$\Rightarrow \left(\frac{z_1}{z_2}\right) = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$



$= \cos\theta \pm i\sin\theta$

$$\Rightarrow \left|\frac{z_1}{z_2}\right| = |\cos\theta \pm i\sin q| - 1$$

$$\Rightarrow \left|\frac{z_1}{z_2}\right| = 1 \Rightarrow |z_1| = |z_2| \text{ or } OA = OB$$

Hence points A(z), B(z) and the origin are the vertices of an isosceles triangle.

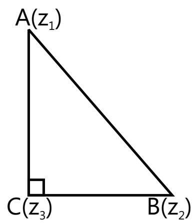
Example 5: Let three vertices A, B, C (taken in clock wise order) of an isosceles right angled triangle with right angle at C, be affixes of complex numbers z_1, z_2, z_3 respectively. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Sol: Here $\frac{z_2 - z_3}{z_1 - z_3} = e^{-i\pi/2}$. Therefore solve it using algebra method.

Given CB = CA and angle $\angle C = \frac{\pi}{2}$

$\frac{z_2 - z_3}{z_1 - z_3} = e^{-i\pi/2}$ or $(z_3 - z_2)^2 = i^2(z_1 - z_3)^2$

$(z_3 - z_2)^2 = -(z_1 - z_3)^2$



or

$z_3^2 + z_2^2 - 2z_2z_3 + z_1^2 + z_3^2 - 2z_1z_3 = 0$

Add and subtract $2z_1z_2$, we get

$z_1^2 + z_2^2 - 2z_1z_2 + 2z_3^2 - 2z_2z_3 - 2z_1z_3 + 2z_1z_2 = 0$, or

$(z_1 - z_2)^2 + 2[z_3(z_3 - z_2) - z_1(z_3 - z_2)] = 0$ or

$(z_1 - z_2)^2 + 2(z_3 - z_1)(z_3 - z_2) = 0$, or

$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Example 6: If A, B, C be the angles of triangle then prove that $\begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix}$ is purely real.

Sol: Here $A+B+C = \pi$, therefore $e^{pi} = \cos\pi + i\sin\pi = -1$. And by using properties of matrices we can solve this problem.

$e^{-pi} = -1$ ... (i)

$e^{i(B+C)} = e^{i(\pi-A)} = e^{pi}\, e^{-iA} = -e^{-iA}$

$e^{-i(B+C)} = -e^{iA}$ ... (ii)

Take e^{iA}, e^{iB} and e^{iC} common from R_1, R_2 and R_3 respectively. $\Delta = e^{i(A+B+C)}$

$$\begin{vmatrix} e^{iA} & e^{-i(A+C)} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{iB} & e^{-i(B+A)} \\ e^{-i(B+C)} & e^{-i(C+A)} & e^{iC} \end{vmatrix}$$

$$= -1\begin{vmatrix} e^{iA} & -e^{iB} & -e^{iC} \\ -e^{iA} & e^{iB} & -e^{iC} \\ -e^{iA} & -e^{iB} & e^{iC} \end{vmatrix}, \text{ by (2)}$$

Take e^{iA}, e^{iB} and e^{iC} common from C_1, C_2 and C_3 and again put $e^{i(A+B+C)} = e^{i\pi} = -1$.

$$\therefore \Delta = (-1)(-1)\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

Now make two zeros and expand

$\Delta = -4$ which is purely real.

Example 7: Prove that $|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2)$. Interpret the result geometrically and deduce that $\left|c + \sqrt{c^2 - d^2}\right| + \left|c - \sqrt{c^2 - d^2}\right| = |c + d| + |c - d|$; all numbers involved being complex

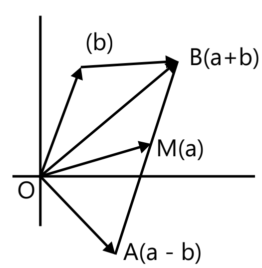
Sol: By using algebra of complex number and modulus property we can prove given expresson. And then by using Appolonius theorem we can interpret the result geometrically.

$S = |a + b|^2 + |a - b|^2$

$= |a|^2 + |b|^2 + 2\text{Re}(a\bar{b}) + |a|^2 + |b|^2 - 2\text{Re}(a\bar{b}) = 2(|a|^2 + |b|^2)$ (proved)

Now $|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2)$

This is nothing but Appolonius theorem. In ΔOAB, M is midpoint of AB on applying Appolonius theorem we get



$OA^2 + OB^2 = 2(AM^2 + OM^2)$

Now take $a = \sqrt{\frac{c+d}{2}}$ and $b = \sqrt{\frac{c-d}{2}}$

Then using result

$|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2)$

$$\text{RHS} = 2\left(\left|\sqrt{\frac{c+d}{2}}\right|^2 + \left|\sqrt{\frac{c-d}{2}}\right|^2\right) = |c + d| + |c - d|$$

$$\text{L.H.S.} = \left(\left|\sqrt{\frac{c+d}{2}}+\sqrt{\frac{c-d}{2}}\right|\right)^2 + \left(\left|\sqrt{\frac{c+d}{2}}-\sqrt{\frac{c-d}{2}}\right|\right)^2$$

On simplifying we get

$$\text{L.H.S.} = \left|c+\sqrt{c^2-d^2}\right| + \left|c-\sqrt{c^2-d^2}\right|$$

Example 8: Show that the triangles whose vertices are z_1, z_2, z_3 and a, b, c are

$$\text{similar if } \begin{vmatrix} z_1 & a & 1 \\ z_2 & b & 1 \\ z_3 & c & 1 \end{vmatrix} = 0.$$

Sol: Consider triangle ABC and DEF are similar, therefore $\frac{AB}{DE} = \frac{BC}{EF}$ and $\angle ABC = \angle DEF$.

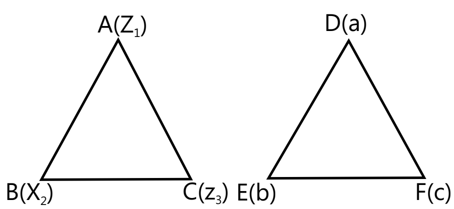
Suppose z_1, z_2, z_3 are given by A, B, C respectively and a, b, c are given by D, E, F respectively. Since the triangle ABC and DEF are similar $\frac{AB}{DE} = \frac{BC}{EF}$ and

$\angle ABC = \angle DEF = \alpha$(say)

We have $\angle B = \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \arg\left(\frac{a-b}{c-b}\right)$

$$\Rightarrow \frac{z_1 - z_2}{z_3 - z_2} = \frac{AB}{BC}(\cos\alpha + i\sin\alpha) \quad \text{... (i)}$$

$$\text{and } \frac{a-b}{c-b} = \frac{DE}{EF}(\cos\alpha + i\sin\alpha) \quad \text{... (ii)}$$



Since $\frac{AB}{DE} = \frac{BC}{EF}$ we have $\frac{AB}{BC} = \frac{DE}{EF}$

Thus, from (i) and (ii) we get

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{a-b}{c-b} \Rightarrow \begin{vmatrix} z_1 - z_2 & a-b \\ z_3 - z_2 & c-b \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} z_1 - z_2 & a-b & 0 \\ z_2 & b & 1 \\ z_3 - z_2 & c-b & 0 \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_2$ and $R_3 \to R_3 + R_2$

$$\text{we get } \begin{vmatrix} z_1 & a & 1 \\ z_2 & b & 1 \\ z_3 & c & 1 \end{vmatrix} = 0$$

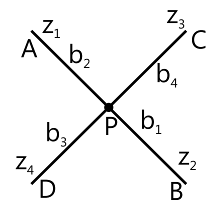
Example 9: If $b_1 + b_2 + b_3 + b_4 = 0$ where b_1 etc. are non-zero real numbers, sum of no two being zero, and $b_1z_1 + b_2z_2 + b_3z_3 + b_4z_4 = 0$ where no three of the points z_1, z_2, z_3, z_4 are collinear then prove that the four point concyclic if $b_1b_2\,|z_1 - z_2|^2 = b_3b_4\,|z_3 - z_4|^2$.

Sol: Here the four points A, B, C, D will be concyclic if PA.PB = PC.CD. therefore obtain PA, PB, PC and CD and simplify.

Here $b_1 + b_2 = -(b_3 + b_4)$

Also $b_1z_1 + b_2z_2 = -(b_3z_3 + b_4z_4)$

Dividing these, $\frac{b_1z_1 + b_2z_2}{b_1 + b_2} = \frac{b_3z_3 + b_4z_4}{b_3 + b_4}$



The left side gives the point that divides the line segment joining $A(z_1)$, $B(z_2)$ in the ratio $b_2 : b_1$ and the right side gives the point that divides the line segment joining the points $C(z_3)$, $D(z_4)$ in the ratio $b_4 : b_3$. So the line segments intersect at P which is

Represented by $\frac{b_1z_1 + b_2z_2}{b_1 + b_2}$ as well as $\frac{b_3z_3 + b_4z_4}{b_3 + b_4}$

Now, $AB = |z_1 - z_2|$

$$\therefore\ PA = \frac{b_2}{b_1 + b_2}(z_1 - z_2)\ \&\ PB = \frac{b_1}{b_1 + b_2}(z_1 - z_2)$$

Also, $CD = |z_3 - z_4|$

$$\therefore\ PC = \frac{b_4}{b_3 + b_4}(z_3 - z_4)\ \&\ PD = \frac{b_3}{b_3 + b_4}(z_3 - z_4)$$

The four points A, B, C, D will be concyclic if PA.PB = PC.CD

$$\text{i.e. } \frac{b_1b_2}{(b_1 + b_2)^2}|z_1 - z_2|^2 = \frac{b_3b_4}{(b_3 + b_4)^2}|z_3 - z_4|^2$$

i.e. $b_1b_2\,|z_1 - z_2|^2 = b_3b_4\,|z_3 - z_4|^2$

$\{\because b_1 + b_2 = -(b_3 + b_4)\}$

Example 10: Show that all the roots of the equation $z^n \cos q_0 + z^{n-1} \cos q_1 + z^{n-2} \cos q_2 + \ldots\ldots\ldots + z \cos q_{n-1} + \cos q_n = 2$ lie outside the circle $|z| = \frac{1}{2}$ where q_0, q_1 etc. are real.

Sol: By using triangle inequality.

Here $|z^n \cos q_0 + z^{n-1} \cos q_1 + z^{n-2} \cos q_2 + \ldots\ldots + z \cos q_{n-1} + \cos q_n| = 2$ .... (i)

By triangle inequality.

$2 = |z^n \cos \theta_n + 2^{n-1} \cos q_1 + 2^{n-1} \cos q_2 + \ldots.. + z$

$\cos q_{n-1} + \cos q_n| \le |z^n \cos q_n| + |z^{n-1} \cos q_1| +$

$|z^{n-2} \cos q_2| + \ldots..+ + |z \cos q_{n-1}| + |\cos q_n|$

$= |z_n|\,|\cos q_n| + |z^{n-1}|\,|\cos q_n| + \ldots\ldots.. + |z|\,|\cos q_{n-1}| +$

$|\cos q_n| \le |z|^n + |z|^{n-1} + \ldots. + |z| + 1$

$(\because |\cos q_1| \le 1 \text{ and } |z^{n+1}| = |z|^{n+1})$

$= \dfrac{1-|z|^{n+1}}{1-|z|} < \dfrac{1}{1-|z|} \quad \therefore\ 2 < \dfrac{1}{1-|z|}$

So $1 - |z|$ is positive and $1-|z| < \dfrac{1}{2}$

$\therefore |z| > 1 - \dfrac{1}{2} = \dfrac{1}{2}$

$\therefore$ All z satisfying (i) lie outside the circle $|z| = \dfrac{1}{2}$

Example 11: If $z + \dfrac{1}{z} = 2\cos\theta$, prove that $\left|\dfrac{z^{2n}-1}{z^{2n}+1}\right| = |\tan nq|$.

Sol: By using formula of roots of quadratic equation, we can solve this problem.

Here $z + \dfrac{1}{z} = 2 \cos \theta$;

$\therefore z^2 - 2 \cos \theta.\ z + 1 = 0$

$\therefore\ z = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos \theta \pm i \sin \theta$

Taking positive sign, $z = \cos \theta + i \sin \theta$

$\therefore\ \dfrac{1}{z} = (\cos \theta + i \sin \theta)^{-1} = \cos \theta - i \sin \theta$

$\therefore\ \dfrac{z^{2n}-1}{z^{2n}+1} = \dfrac{z^n - \dfrac{1}{z^n}}{z^n + \dfrac{1}{z^n}} = \dfrac{(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n}{(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n}$

$= \dfrac{\cos n\theta + i\sin\theta - (\cos n\theta - i\sin n\theta)}{\cos n\theta + i\sin n\theta + (\cos n\theta - i\sin n\theta}$

$= \dfrac{2i \sin n\theta}{2\cos n\theta} = i\tan n\theta$. Taking negative sign,

similarly we get $\dfrac{z^{2n}-1}{z^{2n}+1} = \dfrac{-2i\sin n\theta}{2\cos n\theta} = -\, i \tan n\theta$

$\therefore\ \left|\dfrac{z^{2n}-1}{z^{2n}+1}\right| = |\pm i \tan nq| = |\tan nq|,$

For $|\pm i| = 1$.

Example 12: Find the complex number z which satisfies the condition $|z - 2 + 2i| = 1$ and has the least absolute value.

Sol: Here $z - 2 + 2i = \cos \theta + i \sin \theta$, therefore by obtaing modulus of z we can solve above problem.

$|z - 2 + 2i| = 1$

$\Rightarrow z - 2 + 2i = \cos \theta + i \sin \theta$

Where θ is some real number.

$\Rightarrow z = (2 + \cos \theta) + (\sin \theta - 2)i$

$\Rightarrow |z| = [(2 + \cos \theta)^2 + (\sin \theta - 2)^2]^{1/2}$

$= [8 + \cos^2 \theta + \sin^2 \theta + 4(\cos \theta - \sin \theta)]^{1/2}$

$= \left[9 + 4\sqrt{2}\cos\left(\theta + \dfrac{\pi}{4}\right)\right]^{1/2}$

$|z|$ will be least if $\cos (\theta + \pi/4)$ is least, that is, if $\cos (\theta + \pi/4) = -1$ or $\theta = \dfrac{3\pi}{4}$. Thus, least value of $|z|$ is

$\left(9 - 4\sqrt{2}\right)^{1/2}$ for $z = \left(2 - \dfrac{1}{\sqrt{2}}\right) + i\left(\dfrac{1}{\sqrt{2}} - 2\right)$

Example 13: For every real number $c \ge 0$, find all the complex numbers z which satisfy the equation. $2|z| - 4cz + 1 + ic = 0$.

Sol: Substitute $z = x + iy$ and equate real and imaginary part to zero.

$2\sqrt{x^2 + y^2} - 4c\,(x + iy) + 1 + ic = 0$

$\therefore -4\,cy + c = 0 \Rightarrow y = \dfrac{1}{4}$ ... (i)

$2\sqrt{x^2 + \dfrac{1}{16}} - 4cx + 1 = 0$ or

$4\left(x^2 + \dfrac{1}{16}\right) = (4\,cx - 1)^2$

$4x^2\,(4c^2 - 1) - 8\,cx + \dfrac{3}{4} = 0$

$\therefore\ x = \dfrac{8c \pm \sqrt{64c^2 - 12(4c^2 - 1)}}{8(4c^2 - 1)}$

or $x = \dfrac{4c \pm \sqrt{4c^2 + 3}}{4(4c^2 - 1)}$ ... (ii)

x is real as $c \geq 0$, z = (x, y) as given by (i) and (ii), $c \geq 0$.

Example 14: Consider a square ABCD such that z_1, z_2, z_3, and z_4 represent its vertices A, B, C and D respectively. Express 'z_3' and 'z_4' in terms of z_1 & z_2.

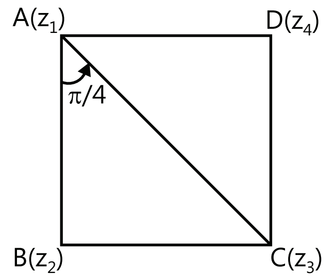
Sol: Consider the rotation of AB about A through an angle $\dfrac{\pi}{4}$.

Therefore $\dfrac{z_3 - z_1}{z_2 - z_1} = \left|\dfrac{z_3 - z_1}{z_2 - z_1}\right| e^{i\frac{\pi}{4}}$.

$\dfrac{z_3 - z_1}{z_2 - z_1} = \left|\dfrac{z_3 - z_1}{z_2 - z_1}\right| e^{i\frac{\pi}{4}} = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

$\Rightarrow z_3 = z_1 + (z_2 - z_1)(1 + i)$

Similarly $z_4 = z_1 + i(z_3 - z_1)$



JEE Main/Boards

Exercise 1

Q.1 Find all non-zero complex numbers z satisfying $\bar{z} = iz^2$.

Q.2 Express $\dfrac{1 + 2i + 3i^2}{1 - 2i + 3i^2}$ in the form A + iB.

Q.3 Find x and y if (x + iy) (2 – 3i) =(4 + i)

Q.4 Find x and y if $\dfrac{(1+i)x - 2i}{3+i} + \dfrac{(2-3i)y + i}{3-i} = i$

Q.5 If x = a + b, y = aα + bβ and z = aβ +bα, where α and β are complex cube roots of unity, show that xyz = $a^3 + b^3$.

Q.6 $\dfrac{1+7i}{(2-i)^2}$ in the polar form.

Q.7 Find the square root of – 8 – 6i.

Q.8 Find the value of smallest positive integer n, for which $\left(\dfrac{1+i}{1-i}\right)^n = 1$.

Q.9 Show that the complex number z = x + iy which satisfies the equation $\left|\dfrac{z - 5i}{z + 5i}\right| = 1$ lies on the x-axis.

Q.10 If z = 1 + i tan α, where $\pi < \alpha < \dfrac{3\pi}{2}$. find the value of |z| cos α.

Q.11 If 1, ω, ω^2 be the cube roots of unity, find the roots of the equation $(x - 1)^3 + 8 = 0$.

Q.12 If |z| < 4, prove that |i z + 3 – 4i| < 9.

Q.13 2 + i 3 is a vertex of square inscribed in circle |z – 1| =2. Find other vertices.

Q.14 Find the centre and radius of the circle formed by the points represented by z = x +iy satisfying the relation $\dfrac{|z - \alpha|}{|z - \beta|}$ = k(k ≠ 1) where α & β are constant complex number's given by $\alpha = \alpha_1 + i\alpha_2$ & $\beta = \beta_1 + i\beta_2$

Q.15 Prove that there exists no complex number z such that $|z| < \dfrac{1}{3}$ and $\sum_{r=1}^{a} a_r z^r = 1$ where $|a_r| < 2$.

Q.16 Let a complex number α, α ≠ 11, be a root of the equation zp + q – zp – zq + 1 = 0, where p, q are distinct primes. Show that either $1 + \alpha + \alpha^2 + \ldots\ldots + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + \ldots\ldots + \alpha^{q-1} = 0$, but not both together.

Q.17 Show that the area of the triangle on the Argand diagram formed by the complex numbers: z, iz and z + iz is: $\frac{1}{2}|z|^2$.

Q.18 If $iz^3 + z^2 - z + i = 0$ then show that $|z| = 1$.

Q.19 Find the value of the expression

$1(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) + ...$

$+ (n-1)(n-\omega)(n-\omega^2)$ where ω is an imaginary cube root of unity.

Q.20 If $x = \frac{1}{2}\left(5-\sqrt{3}i\right)$, then find the value of $x^4 - x^3 - 12x^2 + 23x + 12$.

Q.21 Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Then prove that: $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

Q.22 If z_1, z_2, z_3 are the vertices of an isosceles triangle, right angled at z_2, prove that $z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$.

Q.23 Show that the equation

$$\frac{A^2}{x-a}+\frac{B^2}{x-b}+\frac{C^2}{x-c}+......\frac{H^2}{x-h} = x+\ell,$$

Where A, B, C, ..., a, b, c, ... and ℓ are real, cannot have imaginary roots.

Q.24 Find the common roots of the equation

$z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$.

Q.25 If n is an odd integer greater than 3 but not a multiple of 3, prove that $[(x+y)^n - x^n - y^n]$ is divisible by $xy(x+y)(x^2+xy+y^2)$.

Q.26 If α and β are any two complex numbers,

show that $\left|\alpha+\sqrt{\alpha^2-\beta^2}\right| + \left|\alpha-\sqrt{\alpha^2-\beta^2}\right|$

$= |\alpha+\beta| + |\alpha-\beta|$

Q.27 Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $\frac{z-z_1}{z-z_2}$ is $\frac{\pi}{4}$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$.

Q.28 If $|z| \le 1$, $|w| \le 1$, show that

$|z-w|^2 \le (|z|-|w|)^2 + (\arg z - \arg w)^2$.

Q.29 Let A and B be two complex numbers such that $\frac{A}{B}+\frac{B}{A} = 1$, prove that the origin and the points represented by A and B form the vertices of an equilateral triangle.

Q.30 Let z_1, z_2, z_3 be three complex numbers and a, b, c be real number not all zero, such that a + b + c = 0 and $az_1 + bz_2 + cz_3 = 0$.

Show that z_1, z_2, z_3 are collinear.

Q.31 If $|z - 4 + 3i| \le 2$, find the least and the greatest values of |z| and hence find the limits between which |z| lies.

Q.32 If $|z_1| < 1$ and $\left|\frac{z_1-z_2}{1-\bar{z}_1z_2}\right| < 1$, then show that $|z_2| < 1$.

Q.33 Find the locus of points z if $\log_{\sqrt{3}}\frac{|z|^2-|z|+1}{2+|z|} < 2$.

Q.34 For complex numbers z and ω, prove that $|z|^2\omega - |\omega|^2 z = z - \omega$ if and only if z = w or $z\bar{\omega} = 1$.

Exercise 2

Single Correct Choice Type

Q.1 $|z + 4| \le 3$, $Z \in$ C: then the greatest and least value of $|z + 1|$ are:

(A) (7, 1) (B) (6, 1) (C) (6, 0) (D) None

Q.2 The maximum & minimum values of $|z + 1|$ when $|z + 3| \le 3$ are

(A) (5, 0) (B) (6, 0) (C) (7, 1) (D) (5, 1)

Q.3 The points $z_1 = 3 + \sqrt{3}i$ and $z_2 = 2\sqrt{3}+6i$ are given on a complex plane. The complex number lying on the bisector of the angle formed by the vectors z_1 and z_2 is:

(A) $z = \frac{(3+2\sqrt{3})}{2}+\frac{\sqrt{3}+2}{2}i$

(B) z = 5 + 5i

(C) z = – 1 – I

(D) None of these

Q.4 If z_1, z_2, z_3, z_4 are the vertices of a square in that order, then which of the following do(es) not hold good?

(A) $\frac{z_1 - z_2}{z_3 - z_2}$ is purely imaginary

(B) $\frac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary

(C) $\frac{z_1 - z_2}{z_3 - z_4}$ is purely imaginary

(D) None of these

Q.5 Let z_1, z_2 and z_3 be the complex numbers representing the vertices of a triangle ABC respectively and a, b, c are lengths of BC, CA, AB. If P is a point representing the complex number z_0 satisfying: $a(z_1 - z_0) + b(z_2 - z_0) + c(z_3 - z_0) = 0$, then w.r.t. the triangle ABC, the point P is its:

(A) Centroid (B) Orthocentre

(C) Circumcentre (D) Incentre

Q.6 Three complex numbers α, β & γ are represented in the Argand diagram by the three points A, B, C respectively. The complex number represented by D where A, B, C, D form a parallelogram with BD on a diagonal is:

(A) $\alpha - \beta + \gamma$ (B) $-\alpha + \beta + \gamma$

(C) $\alpha + \beta - \gamma$ (D) $\alpha - \beta - \gamma$

Q.7 If the complex number z satisfies the condition $|z| \geq 3$, then the least value of $\left|z + \frac{1}{z}\right|$ is

(A) $\frac{5}{3}$ (B) $\frac{8}{3}$ (C) $\frac{11}{3}$ (D) None of these

Q.8 Point z_1 & z_2 are adjacent vertices of a regular octagon. The vertex z_3 adjacent to $z_2 (z_3 \neq z_1)$ can be represented by:

(A) $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_1 + z_2)$

(B) $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_1 - z_2)$

(C) $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_2 - z_1)$

(D) None of these

Q.9 If q_1, q_2, q_3 are the roots of the equation, $x^3 + 64 = 0$, then the value of the determinant $\begin{vmatrix} q_1 & q_2 & q_3 \\ q_2 & q_3 & q_1 \\ q_3 & q_1 & q_2 \end{vmatrix}$ is:

(A) 1 (B) 4

(C) 10 (D) none of these

Q.10 $z = (3 + 7i)(p + iq)$ where $p, q \in I - \{0\}$ purely imaginary then minimum value of $|z|^2$ is

(A) 0 (B) 58 (C) $\frac{3364}{3}$ (D) 3364

Q.11 On the complex plane triangles OAP & OQR are similar and ℓ (OA) = 1. If the points P and Q denotes the complex numbers z_1 & z_2 then the complex number 'z' denoted by the point R is given by:

(A) $z_1 z_2$ (B) $\frac{z_1}{z_2}$ (C) $\frac{z_2}{z_1}$ (D) $\frac{z_1 + z_2}{z_2}$

Q.12 If A and B be two complex numbers satisfying $\frac{A}{B} + \frac{B}{A} = 1$. Then the two points represented by A and B and the origin form the vertices of

(A) An equilateral triangle

(B) An isosceles triangle which is not equilateral

(C) An isosceles triangle which is not right angled

(D) A right angled triangle

Q.13 The solutions of the equation in z, $|z|^2 - (z + \bar{z}) + i(z - \bar{z}) + 2 = 0$ are:

(A) $2 + i, 1 - i$ (B) $1 + i, 1 - i$

(C) $1 + 2i, -1 - I$ (D) $1 + i, 1 + i$

Q.14 If $z_1 = -3 + 5i$: $z_2 = -5 - 3i$ and z is a complex number lying on the line segment joining z_1 & z_2 then arg z can be:

(A) $-\frac{3\pi}{4}$ (B) $-\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) $\frac{5\pi}{6}$

Q.15 The points of intersection of the two curves $|z - 3| = 2$ and $|z| = 2$ in an argand plane are:

(A) $\frac{1}{2}(7 \pm i\sqrt{3})$ (B) $\frac{1}{2}(3 \pm i\sqrt{7})$

(C) $\frac{3}{2} \pm i\sqrt{\frac{7}{2}}$ (D) $\frac{7}{2} \pm i\sqrt{\frac{3}{2}}$

Q.16 Let z to be complex number having the argument θ, $0 < \theta < \frac{\pi}{2}$ and satisfying the equality $|z - 3i| = 3$. Then $\cot\theta - \frac{6}{z}$ is equal to:

(A) 1 (B) – 1 (C) i (D) – i

Q.17 The locus represented by the equation, $|z - 1| + |z + 1| = 2$ is:

(A) An ellipse with foci (1, 0): (– 1, 0)

(B) One of the family of circles passing through the points of intersection of the circles $|z + 1| = 1$

(C) The radical axis of the circles $|z - 1| = 1$ and $|z + 1| = 1$

(D) The portion of the real axis between the points (1, 0) and (– 1, 0)

Q.18 Let P denotes a complex number z on the Argand's plane, and Q denotes a complex number $\sqrt{2|z|^2}\cos\left(\frac{\pi}{4}+\theta\right)$ where θ = amp z if 'O' is the origin, then the ΔOPQ is:

(A) Isosceles but not right angled

(B) Right angled but not isosceles

(C) Right isoscles

(D) Equilateral

Q.19 Let z_1, z_2, z_3 be three distinct complex numbers satisfying $|z_1 - 1| = |z_2 - 1| = |z_3 - 1|$.

