# Inverse Trigonometric Functions

## What are Inverse Trigonometric Functions?

The expressions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\text{cosec}^{-1}x$ are called inverse trigonometric function

## Domain and Range of Inverse Trigonometric Function

- $y = \sin^{-1} x$
  - Domain: $-1 \le x \le 1$
  - Range: $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$, i.e., $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
- $y = \cos^{-1} x$
  - Domain: $-1 \le x \le 1$
  - Range: $0 \le y \le \pi$, i.e., $[0, \pi]$
- $y = \tan^{-1} x$
  - Domain: All real numbers $(-\infty < x < \infty)$
  - Range: $\frac{-\pi}{2} < y < \frac{\pi}{2}$, i.e., $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
- $y = \cot^{-1} x$
  - Domain: All real numbers $(-\infty < x < \infty)$
  - Range: $0 < y < \pi$, i.e., $(0, \pi)$
- $y = \sec^{-1} x$
  - Domain: $1 \le x, < \infty$ and $-\infty < x \le -1$ i.e., $x \le -1$ or $x \ge 1$
  - Range: $0 \le y < \frac{\pi}{2}$ and $\frac{\pi}{2} < y \le \pi$. Note that $y \ne \frac{\pi}{2}$, i.e., $[0, \pi] - \left\{\frac{\pi}{2}\right\}$
- $y = \text{cosec}^{-1} x$
  - Domain: $-\infty < x \le -1$ and $1 \le x < \infty$ i.e., $x \le -1$ or $x \ge 1$
  - Range: $\frac{-\pi}{2} \le y < 0$ and $0 < y \le \frac{\pi}{2}$. Note that $y \ne 0$, i.e., $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

## Principal Value Branch and Principal Values

The range of an inverse trigonometric function is the principle value branch and those values which lie in the principal value branch is called the principal value of inverse trigonometric function

## Important Formulae of Inverse Trigonometric Functions

### Subtraction Formulae

(i) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$, if $-1 \le x, y \le 1$ and $x^2 + y^2 \le 1$

(ii) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}[xy + \sqrt{1-x^2}\sqrt{1-y^2}]$, if $-1 \le x, y \le 1$ and $x \le y$

(iii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$, if $xy > -1$

### Addition Formulae

(i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$, if $-1 \le x, y \le 1$ and $x^2 + y^2 \le 1$

(ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$, if $-1 \le x, y \le 1$ and $x + y \ge 0$

(iii) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$, if $xy < 1$

### Other Important Formulae

(i) $\sin^{-1}(-x) = -\sin^{-1}x, x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x, x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1}x, x \in \mathbb{R}$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in \mathbb{R}$

(v) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x, x \in \mathbb{R} - (-1, 1)$

(vi) $\sec^{-1}(-x) = \pi - \sec^{-1}x, x \in \mathbb{R} - (-1, 1)$

(vii) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, x \in [-1, 1]$

(viii) $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}, x \in \mathbb{R} - (-1, 1)$

(ix) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, x \in \mathbb{R}$

(x) $2\sin^{-1}x = \sin^{-1}[2x\sqrt{1-x^2}], \frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

(xi) $2\cos^{-1}x = \cos^{-1}[2x^2 - 1], 0 \le x \le 1$
$= 2\pi - \cos^{-1}(2x^2 - 1), -1 \le x \le 0$

(xii) $2\tan^{-1}x = \begin{cases} \sin^{-1}\frac{2x}{1+x^2}, & x \in [-1, 1] \\ \cos^{-1}\frac{1-x^2}{1+x^2}, & x \ge 0 \\ \tan^{-1}\frac{2x}{1-x^2}, & -1 < x < 1 \end{cases}$

(xiii) $3\sin^{-1}x = \sin^{-1}[3x - 4x^3], \frac{-1}{2} \le x \le \frac{1}{2}$

(xiv) $3\cos^{-1}x = \cos^{-1}[4x^3 - 3x], \frac{-1}{2} \le x \le 1$

(xv) $3\tan^{-1}x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$, if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

## Some important Relations of Inverse Trigonometric Functions

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \text{cosec}^{-1}\frac{1}{x}$

(ii) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}} = \sec^{-1}\frac{1}{x} = \text{cosec}^{-1}\frac{1}{\sqrt{1-x^2}}$

(iii) $\tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \cot^{-1}\frac{1}{x} = \text{cosec}^{-1}\frac{\sqrt{1+x^2}}{x} = \sec^{-1}\sqrt{1+x^2}$

## Elementary Properties of Inverse Trigonometric Function

- $\sin^{-1}x$
  - $\sin^{-1}(\sin x) = x$, for all $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
  - $\sin(\sin^{-1}x) = x$, for all $x \in [-1, 1]$
- $\cos^{-1}x$
  - $\cos^{-1}(\cos x) = x$, for all $x \in [0, \pi]$
  - $\cos(\cos^{-1}x) = x$, for all $x \in [-1, 1]$
- $\tan^{-1}x$
  - $\tan^{-1}(\tan x) = x$, for all $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
  - $\tan(\tan^{-1}x) = x$, for all $x \in R$
- $\cot^{-1}x$
  - $\cot^{-1}(\cot x) = x$, for all $x \in (0, \pi)$
  - $\cot(\cot^{-1}x) = x$, for all $x \in R$
- $\sec^{-1}x$
  - $\sec^{-1}(\sec x) = x$, for all $x \in [0, \pi], x \ne \frac{\pi}{2}$
  - $\sec(\sec^{-1}x) = x$, for all $x \in (-\infty, -1) \cup [1, \infty]$, i.e., $R - (-1, 1)$
- $\text{cosec}^{-1}x$
  - $\text{cosec}^{-1}(\text{cosec}\, x) = x$, for all $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], x \ne 0$
  - $\text{cosec}(\text{cosec}^{-1}x) = x$, for all $x \in [-\infty, -1] \cup (1, \infty)$, i.e., $R - (-1, 1)$

**Trace the Mind Map**

- First Level
- Second Level
- Third Level