13 Oscillations

SIMPLE HARMONIC MOTIONS

- **Periodic Motion**

  Any motion which repeats itself after regular interval of time (i.e. time period) is called periodic motion or harmonic motion.

  Ex. (i) Motion of planets around the sun.

  (ii) Motion of the pendulum of wall clock.

- **Oscillatory Motion**

  The motion of body is said to be oscillatory or vibratory motion if it moves back and forth (to and fro) about a fixed point after regular interval of time.

  The fixed point about which the body oscillates is called mean position or equilibrium position.

  Ex.: (i) Vibration of the wire of 'Sitar'.

  (ii) Oscillation of the mass suspended from spring.

  Note : Every oscillatory motion is periodic but every periodic motion is not oscillatory.

- **Simple Harmonic Motion (S.H.M.)**

  Simple harmonic motion is the simplest form of vibratory or oscillatory motion.

- **Some Basic Terms in SHM**

  - **Mean Position**

    The point at which the restoring force on the particle is zero and potential energy is minimum, is known as its mean position.

  - **Restoring Force**

    The force acting on the particle which tends to bring the particle towards its mean position, is known as restoring force.

    Restoring force always acts in a direction opposite to that of displacement. Displacement is measured from the mean position.

  - **Amplitude**

    The maximum (positive or negative) value of displacement of particle from mean position is defined as amplitude.

  - **Time period (T)**

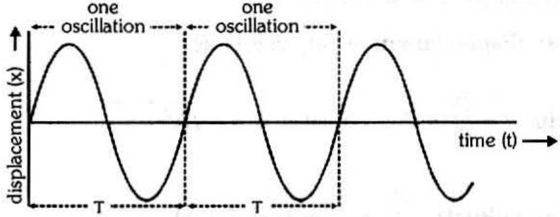
The minimum time after which the particle keeps on repeating its motion is known as time period.

The smallest time taken to complete one oscillation or vibration is also defined as time period.

It is given by $T = \frac{2\pi}{\omega} = \frac{1}{n}$ where ω is angular frequency and n is frequency.

- **One oscillation or One vibration**

When a particle goes on one side from mean position and returns back and then it goes to other side and again returns back to mean position, then this process is known as one oscillation.



- **Frequency (n or f)**

The number of oscillations per second is defined as frequency.

It is given by $n = \frac{1}{T} = \frac{\omega}{2\pi}$

- **Phase**

Phase of a vibrating particle at any instant is the state of the vibrating particle regarding its displacement and direction of vibration at that particular instant.

In the equation $x = A \sin(\omega t + \phi)$, $(\omega t + \phi)$ is the phase of the particle.

The phase angle at time t = 0 is known as initial phase or epoch.

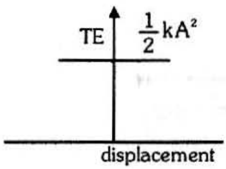
The difference of total phase angles of two particles executing SHM with respect to the mean position is known as phase difference.

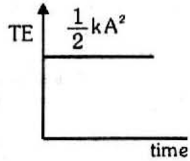
Two vibrating particles are said to be in same phase if the phase difference between them is an even multiple of π, i.e. $\Delta\phi = 2n\pi$ where n = 0, 1, 2, 3,....

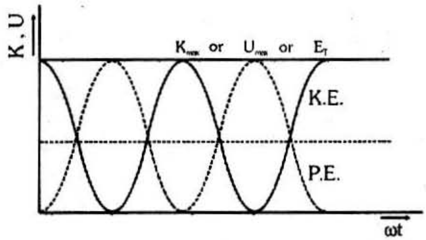
Two vibrating particle are said to be in opposite phase if the phase difference between them is an odd multiple of π i.e., $\Delta\phi = (2n + 1)\pi$ where n = 0, 1, 2, 3,....