If $z_1 + z_2 + z_3 = 3$ then z_1, z_2, z_3 must represent the vertices of:

(A) An equilateral triangle

(B) An isoseles triangles which is not equilateral

(C) A right triangle

(D) Nothing definite can be said

Q.20 If $p = a + b\omega + c\omega^2$; $q = b + c\omega + a\omega^2$; and $r = c + a\omega + b\omega^2$ where a, b, c $\neq$ 0 and ω is the complex cube root of unity, then:

(A) $p + q + r = a + b + c$

(B) $p^2 + q^2 + r^2 = a^2 + b^2 + c^2$

(C) $p^2 + q^2 + r^2 = 2(pq + qr + rp)$

(D) None of these

Previous Years' Questions

Q.1 The smallest positive integer n for which $\left(\frac{1+i}{1-i}\right)^n = 1$, is ***(1980)***

(A) 8 (B) 16

(C) 12 (D) None of these

Q.2 The complex numbers $z = x + iy$ which satisfy the equation $\left|\frac{z-5i}{z+5i}\right| = 1$ lie on ***(1981)***

(A) The x-axis

(B) The straight line y = 5

(C) A circle passing through the origin

(D) None of these

Q.3 If $z = x + iy$ and $w = (1 - iz) / (z - i)$, then $|w| = 1$ implies that, in the complex plane ***(1983)***

(A) z lies on the imaginary axis

(B) z lies on the real axis

(C) z lies on the unit circle

(D) None of these

Q.4 The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order, if and only if ***(1983)***

(A) $z_1 + z_4 = z_2 + z_3$ (B) $z_1 + z_3 = z_2 + z_4$

(C) $z_1 + z_2 = z_3 + z_4$ (D) None of these

Q.5 If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2)$ is equal to ***(1987)***

(A) $-\pi$ (B) $-\frac{\pi}{2}$ (C) 0 (D) $\frac{\pi}{2}$

Q.6 The complex numbers sin x + i cos 2x and cos x – i sin 2x are conjugate to each other, for ***(1988)***

(A) $x = n\pi$ (B) $x = 0$

(C) $x = \left(n+\frac{1}{2}\right)\pi$ (D) No value of x

Q.7 If ω ($\neq$ 1) is a cube root of unity and $(1 + \omega)^7 = A + B\omega$, then A and B are respectively ***(1995)***

(A) 0, 1 (B) 1, 1 (C) 1, 0 (D) – 1, 1

Q.8 Let z and ω be two non-zero complex numbers such that $|z| = |\omega|$ and $\arg(z) + \arg(\omega) = \pi$, then z equals ***(1995)***

(A) ω (B) $-\omega$ (C) $\overline{\omega}$ (D) $-\overline{\omega}$

Q.9 If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ is equal to ***(1998)***

(A) $128\,\omega$ (B) $-128\,\omega$ (C) $128\,\omega^2$ (D) $-128\,\omega^2$

Q.10 The value of sum $\sum_{n=1}^{13}\left(i^n + i^{-n+1}\right)_n$ where $i = \sqrt{-1}$ equals ***(1998)***

(A) i (B) i – 1 (C) – i (D) 0

Q.11 If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then ***(1998)***

(A) x = 3, y = 1 (B) x = 1, y = 1

(C) x = 0, y = 3 (D) x = 0, y = 0

Q.12 If z_1, z_2 and z_3 are complex numbers such that

$|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$,

then $|z_1 + z_2 + z_3|$ is ***(2000)***

(A) Equal to 1 (B) Less than 1

(C) Greater than 3 (D) Equal to 3

Q.13 Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, then value of the determinant

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$ is ***(2002)***

(A) 3ω (B) $3\,\omega\,(\omega - 1)$ (C) $3\omega^2$ (D) $3\,\omega\,(1 - \omega)$

Q.14 If ω ($\neq 1$) be a cube root of unity and $(1 + \omega^2)^n = (1 + \omega^4)^n$, then the least positive value of n is ***(2004)***

(A) 2 (B) 3 (C) 5 (D) 6

Q.15 A man walks a distance of 3 units from the origin towards the North-West (N 45° E) direction. From there, he walks a distance of 4 units towards the North-West (N 45° W) direction to reach a point P. Then, the position of P in the Argand plane is ***(2007)***

(A) $3e^{i\pi/4} + 4i$ (B) $(3 - 4i)e^{i\pi/4}$

(C) $(4 + 3i)e^{i\pi/4}$ (D) $(3 + 4i)e^{i\pi/4}$

Q.16 If $z \neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies ***(2012)***

(A) Either on the real axis or on a circle passing through the origin.

(B) On a circle with centre at the origin.

(C) Either on the real axis or on a circle not passing through the origin.

(D) On the imaginary axis.

Q.17 If z is a complex number of unit modulus and argument θ, then arg $\left(\frac{1+z}{1+\overline{z}}\right)$ equals ***(2013)***

(A) $\frac{\pi}{2} - \theta$ (B) θ (C) $\pi - \theta$ (D) $-\theta$

Q.18 If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z + \frac{1}{2}\right|$ ***(2014)***

(A) Is equal to $\frac{5}{2}$

(B) Lies in the interval (1, 2)

(C) Is strictly greater than $\frac{5}{2}$

(D) Is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$

Q.19 A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\overline{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a: ***(2015)***

(A) Straight line parallel to y-axis

(B) Circle of radius 2

(C) Circle of radius $\sqrt{2}$

(D) Straight line parallel to x-axis

Q.20 A value of θ for which $\frac{2 + 3i\sin\theta}{1 - 2i\sin\theta}$ is purely imaginary, is: ***(2016)***

(A) $\frac{\pi}{6}$ (B) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

(C) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (D) $\frac{\pi}{3}$

JEE Advanced/Boards

Exercise 1

Q.1 Prove that with regard to the quadratic equation $z^2 + (p + ip')z + q + iq' = 0$ where p, p', q, q' are all real.

(i) If the equation has one real root then $q'^2 - pp'q' + qp'^2 = 0$

(ii) If the equation has two equal roots then $p^2 - q'^2 = 4q$ and $pp' = 2q'$.

state whether these equal roots are real or complex.

Q.2 Let z = 18 + 26i where $z_0 = x_0 + iy_0$ ($x_0, y_0 \in R$) is the cube roots of z having least positive argument. Find the value of $x_0y_0(x_0 + y_0)$.

Q.3 Show that the locus formed by z in the equation $z^3 + iz = 1$ never crosses the coordinate axes in the Argand's plane.

Further show that $|z| = \sqrt{\dfrac{-\text{Im}(z)}{2\text{Re}(z)\text{Im}(z)+1}}$

Q.4 Consider the diagonal matrix A_n = dia ($d_1, d_2, d_3, \ldots d_n$) of order where

$D_i = a^{i-1}$, $1 \le i \le n$ and $\alpha = e^{\frac{i2\pi}{n}}$; $i = \sqrt{-1}$, is the n^{th} root of unity.

Let L: represent the value of Tr. $(A_7)^7$.

M: denotes the value of det (A_{2n+1}) + det (A_{2n}).

Find the value of (L + M).

[Note: $T_r(A)$ denotes trace of square matrix A]

Q.5 Let $z_1, z_2 \in C$ such that $z_1^2 + z_2^2 \in R$. If $z_1(z_1^2 - 3z_2^2) = 10$ and $z_2(3z_1^2 - z_2^2) = 30$. Find the value of $(z_1^2 + z_2^2)$.

Q.6 If the equation $(z + 1)^7 + z^7 = 0$ has roots $z_1, z_2, \ldots z_7$, find the value of

(a) $\sum_{r=1}^{7} \text{Re}(Z_r)$ and $\sum_{r=1}^{7} \text{Im}(Z_r)$

Q.7 If z is one of the imaginary 7^{th} roots of unity, then find the equation whose roots are $(z + z^4 + z^2)$ and $(z^6 + z^3 + z^5)$.

Q.8 If the expression $z^5 - 32$ can be factorised into linear and quadratic factors over real coefficients as $(z^5 - 32) = (z - 2)(z^2 - pz + 4)(z^2 - zq + 4)$ then find the value of $(p^2 + 2p)$.

Q.9 Let z_1 & z_2 be any two arbitrary complex numbers then prove that:
$|z_1| + |z_2| \ge \frac{1}{2}(|z_1| + |z_2|)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right|$.

Q.10 Let z_i (i = 1, 2, 3, 4) represent the vertices of a square all of which lie on the sides of the triangle with vertices (0, 0), (2, 1) and (3, 0). If z_1 and z_2 are purely real, then area of triangle formed by z_3, z_4 and origin is m (where m and n are in their lowest form). Find the value of (m + n).

Q.11 (i) Let C_r's denotes the combinatorial coefficients in the expansion of $(1 + x)^n$, $n \in N$. If the integers

$a_n = C_0 + C_3 + C_6 + C_9 + \ldots\ldots$

$b_n = C_1 + C_4 + C_7 + C_{10} + \ldots\ldots$

and $c_n = C_2 + C_3 + C_8 + C_{11} + \ldots\ldots$

then prove that

(a) $a_n^3 + b_n^3 + c_n^3 - 3a_nb_nc_n = 2^n$.

(b) $(a_n - b_n)^2 + (b_n - c_n)^2 + (c_n - a_n)^2 = 2$

(ii) Prove the identity:

$(C_0 - C_2 + C_4 - C_6 + \ldots\ldots\ldots)^2$

$+ (C_1 - C_3 + C_5 - C_7 + \ldots\ldots)^2 = 2^n$.

Q.12 Let z_1, z_2, z_3, z_4 be the vertices A, B, C, D respectively of a square on the Argand diagram taken in anticlockwise direction then prove that:

(i) $2z_2 = (1 + i)z_1 + (1 - i)z_3$ & (ii) $2z_4 = (1 - i)z_1 + (1 + i)z_3$

Q.13 A function f is defined on the complex number by f(z) = (a + bi)z, where 'a' and 'b' are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that |a + bi| = 8 and that $b^2 = \frac{u}{v}$ where u and v are co-primes. Find the value of (u + v).

Q.14 Prove that

(a) $\cos x + {}^nC_1 \cos 2x + {}^nC_2 \cos 3x + \ldots + {}^nC_n \cos(n+1)x = 2^n.\cos^n \frac{x}{2}.\cos\left(\frac{n+2}{2}\right)x$

(b) $\sin x + {}^nC_1 \sin 2x + {}^nC_2 \sin 3x + \ldots + {}^nC_n \sin(n+1)x = 2^n.\cos^n \frac{x}{2}.\sin\left(\frac{n+2}{2}\right)x.$

Q.15 Let $f(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial with real coefficients satisfying $f(i) = 0$ and $f(1+i) = 5$. Find the value of $a^2 + b^2 + c^2 + d^2$.

Q.16 Let $\omega_1, \omega_2, \omega_3, \ldots \omega_n$ be the complex numbers. A line L on the complex plane is called a mean line for the points $\omega_1, \omega_2, \omega_3, \ldots \omega_n$ if L contains the points (complex numbers) $z_1, z_2, z_3, \ldots z_n$ such that $\sum_{k=1}^{n}(z_k - \omega_k) = 0$.

Now for the complex number $\omega_1 = 32 + 170i$, $\omega_2 = -7 + 64i$, $\omega_3 = -9 + 200i$, $\omega_4 = 1 + 27i$ and $\omega_5 = -14 + 43i$, there is a unique mean line with y-intercept 3. Find the slope of the line.

Q.17 A particle start to travel from a point P on the curve C_1: $|z - 3 - 4i| = 5$, where $|z|$ is maximum. From P, the particle moves through an angle $\tan^{-1}\frac{3}{4}$ in anticlockwise direction on $|z - 3 - 4i| = 5$ and reaches at point Q. From Q, it comes down parallel to imaginary axis by 2 units and reaches at point R. Find the complex number corresponding to point R in the Argand plane.

Q.18 Evaluate: $\sum_{p=1}^{32}(3p-2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right)^p$

Q.19 Let a, b, c be distinct complex numbers

such that $\frac{a}{1-b} = \frac{b}{1-c} = \frac{c}{1-a} = k$.

Find the value of k.

Q.20 Let α, β be fixed complex numbers and z is a variable complex number such that

$|z-\alpha|^2 + |z-\beta|^2 = k$

Find out the limits for 'k' such that the locus of z is a circle. Find also the centre and radius of the circle.

Q.21 C is the complex number. f: C $\to$ R is defined by $f(z) = |z^3 - z + 2|$. Find the maximum value of f(z) if $|z| = 1$.

Q.22 Let a, b, c are distinct integers and ω, ω^2 are the imaginary cube roots of unity. If minimum value of $|a + b\omega + c\omega^2| + |a + b\omega^2 + c\omega|$ is $n^{\frac{1}{4}}$ where $n \in N$, then find the value of n.

Q.23 If the area of the polygon whose vertices are the solutions (in the complex plane) of the equation

$x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$

can be expressed in the simplest form as $\frac{a\sqrt{b}+c}{d}$, find the value of $(a + b + c + d)$.

Q.24 If a and b are positive integer such that $N = (a + ib)^3 - 107i$ is a positive integer.

Find N.

Q.25 If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ (a, b, c, d $\in$ R) has 4 non real roots, two with sum $3 + 4i$ and the other two with product $13 + i$. Find the value of 'b'.

Q.26 Resolve $z^5 + 1$ into linear and quadratic

factors with real coefficients. Deduce that:

$4\sin\frac{\pi}{10}\cos\frac{\pi}{5} = 1$.

Q.27 If $x = 1 + i\sqrt{3}$: $y = 1 - i\sqrt{3}$ & $z = 2$,

prove that $x^p + y^p = z^p$ for every prime $p > 3$.

Q.28 Dividing f(z) by $z - i$, we get the remainder i and dividing it by $z + i$ we get the remainder $1 + i$. Find the remainder upon the division of f(z) by $z^2 + 1$.

Q.29 (a) Let $z = a + b$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by

$A = \{z \mid |z| \le 2\}$ and

$B = \{z \mid (1 - i)z + (1 + i)\bar{z} \ge 4\}$.

Find the area of the region $A \cap B$.

(b) For all real numbers x, let the mapping $f(x)=\frac{1}{x-i}$, where $i=\sqrt{-1}$. If there exist real number a, b, c and d for which f(a), f(b), f(c) and f(d) form a square on the complex plane. Find the area of the square.

Q.30

Column I	Column II
(A) Let ω be a non-real cube root of unity then the number of distinct elements in the set $\{(1+\omega+\omega^2+...+\omega^n)^m \mid m,n\in N\}$	(p) 4
(B) Let 1, ω, ω^2 be the cube root of unity. The least possible degree of a polynomial with real coefficients having roots $2\omega,(2+3\omega),(2+3\omega^2),(2-\omega-\omega^2)$, is	(q) 5
(C) $\alpha=6+4i$ and $\beta=(2\ +4i)$ are two complex numbers on the complex plane. A complex number z. A complex number z satisfying amp $\left(\frac{z-\alpha}{z-\beta}\right)=\frac{\pi}{6}$ moves on the major segment of a circle whose radius is	(r) 6
	(s) 7

Exercise 2

Single Correct Choice Type

Q.1 The set of points on the complex plane such that $z^2 + z + 1$ is real and positive. (where $z = x + iy$, $x, y \in R$) is:

(A) Complete real axis only

(B) Complete real axis or all points on the line $2x + 1 = 0$

(C) Complete real axis or a line segment joining points $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)\&\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ excluding both.

(D) Complete real axis or set of points lying inside the rectangle formed by the lines.

$2x + 1 = 0$; $2x - 1 = 0$; $2y - \sqrt{3} = 0$ & $2y + \sqrt{3} = 0$

Q.2 The complex numbers whose real and imaginary parts are integers and satisfy the relation $z\bar{z}^3+z^3\bar{z}=350$ form a rectangle on the Argand plane, the length of whose diagonal is-

(A) 5 (B) 10 (C) 15 (D) 25

Q.3 Let z_1 and z_2 be non-zero complex numbers satisfying the equation, $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$.

The geometrical nature of the triangle whose vertices are the origin and the points representing z_1 & z_2.

(A) An isosceles right angled triangle

(B) A right angled triangle which is not isosceles

(C) An equilateral triangle

(D) An isosceles triangle which is not right angled

Q.4 The set of points on the Argand diagram which satisfy both $|z|\le 4$ & $\operatorname{Arg} z=\frac{\pi}{3}$ is:

(A) A circle and line (B) A radius of a circle

(C) A sector of a circle (D) An infinite part line

Q.5 If z_1 & z_2 are two complex numbers & if $\arg\frac{z_1+z_2}{z_1-z_2}=\frac{\pi}{2}$ but $|z_1 + z_2| \ne |z_1 - z_2|$ then the figure formed by the points represented by 0, z_1, z_2 & $z_1 + z_2$ is:

(A) A parallelogram but not a square rectangle or a rhombus

(B) A rectangle but not a square

(C) A rhombus but not a square

(D) A square

Q.6 If z_1, z_2, z_3 are the vertices of the ΔABC on the complex plane & are also the roots of the equation $z^3 - 3\alpha z^2 + 3\beta z + x = 0$, then the condition for the ΔABC to be equilateral triangle is:

(A) $\alpha^2 = \beta$ (B) $\alpha = \beta^2$

(C) $\alpha^2 = 2\beta$ (D) $\alpha = 3\beta^2$

Q.7 Let A, B, C represent the complex numbers z_1, z_2, z_3 respectively on the complex plane. If the circumcentre of the triangle. ABC lies at the origin then the orthocenter is represented by the complex number:

(A) $z_1 + z_2 - z_3$ (B) $z_2 + z_3 - z_1$

(C) $z_3 + z_1 - z_2$ (D) $z_1 + z_2 + z_3$

Q.8 Which of the following represents a point in an argand's plane, equidistant from the roots of equation $(z + 1)^4 = 16z^4$?

(A) (0, 0) (B) $\left(-\frac{1}{3},0\right)$ (C) $\left(\frac{1}{3},0\right)$ (D) $\left(0,\frac{2}{\sqrt{5}}\right)$

Q.9 The equation of the radical axis of the two circles presented by the equations.

$|z - 2| = 3$ and $|z - 2 - 3i| = 4$ on the complex plane is:

(A) $3y + 1 = 0$ (B) $3y - 1 = 0$

(C) $2y - 1 = 0$ (D) None of these

Q.10 Number of real solution of the equation, $z^3 + iz - 1 = 0$ is

(A) Zero (B) One (C) Two (D) Three

Q.11 A point 'z' moves on the curve $|z - 4 - 3i| = 2$ in an argand plane. The maximum and minimum values of $|z|$ are:

(A) 2, 1 (B) 6, 5 (C) 4, 3 (D) 7, 3

Q.12 Let $z = 1 - \sin\alpha + i\cos\alpha$ where $\alpha \in \left(0,\frac{\pi}{2}\right)$, then the modulus and the principle value of the argument of z are respectively:

(A) $\sqrt{2(1-\sin\alpha)},\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$ (B) $\sqrt{2(1-\sin\alpha)},\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

(C) $\sqrt{2(1+\sin\alpha)},\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$ (D) $\sqrt{2(1+\sin\alpha)},\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

Q.13 z_1 and z_2 are complex numbers. Then

Equ. $\left|\frac{z_1+z_2}{2}+\sqrt{z_1z_2}\right| + \left|\frac{z_1+z_2}{2}-\sqrt{z_1z_2}\right| =$

(A) $2\left|\sqrt{z_1}+\sqrt{z_2}\right|$ (B) $2\left|\sqrt{z_1}-\sqrt{z_2}\right|$

(C) $2\left(\left|\sqrt{z_1}\right|^2+\left|\sqrt{z_2}\right|^2\right)$ (D) $\left(\left|\sqrt{z_1}\right|^2+\left|\sqrt{z_2}\right|^2\right)$

Q.14 If α, β be the roots of the equation $u^2 - 2u + 2 = 0$ and if $\cot\theta = x + 1$, then $\frac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta}$ is equal to:

(A) $\frac{\sin n\theta}{\sin^n\theta}$ (B) $\frac{\cos n\theta}{\cos^n\theta}$ (C) $\frac{\sin n\theta}{\cos^n\theta}$ (D) $\frac{\cos n\theta}{\sin^n\theta}$

Q.15 If A_r (r = 1, 2, 3, ..., n) are the vertices of a regular polygon inscribed in a circle of radius R, then

$(A_1A_2)^2 + (A_1A_3)^2 + (A_1A_4)^2 + \ldots + (A_1A_n)^2 =$

(A) $\frac{nR^2}{2}$ (B) $2nR^2$

(C) $4R^2\cot\frac{\pi}{2n}$ (D) $(2n - 1)R^2$

Q.16 If the equation $z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$, where a_1, a_2, a_3, a_4 are real coefficients different from zero has a pure imaginary root then the expression $\frac{a_3}{a_1a_2}+\frac{a_1a_4}{a_2a_3}$ has the value equal to

(A) 0 (B) 1 (C) – 2 (D) 2

Q.17 All roots of the equation $(1 + z)^6 + z^6 = 0$

(A) Lie on a unit circle with centre at the origin

(B) Lie on a unit circle with centre at (– 1, 0)

(C) Lie on the vertices of a regular polygon with centre at the origin

(D) Are collinear

Q.18 Number of roots of the equation $z^{10} - z^5 - 992 = 0$ with real part negative is:

(A) 3 (B) 4 (C) 5 (D) 6

Q.19 z_1 and z_2 are two distinct points in argand plane. If $a|z_1| = b|z_2|$, then the point $\frac{az_1}{bz_2}+\frac{bz_2}{az_1}$ is a point on the $(a, b \in R)$

(A) Line segment [– 2, 2] of the real axis

(B) Line segment [– 2, 2] of the imaginary axis

(C) Unit circle $|z| = 1$

(D) The line with $\arg z = \tan^{-1} 2$

Q.20 If ω is an imaginary cube root of unity, then the value of $(p + q)^3 + (p\omega + q\omega^2)^3 + (p\omega^2 + q\omega)^3$ is

(A) $p^3 + q^3$

(B) $3(p^3 + q^3)$

(C) $3(p^3 + q^3) - pq(p + q)$

(D) $3(p^3 + q^3) + pq(p + q)$

Q.21 The solution set of the equation $\left(1+i\sqrt{3}\right)^{x}-2^{x}=0$

(A) Form an A.P. (B) Form a G.P.

(C) Form an H.P. (D) Is a empty set

Multiple Correct Choice Type

Q.22 In the quadratic equation $x^2 + (p + iq)x + 3i = 0$, p and q are real. If the sum of the squares of the roots is 8 then

(A) $p = 3, q = -1$ (B) $p = 3, q = 1$

(C) $p = -3, q = -1$ (D) $p = -3, q = 1$

Q.23 Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| - |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\frac{z_1+z_2}{z_1-z_2}$ may be

(A) Zero (B) Real & positive

(C) Real and negative (D) Purely imaginary

Q.24 Given a, b, x, y $\in$ R then which of the following statement(s) hold good?

(A) $(a + ib)(x + iy)^{-1} = a - ib \Rightarrow x^2 + y^2 = 1$

(B) $(1 - ix)(1 + ix)^{-1} = a - ib \Rightarrow a^2 + b^2 = 1$

(C) $(a + ib)(a - ib)^{-1} = x - iy \Rightarrow |x + iy| = 1$

(D) $(y - ix)(a + ib)^{-1} = y + ix \Rightarrow |a - ib| = 1$

Q.25 If $z = x = iy = r(\cos\theta + i\sin\theta)$ then the values of $\sqrt{z}$ is equal to:

(A) $\pm\frac{1}{\sqrt{2}}\left(\sqrt{r+x}+i\sqrt{r-x}\right)$ for $y \geq 0$

(B) $\pm\frac{1}{\sqrt{2}}\left(\sqrt{r+x}-i\sqrt{r-x}\right)$ for $y \geq 0$

(C) $\pm\frac{1}{\sqrt{2}}\left(\sqrt{r+x}+i\sqrt{r-x}\right)$ for $y \leq 0$

(D) $\pm\frac{1}{\sqrt{2}}\left(\sqrt{r+x}-i\sqrt{r-x}\right)$ for $y \leq 0$

Q.26 For two complex numbers z_1 and z_2

$\left(az_1+b\bar{z}_1\right)\left(cz_2+d\bar{z}_2\right) = \left(cz_1+d\bar{z}_1\right)\left(az_2+b\bar{z}_2\right)$

If (a, b, c, d $\in$ R):

(A) $\frac{a}{b}=\frac{c}{d}$ (B) $\frac{a}{d}=\frac{b}{c}$

(C) $|z_1| = |z_2|$ (D) $\arg z_1 = \arg z_2$

Q.27 Let z_1, z_2 be two complex numbers represented by point on the circle $|z_1| = 1$ and $|z_2| = 2$ respectively, then:

(A) Max $|2z_1 + z_2| = 4$ (B) Min $|z_1 - z_2| = 1$

(C) $\left|z_2+\frac{1}{z_1}\right|\leq 3$ (D) None of these

Q.28 If α, β any two complex numbers such that $\left|\frac{\alpha-\beta}{1-\bar{\alpha}\beta}\right|=1$, then

(A) $|\alpha| = 1$ (B) $|\beta| = 1$

(C) $\alpha = e^{i\theta}, \theta \in R$ (D) $\beta = e^{i\theta}, \theta \in R$

Q.29 On the argand plane, let $\alpha = -2 + 3z$, $\beta = -2 - 3z$ and $|z| = 1$. Then the correct statement is:

(A) α moves on the circle, centre at (– 2, 0) and radius 3

(B) α and β describe the same locus

(C) α and β move on different circles

(D) $\alpha - \beta$ moves on a circle concentric with $|z| = 1$

Q.30 The value of $i^n + i^{-n}$, for $i=\sqrt{-1}$ and $n \in I$ is:

(A) $\frac{2^n}{(1-i)^{2n}}+\frac{(1+i)^{2n}}{2^n}$ (B) $\frac{(1+i)^{2n}}{2^n}+\frac{(1-i)^{2n}}{2^n}$

(C) $\frac{(1+i)^{2n}}{2^n}+\frac{2^n}{(1-i)^{2n}}$ (D) $\frac{2^n}{(1+i)^{2n}}+\frac{2^n}{(1-i)^{2n}}$

Q.31 A complex number z satisfying the equation,

$$\log_{14}(13\,|z^2-4i|)+\log_{196}\frac{1}{(13+|z^2+4i|)^2}=0$$

(A) Can be purely real

(B) Can be purely imaginary

(C) Must be imaginary

(D) Must be real or purely imaginary

Q.32 Let S be the set of real values of x satisfying the inequality

$1-\log_2 \dfrac{|x+1+2i|-2}{\sqrt{2}-1} \geq 0$, then S contains:

(A) [– 3, – 1)   (B) (– 1, 1]

(C) [– 2, 2)   (D) [1, 2]

Q.33 If x = cos α; y = cos β ; z = cos γ ; Where α, β, γ ∈ R, then

(A) $\sum x = \Pi x \Rightarrow \cos(\alpha - \beta) = 1$

(B) $\Pi \dfrac{x-y}{z} = 8\Pi \cos \dfrac{\alpha-\beta}{2}$

(C) $\Pi \dfrac{x+y}{z}$ is real

(D) $\sum(\text{Re})x = \cos\left(\sum\alpha\right), \sum \text{Im}(x) = \sin\left(\sum\alpha\right)$

Q.34 If $x_r = \cos\left(\dfrac{\pi}{2^r}\right)$ for $1 \leq r \leq n$; r, n ∈ N then:

(A) $\lim\limits_{n\to\infty} \text{Re}\left(\prod_{r=1}^{n} x_r\right) = -1$ (B) $\lim\limits_{n\to\infty} \text{Re}\left(\prod_{r=1}^{n} x_r\right) = 0$

(C) $\lim\limits_{n\to\infty} \text{Im}\left(\prod_{r=1}^{n} x_r\right) = 1$ (D) $\lim\limits_{n\to\infty} \text{Im}\left(\prod_{r=1}^{n} x_r\right) = 0$

Q.35 If 1, z_1, z_2, z_3 ...... z_{n-1} be the n^{th} roots of unity and ω be a non real complex cube root of unity then the product $\prod_{r=1}^{n-1}(\omega - z_r)$ can be equal to:

(A) 0   (B) 1   (C) – 1   (D) 2

Q.36 Identify the correct statements(s).

(A) No non zero complex number z satisfies the equation, $\bar{z} = -4z$

(B) $\bar{z} = z$ implies that z is purely real

(C) $\bar{z} = -z$ implies that z is purely imaginary

(D) If z_1, z_2 are the roots of the quadratic equation $az^2 + bz + c = 0$ such that Im $(z_1z_2) \neq 0$ then a, b, c must be real numbers

Q.37 If the complex numbers z_1, z_2, z_3 & z_1', z_2' and z_3' are representing the vertices of two triangles such that $z_3 = (1 - z_0)\, z_1 + z_0z_2$ and $z_3' = (1 - z_0)\, z_1' + z_0z_2'$ where z_0 is also a complex number then:

(A) $\begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix} = 0$

(B) The two triangles are congruent

(C) The two triangles are similar

(D) The two triangles have the same area.

Q.38 If 'z' be any complex number in a plane (|z| ≠ 0) then the complex number z for which the multiplication inverse is equal to the additive inverse is:

(A) 0 + i   (B) 0 – i   (C) 1 – i   (D) 1 + i

Q.39 Given $z = a + bi = \dfrac{1-ix}{1+ix}$; a, b, x ∈ R, then which of the following holds good?

(A) $-\dfrac{\pi}{2} < \arg z \leq 0$ (B) $-\pi < \arg z \leq 0$

(C) |z| = 1   (D) arg z = π ; |z| = 1

Previous Years' Questions

Q.1 If z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 + 5i|$ is ...... ***(2011)***

Q.2 Let $\omega = e^{i\frac{2\pi}{3}}$ and a, b, c, x, y, z be non-zero complex number such that a + b + c = x, $a + b\omega + c\omega^2 = y$, $a + b\omega^2 + c\omega = z$. Then, the value of

$\dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is ...... ***(2011)***

Paragraph (for Q.3, 4, 5)

Read the following passage and answer the following questions.

Let A, B, C be three sets of complex number as defined below

A = {z: $|I_m\, z \geq 1|$}

B = {z: |z – 2 – i| = 3}

C = {z: Re ((1 – i)z) + $\sqrt{2}$) ***(2008)***

Q.3 The number of elements in the set $A \cap B \cap C$ is

(A) 0   (B) 1   (C) 2   (D) ∞

Q.4 Let z be any point in $A \cap B \cap C$. The $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

(A) 25 and 29 (B) 30 and 34

(C) 35 and 39 (D) 40 and 44

Q.5 Let z be any point in $A \cap B \cap C$ and let w any point satisfying $|w - 2 - i| < 3$. Then $|z| - |w| + 3$ lies between

(A) – 6 and 3 (B) – 3 and 6

(C) – 6 and 6 (D) – 3 and 9

Q.6 If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then ***(1982)***

(A) Re (z) = 0

(B) Im (z) = 0

(C) Re (z) > 0, Im (z) > 0

(D) Re (z) > 0, Im (z) < 0

Q.7 The inequality $|z - 4| < |z - 2|$ represents the region given by ***(1982)***

(A) Re (z) $\geq$ 0 (B) Re (z) < 0

(C) Re (z) > 0 (D) None of these

Q.8 If a, b, c and u, v, w are the complex numbers representing the vertices of two triangles such that $c = (1-r)a + rb$ and $w = (1 - r)u + rv$, where r is a complex number, then the two triangles ***(1985)***

(A) Have the same area (B) Are similar

(C) Are congruent (D) None of these

Q.9 The value of $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$ is ***(1987)***

(A) – 1 (B) 0 (C) – i (D) i

Q.10 Let z and w be two complex numbers such that $|z| \leq 1$, $|w| \leq 1$ and $|z + iw| = |z - i\overline{w}| = 2$, then z equals ***(1995)***

(A) 1 or i (B) i or – i (C) 1 or – 1 (D) i or –1

Q.11 For positive integers n_1, n_2 the value of expression $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$, Here $i = \sqrt{-1}$ is a real number, if and only if ***(1996)***

(A) $n_1 = n_2 + 1$ (B) $n_1 = n_2 - 1$

(C) $n_1 = n_2$ (D) $n_1 > 0, n_2 > 0$

Q.12 If $i = \sqrt{-1}$, then

$$4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$$

is equal to ***(1999)***

(A) $1 - i\sqrt{3}$ (B) $-1 + i\sqrt{3}$ (C) $i\sqrt{3}$ (D) $-i\sqrt{3}$

Q.13 If arg (z) < 0, then arg (– z) – arg (z) equal ***(2000)***

(A) π (B) $-\pi$ (C) $-\frac{\pi}{2}$ (D) $\frac{\pi}{2}$

Q.14 If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and Re $(z_1\overline{z}_2) = 0$, then the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfied by ***(1985)***

(A) $|w_1| = 1$ (B) $|w_2| = 1$

(C) Re $(w_1\overline{w}_2) = 0$ (D) None of these

Q.15 Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If arg (w) denotes the principal argument of a non-zero complex number w, then ***(2010)***

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

(B) arg $(z - z_1)$ = arg $(z - z_2)$

(C) $\begin{vmatrix} z - z_1 & \overline{z} - \overline{z}_1 \\ z_2 - z_1 & \overline{z}_2 - \overline{z}_1 \end{vmatrix} = 0$

(D) arg $(z - z_1)$ = arg $(z_2 - z_1)$

Q.16 Match the statements in column I with those in column II. ***(2010)***

[Note: Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z.]

Column I	Column II
(A) The set of points z satisfying $\|z-i\|z\|\| = \|z + i\|z\|\|$ Is contained in or equal to	(p) An ellipse with eccentricity $\frac{4}{5}$
(B) The set of points z satisfying $\|z+4\|+\|z-4\|=10$ is contained in or equal to	(q) The set of points z satisfying Im = z = 0
(C) If $\|w\|=2$, then the set of points $z = w - \frac{1}{w}$ is contained in or equal to	(r) The set of points z satisfying $\|\text{Im } z\| \leq 1$

(D) If $\lvert w \rvert = 1$, then the set of points $z = w + \frac{1}{w}$ is contained in or equal to	(s) The set of points z satisfying $\lvert \text{Re } z \rvert \le 2$
	The set of points z satisfying $\lvert z \rvert \le 3$

Q.17 Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex number z satisfying ***(2010)***

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0 \text{ is equal to}$$

Q.18 Let z_1 and z_2 be two distinct complex numbers and let $z = (1-t)z_1 + tz_2$ for some real number t with 0 < t < 1. If Arg (w) denotes the principal argument of a nonzero complex number w, then ***(2010)***

(A) $\lvert z - z_1 \rvert + \lvert z - z_2 \rvert = \lvert z_1 - z_2 \rvert$

(B) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$

(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

(D) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

Q.19 if z is any complex number satisfying $\lvert z - 3 - 2i \rvert \le 2$, then the minimum value of $\lvert 2z - 6 - 5i \rvert$ is ***(2011)***

Q.20 Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non-zero complex numbers such that

$a + b + c = x$

$a + b\omega + c\omega^2 = y$

$a + b\omega^2 + c\omega = z$

Then the value of $\frac{\lvert x \rvert^2 + \lvert y \rvert^2 + \lvert z \rvert^2}{\lvert a \rvert^2 + \lvert b \rvert^2 + \lvert c \rvert^2}$ is ***(2011)***

Q.21 Let $\omega \ne 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$ where each if a, b and c is either ω or ω^2. Then the number of distinct matrices in the set S is ***(2011)***

(A) 2 (B) 6 (C) 4 (D) 8

Q.22 Let z be a complex number such that the imaginary part of z is nonzero and $a = z^2 + z + 1$ is real. Then a cannot take the value ***(2012)***

(A) -1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

Q.23 Let complex numbers α and $\frac{1}{\alpha}$ lies on circle $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$, respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2\lvert z_0 \rvert^2 = r^2 + 2$, then $\lvert \alpha \rvert =$ ***(2013)***

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{3}$

Q.24 Let $\omega = \frac{\sqrt{3} + i}{2}$ and $P = \{\omega^n : n = 1, 2, 3, ...\}$. Further $H_1 = \left\{z \in C : \text{Re}z > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in C : \text{Re}z < \frac{-1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1, z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1 O z_2 =$ ***(2013)***

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

Q.25 Let ω be a complex cube root of unity with $\omega \ne 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \ne 0$, when n = ***(2013)***

(A) 57 (B) 55 (C) 58 (D) 56

Paragraph (for Q.26 and Q.27) ***(2013)***

Let $S = S_1 \cap S_2 \cap S_3$, where

$S_1 = \{z \in C : \lvert z \rvert < 4\}$, $S_2 = \left\{z \in C : \text{Im}\left[\frac{z - 1 + \sqrt{3}i}{1 - \sqrt{3}i}\right] > 0\right\}$

and $S_3 = \{z \in C : \text{Re } Z > 0\}$

Q.26 Area of S =

(A) $\frac{10\pi}{3}$ (B) $\frac{20\pi}{3}$ (C) $\frac{16\pi}{3}$ (D) $\frac{32\pi}{3}$

Q.27 $\underset{z \in S}{\text{Min}} |1-3i-z| =$

(A) $\frac{2-\sqrt{3}}{2}$ (B) $\frac{2+\sqrt{3}}{2}$ (C) $\frac{3-\sqrt{3}}{2}$ (D) $\frac{3+\sqrt{3}}{2}$

Q.28 Match the following: ***(2014)***

Column I	Column II
(i) The number of polynomials f(x) with non-negative integer coefficients of degree ≤ 2, satisfying f (0) = 0 and $\int_0^1 f(x)dx = 1$, is	(p) 8
(ii) The number of points in the interval $\left[-\sqrt{13}, \sqrt{13}\right]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	(q) 2
(iii) $\int_{-2}^{2} \frac{3x^2}{(1+e^x)} dx$ equals	(r) 4
(iv) $\frac{\left(\int_{-1/2}^{1/2} \cos 2x.\log\left(\frac{1+x}{1-x}\right) dx\right)}{\left(\int_{0}^{1/2} \cos 2x.\log\left(\frac{1+x}{1-x}\right) dx\right)}$ equals	(s) 0

Codes:

	(i)	(ii)	(iii)	(iv)
(A)	r	q	s	p
(B)	q	r	s	p
(C)	r	q	p	s
(D)	q	r	p	s

Q.29 For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression.

$\frac{\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3} |\alpha_{4k-1} - \alpha_{4k-2}|}$ is ***(2015)***

Q.30 Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and r, s $\in \{1,2,3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is ***(2016)***

Q.31 Let a, b $\in \mathbb{R}$ and $a^2 + b^2 \ne 0$.

Suppose $S = \left\{ z \in \mathbb{C} : z = \frac{1}{a+ibt'} t \in \mathbb{R}, t \ne 0 \right\}$, where $i = \sqrt{-1}$. If z = x + iy and $z \in S$, then (x, y) lies on ***(2016)***

(A) The circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \ne 0$

(B) The circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \ne 0$

(C) The x-axis for $a \ne 0$, b = 0

(D) The y-axis for a = 0, $b \ne 0$

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q.6	Q.9	Q.15
Q.18	Q.22	Q.24
Q.28	Q.31	Q.34

Exercise 2

Q.2	Q.8	Q.10
Q.13	Q.16	Q.18

Previous Years' Questions

Q.2	Q.4	Q.7
Q.10	Q. 13	Q.15

JEE Advanced/Boards

Exercise 1

Q.7	Q.11	Q.13
Q.16	Q.18	Q.25
Q.29	Q.30	

Exercise 2

Q.2	Q.6	Q.9
Q.15	Q.19	Q.22
Q. 25	Q.27	Q.31
Q. 33	Q. 36	Q. 39

Previous Years' Questions

Q.2	Q.4	Q. 8
Q.11	Q.14	Q.15

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $z = 0, i, \pm \frac{\sqrt{3}}{2} - \frac{i}{2}$

Q.2 – i

Q.3 $x = \frac{5}{13}, y = \frac{14}{13}$

Q.4 x = 3, y – 1

Q.6 $\sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

Q.7 ± (1 – 3i)

Q.8 n = 4

Q.10 – 1

Q.11 $-1, 1-2\omega, 1-2\omega^2$

Q.13 $z_1 = (1-\sqrt{3}) + i$; $z_2 = -i\sqrt{3}$; $z_3 = (1+\sqrt{3}) - i$

Q.14 Centre $\left[\frac{\alpha - K^2\beta}{1-K^2}\right]$, radius $\frac{|\alpha+\beta-K|}{1-K^2}$

Q.19 $\frac{(n-1)n}{4}[n^2+3n+4]$

Q.20 5

Q.24 ω, ω^2

Q.31 $3 \le |z| \le 7$

Q.33 Interior of circle $x^2 + y^2 = 25$

Exercise 2

Single Correct Choice Type

Q.1 C	**Q.2** D	**Q.3** B	**Q.4** C	**Q.5** D	**Q.6** A	**Q.7** B
Q.8 C	**Q.9** D	**Q.10** D	**Q.11** A	**Q.12** A	**Q.13** D	**Q.14** D
Q.15 B	**Q.16** C	**Q.17** D	**Q.18** C	**Q.19** A	**Q.20** C	

Previous Years' Questions

Q.1 D	**Q.2** A	**Q.3** B	**Q.4** B	**Q.5** C	**Q.6** D	**Q.7** B
Q.8 D	**Q.9** D	**Q.10** B	**Q.11** D	**Q.12** A	**Q.13** B	**Q.14** B
Q.15 D	**Q.16** A	**Q.17** B	**Q.18** B	**Q.19** B	**Q.20** C	

JEE Advanced/Boards

Exercise 1

Q.2 12 **Q.4** 7 **Q.5** 10 **Q.6** (a) $-\frac{7}{2}$, (b) zero **Q.7** $x^2 + x + 2 = 0$ **Q.8** 4

Q.10 41 **Q.13** 259 **Q.15** 26 **Q.16** 163 **Q.17** (3 + 7i) **Q.18** 48 (1 – i)

Q.19 $-\omega$ or $-\omega^2$ **Q.20** $k > |\alpha - \beta|^2$

Q.21 If (z) is maximum when z = ω, when ω is the cube root unity v and If (z) = $\sqrt{13}$

Q.22 144 **Q.23** 8 **Q.24** 198 **Q.25** 51

Q.26 $(z + 1)(z^2 - 2z\cos 36° + 1)(z^2 - 2z\cos 108° + 1)$

Q.28 $\frac{iz}{2} + \frac{1}{2} + i$

Q.29 (a) $\pi - 2$; (b) $\frac{1}{2}$

Q.30 A $\rightarrow$ s; B $\rightarrow$ q; C $\rightarrow$ p

Exercise 2

Single Correct Choice Type

Q.1 C	**Q.2** B	**Q.3** A	**Q.4** B	**Q.5** C	**Q.6** A	**Q.7** D
Q.8 C	**Q.9** B	**Q.10** A	**Q.11** D	**Q.12** A	**Q.13** D	**Q.14** A
Q.15 B	**Q.16** B	**Q.17** D	**Q.18** A	**Q.19** A	**Q.20** B	**Q.21** A

Multiple Correct Choice Type

Q.22 B, C	**Q.23** A, D	**Q.24** A, B, C, D	**Q.25** A, D	**Q.26** A, D	**Q.27** A, B, C	**Q.28** A, B, C, D
Q.29 A, B, D	**Q.30** B, D	**Q.31** A, B, D	**Q.32** A, B	**Q.33** D	**Q.34** A, D	**Q.35** A, B, D
Q.36 A, B, C	**Q.37** A, C	**Q.38** A, B	**Q.39** A, B, C, D			

Previous Years' Questions

Q.1 5	**Q.2** 3	**Q.3** B	**Q.4** C	**Q.5** D	**Q.6** B	**Q.7** D
Q.8 B	**Q.9** D	**Q.10** C	**Q.11** D	**Q.12** C	**Q.13** A	**Q.14** A, B, C
Q.15 A, C, D	**Q.16** A → q, r; B → p; C → p, s, t; D → q, r, s, t			**Q.17** 3	**Q.18** A, C, D	**Q.19** 5
Q.20 3	**Q.21** A	**Q.22** D	**Q.23** C	**Q.24** C, D	**Q.25** B, C, D	**Q.26** B
Q.27 C	**Q.28** C	**Q.29** 4	**Q.30** 1	**Q.31** A, C, D		

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $\bar{z} = i(z^2) \Rightarrow$ Let $x = a + ib$

$\Rightarrow a - ib = i(a^2 - b^2 + 2abi)$

$\Rightarrow a = -2ab$ and $-b = a^2 - b^2$

$a(1 + 2b) = 0$ and $a^2 = b^2 - b$

$a = 0$ or $b = -\dfrac{1}{2}$

if $a = 0 \Rightarrow b = 0, 1$

if $b = -\dfrac{1}{2}$, $a = \pm\dfrac{\sqrt{3}}{2}$

Complex numbers are $z = 0, i, \pm\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}$

Sol 2: $\dfrac{1+3i^2+2i}{1+3i^2-2i} = \dfrac{-2+2i}{-2-2i} = \dfrac{1-i}{1+i}$

$= \left(\dfrac{1-i}{1+i}\right)\left(\dfrac{1-i}{1-i}\right) = \dfrac{1+i^2-2i}{1-i^2} = -i$

Sol 3: $(x + iy)(2-3i) = 4 + i$

$\Rightarrow (2x + 3y) + i(2y - 3x) = 4 + i$

$\Rightarrow 2x + 3y = 4$ and $2y - 3x = 1$

$\Rightarrow x = \dfrac{5}{13}$ and $y = \dfrac{14}{13}$

Sol 4: $\dfrac{(1+i)x-2i}{3+i} + \dfrac{(2-3i)y+i}{3-i} = i$

$= \dfrac{[x+i(x-2)][3-i]+[3+i][2y+i(1-3y)]}{10}$

$= \dfrac{[3x+x-2+i(3x-6-x)]+[6y+3y-1+i(2y+3-9y)]}{10}$

$= \dfrac{4x-2+i(2x-6)+(9y-1)+i(-7y+3)}{10}$

$= \dfrac{4x+9y-3+i(2x-7y-3)}{10} = i$

$2x - 7y - 3 = 10$ and $4x + 9y - 3 = 0$

$\Rightarrow x = 3$ and $y = -1$

Sol 5: $x = a + b$

$y = \alpha a + b\beta$

$z = a\beta + b\alpha$

α and β are complex cube roots of unity

$\Rightarrow \alpha\beta = 1, \alpha^2 = \beta, \beta^2 = \alpha$

(as $\alpha = \omega, \beta = \omega^2, \alpha^2 = \beta$) ..... (i)

$xyz = (a + b)(\alpha a + b\beta)(a\beta + b\alpha)$

$= (\alpha a^2 + \alpha ab + \beta ab + b^2\beta)(a\beta + b\alpha)$

$= \alpha a^3\beta + \alpha a^2 b\beta + \alpha^2 b\beta^2 + ab^2\beta^2$

$+ \alpha^2 a^2 b + \alpha^2 ab^2 + ab^2 a\beta + b^3\alpha\beta$

$= \alpha\beta(a^3 + b^3 + a^2b + ab^2) + \alpha^2(a^2b + b^2a)$

$+ \beta^2(a^2b + b^2a)$

from eq^n. (i)

$= a^3 + b^3 + a^2b + ab^2 + (a^2b + b^2a)(\alpha^2 + b^2)$

$= a^3 + b^3 + a^2b + ab^2 + (a^2b + b^2a)(-1)$

$= a^3 + b^3 + a^2b + ab^2 - a^2b - ab^2$

$= a^3 + b^3$ hence proved.

Sol 6: $\dfrac{1+7i}{(2-i)^2} \times \dfrac{(2+i)^2}{(2+i)^2} = \dfrac{(1+7i)(2+i)^2}{25}$

$= \dfrac{(1+7i)(3+4i)}{25} = \dfrac{-25+25i}{25}$

$z = -1 + i$

$z(\theta) = |z|e^{iq}$

$|z| = \sqrt{1+1} = \sqrt{2}$

$\tan\theta = \dfrac{1}{-1} = -1$

$\Rightarrow \theta = \tan^{-1}(-1) = \dfrac{3\pi}{4}$

$\Rightarrow z(\theta) = \sqrt{2}e^{i\frac{3\pi}{4}} = \sqrt{2}\left[\cos\left(\dfrac{3\pi}{4}\right) + i\sin\left(\dfrac{3\pi}{4}\right)\right]$

Sol 7: $\sqrt{-8-6i} = a + ib$

$-8 - 6i = a^2 - b^2 + i(2ab)$

$\Rightarrow a^2 - b^2 = -8$ ...... (i)

$2ab = -6$

$\Rightarrow ab = -3$ ...... (ii)

Solving (i) and (ii), we get

$a = -1$ & $b = +3$

$a = 1$ & $b = -3$

So square root

$= (-1 + 3i)$ and $(1 - 3i)$

Sol 8: $\left(\dfrac{1+i}{1-i}\right)^n = 1$

$\dfrac{(1+i)^{2n}}{[(1+i)(1-i)]^n} = 1$

$\dfrac{(1+i)^{2n}}{2^n} = 1 \Rightarrow \dfrac{\left[(1+i)^2\right]^n}{2^n} = 1 \Rightarrow \dfrac{[2i]^n}{2^n} = 1$

$\Rightarrow i^n = 1 \Rightarrow n = 4, 8, 12$

Minimum value of n is 4

Sol 9: $\left|\dfrac{z-5i}{z+5i}\right| = 1$

$z = x + iy$

$\Rightarrow |x + i(y-5)| = |x + i(y+5)|$

$\Rightarrow x^2 + (y-5)^2 = x^2 + (y+5)^2$

$\Rightarrow y = 0$

i.e. complex part of z is zero.

z is pure real i.e. it lies on x axis.

Sol 10: $z = 1 + i\tan\alpha$

$|z| = \sqrt{1+\tan^2\alpha} = \sqrt{\sec^2\alpha} = |\sec\alpha|$

For $\alpha \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

$\sec\alpha < 0 \Rightarrow |\sec\alpha| = -\sec\alpha$

$|z| = -\sec\alpha$

$|z|\cos\alpha = -1$

Sol 11: $(x - 1)^3 = -8$

$x - 1 = (-8)^{1/3}$

$x - 1 = (8^{1/3})(-1)^{1/3} \Rightarrow 1 - x = (8)^{1/3}(1)^{1/3}$

$(-x + 1) = 2, 2\omega, 2\omega^2$

$x = -1, 1 - 2\omega, 1 - 2\omega^2$

Sol 12: $|z| < 4$

$|3 + i(z - 4)| < 9$(To prove)

We know that $|z_1 + z_2| \le |z_1| + |z_2|$

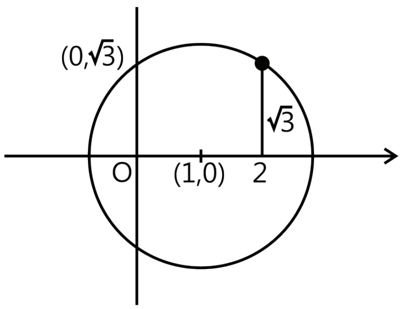
$|iz + (3 - 4i)| \le |z| + |3 - i4|$

$|z| < 4$

$\Rightarrow |iz + (3 - 4i)| < 4 + 5$

$\Rightarrow |iz + (3 - 4i)| < 9$ Hence proved

Sol 13: $2 + i\sqrt{3}$ is vertex of square inscribed in $|z - 1| = 2$



Here one vertex is $A(2,\sqrt{3})$ and equation of circle is $(x-1)^2 + y^2 = 4$

Radius of the circle is 2. Hence side of the square will be $2\sqrt{2}$.

Points that lie on the circle and are at a distance $2\sqrt{2}$ from A are $B(1-\sqrt{3},1)$ and $D(1+\sqrt{3},-1)$.

The point C will be the other end of diameter of A. Hence $C(0,-\sqrt{3})$.

Hence the four vertices are

$2+i\sqrt{3}$, $1-\sqrt{3}+i$, $-\sqrt{3}i$ and $1+\sqrt{3}-i$

Sol 14: $\left|\dfrac{z-\alpha}{z-\beta}\right| = k$

$\Rightarrow |z - \alpha|^2 = k^2|z - \beta|$

$\alpha = \alpha_1 + i\alpha_2$, $\beta = b_1 + i\beta_2$

$\Rightarrow (x - \alpha_1)^2 + (y - \alpha_2)^2 = k^2[(x - B_1)^2 + (y - \beta_2)^2]$

$\Rightarrow x^2 + \alpha_1^{\,2} + y^2 + \alpha_2^{\,2} - 2x\alpha_1 - 2y\alpha_2$

$= k^2x^2 + k^2y^2 + k^2\beta_1^{\,2} + k^2\beta_2^{\,2} - 2x\beta_1k^2 - 2y\beta_2k^2$

$\Rightarrow x^2(k^2-1) + y^2(k^2-1) + x(2\alpha_1 - 2\beta_1k^2)$

$+y(2\alpha_2-2\beta_2k^2)+k^2\beta_1^{\,2}+k^2\beta_2^{\,2}-\alpha_1^{\,2}-\alpha_2^{\,2}=0$

$$\Rightarrow x^2 + y^2 + x\left[\frac{2\alpha_1 - 2\beta_1k^2}{k^2-1}\right] + y\left[\frac{2\alpha_2 - 2\beta_2k^2}{k^2-1}\right] + \left[\frac{k^2\beta_1^2 + k^2\beta_2^2 - \alpha_1^2 - \alpha_2^2}{k^2-1}\right] = 0$$

Eq^n. of circle

with centre as $\left[\dfrac{\beta_1k^2-\alpha_1}{k^2-1}, \dfrac{\beta_2k^2-\alpha_2}{k^2-1}\right]$

or $\left[\dfrac{\alpha - k^2\beta}{1-k^2}\right]$

and radius $= \dfrac{(\alpha_1+\beta_1-k)^2 + (\alpha_2+\beta_2)^2}{1-k^2} = \dfrac{|\alpha+\beta-k|}{1-k^2}$

Sol 15: $|z| < \dfrac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$ ... (i)

$\Rightarrow |a_1z^1 + a_2z^2 + a_3z^3 \ldots\ldots a_nz^n| = 1$

$|z_1 + z_2 + z_3 + \ldots..z_n| \le |z_1| + |z_2| + \ldots..|z_n|$ ... (ii)

$\Rightarrow 1 \le |a_1z| + |a_2z^2| + |a_3z^3| \ldots..|a_nz^n|$

$\Rightarrow |a_1||z| + |a_2||z|^2 + |a_3||z|^3 \ldots..|a_n||z|^n \ge 1$

$\Rightarrow |z| + |z|^2 \ldots\ldots.|z|^n \ge \dfrac{1}{2}$

$\Rightarrow$ (Limiting case $n \to \infty$)

$\dfrac{|z|}{1-|z|} \ge \dfrac{1}{2} \Rightarrow |z| \ge \dfrac{1}{3} - (2)$

From (i) and (ii), we can say that there is no 'z' satisfying both conditions.