- **Angular frequency (ω)** :The rate of change of phase angle of a particle with respect to time is defined as its angular frequency. $\omega = \sqrt{\frac{k}{m}}$

- **For linear SHM** ($F \propto -x$) : $F = m\frac{d^2x}{dt^2} = -kx = -m\omega^2 x$ where $\omega = \sqrt{\frac{k}{m}}$
- **For angular SHM** ($\tau \propto -\theta$) : $\tau = I\frac{d^2\theta}{dt^2} = I\alpha = -k\theta = -m\omega^2\theta$ where $\omega = \sqrt{\frac{k}{m}}$
- **Displacement** $x = A\sin(\omega t + \phi)$,
- **Angular displacement** $\theta = \theta_0 \sin(\omega t + \phi)$
- **Velocity** $v = \frac{dx}{dt} = A\omega\cos(\omega t + \phi) = \omega\sqrt{A^2 - x^2}$
- **Angular velocity** $\frac{d\theta}{dt} = \theta_0\,\omega\cos(\omega t + \phi)$
- **Acceleration** $a = \frac{d^2x}{dt^2} = -A\omega^2\sin(\omega t + \phi) = -\omega^2 x$
- **Angular acceleration** $\frac{d^2\theta}{dt^2} = -\theta_0\omega^2\sin(\omega t + \phi) = -\omega^2\theta$
- **Kinetic energy** $K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2\cos^2(\omega t + \phi)$
- **Potential energy** $U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 A^2\sin^2(\omega t + \phi)$
- **Total energy** $E = K + U = \frac{1}{2}m\omega^2 A^2 = \text{constant}$







Note :

(i) Total energy of a particle in S.H.M. is same at all instant and at all displacement.

(ii) Total energy depends upon mass, amplitude and frequency of vibration of the particle executing S.H.M.

- **Average energy in SHM**

(i) The time average of P.E. and K.E. over one cycle is

(a) $\langle K \rangle_t = \frac{1}{4}kA^2$ (b) $\langle PE \rangle_t = \frac{1}{4}kA^2$ (c) $\langle TE \rangle_t = \frac{1}{2}kA^2 + U_0$

(ii) The position average of P.E. and K.E. between x = – A to x=A

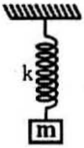
(a) $\langle K \rangle_x = \frac{1}{3}kA^2$ (b) $\langle PE \rangle_x = U_0 + \frac{1}{6}kA^2$ (c) $\langle TE \rangle_x = \frac{1}{2}kA^2 + U_0$

- **Differential equation of SHM**

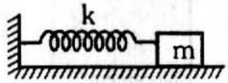
□ Linear SHM: $\frac{d^2x}{dt^2} + \omega^2 x = 0$

□ Angular SHM: $\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$

- **Spring block system**



$T = 2\pi\sqrt{\frac{m}{k}}$



$T = 2\pi\sqrt{\frac{m}{k}}$

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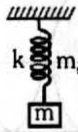
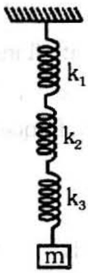
$T = 2\pi\sqrt{\frac{\mu}{k}}$ where μ=reduced mass $= \frac{m_1 m_2}{m_1 + m_2}$

- When spring mass is not negligible :



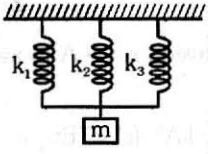
$T = 2\pi\sqrt{\frac{m + \frac{m_s}{3}}{k}}$

- **Series combination of springs**



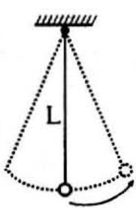
$T = 2\pi\sqrt{\frac{m}{k_{eff}}}$ where $\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$

- **Parallel combination of springs**

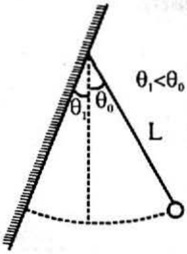


$T = 2\pi\sqrt{\frac{m}{k_{eff}}}$ where $k_{eff} = k_1 + k_2 + k_3$

- **Time period of simple pendulum**



Time period $T = 2\pi\sqrt{\frac{L}{g}}$



Time period $T = \left\{\pi + 2\sin^{-1}\left(\frac{\theta_1}{\theta_0}\right)\right\}\sqrt{\frac{L}{g}}$

If length of simple pendulum is comparable to the radius of the earth R, then

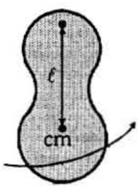
$T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{\ell} + \frac{1}{R}\right)}}$. If $\ell << R$ then $T = 2\pi\sqrt{\frac{\ell}{g}}$

If $\ell >> R$ then $T = 2\pi\sqrt{\frac{R}{g}} \approx 84$ minutes

- **Second pendulum**

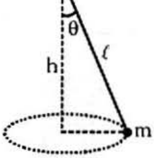
  Time period = 2 seconds, Length ≈ 1 meter (on earth's surface)