Sol 16: $z^{p+q} - z^p - z^q + 1 = 0$

$z^p(z^q - 1).(z^q - 1) = 0$

$(z^p - 1).(z^q - 1) = 0$

$z^p = 1$ or $z^q = 1$

If α is roots, then $\alpha^o, \alpha, \alpha^2, \ldots\ldots\ldots\ldots. \alpha^{n-1}$ also roots of $z^p = 1$ or $z^q = 1$

Sum of roots $= 1 + \alpha + \alpha^2 + \ldots\ldots\ldots\ldots\ldots\ldots..+ \alpha^{n-1} = 0$

Sol 17: z, iz, z + iz

(x, y) (–y, x) (x – y, y + x)

$$\Delta = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y & x & 1 \\ x-y & x+y & 1 \end{vmatrix} = \frac{1}{2}\,|-x^2 - y^2|$$

$\Rightarrow |D| = \frac{1}{2}(x^2 + y^2) = \frac{1}{2}|z|^2$

Sol 18: $iz^3 + i + z^2 - z = 0$

$iz(z^2 + i) + i(z^2 + i) = 0$

$(iz + 1)(z^2 + i) = 0$

Either $z = i$ or $z^2 = i$

If $z = I \Rightarrow |z| = |i| = 1$

If $z^2 = i \Rightarrow |z|^2 = 1 \Rightarrow |z| = 1$

Hence, $|z| = 1$

Sol 19: $1(2 - \omega)(2 - \omega^2) + 2(3 - \omega)(3 - \omega^2) + \ldots\ldots (n - 1)(n - \omega)(n - \omega^2)$

$T_n = n(n + 1 - \omega)(n + 1 - \omega^2)$

$= (n^2 + n - n\omega)(n + 1 - \omega^2)$

$= n^3 + n^2 - n^2\omega^2 + n^3 + n - n\omega^2 - n^2\omega - n\omega + n\omega^3$

$= n^3 + n^2(2 - \omega^2 - \omega) + n(1 - \omega - \omega^2 + 1)$

$= n^3 + 3n^2 + 3n$

$S_n = \Sigma T_n = \Sigma(n^3 + 3n^2 + 3n)$

$= \Sigma n^3 + 3\Sigma n^2 + 3\Sigma n$

$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + (2n+1) + 3\right]$$

$$= \frac{n(n+1)}{4}[n^2 + n + 6 + 4n + 2]$$

$$Sn = \Sigma T_n = \frac{n(n+1)}{4}[n^2 + 5n + 8]$$

$$S_{n-1} = \frac{(n-1)n}{4}[n^2 + 1 - 2n + 5n - 5 + 8]$$

$$= \frac{(n-1)n}{4}[n^2 + 3n + 4]$$

Sol 20: $x = \frac{1}{2}(5 - i\sqrt{3}) = \frac{1}{2}(6 - 1 - i\sqrt{3}) = 3 + w$

$x^4 - x^3 - 12x^2 + 23x + 12$

$= x^3(x - 3) + 2x^2(x - 3) - 6x(x - 3)$

$+ 5(x - 3) + 27$

$= (x - 3)[x^3 + 2x^2 - 6x + 5] + 27$

$= (x - 3)[(x - 3)(x^2 + 5x + 9) + 32] + 27$

$= (x - 3)^2[x^2 + 5x + 9] + 32(x - 3) + 27$

$= (x - 3)^2[x^2 - 3x + 8x - 24 + 33] + 32(x - 3) + 27$

$= (x - 3)^3(x + 8) + 33(x - 3)^2 + 32(x - 3) + 27$

$= \omega^3(x + 8) + 33(\omega^2) + 32\omega + 27$

$= \omega + 1 + 32\omega + 33\omega^2 + 27$

$= \omega + \omega^2 + 32\omega + 32\omega^2 + 38$

$= -1 - 32 + 38 = 5$

Sol 21: $z_2 - z_1 = (z_3 - z_1)e^{-i\pi/3}$

$z_3 - z_2 = (z_1 - z_2)e^{-i\pi/3}$

$$\frac{z_2 - z_1}{z_3 - z_2} = \frac{z_3 - z_1}{z_1 - z_2}$$

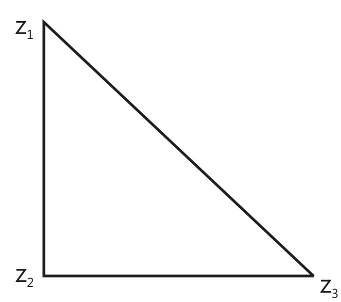
$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

$$\left(\frac{z_1 + z_2 + z_3}{3}\right)^2 = z_0^2$$

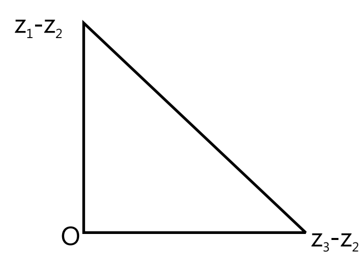
$$\Rightarrow \frac{z_1^2 + z_2^2 + z_3^2 + 2(z_1^2 + z_2^2 + z_3^2)}{9} = z_0^2$$

$\Rightarrow z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

Sol 22: z_1 z_2 z_3 of an isosceles ΔL at z_2



Considering z_2 at origin



$(z_1 - z_2) = (z_3 - z_2)e^{i\pi/2}$

$z_1 - z_2 = (z_3 - z_2)i$

$(z_1 - z_2)^2 = -(z_3 - z_1)^2$

$\Rightarrow z_1^2 + z_3^2 + 2z_2^2 = 2z_1z_3 + 2z_1z_2$

$\Rightarrow z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$

Sol 23: $\frac{A^2}{x-a} + \frac{B^2}{x-b} \ldots\ldots\ldots \frac{H^2}{x-H} = x + \ell$ Let $x = p + iq$

$$(p+\ell) + iq = \sum \frac{A^2}{(p-a)+iq} = \sum \frac{A^2[(p-a)-iq]}{(p-a)^2+q^2}$$

$$= \sum \frac{A^2(p-a)}{(p-a)^2+q^2} - i\sum \frac{A^2 q}{(p-a)^2+q^2}$$

Equating the imaginary terms we get

$$q + \sum \frac{A^2 q}{(p-a)^2+q^2} = 0$$

$$\Rightarrow q\left(1 + \sum \frac{A^2}{(p-a)^2+q^2}\right) = 0$$

$$\Rightarrow q = 0 \text{ or } \sum \frac{A^2}{(p-a)^2+q^2} = -1$$

Here A, p, a and q are all real hence q = 0 is the only solution.

$\Rightarrow$ x cannot have imaginary roots.

Sol 24: $z^3 + 2z^2 + 2z + 1 = 0$

$\Rightarrow z = -1, \omega, \omega^2$

$z^{1985} + z^{100} + 1 = 0$

$\Rightarrow z = \omega, \omega^2$

Common roots are ω, ω^2

Sol 25: Let $f(x,y) = (x+y)^n - x^n - y^n$

$xy(x+y)(x^2+xy+y^2) = (x-0)(y-0)(x+y)(x-wy)(x-w^2y)$

$f(x=0)=0$ and $f(y=0)=0$

$f(y=-x)=(x-x)^n-(x)^n-(-x)^n=0$

$f(x=wy)=(wy+y)n-w^ny^n-y^n$; $= yn[w2n + wn + 1] = 0$

Similarly $f(x=w^2y)=0$

$\therefore$ f(x) is divisible by $xy(x+y)(x^2+xy+y^2)$

Sol 26: $T = \left|\alpha + \sqrt{\alpha^2-\beta^2}\right| + \left|\alpha - \sqrt{\alpha^2-\beta^2}\right|$

$T^2 = \left|\alpha + \sqrt{\alpha^2-\beta^2}\right|^2 + \left|\alpha - \sqrt{\alpha^2-\beta^2}\right|^2 +$

$$2\left|\alpha + \sqrt{\alpha^2-\beta^2}\right|\left|\alpha - \sqrt{\alpha^2-\beta^2}\right|$$

$$= \left(\alpha + \sqrt{\alpha^2+\beta^2}\right)\left(\overline{\alpha + \sqrt{\alpha^2-\beta^2}}\right) + \left(\alpha + \sqrt{\alpha^2-\beta^2}\right)$$

$$\left(\overline{\alpha - \sqrt{\alpha^2-\beta^2}}\right) + 2|\alpha^2 - a^2 + \beta^2|$$

$$= 2\left[\left|\alpha^2\right| + \left|\sqrt{\alpha^2-\beta^2}\right|^2\right] + 2|\beta|^2$$

$$= 2|\alpha|^2 + 2|\beta|^2 + 2\left|\alpha^2-\beta^2\right|$$

$$= 2|\alpha|^2 + 2|\beta|^2 + 2|\alpha+\beta||\alpha-\beta|$$

$$= \alpha\bar{\alpha} + \beta\bar{\beta} + \alpha\bar{\beta} + \bar{\alpha}\beta + \alpha\bar{\alpha} + \beta\bar{\beta} -$$

$$\alpha\bar{\beta} - \beta\bar{\alpha} + 2|\alpha+\beta||\alpha-\beta|$$

$$= |\alpha+\beta|^2 + |\alpha-\beta|^2 + 2|\alpha+\beta||\alpha-\beta|$$

$$= \left[|\alpha+\beta| + |\alpha-\beta|\right]^2$$

$T = |\alpha+\beta| + |\alpha-\beta|$ Hence proved

Sol 27: $z_1 = 10 + 6i$; $z_2 = 4 + 6i$

$$\frac{z-z_1}{z-z_2} = \frac{x-10+(y-6)i}{x-4+(y-6)i}\left(\frac{(x-4)-i(y-6)}{(x-4)-i(y-6)}\right)$$

$$= \frac{(x-10)(x-4)+(y-6)^2+i[(y-6)(x-4)+(x-10)(6-y)]}{(x-4)^2+(y-6)^2}$$

$$\arg\frac{z-z_1}{z-z_2} = \frac{\pi}{4}$$

$\Rightarrow (y-6)(x-4)+(x-10)(6-y)$

$= (x-10)(x-4) + (y-6)^2$

$\Rightarrow (y-6)(x - 4 - x + 10) = x^2 - 14x + 40 + (y-6)^2$

$\Rightarrow 6(y-6) = x^2 - 14x + 40 + y^2 + 36 - 12y$

$\Rightarrow x^2 + y^2 - 18y - 14x + 112 = 0$

$\Rightarrow (x-7)^2 + (y-9)^2 = 18$

$\Rightarrow |z - 7 - 9i| = 3\sqrt{2}$

Sol 28: $|z-w|^2 = (z-w)(\bar{z}-\bar{w})$

$$= |z|^2 + |w|^2 - \bar{z}w - z\bar{w} + 2|z||w| - 2|z||w|$$

$$= (|z| - |w|)^2 - \bar{z}w - z\bar{w} + 2|z||w|$$

Let $z = r_1\cos\theta_1$; $w = r_2\cos\theta_2$

$$= \left(|z|-|w|\right)^2 - 2r_1r_1\cos\left(\theta_1-\theta_2\right)+2r_1r_2$$

$$= \left(|z|-|w|\right)^2 + 2r_1r_1\left(2\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)\right)$$

$$= \left(|z|-|w|\right)^2 + 4r_1r_1\left(\sin\frac{\theta_1-\theta_2}{2}\right)^2$$

We know, $\sin\theta \le \theta$ and $r_1, r_2, \le 1$

$$\le \left(|z|-|w|\right)^2 + 4\times\left(\frac{\theta_1-\theta_2}{2}\right)^2$$

$$\le \left(|z|-|w|\right)^2 + \left(\theta_1-\theta_2\right)^2$$

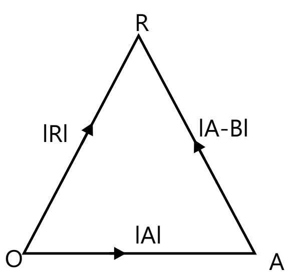
$$\le \left(|z|-|w|\right)^2 + \left(\arg|z|-\arg|w|\right)^2$$

Sol 29: $\frac{A}{B}+\frac{B}{A}=1$

$A^2 + B^2 = AB$

$$\frac{A}{B}=1-\frac{B}{A}$$

$$\frac{|A|}{|B|}=\frac{|A-B|}{|A|}$$



$$\frac{|A^2|}{|B|}=|A-B|$$

$$\frac{|B^2|}{|A|}=|B-A|$$

$$|A-B|=|B-A| \Rightarrow \left|\frac{A^2}{B}\right|=\left|\frac{B^2}{A}\right|$$

$\Rightarrow |A^3| = |B^3|$

$\Rightarrow |A| = |B| = |A - B|$

i.e. all sides are equal it forms an equilateral D

Sol 30: $a + b + c = 0 \Rightarrow b = -a - c$

$az_1 + bz_2 + cz_3 = 0$

$az_1 + cz_3 - (a + c)z_2 = 0$

$$z_2 = \frac{az_1+cz_3}{a+c}$$

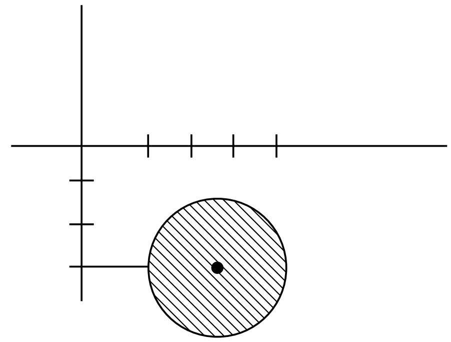
$\Rightarrow a(z_1 - z_2) + c(z_3 - z_2) = 0$

$$\Rightarrow (z_1 - z_2) = \frac{-c}{a}(z_3 - z_2)$$

$z_1 - z_2 = k_1(z_3 - z_2)$

$\Rightarrow z_1, z_2$ and z_3 are collinear (by vector)

Sol 31: $|z - 4 + 3i| \le 2$



The shaded area show z

Min and max value = Distance of centre from origin ± radius

$= 5 \pm 2 = 3, 7$

$\Rightarrow 3 \le |z| \le 7$

Sol 32: $\left|\frac{z_1-z_2}{1-\bar{z}_1z_2}\right|<1$

$$|z_1-z_2|^2<|1-\bar{z}_1z_2|^2$$

$$(z_1-z_2)\ (\bar{z}_1-\bar{z}_2)\ <\ (1-\bar{z}_1z_2)(1-z_1\bar{z}_2)$$

$\Rightarrow |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \bar{z}_1Z_2$

$< -z_1\bar{z}_2 - z_2\bar{z}_1 + |z_1|^2|z_2|^2 + 1$

$\Rightarrow |z_1|^2 - 1 < (|z_1|^2 - 1)\ |z_2|^2$

$$|z_2|^2 < \frac{|z_1|^2-1}{|z_1|^2-1}$$

$|z_2|^2 < 1 \Rightarrow |z_2| < 1$

Sol 33: $\log_{\sqrt{3}}\left[\frac{|z|^2-|z|+1}{|z|+2}\right]<2$

$$\frac{|z|^2-|z|+1}{|z|+2}<3$$

$\Rightarrow |z|^2 - |z| + 1 < 3|z| + 6$

$\Rightarrow |z^2| - 4|z| - 5 < 0$

$\Rightarrow (|z| + 1)\ (|z| - 5) < 0$

$\Rightarrow |z| + 1 \geq 0$

So $(|z| - 5) < 0$

Interior of circle $x^2 + y^2 = 25$

Sol 34: $|z|^2 \omega - |\omega^2|z = z - \omega$

$\Rightarrow z\bar{z}\omega - \omega\bar{\omega}z = z = \bar{\omega}$

$\Rightarrow z\,\omega\,(\bar{z} - \bar{\omega}) = z - \omega$

$\Rightarrow z(\bar{z}\omega - 1) = \omega(z\bar{\omega} - 1)$ ..... (i)

$\Rightarrow z(z\bar{\omega} - 1) = \bar{\omega}(\bar{z}\bar{\omega} - 1)$

$\Rightarrow$ Either $z\bar{\omega} = \bar{z}\omega = 1$ or $\bar{z} = \bar{\omega}$

$\frac{x}{\omega} = \frac{\bar{\omega}}{z} \Rightarrow z\bar{z} = \omega\bar{\omega} \Rightarrow |z| = |\omega|$

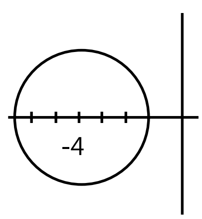
$|z|^2(\omega - z) = (z - \omega)$

$(\omega - z)(|z|^2 + 1) = 0 \Rightarrow \omega = z$

Exercise 2

Single Correct Choice Type

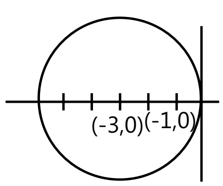
Sol 1: (C) $|z + 4| \leq 3$ Least and greatest value of $|z + 1|$



i.e. distance of z from (–1, 0)

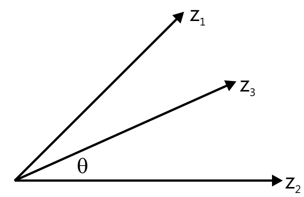
Least is 0; maximum is 2r = 6

Sol 2: (D) $|z + 3| \leq 3$ Least & greatest value of $|z + 1|$ ie its minimum and maximum distance from (–1,0) is 1 and 5.



Sol 3: (B) $z_1 = 3 + i\sqrt{3}$

$z_2 = 2\sqrt{3} + 6i$



$\hat{z}_3 = \hat{z}_2 e^{i\theta}$; $\hat{z}_1 = \hat{z}_3 e^{i\theta}$

$\hat{z}_3^2 = \hat{z}_1\hat{z}_2 = \frac{24i}{24}$

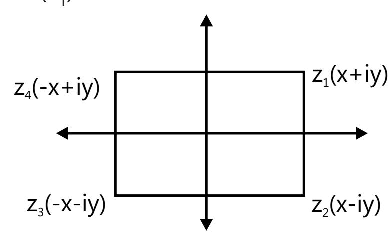
$\hat{z}_3 = \sqrt{i} = \frac{1+i}{\sqrt{2}}$

Unit vector along bisector is $\frac{1+i}{\sqrt{2}}$, complex number lying along this vector is 5(1+i)

Sol 4: (C) z_1, z_2, z_3, z_4 vertices of square

$z_1 - z_2 = i2\text{Im}(z_1)$

$z_3 - z_2 = -2\text{Re}(z_1)$



$\frac{z_1 - z_2}{z_3 - z_2}$ is imaginary

$z_2 - z_4 = [-x + iy - (x - iy)]$

$= 2\text{Re}(z) - i2\text{Im}(z)$

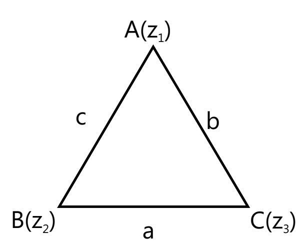
$z_1 - z_3 = 2x + 2iy = 2\text{Re}(z) + i2\text{Im}(z)$

$\frac{z_1 - z_3}{z_2 - z_4} = \frac{2x + 2iy}{2x - 2iy} = \frac{x + iy}{x - iy} = \frac{x^2 - y^2 + 2ixy}{x^2 + y^2}$

(x = y as it is aqueous)

$= \frac{2ixy}{x^2 + y^2}$ (Purely Imaginary)

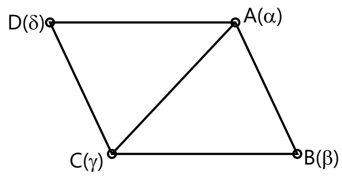
Sol 5: (D) $\frac{az_1 + bz_2 + cz_3}{a + b + c} = z_0$



z_0 is incentre.

Sol 6: (A) $\delta - \gamma = \alpha - \beta$[parallel vector]

$\delta = \alpha + \gamma - \beta$

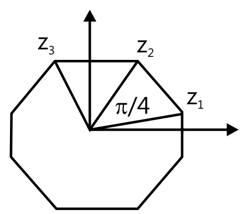


Sol 7: (B) $|z| \geq 3$

$$\left|z + \frac{1}{z}\right| \geq |z| - \frac{1}{|z|} = 3 - \frac{1}{3} = \frac{8}{3}$$

Sol 8: (C)

$$z_2 = z_1 e^{\pm\frac{i\pi}{4}} \qquad z_3 = z_2 e^{\pm\frac{i\pi}{4}}$$



$$z_3 - z_2 = (z_2 - z_1)\,\frac{(1 \pm i)}{\sqrt{2}}$$

$$z_3 = z_2 + (z_2 - z_1)\,\frac{(1 \pm i)}{\sqrt{2}}$$

Sol 9: (D) $x^3 = (4)^3\,(-1)^{1/3}$

$x = -4, -4\omega, -4\omega^2$

$$\begin{vmatrix} q_1 & q_2 & q_3 \\ q_2 & q_3 & q_1 \\ q_3 & q_1 & q_2 \end{vmatrix} = \begin{vmatrix} -4 & -4\omega & -4\omega^2 \\ -4\omega & -4\omega^2 & -4 \\ -4\omega^2 & -4 & -4\omega \end{vmatrix}$$

$$= -64\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = -64\begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$$

Sol 10: (D) $z = (3 + 7i)(p + iq)$ is purely imaginary.

$\Rightarrow 3p - 7q = 0$

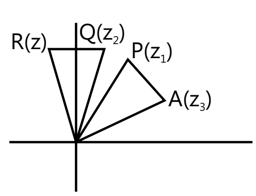
$\Rightarrow p = \frac{7q}{3}$

$|z|^2 = |3 + 7i|^2\,|p + iq|^2 = 58\left(\sqrt{p^2 + q^2}\right)^2$

for minimum $|z|$, $q = 3$, $p = 7$

$|z|^2 = 58(49+9) = 3364$

Sol 11: (A)



$$\frac{z_1}{z_3} = \frac{z}{z_2} \quad \Rightarrow z = \frac{z_1 z_2}{z_3}$$

$|z_3| = 1 \Rightarrow |z| = |z_1 z_2|$

$\Rightarrow z = z_1 z_2$

Sol 12: (A) $\frac{A}{B} + \frac{B}{A} = 1$

Let $y = \frac{A}{B}$

$y + \frac{1}{y} = 1$

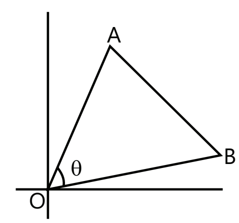
$\Rightarrow y^2 - y + 1 = 0$

$\Rightarrow y = \frac{1 + i\sqrt{3}}{2}$

$\Rightarrow \frac{A}{B} = \frac{1 + i\sqrt{3}}{2}$

$\Rightarrow \left|\frac{A}{B}\right| = 1$

From Rotation Theorem



$\frac{A}{B} = \left|\frac{A}{B}\right| e^{i\theta}$

$\Rightarrow e^{i\theta} = \frac{1 + i\sqrt{3}}{2}$

$\Rightarrow \theta = 60^o$

and $|A| = |B| \Rightarrow \angle OAB = \angle OBA = 60^o$

$\Rightarrow$ AOB is equilateral triangle

Sol 13: (D) $|z|^2 - (z + \bar{z}) + i(z - \bar{z}) + 2 = 0$

$z\bar{z} - (z + \bar{z}) + i(z - \bar{z}) + 2 = 0$

Let $z = a + ib$

$a^2 + b^2 - 2a + i(i2b) + 2 = 0$

$a^2 - 2a + 1 + b^2 - 2b + 1 = 0$

$(a - 1)^2 + (b - 1)^2 = 0$

$\Rightarrow a = b = 1$

$\Rightarrow z = 1 + i$

Sol 14: (D) $z_1 = -3 + 5i$

$z_2 = -5 - 3i$

Eq^n of line $y = 4x + 17$

$\arg z = \tan^{-1}\left(\frac{4x + 17}{x}\right) = \tan^{-1}\left(4 + \frac{17}{x}\right)$

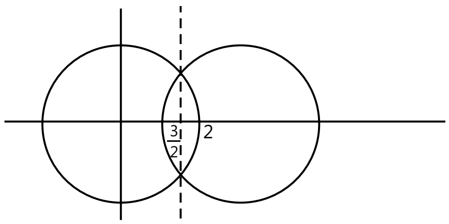
$x \in [-3, -5] \Rightarrow \arg z \in \left[\tan^{-1}\frac{-5}{3}, \tan^{-1}\frac{-3}{5}\right]$

Only option left is $\left(\frac{5\pi}{6}\right)$

Sol 15: (B) $|z-3| = 2 : (x-3)^2 + y^2 = 4$

$|z| = 2 : x^2 + y^2 = 4$

Points of intersection lie on the radical axis $S_1 - S_2 = 0$



Radical axis is $x = \frac{3}{2}$

$x^2 + y^2 = 4$

$\Rightarrow \frac{9}{4} + y^2 = 4 \Rightarrow y^2 = 4 - \frac{9}{4}$

$\Rightarrow y^2 = \frac{7}{4} \Rightarrow y = \frac{\pm\sqrt{7}}{2}$

Complex no. is $\frac{3}{2} \pm \frac{i\sqrt{7}}{2}$

Sol 16: (C) $|z - 3i| = 3 \Rightarrow (x)^2 + (y-3)^2 = 9$

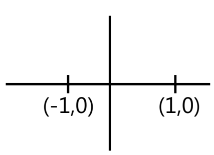
$\Rightarrow \tan\theta = \frac{y}{x}$

$\cot\theta \ \frac{6}{z} = \frac{x}{y} - \frac{6(x-iy)}{(x+iy)(x-iy)} = \frac{x}{y} - \frac{6(x-iy)}{x^2+y^2}$

$x^2 + y^2 = 6y$

$\Rightarrow \frac{x}{y} - \frac{6(x-iy)}{6y} = i$

Sol 17: (D) $|z - 1| + |z + 1| = 2$



The portion of real axis between (–1,0) & (1,0) as the distance between both the point is 2

Sol 18: (C) $Q = \sqrt{2|z|^2}\ \text{cis}\left(\frac{\pi}{4} + \theta\right)$

$= |z|\ \sqrt{2}\left[\cos\left(\frac{\pi}{4} + \theta\right) + i\sin\left(\frac{\pi}{4} + \theta\right)\right]$

$= |z|[(\cos\theta - \sin\theta) + i(\sin\theta + \cos\theta)]$

$= |z|[(\cos\theta + i\sin\theta) + \frac{|z|i}{i}(-\sin\theta + i\cos\theta)]$

$= |z|\ [\cos\theta + i\sin\theta] + \frac{|z|}{i}[-\cos\theta - i\sin\theta]$

$= P + iP$

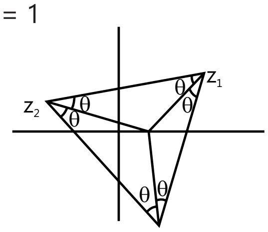
$\Rightarrow \arg\left(\frac{Q-P}{P}\right) = \frac{\pi}{2}$

This is right angle triangle with |P| = |Q–P| and right angle at P.

Sol 19: (A) $|z_1 - 1| = |z_2 - 1| = |z_3 - 1|$

$z_1 + z_2 + z_3 = 3$

Centroid is at z = 1



Distance of vertical z_1, z_2 and z_3 from centroid is same, which mean centroid coincides with circumcenter.

Therefore, Δ is equilateral.

Sol 20: (C) $p = a + b\omega + c\omega^2$

$q = b + c\omega + a\omega^2$

$r = c + a\omega + b\omega^2$

$p + q + r = a(1 + \omega + \omega^2) + b(1 + \omega + \omega^2)$

$+ c(1 + \omega + \omega^2)$

$= (a + b + c)(1 + \omega + \omega^2) = 0$

$p^2+q^2+r^2=(p + q + r)^2 - 2pq - 2qr - 2rp$

$= -2(pq + qr + rp)$

$pq + qr + rp = ab + ac\omega + a^2\omega^2 + b^2\omega^2 + b^2\omega + bc\omega^2 + ab\omega^3 + cb\omega^2 + c^2\omega^3 + ca\omega + \ldots\ldots.$

$= ab+2ac\omega+ab+c^2+(a^2+2bc)\omega^2+b^2\omega$

$= 2ab + c^2 + 2ac\omega + (b^2 + 2bc)\omega^2 + a^2\omega + 2bc + a^2 + 2ba\omega + (b^2 + 2ac)\omega^2 + c^2\omega + 2ac + b^2 + 2bc\omega + (c^2 + a^2 + 2ab)\omega^2$

$= (2ab + 2bc + 2ac)(1 + \omega + \omega^2)$

$+ (c^2 + b^2 + a^2)(1 + \omega^2 + \omega)$

$p^2 + q^2 + r^2 = 0 = 2(pq + qr + rp)$

Previous Years' Questions

Sol 1: (D) Since, $\left(\frac{1+i}{1-i}\right)^n = 1$

$\Rightarrow \left(\frac{1+i}{1-i}\times\frac{1+i}{1+i}\right)^n = 1$

$\Rightarrow \left(\frac{2i}{2}\right)^n = 1 \Rightarrow i^n = 1$

The smallest positive integer n for which $i^n = 1$ is 4

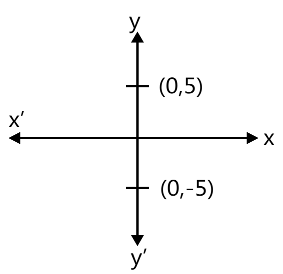
$\therefore$ n = 4

Sol 2:(A) Given, $\left|\frac{z-5i}{z+5i}\right| = 1$

$\Rightarrow |z - 5i| = |z + 5i|$

(if $|z - z_1| = |z - z_2|$,

Then it is a perpendicular bisector of z_1



$\therefore$ Perpendicular bisector of (0, 5) and (0, – 5) is x-axis.

Sol 3: (B) Since, |w| = 1

$\Rightarrow \left|\frac{1-iz}{z-i}\right| = 1 \Rightarrow |z - i| = |1 - iz|$

$\Rightarrow |z - i| = |z + i|$

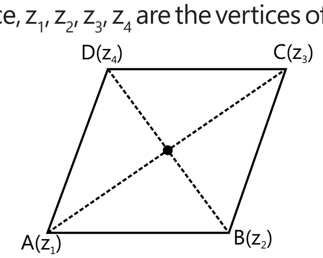
$(\because |1 - iz| = |- i|\,|z + i| = |z + i|)$

$\therefore$ It is a perpendicular bisector of (0, 1) and (0, –1)

i.e., x-axis

Thus, z lies on real axis.

Sol 4: (B) Since, z_1, z_2, z_3, z_4 are the vertices of parallelogram.



$\therefore$ Mid-point of AC = mid-point of BD

$\Rightarrow \frac{z_1+z_3}{2} = \frac{z_2+z_4}{2} \Rightarrow z_1 + z_3 = z_2 + z_4$

Sol 5: (C) Given, $|z_1 + z_2| = |z_1| + |z_2|$

On squaring both sides, we get

$|z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2|\cos(\arg z_1 - z_2)$

$= |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2|$

$\Rightarrow 2|z_1|\,|z_2| \cos(\arg z_1 - \arg z_2) = 2\,|z_1|\,z_2|$

$\Rightarrow \cos(\arg z_1 - \arg z_2) = 1$

$\Rightarrow \arg(z_1) - \arg(z_2) = 0$

Sol 6: (D) Since, $\overline{\sin x + i\cos 2x}$

$= \cos x - i \sin 2x$

$\Rightarrow \sin x - i\cos 2x = \cos x - i \sin 2x$

$\Rightarrow \sin x = \cos x$ and $\cos 2x = \sin 2x < x$

$\Rightarrow \tan x = 1$ and $\tan 2x = 1$

$\Rightarrow x = \frac{\pi}{4}$ and $x = \frac{\pi}{8}$ which is not possible at same time.

Hence, no such value exists.

Sol 7: (B) $(1 + \omega)^7 = (1 + \omega)(1 + \omega)^6$

$= (1 + \omega)(-\omega^2)^6$

$= 1 + \omega$

$\Rightarrow A + B\omega = 1 + 0 \Rightarrow A = 1, B = 1$

Sol 8: (D) Since, $|z| = |\omega|$ and $\arg(z) = \pi - \arg(\omega)$

Let $\omega = re^{i\theta}$, then $\bar{\omega} = re^{-i\theta}$

$\therefore z = re^{(\pi-\theta)} = re^{i\pi}.e^{-i\theta}$

$= -re^{-i\theta} = -\bar{\omega}$

$\therefore$ z lies on the perpendicular bisector of the line joining $-i\omega$ and $-i\bar{\omega}$ is the mirror image of $-i\omega$ in the x-axis, the locus of z is the x-axis

Let $z = x + iy$ and $y = 0$

Now, $|z| \le 1 \Rightarrow x^2 + 0^2 \le 1$

$\Rightarrow -1 \le x \le 1$ ..... (i)

$\therefore$ z may take values given in (i).

Sol 9: (D)

$(1+\omega-\omega^2)^7 = (-\omega^2-\omega^2)^7 \qquad (\because 1+\omega+\omega^2 = 0)$

$= (-2\omega^2)^7 = (-2)^7\omega^{14} = -128\omega^2$

Sol 10: (B)

$$\sum_{n=1}^{13}(i^n+i^{n+1})=\sum_{n=1}^{13}i^n(1+i)=(1+i)\sum_{n=1}^{13}i^n$$

$$=(1+i)(i+i^2+i^3+\ldots..+i^{13})=(1+i)\left[\frac{i(1-i)}{1-i}\right]$$

$$=(1+i)i=-1+i$$

Alternate solution:

Since, sum of any four consecutive powers of iota is zero.

$$\therefore \sum_{n=1}^{13}(i^n+i^{n+1})=(i+i^2+\ldots..+i^{13})+(i^2+i^3+\ldots..+i^{14})$$

$$=i+i^2=i-1$$

Sol 11: (D)

Given, $\begin{vmatrix}6i & -3i & 1\\ 4 & 3i & -1\\ 20 & 3 & i\end{vmatrix}=x+iy$

$\Rightarrow\quad -3i\begin{vmatrix}6i & 1 & 1\\ 4 & -1 & -1\\ 20 & i & i\end{vmatrix}=x+iy$

$\Rightarrow\quad x+iy=0 \qquad (\because C_2 \text{ and } C_3 \text{ are identical})$

$\Rightarrow\quad x=0, y=0$

Sol 12: (A)

Given, $|z_1|=|z_2|=|z_3|=1$

Now, $|z_1|=1$

$\Rightarrow\quad |z_1|^2=1$

$\Rightarrow\quad z_1\bar{z}_2=1, z_3\bar{z}_3=1$

Again now, $\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|=1$

$\Rightarrow\quad |\bar{z}_1+\bar{z}_2+\bar{z}_3|=1$

$\Rightarrow\quad |\overline{z_1+z_2+z_3}|=1$

$\Rightarrow\quad |z_1+z_2+z_3|=1$

Sol 13: (B)

Let $\Delta=\begin{vmatrix}1 & 1 & 1\\ 1 & -1-\omega^2 & \omega^2\\ 1 & \omega^2 & \omega\end{vmatrix}$

Applying $R_2\to R_1 : R_3\to R_3-R_1$

$$=\begin{vmatrix}1 & 1 & 1\\ 0 & -2-\omega^2 & \omega^2-1\\ 0 & \omega^2-1 & \omega-1\end{vmatrix}$$

$$=(-2-\omega^2)(\omega-1)-(\omega^2-1)^2$$

$$=-2\omega+2-\omega^3+\omega^2-(\omega^4-2\omega^2+1)$$

$$=3\omega^2-3\omega=3\omega(\omega-1) \qquad (\because \omega^4=\omega)$$

Sol 14: (B)

Given, $(1+\omega^2)^n=(1+\omega^4)^n$

$\Rightarrow\quad (-\omega)^n=(-\omega^2)^n \qquad (\because \omega^3=1 \text{ and } 1+\omega+\omega^2=0)$

$\Rightarrow\quad \omega^n=1$

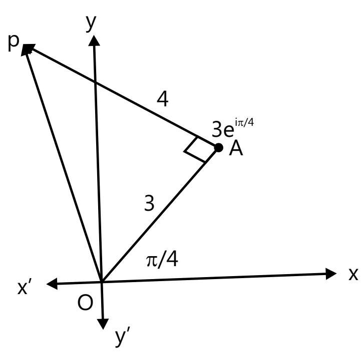
$\Rightarrow\quad n=3$ is the least positive value of n.

Sol 15: (D) Let OA=3, so that the complex number associated with A is $3e^{i\pi/4}$. If z is the complex number associated with P, then

$$\frac{z-3e^{i\pi/4}}{0-3e^{i\pi/4}}=\frac{4}{3}e^{-i\pi/2}=-\frac{4i}{3}$$

$\Rightarrow\quad 3z-9e^{i\pi/4}=12ie^{i\pi/4}$

$\Rightarrow\quad z=(3+4i)e^{i\pi/4}$



Sol 16: (A) Given: $\frac{z^2}{z-1}$ is real

Let z = a + ib, then

$$\frac{z^2}{z-1}=\frac{(a+ib)^2}{a+ib-1}=\frac{a^2-b^2+2aib}{[(a-1)+ib]}\times\frac{[(a-1)-ib]}{[(a-1)-ib]}$$

$$\Rightarrow \frac{-b(a^2-b^2)+2ab(a-1)}{(a-1)^2+b^2}=0$$

$$\Rightarrow -a^2b+b^3+2a^2b-2ab=0$$

$$\Rightarrow a^2b+b^3-2ab=0$$

$$\Rightarrow b(a^2+b^2-2a)=0$$

$\Rightarrow b = 0 \text{ or } a^2 + b^2 - 2a = 0$

$\Rightarrow$ Either real axis or circle passing through origin.

Sol 17: (B) Let $\theta = \arg\left(\frac{1+z}{1+\bar{z}}\right)$

$\Rightarrow \theta = \arg\left(\frac{1+z}{1+\frac{1}{z}}\right) \{|z| = 1 \Rightarrow z\,\bar{z} = 1\}$

$\Rightarrow \theta = \arg(z)$

Sol 18: (B) Given: The expression $\left|z + \frac{1}{2}\right|$ and $|z| \geq 2$

Using triangle in equality

$|z_1 + z_2| \geq ||z_1| - |z_2||$

$\Rightarrow \left|z + \frac{1}{2}\right| \geq \left||z| - \left|\frac{1}{2}\right|\right|$

$\Rightarrow \left|z + \frac{1}{2}\right| \geq \left|z - \frac{1}{2}\right|$

$\Rightarrow \left|z + \frac{1}{2}\right| \geq \frac{3}{2}$ lies in (1, 2)

Sol 19: (B) $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2} = 1$

$\Rightarrow |z_1 - 2z_2| = |2 - z_1\,\bar{z}_2|$

$\Rightarrow (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\,\bar{z}_2)(2 - \bar{z}_1\,z_2)$

$\Rightarrow |z_1|^2 - 2\,z_1\,\bar{z}_2 - 2\bar{z}_1\,z_2 + 4|z_2|^2$

$= 4 - 2\bar{z}_1\,z_2 - 2z_1\,\bar{z}_2 + |z_1|^2\,|z_2|^2$

$\Rightarrow |z_1|^2 + 4|z_2|^2 = 4 + |z_2|^2\,|z_1|^2$

$\Rightarrow |z_1|^2 + \left(1 - |z_2|^2\right) - 4\left(1 - |z_2|^2\right) = 0$

$\Rightarrow \left(1 - |z_2|^2\right)\left(|z_1|^2 - 4\right) = 0$

$\Rightarrow |z_1|^2 = 4 \quad \{|z_2| \neq 1\}$

$\Rightarrow |z_1| = 2$

Sol 20: (C) $\frac{2 + 3i\sin\theta}{1 - 2i\sin\theta}$

$= \frac{(2 + 3i\sin\theta)(1 + 2i\sin\theta)}{1 + 4\sin^2\theta}$

$= \frac{2 + 4i\sin\theta + 3i\sin\theta - 6\sin^2\theta}{1 + 4\sin^2\theta}$

Given $\frac{2 - 6\sin^2\theta}{1 + 4\sin^2\theta} = 0$

$\Rightarrow \sin^2\theta = \frac{1}{3}$

$\Rightarrow \sin\theta = \pm\frac{1}{\sqrt{3}}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right), -\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

JEE Advanced/Boards

Exercise 1

Sol 1: $z^2 + (p + ip')z + (q + iq') = 0$

One real root

$(z^2 + pz + q) + i(p'z + q') = 0$

If z is real

$P'z = -q' \Rightarrow z = \frac{-q'}{p'}$

$z^2 + pz + q = 0$

$\frac{q'^2}{p'^2} - \frac{pq'}{p'} + q = 0$

$q'^2 - pp'q' + qp'^2 = 0$

If eq^n. has 2 equal roots

$(p + ip')^2 = 4(q + iq')$

$p^2 - p'^2 = 4q$ and $p'^2 = 4q^2$

The roots are $\frac{-(p + ip')}{2}$ i.e. roots (equal) are imaginary.

Sol 2: $z = 18 + 26i$

$z_0 = x_0 + iy_0$ is cube root of z

$z_0 = (r\{\cos\theta + i\sin q\})^{1/3} = r^{1/3}\,e^{i\theta/3}$

$= r^{1/3}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)$

$r = (1000)^{1/2}$

$r^{1/3} = (10^{3/2})^{1/3} = 10^{1/2}$

$\cos\theta = \frac{18}{\sqrt{1000}}$ $\sin\theta = \frac{26}{\sqrt{1000}}$

$\cos\theta = 0.57, \sin\theta = 0.82$

$\cos\theta = 3\cos\frac{\theta}{3} - 4\cos^3\left(\frac{\theta}{3}\right)$

$0.57 = 3t - 4t^3$

$\cos\left(\frac{\theta}{3}\right) = 0.2, 0.74, 0.95$

$\cos\frac{\theta}{3} = 0.95$

$\sin\frac{\theta}{3} = 0.3162$

$z = \sqrt{10}\,(0.95 + i\,0.3162) = 3 + i$

$x_0 = 3, y_0 = 1$

$x_0y_0(x_0 + y_0) = 3.4 = 12$

Sol 3: $z^3 + iz = 1$

$z(z^2 + i) = 1$

z can never be Purely real as $(z^2 + i)$ will be imaginary which multiplied with z will not be a real number, hence its locus will never cut x-coo axis Z can never be imaginary (pure) as $(z^2 + i)$ will be of form (a + ib) which multiplied by z, cannot form 1 (real) therefore its locus can never cut y axis.

$z(z^2 + i) = 1$

$(x + iy)\,[x^2 - y^2 + i(2xy + 1)] = 1$

$x^3 - xy^2 + i(2x^2y + x + x^2y - y^3) - 2xy^2 - y = 1$

$x^3 - xy^2 - 2xy^2 - y + i(3x^2y + x - y^3) = 1$

$x^3 - 3xy^2 - y = 1\ \&\ 3x^2y + x - y^3 = 0$

$x(x^2 - 3y^2) = 1 + y\ \&\ y(y^2 - 3x^2) = x$

$|z|^2 = -\frac{1}{2}\left[\frac{1+y}{x} + \frac{x}{y}\right] \Rightarrow |z|^2 = -\frac{1}{2}\left[\frac{y + x^2 + y^2}{xy}\right]$

$|z|^2 = -\frac{1}{2xy}\left[y + |z|^2\right] \Rightarrow |z|^2\left[\frac{2xy+1}{2xy}\right] = \frac{-y}{2xy}$

$\Rightarrow |z| = \sqrt{\frac{-\mathrm{Im}(z)}{2\mathrm{Re}(z)\mathrm{Im}(z)+1}}$

Sol 4: $A_n = \mathrm{dia}(d_1, d_2, \ldots\ldots d_n)$

$d_i = a^{i-1};\ \alpha = e^{i2\pi/n}$

$\Rightarrow d_n = e^{i\frac{2\pi}{n}(n-1)}$

$L = \mathrm{Tr}(A_7)^7$

$M = \det A_{(2n+1)} + \det(A_{2n})$

$(A_7)^7 = e^{i\frac{2\pi 7(0)}{7}} + e^{i2\pi(1)} + \ldots\ldots e^{i2\pi(6)}$

$= \cos 0 + i\sin 0 + \cos 2\pi + i\sin 2\pi \ldots \cos 6\pi + i\sin 6p$

$= [1+1 + \ldots\ldots\ 7 \text{ times} + i\,(0 + 0\ldots\ldots\ldots)] = 7$

$\det A_{2n} = \prod e^{i\frac{2\pi}{2n}(K-1)} = e^{i\frac{\pi}{n}\left(2n\frac{(2n+1)}{2} - 2n\right)}$

$= e^{\frac{\pi i}{n}\left(\frac{4n^2-2n}{2}\right)} = e^{\frac{\pi}{n}i(2n^2-n)} = e^{i\pi(2n-1)}$

$\det A_{2n+1} = e^{i2pn}$

$\det A_{2n} + \det A_{2n+1} = e^{i\pi(2n-1)} + e^{i2pn}$

$= \cos(2n-1)\pi + i\sin(2n-1)\pi + \cos 2n\pi + i\sin 2np$

$M = 0 \Rightarrow L + M = 7$

Sol 5: $z_1(z_1^2 - 3z_2^2) = 10$

$z_2(3z_1^2 - z_2^2) = 30$

$z_1 = a + ib, Z_2 = c + id$

$ab + cd = 0$ as $z_1^2 + z_2^2$ is real

$\Rightarrow z_1^3 - 3z_1z_2^2 + 3z_1^2z_2 - z_2^3 = 40$

$\Rightarrow z_1^2(z_1 + 3z_2) - z_2^2(z_2 + 3z_1) = 40$

$\Rightarrow (z_1 + iz_2)^3 = z_1^3 + iz_2^3 - 3z_1^2iz_2 - 3z_1z_2^2$

$= 10 - 30i$

$\Rightarrow (z_1 + iz_2)^3 = 10 + 30i$

$\Rightarrow ((z_1 + iz_2)^2)^3 = 1000 \Rightarrow z_1^2 + z_2^2 = 10$

Sol 6: $(z + 1)^7 + z^7 = 0$ has roots $z_1 \ldots\ldots z_7$

$\mathrm{Re}(z_1) + \mathrm{Re}(z_2) \ldots\ldots\ \mathrm{Re}(z_7) = ?$

$(z + 1)^7 = (-z)^7$

$\Rightarrow \left((a+1) + ib\right)^7 = (-a - ib)^7$

One of the solution is

$\Rightarrow a + 1 = -a \Rightarrow a = \frac{-1}{2}$ and $b = 0$

$\Rightarrow z = \frac{-1}{2}$

$z_1 + z_2 + z_3 \ldots\ldots z_7 = \frac{-7}{2} + i(0)$

$\mathrm{Re}(z_1 + \ldots\ldots\ldots + z_7) = \frac{-7}{2}$

$\mathrm{Im}\,(z_1 + z_2 \ldots + z_7) = 0$

Sol 7: $z = (1)^{1/7}$

$\alpha + \beta = (z + z^2 + z^3 + z^4 + z^5 + z^6) = (-1) = -1$

$\alpha\beta = (z + z^4 + z^2)(z^3 + z^5 + z^6)$

$= z^4 + z^6 + z^7 + + z^7 + z^9 + z^{10} + z^5 + z^7 + z^8$

$= z^4 + z^6 + 3 + z^5 + z + z^3 + z^2$

$= 3 + z + z^4 + z^6 + z^5 + z^3 + z^2 = 3 - 1 = 2$

$Eq^n \to x^2 + x + 2 = 0$

Sol 8: $z^5 - 32 = z^5 - 2^5$

$= (z - 2)(z^2 - pz + 4)(z^2 - qz + 4)$

$= z^5 + z^4(-q - p - 2) + z^3(+pq + 8 + 2p + 2q) + z^2(-4p - 4q + (pq + 8) - 2)$

$+ z(16 + 8p + 8q) - 32 = 0$

$\Rightarrow p + q + 2 = 0$

$2p + 2q + 8 + pq = 0$

$\Rightarrow pq + 4 = 0 \Rightarrow pq = -4$

$p + q = -2 \Rightarrow p - \frac{4}{p} = -2$

$\Rightarrow p^2 + 2p - 4 = 0$

$\Rightarrow p^2 + 2p = 4$

Sol 9: $|z_1| + |z_2| \ge \frac{1}{2}(|z_1| + |z_2|)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right|$

RHS

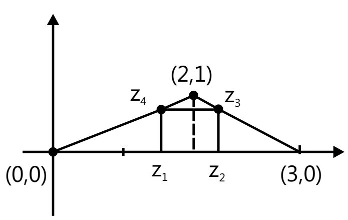
$$\frac{1}{2}(|z_1| + |z_2|)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right|$$

$$\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right| \le \left|\frac{z_1}{|z_1|}\right|\left|\frac{z_2}{|z_2|}\right| = 2$$

$$\frac{1}{2} \times 2(|z_1| + |z_2|) \ge \frac{1}{2}(|z_1| + |z_2|)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right|$$

$$|z_1| + |z_2| \ge \frac{1}{2}(|z_1| + |z_2|)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right|$$

Sol 10:



$z_1 = (a, 0), z_2 = (b, 0), z_3 = (b, c), z_4 = (a, c)$

$b - a = c$(square)

$a = 2c; b = 3c; b + c = 3$

$4c = 3 \Rightarrow c = \frac{3}{4}, a = \frac{3}{2}, b = \frac{9}{4}$

$$\Delta(0, z_3, z_4) = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{9}{4} & \frac{3}{4} & 1 \\ \frac{3}{2} & \frac{3}{4} & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{32} = \frac{m}{n}$$

$\Rightarrow m + n = 32 + 9 = 41$

Sol 11: $(1 + x)^n$

$a_n = C_0 + C_3 + C_6 + ...C_9$

$b_n = C_1 + C_4 + ...$

$c_n = C_2 + C_5 + C_8 + ...$

$a^3 + b^3 + c^3 - 3abc$

$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3........C_nx^n$

$2^n = C_0 + C_1 + C_2 + ...C_n$

$(1+\omega)^n = C_0 + \omega(C_1 + C_4 + C_7 + ...)$

$+ \omega^2(C_2 + C_5 + C_8 + ...) + C_3 + C_6 ......$

$(1 + \omega^2)^n = C_0 + \omega^2(C_1 + C_4 + C_7 ......)$

$+ \omega(C_2 + C_5 + C_8 + ...) + C_3 + C_6 ....$

$2^n + (1 + \omega)^n + (1 + \omega^2)^n = 3(C_0 + C_3 + C_6 ...)$

$2^n + (-\omega^2)^n + (-\omega)^n = 3(C_0 + C_3 + C_6 ...)$

$2^n\omega^2 + \omega(-\omega^2)^n + (-\omega)^n = 3\omega^2(C_1 + C_2 + C_7 ...)$

$$a_n = \frac{2^n + (-\omega^2)^n + (-\omega)^n}{3}$$

$$b_n = \frac{2^n\omega^2 + \omega(-\omega^2)^n + (-\omega)^n}{3\omega^2}$$

$$c_n = \frac{\omega^2 2^n + (-\omega^2)^n + \omega(-\omega)^n}{3\omega^2}$$

Sol 12: $\frac{z_1 - z_2}{z_3 - z_2} = e^{i\pi/2} = i$

$\frac{z_1 - iz_3}{1 - i} = z_2 \Rightarrow Z_2 = \frac{z_1(1+i) + z_3(1-i)}{2}$

$\frac{z_1 - z_4}{z_3 - z_4} e^{-i\pi/2} = -i \Rightarrow z_1 - z_4 = -i(z_3 - z_4)$

$\Rightarrow z_4 = \dfrac{z_1(1-i)+z_3(1+i)}{2}$

Sol 13: f(z) = (a + ib)(z) = c + id

z = (x + iy)

Image = c – id

$(c-x)^2 + (y+d)^2 = c^2 + d^2$

$x^2 + y^2 - 2cx + 2dy = 0$

$a^2 + b^2 = 64$

c = ax – by

d = bx – ay

$\Rightarrow x^2 + y^2 - 2x(ax - by) + 2y(bx - ay) = 0$

$\Rightarrow x^2(1 - 2a) + y^2(1 - 2a) - 2bxy + 2bxy = 0$

$\Rightarrow (x^2 + y^2)(1 - 2a) = 0$

$\Rightarrow a = \dfrac{1}{2} \Rightarrow b^2 = 64 - \dfrac{1}{4} = \dfrac{255}{4} = \dfrac{u}{v}$

$\Rightarrow u + v = 259$

Sol 14:

$\cos x + {}^nC_1\cos 2x + {}^nC_2\cos 3x + \dots\dots\dots + {}^nC_n\cos(n+1)x$

$= \cos x + {}^nC_1\cos^2 x + {}^nC_2\cos^3 3x + \dots\dots\dots + {}^nC_n\cos^{n+1}x$