- **Time period of Physical pendulum**



$$T = 2\pi\sqrt{\frac{I}{mg\ell}} = 2\pi\sqrt{\frac{\frac{k^2}{\ell} + \ell}{g}}$$ where $I_{cm} = mk^2$

- **Time period of Conical pendulum**



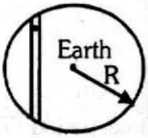
$$T = 2\pi\sqrt{\frac{\ell\cos\theta}{g}} = 2\pi\sqrt{\frac{h}{g}}$$

- **Time period of Torsional pendulum** $T = 2\pi\sqrt{\frac{I}{k}}$

  where k = torsional constant of the wire

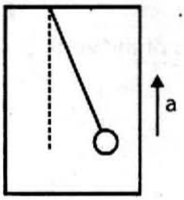
  I=moment of inertia of the body about the vertical axis

- **SHM of a particle in a tunnel inside the earth.**



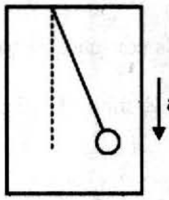
$$T = 2\pi\sqrt{\frac{R}{g}}$$

In accelerating cage



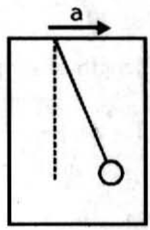
$g_{eff} = g + a$

$$T = 2\pi\sqrt{\frac{\ell}{g+a}}$$



$g_{eff} = g - a$

$$T = 2\pi\sqrt{\frac{\ell}{g-a}}$$



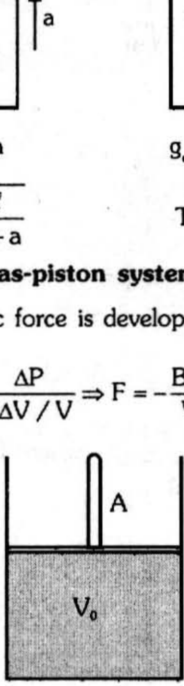
$g_{eff} = \sqrt{g^2 + a^2}$

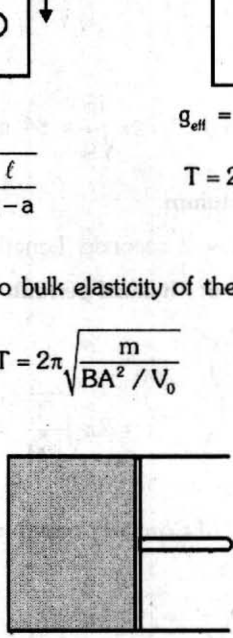
$$T = 2\pi\sqrt{\frac{\ell}{(g^2+a^2)^{1/2}}}$$

- **SHM of gas-piston system**

Here elastic force is developed due to bulk elasticity of the gas

$$B = \frac{\Delta P}{-\Delta V / V} \Rightarrow F = -\frac{BA^2}{V_0}x \Rightarrow T = 2\pi\sqrt{\frac{m}{BA^2/V_0}}$$





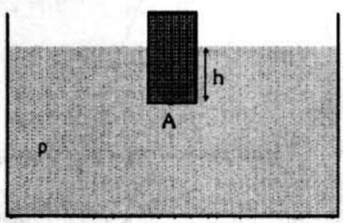
- **SHM of Floating Body**

Restoring force → Thurst

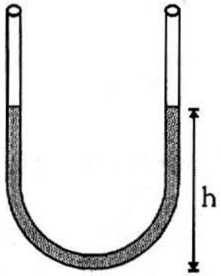
mg =ρAhg → Equilibrium

Restoring force F= − (ρAg)x

$$T = 2\pi\sqrt{\frac{h}{g}}$$



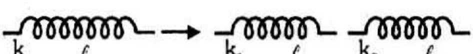
- **SHM in U-tube**



$$T = 2\pi\sqrt{\frac{h}{g}}$$

KEY POINTS

- SHM is the projection of uniform circular motion along one of the diameters of the circle.
- The periodic time of a hard spring is less as compared to that of a soft spring because the spring constant is large for hard spring.
- For a system executing SHM, the mechanical energy remains constant.
- Maximum kinetic energy of a particle in SHM may be greater than mechanical energy as potential energy of a system may be negative.
- The frequency of oscillation of potential energy and kinetic energy is twice as that of displacement or velocity or acceleration of a particle executing S.H.M.
- **Spring cut into two parts :**