$= \cos\overline{x}\,[1 + {}^nC_1\cos x + {}^nC_2\cos^2 x + \dots\dots\dots + {}^nC_n\cos^n x]$

$= \cos x[1 + \cos x]^n$

$= \cos x\left[2\cos\dfrac{x}{2}\cos x\dfrac{x}{2}\right]^n$

$= 2^n \text{Cos}^n\dfrac{x}{2}\cos\dfrac{n}{2}\cos$

$= 2^n\cos^n\dfrac{x}{2}\cos\left[\dfrac{n+2}{2}\right]x$

$= 2^n\cos^n\dfrac{x}{2}\left[\cos\dfrac{(n+2)x}{2} + i\sin\left[\dfrac{n+2}{2}\right]x\right]$

By comparing, we get the desired results. Hence, proved

Sol 15: $f(x) = ax^3 + bx^2 + cx + d \;\; f(i) = 0$

$\Rightarrow -ai - b + ci + d = 0 + io \;\; \Rightarrow -a + c = 0$ and d – b = 0

$\Rightarrow a = c$ and b = d ...(i)

And f (1 + i) = 5

$a(1+i)^3 + b(1+i)^2 + c(1+i) + d = 5$

$\Rightarrow a\left(1 + i^3 + 3i^2 + 3i\right) + b\left(1 + i^2 + 2i\right) + c\left(1+i\right) + d = 5$

$\Rightarrow a\left(1 - i - 3 + 3i\right) + b\left(1 - 1 + 2i\right) + c\left(1+i\right) + d = 5$

$\Rightarrow a\left(-2 + 2i\right) + b\left(2i\right) + c\left(1+i\right) + d = 5$

$\Rightarrow -2a + c + d = 5$ and $2a + 2b + c = 0$

From (i), we have

d – a = 5 and 3a + 2d = 0

$\Rightarrow a = c = -2$ and $b = d = 3$

$\therefore a^2 + b^2 + c^2 + d^2 = (-2)^2 + 3^2 + (-2)^2 + 3^2$

= 4 + 9 + 4 + 9

= 26

Sol 16: $\sum_{k=1}^{n}\left(z_k - w_k\right) = 0$

$\Rightarrow z_1 + z_2 + z_3 + z_4 + z_5$

$= 32 + 170i - 7 + 64i - 9 + 200i + 1 + 27i - 14 + 43i$

$= 3 + 504i$

If y-intercept is 3, then eq. of line

y = mx + 3

yi = mxi + 3 for i=1,2.......5

Now, $z_1 + \dots\dots\dots\dots\dots z_5 = 3 + 504i$

$(x_1 + x_2 + \dots\dots\dots + x_5) + i(y_1 + y_2 + \dots\dots\dots y_5) = 3 + 504i + i\{m(x_1 + x_2 \dots\dots\dots\dots + x_5) + 15\} = 3 + 504i$

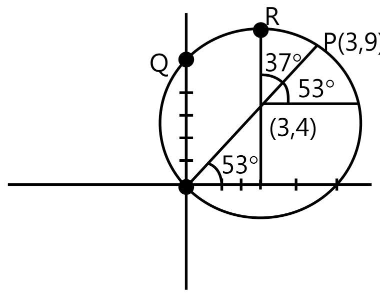
$x_1 + x_2 + \dots\dots\dots\dots + x_5 = 3$

$m(x_1 + x_2 + \dots\dots\dots\dots + x_5) + 15 = 504$

$\Rightarrow 3m + 15 = 504$

$\Rightarrow m = 163$

Sol 17:



$\left|z - 3 - 4i\right| = 5\tan^{-1}\dfrac{3}{4}$ (37°) in clockwise direction

It reaches Q (3, 9) above centre

Then it moves 2 unit downwards i.e. R (3, 7)

Sol 18: $\sum_{p=1}^{32}(3p+2)\left[\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11}-i\cos\frac{2q\pi}{11}\right)\right]^p$

$= \sum(3p+2)\left[-i\left(\sum_{q=0}^{10}e^{\frac{2q\pi i}{11}}-1\right)\right]^p$

$= \sum_{p=1}^{32}(3p+2)i^p = \sum_{p=1}^{32}(3p)i^p \quad \left[2\sum_{1}^{32}i^p=0\right]$

$= 3i(1-3+\cdots-31)-3(2-4+\cdots-32)=48(1-i)$

Sol 19: $\frac{a}{1-b}=\frac{b}{1-c}=\frac{c}{1-a}=k$

a = k – kb

b = k – kc

c = k – ka

$a = k - k^2 + k^2(k - ka)$

$a = k - k^2 + k^3 - k^3a)$

$a = \frac{k-k^2+k^3}{1+k^3}=b=c$

but $a \neq b \neq c$

i.e. $k^3 = -1 \Rightarrow k = -w, -w^2$

Sol 20: $|z-\alpha|^2 + |z-\beta|^2 = k$

Locus of z is a circle

$(x-\alpha_1)^2+(x-\beta_1)^2+(y-\alpha_2)^2+(y-\beta_2)^2 = k$

$\Rightarrow 2x^2 + 2y^2 - 2x(\alpha_1 + \beta_1) - 2y_2(\alpha_2 + \beta_2)$

$+ \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 - k = 0$

$\Rightarrow x^2 + y^2 - 2x\left(\frac{\alpha_1+\beta_1}{2}\right)-2y\left(\frac{\alpha_2+\beta_2}{2}\right)$

$+\left(\frac{\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2-k}{2}\right)=0$

$r > 0$ i.e. $g^2 + f^2 - c > 0$

$\left(\frac{\alpha_1+\beta_1}{2}\right)^2+\left(\frac{\alpha_2+\beta_2}{2}\right)^2-2\left(\frac{\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2-k}{4}\right)>0$

$\Rightarrow -\alpha_1^2+\alpha_2^2-\beta_1^2-\beta_2^2+2\alpha_1\beta_1+2\alpha_2\beta_2+k>0$

$\Rightarrow k > \alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2-2\alpha_1\beta_1-2\alpha_2\beta_2$

$\Rightarrow k > (\alpha_1-\beta_1)^2+(\alpha_2-\beta_2)^2 \Rightarrow k > |\alpha-\beta|^2$

Centre = $\left(\frac{\alpha_1+\beta_1}{2},\frac{\alpha_2+\beta_2}{2}\right)$

Radius = $\sqrt{\frac{2k+2\alpha_1\beta_1+2\alpha_2\beta_2-\alpha_1^2-\alpha_2^2-\beta_1^2-\beta_2^2}{4}}$

Sol 21: $f(z) = |z^3 - z + 2|\ |z| = 1$

$(f(z))^2 = |z^3 - z + 2|^2=(z^3 - z + 2)(\bar{z}^3 - \bar{z} + 2)$

$= 1-z^2+2z^3-\bar{z}^2+z\bar{z}-2z+2\bar{z}^3-2\bar{z}+4$

$= 6 - (z^2 - \bar{z}^2) -2(z + \bar{z}) + 2(Z^3 + \bar{z}^3)$

$= 6-2(a^2-b^2)-4(a)+2(z+\bar{z})(z^2+\bar{z}^2-z\bar{z})$

$= 6-2(a^2-b^2)-4(a)+4a(2a^2-2b^2-1)$

$= 6-8a-2(2a^2-1) + 8a(2a^2-1)$

$f(z) = 16a^3 - 4a^2 - 16a + 8$

$f'(z) = 48a^2 - 8a - 16 = 8(6a^2 - a - 2)$

$= 8(6a^2-4a + 3a - 2)$

$= 8(2a(3a - 2) + 1(3a - 1))$

$= 8(2a + 1)(3a - 2)$

$a = \frac{-1}{2}\ a = \frac{2}{3}$

For $a = \frac{2}{3}$

$f(z) = 16 \times \frac{8}{27} - 4 \times \frac{4}{9} - 16 \times \frac{2}{3} + 8$

$= \frac{128-48-288+202}{27}<0$

For $a = \frac{-1}{2}$; $f(z) = -2 - 1 + 8 + 8 = 13$

Maximum value of f(z) = $\sqrt{13}$

Sol 22: $|a + b\omega + c\omega^2| + |a + b\omega^2 + c\omega| \geq$

$|a + b\omega + c\omega^2 + a + b\omega^2 + c\omega|$

$= |2a - b - c| = |a - b + a - c|$

Minimum value will occur when

a – b = k, a – c = –k

i.e. k = 1

a – b = 1 and a – c = –1

with a & b being least possible value integer values

$\Rightarrow a = 2, b = 1, c = 3$

$|a + b\omega + c\omega^2| + |a + b\omega^2 + c\omega|$

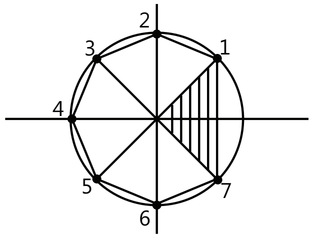
$= 2|2 + \omega + 3\omega^2| = 2|1 + 2\omega^2|$

$= 2|\omega^2 - \omega| = 2|(-\sqrt{3})| = 2\sqrt{3}$

$= \sqrt{12} = n^{1/4} \Rightarrow n = 144$

Sol 23: $x^7 + x^6 + \dots\dots\dots 1 = 0$

$\left(\frac{x^8-1}{x-1}\right) = 0$



Total area = unshaded area + shaded area

Unshaded area $= 6\times 1\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{vmatrix}$

$= \frac{6}{2}\times\frac{1}{\sqrt{2}} = \frac{6}{2\sqrt{2}}$

Shaded area $= 2\times\frac{1}{2}\times\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}} = \frac{1}{2}$

Total area $= \frac{6}{2\sqrt{2}}+\frac{1}{2} = \frac{6+\sqrt{2}}{2\sqrt{2}} = \frac{6\sqrt{2}+2}{4}$

$= \frac{3\sqrt{2}+1}{2} = \frac{a\sqrt{b}+c}{d}$

$a + b + c + d = 3 + 2 + 1 + 2 = 8$

Sol 24: $N = (a + ib)^3 - 107i$

$= a^3 - ib^3 + 3a^2ib - 3ab^2 - 107i$

$N = (-b^3 + 3a^2b - 107)i + a^3 - 3ab^2$

$\Rightarrow 3a^2b - b^3 = 107 \Rightarrow b(3a^2 - b^2) = 107$

$\Rightarrow b = 1$ and $3a^2 - b^2 = 107 \Rightarrow a^2 = 36$

$\Rightarrow N = a^3 - 3ab^2 = 216 - 18 = 198$

Sol 25: $x^4 + ax^3 + bx^2 + cx + d$ has 4 non-real roots.

$\alpha + \beta = 3 + 4i$

$\gamma\delta = 13 + i(\gamma = \bar{\alpha}, \delta = \bar{\beta})$

$\Rightarrow \bar{\alpha}\bar{\beta} = 13 + i$, $a\beta = 13 - i$

$\alpha + \beta = 3 + 4i$

$\Rightarrow a\beta + b\gamma + g\delta + d\alpha + a\gamma + b\delta = b$

$\Rightarrow a\beta + \beta\bar{\alpha} + \bar{\alpha}\bar{\beta} + \alpha\bar{\alpha} + \beta\bar{\beta} + \alpha\bar{\beta} = b$

$\Rightarrow 13 - i + 13 + i + (\bar{\alpha}+\bar{\beta})(\alpha + \beta) = b$

$= 26 + (3 + 4i)(3 - 4i) = b$

$\Rightarrow b = 26 + 25 = 51$

Sol 26: $z^5 + 1$

$= (z^3 + 1) z^2 + (1 - z^2)$

$= (z + 1)(z^2 - z + 1) z^2 + (1 - z^2)$

$= (1 + z)[(z^2 - z + 1) z^2 + 1 - z]$

$= (1 + z)[z^4 - z^3 + z^2 - z + 1]$

$= (1 + z)[z^2 + az + 1][z^2 + bz + 1]$

$\Rightarrow b + a = -1$

$ba + 2 = 1$

$ab = -1$

$\Rightarrow a - \frac{1}{a} = -1 \Rightarrow a^2 + a - 1 = 0$

$\Rightarrow a = \frac{-1\pm\sqrt{5}}{2} \Rightarrow a = \frac{-1+\sqrt{5}}{2} = -2\cos 108°$

And $b = \frac{-1-\sqrt{5}}{2} = -2\cos 36°$

$\therefore$ Factors are $(1 + z)(z^2 - 2z\cos 36° + 1)(z^2 - 2\cos 108° + 1)$

Since $ab = -1$

$\Rightarrow 4\cos 36°\cos 108° = -1 \Rightarrow 4\cos\frac{\pi}{5}\cos\frac{\pi}{10} = 1$

Sol 27: $x = 1 + i\sqrt{3}$

$y = 1 - i\sqrt{3}$

$z = 2$

$x = -2\omega y = -2\omega^2 z = 2$

$x^p + y^p p > 3$ prime (P is odd)

$= -2^p\omega^p - 2^p\omega^{2p} = -2^p(\omega^p + \omega^{2p}) = 2^p = z^p$

Sol 28: $f(z) = a(z - i) + i \Rightarrow f(i) = i$ ... (i)

$f(z) = b(z + i) + (1 + i) \Rightarrow f(-i) = 1 + i$ ... (ii)

$f(z) = c(z^2 + 1) + k_1 z + k_2$ ... (iii)

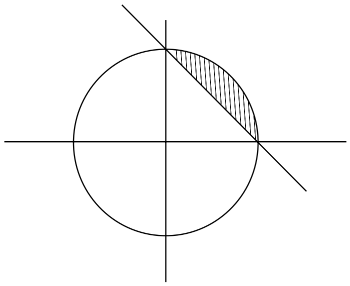
Substituting values from (i) and (ii) in (iii)

$\Rightarrow i = k_1 i + k_2$ and $i + 1 = -k_1 i + k_2$

$\Rightarrow k_2 = i + \frac{1}{2}$ and $k_1 = \frac{i}{2}$

$\therefore$ Remainder $= \frac{iz}{2} + i + \frac{1}{2}$

Sol 29: (a)



A = {z| $|z| \le 2$}

B = {z| $(1 - i)z + (1 + i)\bar{z} \ge 4$}

Let z = a + ib

$\Rightarrow (1 - i)(a + ib)+(1 + i)(a - ib) \ge 4$

$\Rightarrow a + b + a + b \ge 4 \Rightarrow a + b \ge 2$

area = $\frac{\pi r^2}{4} - \frac{1}{2}r^2 = \left(\frac{\pi}{4} - \frac{1}{2}\right)r^2 = (\pi - 2)$

(b) f(x) = $\frac{1}{x-i} = \frac{1}{x-i}\left(\frac{x+i}{x+i}\right)$

$= \frac{x}{x^2+1} + i\left(\frac{1}{x^2+1}\right)$

x – coordinate = $\frac{k}{k^2+1}$

y – coordinate = $\frac{1}{k^2+1}$

$\therefore$ Locus of the function will be

$x^2 + y^2 - y = 0$

This is circle with diameter 1

Hence the area of the square = $\frac{1}{2}$

Sol 30: (a) $(1 + \omega + ...\omega^n)^m$, m, n $\in$ N

$n \in 3k \Rightarrow 1$

$n \in 3k + 1 \Rightarrow (-\omega^2)^m$

$n \in 3k + 2 \Rightarrow 0$

$m = 1 \Rightarrow (-\omega^2)$, $m = 2 \Rightarrow (\omega)$, $m = 3 \Rightarrow (-1)$

$m = 4 \Rightarrow (\omega^2)$, $m = 5 \Rightarrow (-\omega)$, $m = 6 \Rightarrow (1)$

No. of distinct elements are 7(S)

(b) Real coefficient,

Root $\to 2\omega, 2+3\omega, 2+3\omega^2, 2-\omega-\omega^2$

Roots are $2\omega, 2 + 3\omega, 2 + 3\omega^2, 3$

Other root will be $2\bar{\omega}$

Total no. of roots are 5 (q)

(c) $\alpha = 6 + 4i\beta = 2 + 4i$

$\text{amp}\left(\frac{z-\alpha}{z-\beta}\right) = \frac{\pi}{6}$

$\tan^{-1}\left[\frac{(x-6)+(y-4)i}{(x-2)+i(y-4)}\frac{(x-2)-i(y-4)}{(x-2)-i(y-4)}\right] = \frac{\pi}{6}$

$\text{Re}(z) = (x - 6)(x - 2) + (y - 4)^2$

$\text{Im}(z) = (x - 2)(y - 4) + (x - 6)(y - 4)$

$\Rightarrow \frac{(x-2)(y-4)+(x-6)(4-y)}{(x-6)(x-2)+(y-4)^2} = \frac{1}{\sqrt{3}}$

$\Rightarrow \frac{-4x-2y+8+4x+6y-24}{x^2+y^2-8x-8y+12+16} = \frac{1}{\sqrt{3}}$

$\Rightarrow x^2 + y^2 - 8x - 8y + 28 = 4\sqrt{3}y - 16\sqrt{3}$

$\Rightarrow x^2+y^2-8x+y(-8-4\sqrt{3})+28+16\sqrt{3} = 0$

$\Rightarrow r = \sqrt{16+(4+2\sqrt{3})^2-28-16\sqrt{3}}$

$= \sqrt{16+16+12-28+16\sqrt{3}-16\sqrt{3}}$

$= \sqrt{16} = 4$(p)

Exercise 2

Single Correct Choice Type

Sol 1: (C) $z^2 + z + 1$ is real and +ve

$x^2 - y^2 + x + 1 + i(2xy + y)$

$x^2 - y^2 + x + 1 > 0$ and $2xy + y = 0$

$\Rightarrow y(2x + 1) = 0 \Rightarrow y = 0$ or $x = \frac{-1}{2}$

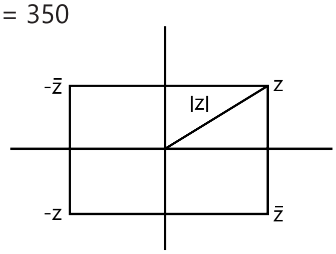
If y = 0 $x^2 + x + 1 > 0 \Rightarrow$ z represents real axis

If x = $\frac{-1}{2}\ \frac{3}{4} - y^2 > 0$, Line segment joining

$\left[-\frac{1}{2}, \frac{-\sqrt{3}}{2}\right]$ to $\left[-\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$

Sol 2: (B) $z\bar{z}^3 + z^3\bar{z} = 350$

$|z|^2 (z^2 + \bar{z}^2) = 350$



Length of diagonal $2|z|$

$|z|^2\ (x^2 - y^2 + 2ixy + x^2 - y^2 - 2ixy) = 350$

$(x^2 + y^2)(x^2 - y^2) = \dfrac{350}{2} = 175$

As x and y are integers

$x = \pm 4;\ y = \pm 3$

$\Rightarrow |z| = 5$

Length of diagonal is 10

Sol 3: (A) $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$

$(z_1 - z_2)^2 + z_2^2 = 0$

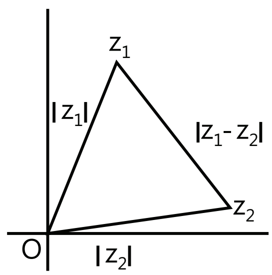
$\Rightarrow (z_1 - z_2)^2 = -z_2^2 = |z_1 - z_2| = |z_2|$ and $(z_1 - z_2) = iz_2$

and

$$\left(\frac{z_1}{z_2}\right)^2 - 2\frac{z_1}{z_2} + 2 = 0$$

$$\Rightarrow \frac{z_1}{z_2} = 1 + i$$

$$\Rightarrow \left|\frac{z_1}{z_2}\right| = \sqrt{2}$$



Now,

$$\frac{z_1 - 0}{z_2 - 0} = \left|\frac{z_1}{z_2}\right| e^{i\angle z_1 o z_2}$$

$$e^{i\angle z_1 o z_2} = \frac{1+i}{\sqrt{2}} = e^{i\pi/4}$$

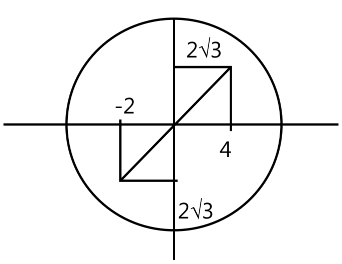
$$\Rightarrow \angle z_1 o z_2 = 45^o = \angle o z_1 z_2$$

Also,

$$\frac{z_2 - z_1}{z_2} = \left|\frac{z_2 - z_1}{z_2}\right| e^{i\angle 0 z_2 z_1}$$

$$\Rightarrow e^{i\angle o z_2 z_1} = i \Rightarrow \angle o z_2 z_1 = 90^o$$

Sol 4: (B)



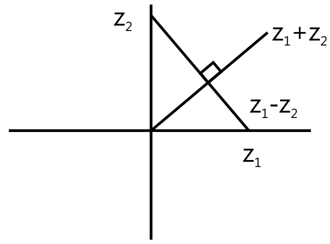
$|Z| \le 4$ and Arg $(z) = \dfrac{\pi}{3}$

$x^2 + y^2 \le 16$ and $y = \sqrt{3}x$

$\Rightarrow x \in [-2, 2]$ & $y \in \left[-2\sqrt{3}, 2\sqrt{3}\right]$

Set of points lie on radius of circle

Sol 5: (C)



$$\arg \frac{z_1 + z_2}{z_1 - z_2} = \frac{\pi}{2}$$

$$\tan^{-1}\left(\frac{z_1 + z_2}{z_1 - z_2}\right) = \frac{\pi}{2}$$

$|z_1 + z_2| \ne |z_1 - z_2|$

Diagonals are perpendicular but not equal ie figure represented is rhombus but not a square.

Sol 6: (A) Condition for equilateral

$\Delta : z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

$z^3 - 3\alpha z^2 + 3\beta z + x = 0$

z_1, z_2, z_3

$\Rightarrow z_1 + z_2 + z_3 = 3\alpha$

$\Rightarrow z_1z_2 + z_2z_3 + z_3z_1 = 3\beta$

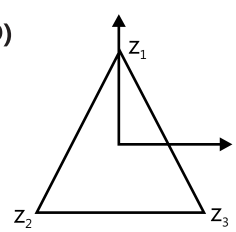
$\Rightarrow z_1, z_2, z_3 = -x$

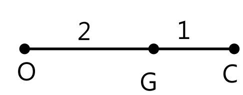
$(z_1 + z_2 + z_3)^2 = z_1^2 + z_2^2 + z_3^2 + 2(3\beta)$

$(3\alpha)^2 = 9\beta$

$\alpha^2 = \beta$

Sol 7: (D)

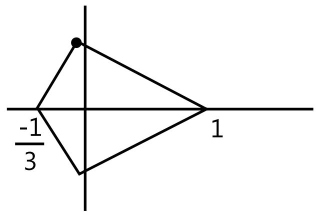




$$\frac{z_1 + z_2 + z_3}{3} = \text{Centroid},\quad z_c = 0$$

$$\frac{z_1 + z_2 + z_3}{3} = \frac{2z_C + z_O}{3} \Rightarrow z_O = z_1 + z_2 + z_3$$

Sol 8: (C)



$(z + 1)^4 = 16z^4$

$\Rightarrow z + 1 = 2z, 2iz, -2z, -2iz$

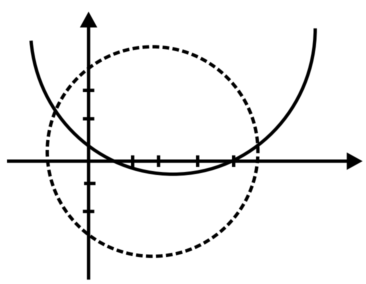
$\Rightarrow z = 1, \dfrac{1}{2i-1}, -\dfrac{1}{3}, \dfrac{1}{-1-2i} = -\dfrac{1}{3}, 1, \dfrac{-1-2i}{5}, \dfrac{-1+2i}{5}$

Point equidistant is $z = \dfrac{z_1+z_2}{2}$

Where $z_1 = (1, 0)$, $z_2 = \left(\dfrac{-1}{3}, 0\right)$

$$z = \frac{1-\frac{1}{3}}{2} = \frac{1}{3}$$

Sol 9: (B) $|z - 2| = 3$



$|z - 2 - 3i| = 4$

$\Rightarrow S_1 = (x - 2)^2 + y^2 = 9$

$\Rightarrow S_2 = (x - 2)^2 + (y - 3)^2 = 16$

Both circles are intersecting. So, radical axis will be

$S_1 - S_2 = 0 \Rightarrow 9 - 6y = 7 \Rightarrow 3y - 1 = 0$

Sol 10: (A) $z^3 + iz - 1 = 0$

Let z=k (real)

$\Rightarrow k^3 + ik - 1 = 0$

$k^3 + ik - 1 = 0$ and $k = 0$

If k = 0, then 0 – 1 = 0

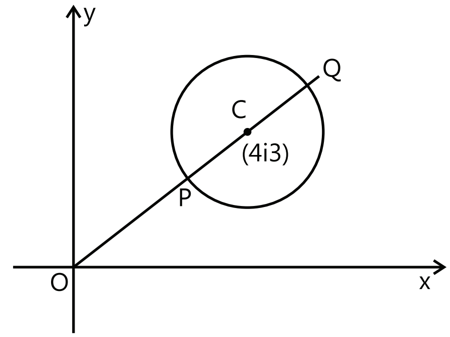
Not possible

Therefore $z \neq k$ (real)

Hence, no real solution

Sol 11: (D) $|z-4-3i|=2$

z lies on a circle shown in the figure $|z|=|z-0|$ is nothing but distance from origin



Minimum = OP = OC-CP

$= \sqrt{4^2+3^2} - 2 = 5-2=3$

Maximum = OC + CQ =5+9=7

Sol 12: (A) $z = 1 - \sin\alpha + i\cos\alpha$

$|z| = \sqrt{1+\sin^2\alpha - 2\sin\alpha + \cos^2\alpha}$

$= \sqrt{2-2\sin\alpha} = \sqrt{2(1-\sin\alpha)}$

$$\arg z = \tan^{-1}\frac{\cos\alpha}{1-\sin\alpha} = \tan^{-1}\left(\frac{1}{\sec\alpha - \tan\alpha}\right)$$

$$= \tan^{-1}\frac{1+\sin\alpha}{\cos\alpha} = \tan^{-1}\frac{\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)^2}{\left(\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\right)^2}$$

$$= \tan^{-1}\left(\frac{\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}\right) = \tan^{-1}\left(\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right] = \frac{\pi}{4}+\frac{\alpha}{2}$$

Sol 13: (D) $\left|\dfrac{z_1+z_2}{2}+\sqrt{z_1z_2}\right| + \left|\dfrac{z_1+z_2}{2}-\sqrt{z_1z_2}\right|$

$$= \frac{\left|z_1+z_2+2\sqrt{z_1z_2}\right| + \left|z_1+z_2-2\sqrt{z_1}\sqrt{z_2}\right|}{2}$$

$$= \frac{\left(\sqrt{z_1}+\sqrt{z_2}\right)^2 + \left(\sqrt{z_1}-\sqrt{z_2}\right)^2}{2}$$

$$= \frac{2(z_1)^2+2(z_2)^2}{2} = \left|\sqrt{z_1}\right|^2 + \left|\sqrt{z_2}\right|^2$$

Sol 14: (A) $u^2 - 2u + 2 = 0$

$\Rightarrow (u-1)^2 = -1$

$\Rightarrow u - 1 = \pm i$

$\Rightarrow u = 1 + i, 1 - i = (\alpha, \beta)$

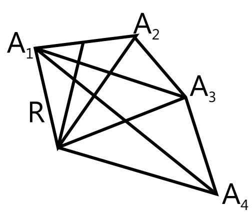
$\cot\theta = x + 1$

$$\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha-\beta} = \frac{(\cot\theta+i)^n - (\cot\theta-i)^n}{2i}$$

$$= \frac{(\cos\theta+i\sin\theta)^n - (\cos\theta-i\sin\theta)^n}{2i\sin^n\theta}$$

$$= \frac{e^{i\theta n} - e^{-i\theta n}}{2i\sin^n\theta} = \frac{\sin n\theta}{\sin^n\theta}$$

Sol 15: (B) $A_1 \ldots\ldots A_n$ vertices of regular polygon in a circle of radius R



$(A_1A_2)^2 + (A_1A_3)^2 \ldots\ldots (A_1A_n)^2$

$A_1A_2 = 2R\sin\frac{\pi}{n}$

$A_1A_3 = 2R\sin\frac{2\pi}{n}$

$A_1A_2 = 2R\frac{3\pi}{n}$

$$4R^2\left[\sin^2\frac{\pi}{n} + \sin^2\frac{2\pi}{n} + \sin^2\frac{3\pi}{n} \ldots \sin^2\frac{(n-1)\pi}{n}\right]$$

$$= -2R^2\left[1 - n + \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \ldots \cos(2n-1)\frac{\pi}{n}\right]$$

$$= -2R^2\left[1 - n + \frac{\sin \pi k}{\sin n\left(\frac{\pi}{n}\right)} - \cos\frac{2n\pi}{n}\right] = 2nR^2$$

Sol 16: (B) $z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$

$z = ib$

$b^4 - ib^3a_1 - b^2a_2 + iba_3 + a_4 = 0$

$b^4 - b^2a_2 + a_4 = 0 \Rightarrow b^4a_2 + a_4a_2 = b^2a_2^2$

$\Rightarrow ba_3 - b^3a_1 = 0 \Rightarrow a_3 = b^2a_1$

$$\frac{a_3}{a_1a_2} + \frac{a_1a_4}{a_2a_3} = \frac{b^2}{a_2} + \frac{a_4}{a_2b^2} = \frac{b^4a_2 + a_4a_2}{a_2^2b^2} = \frac{b^2a_2^2}{b^2a_2^2} = 1$$

Sol 17: (D) $(1+z)^6 + z^6 = 0$

$\Rightarrow (1+z)^6 = -z^6$

Now,

$$\Rightarrow \left(\frac{1+z}{z}\right)^6 = -1$$

(4, 3)

$$\Rightarrow 1 + \frac{1}{z} = (-1)^{1/6}$$

$$\Rightarrow \frac{1}{z} = \cos\frac{2K\pi+\pi}{6} - 1$$

$$\Rightarrow \frac{1}{z} = -\left\{2\sin\frac{2K\pi+\pi}{12}\cos\left(\frac{2K\pi+\pi}{12} - \frac{\pi}{2}\right)\right\}$$

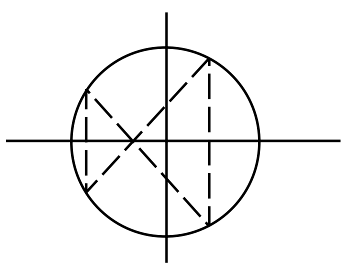
$$\Rightarrow z = -\frac{1}{2\sin\frac{2K\pi+\pi}{12}}\left\{\cos\left(\frac{\pi}{2} - \frac{2K\pi+\pi}{12}\right)\right\}$$

$$z = -\frac{1}{2}\left\{1 + i\cot\frac{2K\pi+\pi}{12}\right\}$$

$\Rightarrow z$ lies on line $x = -\frac{1}{2}$

$\Rightarrow$ All roots are collinear.

Sol 18: (A)



$z^{10} - z^5 = 992$

$z^5(z^5 - 1) = 32(31)$

$z^5 = t$

$\Rightarrow t^2 - t = 992 \Rightarrow t^2 - 32t + t - 992 = 0$

$\Rightarrow t = 32, -31 \Rightarrow z^5 = 32, -31$

$\Rightarrow z^5 = 32$ has 2 roots with –ve real Part

and $z^5 = -31$ has 3 roots with –ve real part

Sol 19: (A) $a|z_1| = b|z_2|$

$\frac{a}{b} = \frac{|z_2|}{|z_1|}$ $T = \frac{az_1}{bz_2} + \frac{bz_2}{az_1}$ let $\frac{az_1}{bz_2} = z$

$T = z + \frac{1}{z}\frac{\bar{z}}{\bar{z}} = z + \frac{\bar{z}}{|z|^2}$

$|z| = 1 \Rightarrow T = z + \bar{z} = 2\text{Re}(z)$

$\text{Re}(z) \in (-1, 1)$

$\Rightarrow T \in [-2, 2]$

Sol 20: (B) $(p+q)^3 + (p\omega+q\omega^2)^3 + (p\omega^2+q\omega)^3$

$= p^3+q^2+3p^2q+3pq^2+p^3+q^3+3pq^2\omega^5$

$+3p^2q\omega^4+p^3+q^3+3p^2q\omega^5+3pq^2\omega^4$

$= 3(p^3+q^3)$

Sol 21: (A) $\left(1+i\sqrt{3}\right)^x = 2^x \Rightarrow \left(\frac{1+i\sqrt{3}}{2}\right)^x = 1$

$\Rightarrow (-\omega)^x = 1$

$\Rightarrow x = 6, 12, 18, \ldots\ldots\ldots$

It forms an AP

Multiple Correct Choice Type

Sol 22: (B, C) $x^2+(p+iq)x + 3i = 0$

$a^2 + b^2 = 8$

$(\alpha + \beta) = -(p + iq)$

$a\beta = 3i$

$\Rightarrow (\alpha + \beta)^2 = 8 + 6i = p^2 - q^2 + 2ipq$

$p^2 - q^2 = 8$

$2pq = 6$

$\Rightarrow p = 3, q = +1$ or $p = -3, q = -1$

Sol 23: (A, D) $|z_1| = |z_2| \Rightarrow a^2 + b^2 = c^2 + d^2$

$z_1 = a + ib, a > 0$

$z_2 = c + id, d < 0$

$\frac{z_1+z_2}{z_1-z_2} = \frac{(a+c)+i(b+d)}{(a-c)+i(b-d)}$

(a + c) & (b + d) can be zero, so value can be zero

(a − c) & (b − d) can be simultaneously zero ⇒ purely imaginary.

Sol 24: (A, B, C, D) (a) $a, b, x\ y$ is $\in R$

$\frac{a+ib}{x+iy} = a - ib$

$\Rightarrow x + iy = \frac{a^2-b^2+2abi}{a^2+b^2}$

$\Rightarrow x = \frac{a^2-b^2}{a^2+b^2}, y = \frac{2ab}{a^2+b^2}$

$\Rightarrow x^2 + y^2 = 1$ A is correct

(b) $\frac{1-ix}{1+ix} = a - ib$

$\Rightarrow \frac{1-x^2-2ix}{1+x^2} = a - ib$

$\Rightarrow a = \frac{1-x^2}{1+x^2}, b = \frac{2x}{1+x^2}$

$\Rightarrow a^2 + b^2 = 1$ B is correct

(c) $\frac{a+ib}{a-ib} = x - iy$

$\Rightarrow \frac{(a+ib)^2}{a^2+b^2} = x - iy$

$\Rightarrow x + iy = \frac{a^2-b^2+2iab}{a^2+b^2}$

$\Rightarrow x + iy = \frac{a^2-b^2-2iab}{a^2+b^2} \Rightarrow |x + iy| = 1$

(d) $\frac{y-ix}{a+ib} = y + ix$

$\Rightarrow \frac{y^2-x^2-2ixy}{y^2+x^2} = a + ib$

$\Rightarrow a - ib = \frac{y^2-x^2+2ixy}{x^2+y^2} \Rightarrow |a - ib| = 1$

Sol 25: (A, D) $z = x + iy = r(\cos\theta + i\sin\theta)$

$\sqrt{z} = \sqrt{x+iy} = \sqrt{r(\cos\theta+i\sin\theta)}$

$= \sqrt{r}\,e^{i\theta/2} = \sqrt{r}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]$ for $y > 0$

$= \sqrt{r}\left[\sqrt{\frac{1+\cos\theta}{2}} + i\sqrt{\frac{1-\cos\theta}{2}}\right]$

$= \frac{\sqrt{r}}{\sqrt{2}}\left[\frac{\sqrt{r+x}+i\sqrt{r-x}}{\sqrt{r}}\right] = \frac{\sqrt{r+x}+i\sqrt{r-x}}{\sqrt{2}}$ [A]

Similarly

$= \frac{1}{\sqrt{2}}\left[\sqrt{r+x} - i\sqrt{r-x}\right]$ for $y < 0$

Sol 26: (A, D)

$\left(az_1 + b\bar{z}_1\right)\left(cz_2 + d\bar{z}_2\right) = \left(cz_1 + d\bar{z}_1\right)\left(az_2 + b\bar{z}_2\right)$

$adz_1\bar{z}_2 + bc\bar{z}_1 z_2 = bcz_1\bar{z}_2 + ad\bar{z}_1 z_2$

$\Rightarrow ad = bc$

$\frac{a}{b} = \frac{c}{d}$ (A is correct)

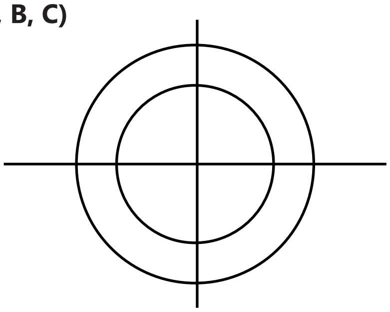
$\Rightarrow \arg z_1 = \tan^{-1}\frac{b}{a}$, $\arg z_2 = \tan^{-1}\frac{d}{c}$

$\Rightarrow \arg z_1 = \arg z_2$

or $z_1\bar{z}_2 = \bar{z}_1 z_2 \Rightarrow \frac{z_1}{\bar{z}_1} = \frac{z_2}{\bar{z}_2}$

$\Rightarrow \frac{a+ib}{a-ib} = \frac{c+id}{c-id} \Rightarrow -ad + bc = 0$

Sol 27: (A, B, C)



$|z_1| = 1$ and $|z_2| = 2$

Max $|2z_1 + z_2|$

$|2z_1 + z_2| \le 2|z_1| + |z_2| = 4$ (A is correct)

Min $|z_1 - z_2| \ge |z_1| - |z_2|$

$|z_1 - z_2| \ge 1$(B is correct)

$\left|z_2 + \frac{1}{|z_1|}\right| \le |z_2| + \frac{1}{|z_1|} \le 2$ (C is correct)

Sol 28: (A, B, C, D) $\left|\frac{\alpha-\beta}{1-\bar{\alpha}\beta}\right| = 1$

$|\alpha - \beta|^2 = |1 - \bar{\alpha}\beta|^2$

$\Rightarrow (\alpha - \beta)(\bar{\alpha} - \bar{\beta}) = (1 - \bar{\alpha}\beta)\left(1 - \alpha\bar{\beta}\right)$

$|\alpha|^2 + |\beta|^2 - \alpha\bar{\beta} - \beta\bar{\alpha} = 1 - \alpha\bar{\beta} - \beta\bar{\alpha} + |\alpha|^2|\beta|^2$

$|\alpha|^2 (\alpha - |\beta|^2) = (1 - |\beta|^2) \Rightarrow (|\alpha|^2 - 1)(1 - |\beta|^2) = 0$

$\Rightarrow |\alpha| = 1$ A is correct

$\Rightarrow |\beta| = 1$ B is correct

$\alpha = e^{i\theta} = \beta$ [C and D are correct]

Sol 29: (A, B, D) $\alpha = 3z - 2$

$\beta = -3z - 2$

$|z| = 1$

$|\alpha + 2| = 3 \to$ A is correct

$|\beta + 2| = 3 \to$ B is correct

$\alpha - \beta = 6Z$ D is correct

$\Rightarrow \alpha - \beta$ and z, both move on same circle

Sol 30: (B, D) $(i^n + i^{-n})$ is

(A) $\frac{2^{2n} + 2^{2n}}{2^n(1-i)^{2n}} = \frac{2.2^n}{(1-i)^{2n}} = \frac{2^{n+1}}{2^{2n}}(1+i)^{2n} =$

$\frac{(2i)^n}{2^{n-1}} = 2i^n = i^n + i^n$

(B) $\frac{(1+i)^{2n} + (1-i)^{2n}}{2^n} = \frac{(2i)^n + (-2i)^n}{2^n} = i^n + (-i)^n$

$= i^n + i^{-n}$

(C) $\frac{-(2i)^n}{2^n} + \frac{2^n}{(-2i)^n} = -i^n + i^n = 0$

(D) $\frac{2^n}{(2i)^n} + \frac{2^n}{(-2i)^n} = i^{-n} + i^n$

Sol 31: (A, B, D)

$$\log_{14}\left(13 + |z^2 - 4i|\right) + \log_{196}\left(\frac{1}{\left(13 + |z^2 + 4i|\right)^2}\right) = 0$$

$$\log_{14}\frac{13 + |z^2 - 4i|}{13 + |z^2 + 4i|} = 0$$

$\Rightarrow 13 + |z^2 - 4i| = 13 + |z^2 + 4i|$

$\Rightarrow |z^2 - 4i| = |z^2 + 4i|$

$\Rightarrow |x^2 - y^2 + 2ixy - 4i| = |x^2 - y^2 + 2ixy + 4i|$

$\Rightarrow (x^2 - y^2)^2 + (2xy - 4)^2 = (x^2 - y^2)^2 + (2xy + 4)^2$

$\Rightarrow -16xy = 16xy \Rightarrow$ either $x = 0$ or $y = 0$

If $y = 0$ can be purely real

$x = 0$ can be purely imaginary

Must be real or purely imaginary.

Sol 32: (A, B) $1-\log_2\left[\frac{|x+1+2i|-2}{\sqrt{2}-1}\right]\ge 0$

$\Rightarrow \log_2 \frac{|x+1+2i|-2}{\sqrt{2}-1} \le 1$

$\Rightarrow |x + 1 + 2i| - 2 \le 2(\sqrt{2}-1)$

$\Rightarrow |x + 1 + 2i| \le 2\sqrt{2} \Rightarrow (x + 1)^2 + 4 \le 8$

$\Rightarrow (x + 1)^2 \le 4 \Rightarrow -2 \le x + 1 \le 2$

$\Rightarrow -3 \le x \le 1 \Rightarrow x \in [-3, 1]$

Sol 33: (D) $x = \cos \alpha$

$y = \cos \beta$

$z = \cos \gamma$

$\Sigma x = \cos\alpha + \cos\beta + \cos\gamma$

$= \cos\alpha+\cos\beta+\cos\gamma+i(\sin\alpha+\sin\beta+\sin\gamma)$

$\Pi x=\cos\alpha \cos\beta \cos\gamma = e^{i\alpha} e^{i\beta} e^{i\gamma} = e^{i(\alpha + \beta + \gamma)}$

Sol 34: (A, D) $x_r = \cos\left(\frac{\pi}{2^r}\right)$

$\lim\limits_{n\to\infty}\prod\limits_{r=1}^{n} x_r = e^{i\left(\frac{\pi}{2^r}+\frac{\pi}{2^{r+1}}......\right)} = e^{i\left(\frac{\pi}{2}+\frac{\pi}{4}+\cdots\right)}$

$= e^{i\frac{\pi}{2\left(1-\frac{1}{2}\right)}} = e^{i\pi}$

$= \cos\pi + i\sin\pi = -1 + 0i$

Real part is –1

Imaginary part is 0

Sol 35: (A, B, D) $\prod\limits_{r=1}^{n-1}(\omega-z_r)$

$(\omega - z_1)(\omega - z_2)(\omega - z_3)..... (\omega - z_{n-1}) = \frac{\omega^n-1}{\omega-1}$

If n is a multiple of 3, value can be zero as z_1 can be ω or ω^2

If $n = 3k + 1$ value is 1

If $n = 3k + 2$ value is $1 + \omega$

Sol 36: (A, B, C) $\bar{z} = -4z$

$x - iy= -4x - 4iy$

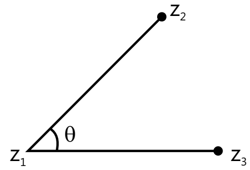
$5x + i(3y) = 0$

$x = y = 0$ (A is true)

$\bar{z} = z$ implies z is purely real (B is correct)

$\bar{z} = -z$ implies z is purely imaginary (C is correct)

Sol 37: (A, C)



$z_3 = (1 - z_0)z_1 + z_0z_2$

$z'_3 = (1 - z_0)z'_1 + z_0z'_2$

$z_0 = \frac{z_3-z_1}{z_2-z_1} = \frac{z'_3-z'_1}{z'_2-z'_1} \Rightarrow \begin{vmatrix} z_1 & z'_1 & 1 \\ z_2 & z'_2 & 1 \\ z_3 & z'_3 & 1 \end{vmatrix} = 0$

angle is same

The two triangles are similar

Sol 38: (A, B) Z multiplicative inverse is same as additive inverse

$\frac{1}{a+ib} = -a-ib$

$\frac{a-ib}{a^2+b^2} = -a-ib$

$\Rightarrow a = -a^3 - ab^2 \Rightarrow a(1 + a^2 + b^2) = 0$

$\Rightarrow a = 0$ and

$b = a^2b + b^3 \Rightarrow b(1 - a^2 - b^2) = 0$

$b \ne 0$ so $a^2 + b^2 = 1 \Rightarrow b = \pm 1$

$z = 0 \pm i$

Sol 39: (A, B, C, D) $Z = a + bi = \frac{(1-ix)(1-ix)}{(1+ix)(1-ix)}$

$a + bi = \frac{1-x^2-2ix}{1+x^2}$

$\Rightarrow |z| = 1$

$\text{Arg } z = \tan^{-1}\left(\frac{2x}{x^2-1}\right)$

$\text{Arg } (z) \equiv (-\pi, \pi]$

Previous Years' Questions

Sol 1: Given, $|z-3-2i| \le 2$ .... (i)

To find minimum of $|2z-6+5i|$

Or $2\left|z-3+\frac{5}{2}i\right|$,

By using triangle inequality

i.e., $\left||z_1|-|z_2|\right| \le \left|z_1+\bar{z}_2\right|$

$\therefore \left|z-3+\frac{5}{2}i\right|$

$= \left|z-3-2i+2i+\frac{5}{2}i\right| = \left|(z-3-2i)+\frac{9}{2}i\right|$

$\ge \left||z-3-2i|-\frac{9}{2}\right|$

$\ge \left|2-\frac{9}{2}\right| \ge \frac{5}{2}$

$\Rightarrow \left|z-3+\frac{5}{2}i\right| \ge \frac{5}{2}$ or $|2z-6+5i| \ge 5$

Sol 2: $\omega = e^{i\frac{2\pi}{3}}$

Then, $\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b^2|+|c|^2} = 3$

Note: Here, $\omega = e^{i2\pi/3}$, then only integer solution exists.

Sol 3: (B)

$A = \{z : \text{Im}(z) \ge 1\}$

$B = \{z : |z-2-i| = 3\}$

$C = \{z : \text{Re}[(1-i)z+\sqrt{2}]\}$

Taking $|z-2-i| = 3$

Let $z = z+iy$

$|x+iy-2-i| = 3$

$\Rightarrow |(x-2)+i(y-1)|^2 = (3)^2$

$\Rightarrow (x-2)^2+(y-1)^2 = 9$ ...(i)

And

$\text{Re}[(1-i)z] = \sqrt{2}$

$\text{Re}[(1-i)(x+iy)] = \sqrt{2}$

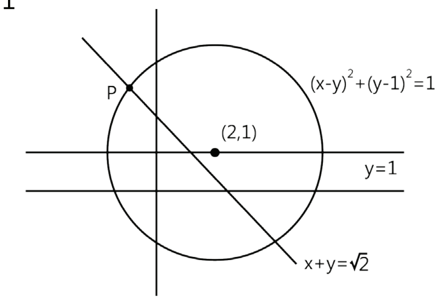
$\Rightarrow x+y = \sqrt{2}$ ....(ii)

And

$\text{Im}(z) \ge 1$

$\Rightarrow \text{Im}(x+iy) \ge 1$

$\Rightarrow y \ge 1$ ....(iii)



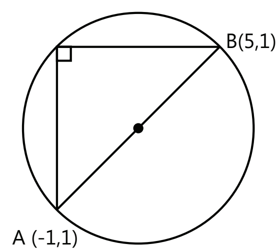
From (i), (ii) and (iii) $A \cap B \cap C$ has only one point P shown in the figure.

Sol 4: (C) $|z+1-i|^2+|z-5-i|^2$

$= |z-(-1+i)|^2+|z-(5+i)|^2$

The point A (-1, 1) and dB (5, 1) are end points of one of the diameter

In right angle Δ APB



$AP^2+BP^2 = (AB)^2$

$\Rightarrow |z+1-i|^2+|z-5-i|^2$

$= (6)^2 = 36$

Sol 5: (D) $|w-2-i| < 3$

From triangle in equality

$|z-w| > \left||z|-|w|\right|$

$\Rightarrow -|z-w| < |z|-|w| < |z-w|$ ...(i)

Given $|w-2-i| < 3$, which means that w lies inside the given circle. Since z is presented by point P and W has to be the other end of diameter

$|z-w|$ = length of diameter

From (i), we get

$-6 < |z| - |w| < 6$

$\Rightarrow -6+3 < |z| - |w| + 3 < 6+3$

$\Rightarrow -3 < |z| - |w| + 3 < 9$

Sol 6: (B) Given, $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$

$\left(\because \omega = \frac{-1+i\sqrt{3}}{2} \text{ and } \omega^2 = \frac{-1-i\sqrt{3}}{2}\right)$

Now, $\frac{\sqrt{3}+i}{2} = -i\left(\frac{-1+i\sqrt{3}}{2}\right) = -i\omega$

And $\frac{\sqrt{3}-i}{2} = i\left(\frac{-1-i\sqrt{3}}{2}\right) = i\omega^2$

$\therefore z = (-i\omega)^5 + (i\omega^2)^5 = -i\omega^2 + i\omega$

$= i(\omega - \omega^2) = i\left(i\sqrt{3}\right) = -\sqrt{3}$

$\Rightarrow$ Re (z) < 0 and Im (z) = 0

Alternate solution:

We known $z + \bar{z} = 2\mathrm{Re}(z)$

If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then z is purely real,

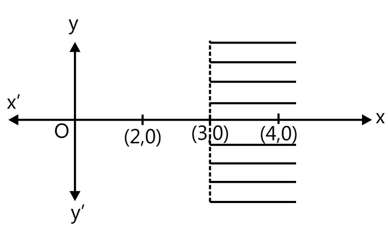
i.e. Im (z) = 0

Sol 7: (D) Given, $|z - 4| < |z - 2|$

Since, $|z - z_1| > |z - z_2|$ represents the region on right side of perpendicular bisector of z_1 and z_2

$\therefore |z - 2| > |z - 4|$

$\Rightarrow \mathrm{Re}(z) > 3$ and $\mathrm{Im}(z) \in R$



Sol 8: (B) Since a, b, c and u, v, w are the vertices of two triangles.

Also, c = (1 – r) a + rb and w = (1 – r) u + rv ... (i)

Consider $\begin{vmatrix} a & u & 1 \\ b & v & 1 \\ c & w & 1 \end{vmatrix}$

Applying $R_3 \to R_3 - \{(1-r)R_1 + rR_2\}$

$= \begin{vmatrix} a & u & 1 \\ b & v & 1 \\ c-(1-r)a-rb & w-(1-r)u-rv & 1-(1-r)-r \end{vmatrix}$

$= \begin{vmatrix} a & u & 1 \\ b & v & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$ (from eq. (i))

Hence, two triangles are similar.

Sol 9: (D) $\sum_{k=1}^{6}\left(\sin\frac{2k\pi}{7} - i\cos\frac{2k\pi}{7}\right)$

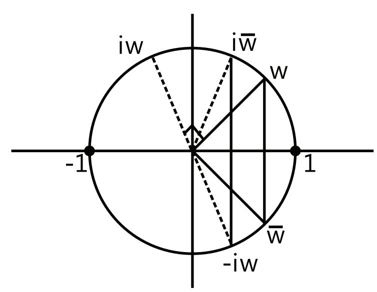
$\sum_{k=1}^{6} i\left(\cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}\right) = -i\left\{\sum_{k=1}^{6} e^{\frac{i2k\pi}{7}}\right\}$

$\left\{\sum_{k=0}^{6} e^{\frac{i2k\pi}{7}} = 0\right\} = -i\left\{\sum_{k=0}^{6} e^{\frac{i2k\pi}{7}} - 1\right\} = -i\left[0 - 1\right] = i$

Sol 10: (C) Given, $|z + iw| = |z - i\bar{w}|\ 2 = 2$

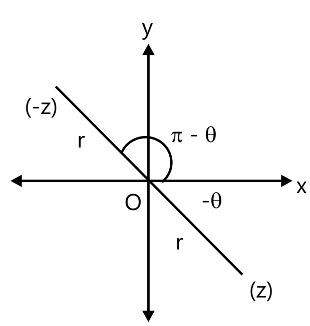
$\Rightarrow \quad |z - (-iw)| = |z - (i\bar{w})| = 2$

$\Rightarrow \quad |z - (-iw)| = |z - (-i\bar{w})|$.



$\therefore$ z may take values given in (c).

Alternate solution



$|z+iw| \le |z| + |iw| = |z| + |w| \le 1+1 = 2$

$\therefore \quad |z+iw| \le 2$

$\Rightarrow \quad |z+iw| = 2$ holds when

Arg z – arg iw = 0

Similarly, when $|z - i\overline{w}| = 2$

Then $\frac{z}{\overline{w}}$ is purely imaginary

Now, given relation

$|z+iw| = |z - i\overline{w}| = 2$

Put w = i, we get

$|z+i^2| = z + i^2| = 2$

$\Rightarrow \quad |z-1| = 2$

$\Rightarrow \quad z = -1 \qquad (\because |z| \le 1)$

Put w = - i, we get

$|z-i^2| = |z-i^2| = 2$

$\Rightarrow \quad |z+1| = 2$

$\Rightarrow \quad z = 1 \qquad (\because |z| \le 1)$

$\therefore$ z = 1 or – 1 is the one correct option given.