Here $\frac{\ell_1}{\ell_2} = \frac{m}{n}$

$$\ell_1 = \left(\frac{m}{m+n}\right)\ell, \quad \ell_2 = \left(\frac{n}{m+n}\right)\ell \qquad \text{But } k\ell = k_1\ell_1 = k_2\ell_2$$

$$\Rightarrow k_1 = \frac{(m+n)}{m}k; \quad k_2 = \frac{(m+n)}{n}k$$

FREE, DAMPED, FORCED OSCILLATIONS AND RESONANCE

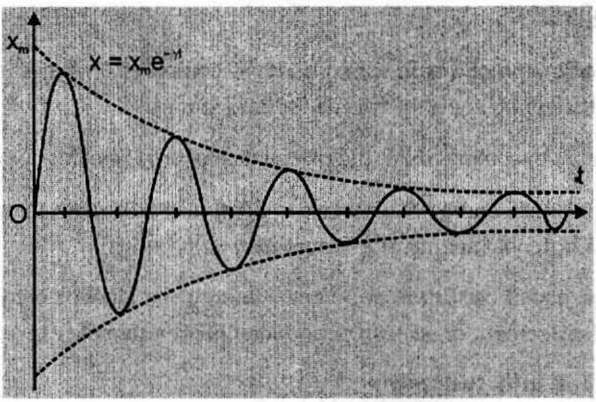
Free oscillation

- The oscillations of a particle with fundamental frequency under the influence of restoring force are defined as free oscillations.

Damped oscillations

- The oscillations of a body whose amplitude goes on decreasing with time are defined as damped oscillations.
- In these oscillations the amplitude of oscillations decreases exponentially due to damping forces like frictional force, viscous force etc.
- If initial amplitude is X_m then amplitude after time t will be $x = x_m e^{-\gamma t}$ where

  γ = Damping coefficient



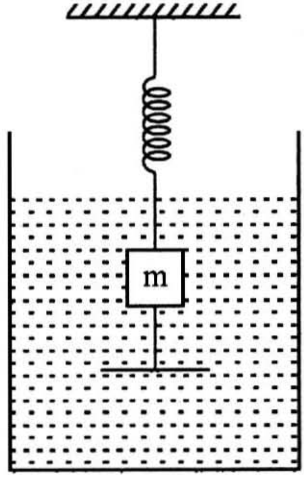
FORCED OSCILLATIONS

- The oscillations in which a body oscillates under the influence of an external periodic force (driver) are known as forced oscillations.
- The driven body does not oscillate with its natural frequency rather it oscillates with the frequency of the driver.
- The amplitude of oscillator decreases due to damping forces but on account of the energy gained from the external source (driver) it remains constant.

RESONANCE

- When the frequency of external force (driver) is equal to the natural frequency of the oscillator (driven), then this state of the driver and the driven is known as the state of resonance.

- In the state of resonance, there occurs maximum transfer of energy from the driver to the driven. Hence the amplitude of motion becomes maximum.
- In the state of resonance the frequency of the driver (ω) is known as the resonant frequency.



Damped Oscillations :

Damping force $F_d = -bv$

where v = velocity,

b = damping constant

Restoring force on block $F = -kx$

So net force on block

$$F_{net} = -kx - bv$$

$$\Rightarrow \quad ma = -kx - bv$$

$$\Rightarrow \quad \boxed{m\frac{d^2x}{dt^2} + kx + bv = 0}$$

It is the differential equation of damped oscillation.

Solution of this equation is given by

$$\boxed{x = A_0 e^{\left(-\frac{bt}{2m}\right)} \sin(\omega' t + \phi)}$$

where $A(t) = A_0 e^{\left(-\frac{bt}{2m}\right)}$

$$\Rightarrow \quad A(t) = A_0 e^{-\gamma t}$$

so $\boxed{\gamma = \frac{b}{2m}}$ and $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

Energy in damped oscillation

$$E(t) = \frac{1}{2}kA^2(t) = \frac{1}{2}k\left[A_0 e^{-\frac{bt}{2m}}\right]^2$$

$$\Rightarrow \quad E(t) = \frac{1}{2}kA_0^2 e^{(-bt/m)} \Rightarrow \boxed{E(t) = E_0 e^{(-bt/m)}}$$