Sol 11: (D)

$(1+i)^{n_1} + (1-i)^{n_1} + (1+i)^{n_2} + (1-i)^{n_2}$

$= [{}^{n_1}C_0 + {}^{n_1}C_1 i + {}^{n_1}C_2 i^2 + {}^{n_1}C_3 i^3 + ...]$

$+ [{}^{n_1}C_0 - {}^{n_1}C_1 i + {}^{n_i}C_2 i^2 - {}^{n_1}C_3 i^3 + ...]$

$+ [{}^{n_2}C_0 + {}^{n_2}C_1 i + {}^{n_2}C_2 i^2 + {}^{n_2}C_3 i + ...]$

$+ [{}^{n_2}C_0 - {}^{n_2}C_1 i + {}^{n_2}C_2 i^2 - {}^{n_2}C_3 i^3 + ...]$

$= 2[{}^{n_1}C_0 + {}^{n_1}C_2 i^2 + {}^{n_1}C_4 i^4 + ...]$

$+ 2[{}^{n_2}C_0 + {}^{n_2}C_2 i^2 + {}^{n_2}C_4 i^4 + ...]$

$= 2[{}^{n_1}C_0 - {}^{n_1}C_2 + {}^{n_1}C_4 = ...]$

$+ 2[{}^{n_2}C_0 - {}^{n_2}C_2 + {}^{n_2}C_4 - ....]$

This is a real number irrespective of the values of n_1 and n_2

Alternate solution

$\{(1+i)^{n_1} + (1-i)^{n_1}\} + \{(1+i)^{n_2} + (1-i)^{n_2}\}$

$\Rightarrow$ a real number for all n_1 and $n_2 \in R$.

$[\because z + \overline{z} = 2\text{Re}(z) \Rightarrow (1+i)^{n_1} + (1-i)^{n_1}$

is real number for all $n \in R]$

Sol 12: (C) If in a complex number a + ib, the ratio a: b is $1:\sqrt{3}$, then always convert the complex number in the form of ω.

Since, $\quad \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$\therefore 4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$

$= 4 + 5\omega^{334} + 3\omega^{365}$

$= 4 + 5\cdot(\omega^3)^{111}\cdot\omega + 3\cdot(\omega^3)^{121}\cdot\omega^2$

$= 4 + 5\omega + 3\omega^2 \qquad (\because \omega^3 = 1)$

$= 1 + 3 + 2\omega + 3\omega + 3\omega^2$

$= 1 + 2\omega + 3(1+\omega+\omega^2) = 1 + 2\omega + 3\times 0$

$= 1 + (-1 + \sqrt{3}i) = \sqrt{3}i.$

Sol 13: (A) Since, arg (z) < 0

$\Rightarrow \quad \arg(z) = -\theta$

$\Rightarrow \quad z = r\cos(-\theta) + i\sin(-\theta) = r(\cos\theta - i\sin\theta)$

and $-z = -r[\cos\theta - i\sin\theta]$

$= r[\cos(\pi-\theta) + i\sin(\pi-\theta)]$

$\therefore \quad \arg(-z) = \pi - \theta$

Thus, arg (-z) –arg (z)

$= \pi - \theta - (-\theta) = \pi$

Alternate solution:

Reason: arg (-z) – arg z $= \arg\left(\frac{-z}{z}\right) = \arg(-1) = \pi$

And also arg z – arg (-z) $= \arg\left(\frac{z}{-z}\right) = \arg(-1) = \pi$

Sol 14: (A, B,C) Since, $z_1 = a + ib$ and $z_2 = c + id$

$\Rightarrow \quad |z_1|^2 = a^2 + b^2 = 1$ and $|z_2|^2 = c^2 + d^2 = 1 \qquad ...(i)$

$(\because |z_1| = |z_2| = 1)$

Also, Re $(z_1\overline{z}_2) = 0 \Rightarrow \quad ac + bd = 0$

$\Rightarrow \quad \frac{a}{b} = -\frac{d}{c} = \lambda \qquad$ (say)...(ii)

From Eqs. (i) and (ii), $b^2\lambda^2 + b^2 = c^2 + \lambda^2 c^2$

$\Rightarrow \quad b^2 = c^2$ and $a^2 = d^2$

Also, given $w_1 = a + ic$ and $w_2 = b + id$

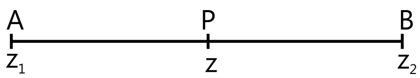
Now, $\quad |w_1| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1$

$|w_2| = \sqrt{b^2 + d^2} = \sqrt{a^2 + b^2} = 1$

and $\text{Re}(w_1\overline{w_2}) = ab + cd = (b\lambda)b + c(-\lambda c)$

$= \lambda(b^2 - c^2) = 0$

Sol 15: (A, C, D) Given, $z = \dfrac{(1-t)z_1 + tz_2}{(1-t)+t}$



t : (1 - t)

Clearly, z divides z_1 and z_2 in the ratio of t:

(1-t), 0 < t < 1

$\Rightarrow$ AP + BP = AB

ie, $|z - z_1| + |z - z_2| = |z_1 - z_2|$

$\Rightarrow$ Option (a) is true.

And arg $(z - z_1) = \arg(z_2 - z) = \arg(z_2 - z_1)$

$\Rightarrow$ (b) is false and (d) is true.

Also, $\arg(z - z_1) = \arg(z_2 - z_1)$

$\Rightarrow \arg\left(\dfrac{z - z_1}{z_2 - z_1}\right) = 0$

$\therefore \dfrac{z - z_1}{z_2 - z_1}$ is purely real.

$\Rightarrow \dfrac{z - z_1}{z_2 - z_1} = \dfrac{\overline{z} - \overline{z_1}}{\overline{z_2} - \overline{z_1}}$ or $\begin{vmatrix} z - z_1 & \overline{z} - \overline{z_1} \\ z_2 - z_1 & \overline{z_2} - \overline{z_1} \end{vmatrix} = 0$

Sol 16: (A) $|z - i|z|| = |z + i|z||$

$\Rightarrow |z - i|z||^2 = |z + i|z||^2$

$\Rightarrow (z - i|z|)(\overline{z} + i|z|) = (z + i|z|)(\overline{z} - i|z|)$

$\Rightarrow z\overline{z} + iz|z| - i|z|\overline{z} + |z|^2$

$= z\overline{z} - iz|z| + i|z|\overline{z} + |z|^2$

$\Rightarrow 2iz|z| = 2i|z|\overline{z}$

$\Rightarrow 2i\,|z|(z - \overline{z}) = 0$

$\Rightarrow |z| = 0$ or $z - \overline{z} = 0$

$\Rightarrow \text{Im}(z) = 0$

Also $|\text{Im}(z)| \le 1$

$A \to$ q, r

(B) |z + 4| + |z - 4| = 0

Its an equation of ellipse having

aq = 4 and 2a = 10

$\Rightarrow e = \dfrac{4}{5}$

$B \to p$

(C) $z = w - \dfrac{1}{w}$

$\Rightarrow z = w - \dfrac{1}{w\overline{w}} \times \overline{w}$

$= w - \dfrac{\overline{w}}{|w|^2}$

$= w - \dfrac{\overline{w}}{4}$

Let w = p + I q, then

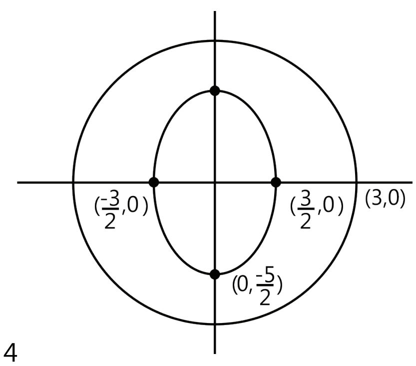
$z = \dfrac{3P}{4} + i\dfrac{5q}{4}$

Now, let z = x + iy

$\Rightarrow x = \dfrac{3P}{4} \quad \Rightarrow P = \dfrac{4x}{3}$

$\Rightarrow y = \dfrac{5q}{4} \quad \Rightarrow q = \dfrac{4y}{5}$

$|w|^2 = P^2 + q^2 = 4$



$\Rightarrow \dfrac{16x^2}{9} + \dfrac{16y^2}{25} = 4$

$\Rightarrow \dfrac{x^2}{9/4} + \dfrac{y^2}{25/4} = 1$, its an ellipse

$e^2 = 1 - \dfrac{b^2}{a^2}$

$= 1 - \dfrac{9}{25} = \dfrac{16}{25}$

$\Rightarrow e = \dfrac{4}{5}$

From figure, we can conclude that

$|\text{Re}(z)| \le 2$

And $|z| \le 3$

$C \to$ p, s, t

(D) $z = w + \frac{1}{w} = w + \frac{\bar{w}}{w\,\bar{w}} = w + \frac{\bar{w}}{|w|^2} = w + \bar{w}$

Let w = a + i b

z = a + i b + a – i b = 2a

Now, |z| = 2 |a| and Im (z) = 0, $\mathrm{Im}(z) \le 1$

$\Rightarrow \mathrm{Re}(z) \le 2$ and $|z| \le 3$

$D \to q, r, s$ and t

Sol 17: (B)

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} z & \omega & \omega^2 \\ z & z+\omega^2 & 1 \\ z & 1 & z+\omega \end{vmatrix} = 0$$

$$\left\{\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right\}$$

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

Expanding the determinant, we get

$$z\left[\left(z+\omega^2\right)\left(z+\omega\right) - 1 - \omega\left(z+\omega-1\right) + \omega^2\left(1-z-\omega^2\right)\right] = 0$$

$$\Rightarrow z\left[z^2 + z\omega + z\omega^2 + \omega^2 - 1 - z\omega - \omega^2 + \omega + \omega^2 - \omega^2 z - \omega^4\right] = 0$$

$$\Rightarrow z\left[z^2\right] = 0$$

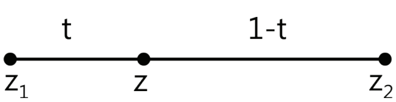
$$\Rightarrow z = 0$$

Sol 18: (A, C, D) Given $z = (1-t)z_1 + tz_2$

This equation represents line segment between z_1 and z_2

From fig.

$|z - z_1| + |z - z_2| = |z_1 - z_2|$



$\mathrm{Arg}\left(z - z_1\right) = \mathrm{Arg}\left(z_2 - z_1\right)$

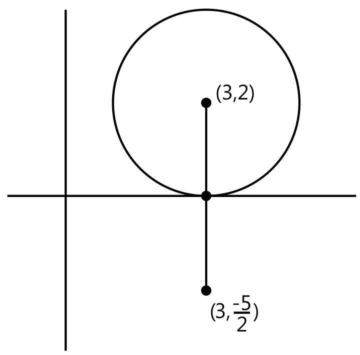
$$\frac{z - z_1}{\bar{z} - \bar{z}_1} = \frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1}$$

$$\Rightarrow \begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$$

Sol 19: $|2z - 6 - 5i|$

$= 2\left|z - 3 + \frac{5}{2}i\right|$

Here $\left|z - \left(3 - \frac{5}{2}\,i\right)\right|$ is nothing but the distance between any point on the circle $|z - 3 - 2i| \le 2$ and point $\left(3, \frac{-5}{2}\right)$



$\therefore$ Minimum value $= \frac{5}{2}$

$|2z - 6 + 5i|_{min} = 2 \times \frac{5}{2} = 5$

Sol 20: Given: a + b + c = x

$a + b\omega + c\omega^2 = y$

$a + b\omega^2 + c\omega = z$

$$\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = \frac{x\bar{x} + y\bar{y} + z\bar{z}}{|a|^2 + |b|^2 + |c|^2}$$

$$= \frac{\left(a+b+c\right)\left(\bar{a}+\bar{b}+\bar{c}\right) + \left(a + b\omega + c\omega^2\right) + \left(\bar{a} + \bar{b}\bar{\omega} + \bar{c}\bar{\omega}^2\right) + \left(a + b\omega^2 + c\omega\right)\left(\bar{a} + \bar{b}\bar{\omega}^2 + \bar{c}\bar{\omega}\right)}{|a|^2 + |b|^2 + |c|^2}$$

$$\left\{\begin{array}{l} \omega = e^{i\pi/3} \\ \bar{\omega} = e^{-i\pi/3} \\ \omega^2 = e^{2i\frac{\pi}{3}} = -\bar{\omega} \end{array}\right.$$

$$= \frac{\left(a+b+c\right)\left(\bar{a}+\bar{b}+\bar{c}\right) + \left(a + b\omega - \bar{\omega}c\right)\left(\bar{a} + \bar{b}\bar{\omega} + \bar{c}\bar{\omega}^2\right) + \left(a - b\bar{\omega} + c\omega\right)\left(\bar{a} + \bar{b}\bar{\omega}^2 + \bar{c}\bar{\omega}\right)}{|a|^2 + |b|^2 + |c|^2}$$

$$= \frac{3\left(|a|^2+|b|^2+|c|^2\right)}{|a|^2+|b|^2+|c|^2} = 3$$

Sol 21: (A) Given $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$

Determinant $(D) = \begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$

$|D| = 1(1-c\omega) - a(\omega - c\omega^2) + b(\omega^2 - \omega^2)$

$\Rightarrow |D| = 1 - (a+c)\omega + ac\omega^2$

$|D| \neq 0$, only when a = ω and c = ω

$\therefore (a, b, c) \equiv (\omega, \omega, \omega)$ or $(\omega, \omega^2, \omega)$

$\therefore$ Two non-singular matrices are possible.

Sol 22: (D) $a = z^2 + z + 1 = 0$

$\Rightarrow z^2 - z + 1 - a = 0$

Discriminant < 0 {Since imaginary part of z us not zero}

1 – 4 (1 - a) < 0

$\Rightarrow 1 - 4 + 4a < 0$

$\Rightarrow 4a < 3$

$\Rightarrow a < \frac{3}{4}$

Sol 23: (C) $(x - x_0)^2 + (y - y_0)^2 = r$

$\Rightarrow |z - z_0| = r$

α lies on it, then

$|\alpha - z_0| = r$

$\Rightarrow (\alpha - z_0)(\bar{\alpha} - \bar{z}_0) = r^2$

$= |\alpha|^2 - \bar{z}_0\,\alpha - z_0\,\bar{\alpha} + |z_0|^2 = r^2$ ....(i)

Similarly,

$(x - x_0)^2 + (y - y_0)^2 = 2r$

$\Rightarrow |z - z_0| = 2r$

$\frac{1}{\alpha}$ lies on it, then

$\left|\frac{1}{\alpha} - z_0\right| = 2r$

$\Rightarrow \left(\frac{1}{\alpha} - z_0\right)\left(\frac{1}{\alpha} - \bar{z}_0\right) = 4r^2$

$\Rightarrow \left(\frac{1}{\bar{\alpha}\,\alpha} - \frac{\bar{z}_0}{\bar{\alpha}} - \frac{z_0}{\alpha} + |z_0|^2\right) = 4r^2$

$\left\{\alpha\,\bar{\alpha} = |\alpha|^2 \Rightarrow \bar{\alpha} = \frac{|\alpha|^2}{\alpha}\right\}$

$\Rightarrow \frac{1}{|\alpha|^2} - \frac{\bar{z}_0\alpha}{|\alpha|^2} - \frac{z_0\,\bar{\alpha}}{|\alpha|^2} + |z_0|^2 = 4r^2$

$\Rightarrow 1 - z_0\,\bar{\alpha} - \bar{z}_0\,\alpha + |\alpha|^2|z_0|^2 = 4r^2\,|z|^2$ ...(ii)

Subtracting (ii) from (i), we get

$1 - |\alpha|^2 + |\alpha|^2|z_0|^2 - |z_0|^2 = r^2\left(4|\alpha|^2 - 1\right)$

$1 - |\alpha|^2 + \frac{r^2+2}{2}\left(|\alpha|^2 - 1\right) = r^2\left(4|\alpha|^2 - 1\right)$

$\Rightarrow \left(|\alpha|^2 - 1\right)\left[\frac{r^2+2}{2} - 1\right] = r^2\left(4|\alpha|^2 - 1\right)$

$\Rightarrow \left(|\alpha|^2 - 1\right)\left[\frac{r^2}{2}\right] = r^2\left(4|\alpha|^2 - 1\right)$

$\Rightarrow |\alpha|^2 - 1 = 8|\alpha|^2 - 2$

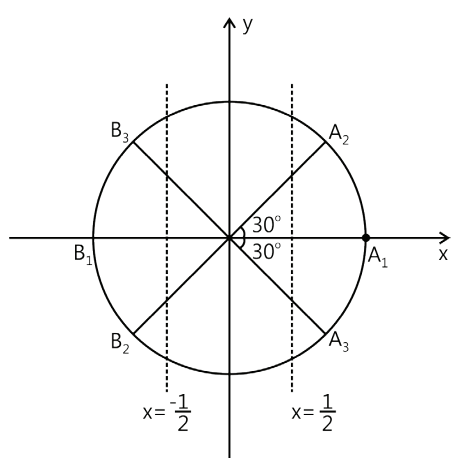
$\Rightarrow 7|\alpha|^2 = 1$

$\Rightarrow |\alpha| = \frac{1}{\sqrt{7}}$

Sol 24: (C, D) $P = \{w^n : n = 1, 2, 3, ...\}$

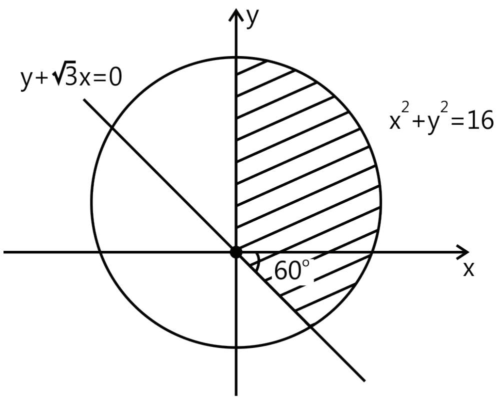
$w = \frac{\sqrt{3}}{2} + \frac{1}{2} \Rightarrow |w| = 1$

$\Rightarrow$ All the complex number belong to set P lie on circle of unit radius, centre at origin.



All the complex number belong to PnH_1 lie right of the line $x = \frac{1}{2}$ on circle. Possible positions are shown in the figure as A_1, A_2, A_3.

Similarly, all the complex number belong to PnH_2 lie left of the line $x = \frac{-1}{2}$ on circle. Possible positions are shown in the figure as $B_1, B_2, B_2 < z_1\ 0\ z_2 =$ angle between A_2 and $B_3 = \frac{2\pi}{3}$ or angle between A_1 and $B_3 = \frac{5\pi}{6}$



Sol 25: (B, C, D) $P = \left[p_{ij}\right]_{n \times n}$ $\quad p_{ij} = \omega^{i+j}$

$$P = \begin{bmatrix} \omega^2 & \omega^3 & \omega^4 & \cdots & \omega^{n+1} \\ \omega^3 & \omega^4 & \omega^5 & \cdots & \omega^{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega^{2n+1} & \omega^{n+2} & \omega^{n+3} & \cdots & \omega^{n+n} \end{bmatrix}$$

$$\Rightarrow P^2 = \begin{bmatrix} \omega^4 + \omega^6 + \omega^8 + \ldots & \omega^5 + \omega^7 + \omega^9 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \omega^{n+3} + \omega^{n+5} + \ldots & \cdots & \cdots & \cdots \end{bmatrix}$$

Lets take element $P_{11} = \omega^4 + \omega^6 + \omega^8 + \ldots n$ terms

$$= \frac{\omega^4\left(\omega^{2n} - 1\right)}{\omega - 1}$$

If n is multiple of 3, then this element will vanish. Which is the case for every element.

$\therefore$ n can not be multiple of 3, for $P^2 \neq 0$

$\therefore$ Possible values of n are 55, 58, 56

Sol 26: (B) $S_1 = \{z \in C : |z| < 4\}$

z lies inside of a circle given by |z| = 4

$$S_2 = \left\{z \in C : \text{Im}\left[\frac{z - 1 + \sqrt{3}i}{1 - \sqrt{3}i}\right] > 0\right\}$$

$$\text{Im}\left[\frac{\left(z - 1 + \sqrt{3}i\right)\left(1 + \sqrt{3}i\right)}{4}\right] > 0$$

Let z = x + i y

$$\Rightarrow \text{Im}\left[\frac{(x + iy)\left(1 + \sqrt{3}i\right) - \left(1 - \sqrt{3}i\right)\left(1 + \sqrt{3}i\right)}{4}\right] > 0$$

$$\Rightarrow \text{Im}\left[\frac{\left(x + \sqrt{3}y\right) + i\left(\sqrt{3}x + y\right) - 4}{4}\right] > 0$$

$$\Rightarrow \frac{\sqrt{3}x + y}{4} > 0 \Rightarrow \sqrt{3}x + y > 0$$

$S_3 \{z \in C : \text{Re}(z) > 0\}$

$\Rightarrow$ z lies in either first or fourth quadrant.

Now, points of intersection of circle $x^2 + y^2 = 16$ and $\sqrt{3}x + y = 0$ is $x^2 + 3x^2 = 16$

$\Rightarrow 4x^2 = 16 \Rightarrow x = \pm 2$

$\Rightarrow y = \pm 2\sqrt{3}$

$$\text{Area} = \frac{1}{2}(4)^2 \times \frac{5\pi}{6}$$

$$= \frac{1}{2} \times 16 \times \frac{5\pi}{6} = \frac{20\pi}{3} \text{ .sq. unit}^2$$

Sol 27: (C) $|1 - 3i - z|$

$= |z - 1 + 3i| = |z - (1 - 3i)|$

Min $\left(\left|z - (1 - 3i)\right|\right) =$ distance of point (1, -3) from line $\sqrt{3}\,x + y = 0$

$$= \left|\frac{\sqrt{3} \times 1 - 3}{\sqrt{3 + 1}}\right| = \left|\frac{\sqrt{3} - 3}{2}\right|$$

$$= \frac{3 - \sqrt{3}}{2}$$

Sol 28: (C) $z_k = \cos\frac{2\pi x}{10} + i\sin\frac{2k\pi}{10}, k = 1, 2, \ldots 9$

(p) $z_k = e^{i\frac{2\pi k}{10}}$

$\Rightarrow z_k.z_j = e^{i\frac{2\pi}{10}(k+j)} = 1$

$\Rightarrow \cos\frac{2\pi}{10}(k+j) + i\sin\frac{2\pi}{10}(k+j) = 1$

If k + j = 10 m (multiple of 10), then above equation is True.

(q) $z_1.z = z_k$

$\Rightarrow z = \frac{z_k}{z_1} = \frac{e^{i\frac{2\pi k}{10}}}{e^{i\frac{2\pi}{10}}} = e^{i\frac{(k-1)}{10}2\pi}$

Clearly, this equation has many solutions

$\Rightarrow$ Q is False

(r) Consider, $z^{10} = 1$

$z^{10} - 1 = (z-1)(z-z_1)(z-z_2)(z-z_3)....(z-z_9)$

Where z_k represents the roots of equation $z^{10} = 10$

$= 1 + z + z^2 + ... + z^9$

$\Rightarrow \frac{Z^{10}-1}{z-1} = (z-z_1)(z-z_2)......(z-z_9)$

$= 1 + z + z^2 + z^3 + ... + z^9$

$\frac{z^{10}-1}{z-1} = (z-z_1)(z-z_2)....(z-z_9)$

= 1 + 1 + 1 10

= 10

(s) We know that sum of roots of $z^{10} - 1 = 0$ is zero

$\Rightarrow 1 + \sum_{k=1}^{9}\cos\frac{2k\pi}{10} = 0 \Rightarrow \sum_{k=1}^{9}\cos\frac{2k\pi}{10} = -1$

$\Rightarrow 1 - \sum_{k=1}^{9}\cos\frac{2k\pi}{10} = 2$

Sol 29: Given $\alpha_k = \cos\frac{k\pi}{7} + i\sin\frac{k\pi}{7} = e^{i\frac{k\pi}{7}}$

$$\frac{\sum_{k=1}^{12}|\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1} - \alpha_{4k-2}|} = \frac{\sum_{k=1}^{12}\left|e^{i\frac{(k+1)\pi}{7}} - e^{i\frac{k\pi}{7}}\right|}{\sum_{k=1}^{3}\left|e^{i(4k-1)\frac{\pi}{7}} - e^{i\frac{(4k-2)}{7}}\right|}$$

$$= \frac{\sum_{k=1}^{12}\left|e^{ik\frac{\pi}{7}}\right|\left|e^{i\frac{\pi}{7}} - 1\right|}{\sum_{k=1}^{3}\left|e^{i\frac{(4k-2)}{7}}\right|\left|e^{i\frac{\pi}{7}} - 1\right|} = \frac{\sum_{k=1}^{12}\left|e^{i\frac{\pi}{7}} - 1\right|}{\sum_{k=1}^{3}\left|e^{i\frac{\pi}{7}} - 1\right|}$$

$$= \frac{\sum_{k=1}^{12} 2\sin\frac{\pi}{14}}{\sum_{k=1}^{3} 2\sin\frac{\pi}{14}} = \frac{12 \times 2\sin\frac{\pi}{4}}{3 \times 2\sin\frac{\pi}{4}} = 4$$

Sol 30: $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$

$$P^2 = \begin{bmatrix} (-z)^{2r} + (z^{4s}) & (-z)^r z^{2s} + z^r z^{2s} \\ (-z)^r z^{2s} + z^r z^{2s} & z^{2r} + z^{4s} \end{bmatrix}$$

$$= \begin{bmatrix} z^{2r} + z^{4s} & z^{2s}\left[(-z)^r + z^r\right] \\ z^{2s}\left((-z)^r + z^r\right) & z^{2r} + z^{4s} \end{bmatrix}$$

Given that $P^2 - I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow z^{2r} + z^{4s} = -1$

And $z^{2s}\left[(-z)^r + z^r\right] = 0$

Now, we have $z = \frac{-1 + i\sqrt{3}}{2} = \omega$

$\Rightarrow \omega^{2r} + \omega^{4s} = -1$ and $\omega^{2s}\left[(-\omega)^r + \omega^r\right] = 0$

Only $(r, s) \equiv (1, 1)$ satisfies both the equation.

Only one pair exists.

Sol 31: (A, C, D) $S = \left\{z \in C : z = \frac{1}{a + ibt}, t \in R, t \neq 0\right\}$

$z = \frac{1}{a + ibt} \times \frac{a - ibt}{a - ibt}$

$\Rightarrow z = \frac{a - ibt}{a^2 + b^2t^2} = x + iy$ (Let)

$\Rightarrow x = \frac{a}{a^2 + b^2t^2}$ and $y = \frac{bt}{a^2 + b^2t^2}$

$\Rightarrow \frac{x}{y} = \frac{a}{bt} \Rightarrow t = \frac{ay}{bx}$

Substituting 't' in $x = \frac{a}{a^2 + b^2 t^2}$

$$x = \frac{a}{a^2 + b^2 \frac{a^2y^2}{b^2x^2}}$$

$$\Rightarrow x = \frac{ax^2}{a^2x^2 + a^2y^2} \Rightarrow a^2x^2 + a^2y^2 - ax = 0$$

$$\Rightarrow x^2 + y^2 - \frac{x}{a} = 0$$

$$\left(x - \frac{1}{2a}\right)^2 + y^2 = \left(\frac{1}{2a}\right)^2$$

Centre $\equiv \left(\frac{1}{2a}, 0\right)$

Radius $= \frac{1}{2a}$, when $a > 0$, $b \neq 0$

If $b = 0, a \neq 0$

$y = 0 \Rightarrow x - \text{axis}$

If $a = 0, b \neq 0$

$x = 0 \Rightarrow y - \text{axis}$