Determinants Exercise 1 :
Single Option Correct Type Questions

- This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. If $f(n)=\alpha^n+\beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$ $=k(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, k^2d is equal to

(a) 1 (b) -1
(c) $\alpha\beta$ (d) $\alpha\beta\gamma$

2. Let $\Delta(x)=\begin{vmatrix} x+a & x+b & x+a-c \\ x+b & x+c & x-1 \\ x+c & x+d & x-b+d \end{vmatrix}$ and $\int_0^2 \Delta(x)\,dx=-16$, where a, b, c and d are in AP, then the common difference of the AP is equal to

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 4

3. If $\Delta(x)=\begin{vmatrix} x & 1+x^2 & x^3 \\ \log_e(1+x^2) & e^x & \sin x \\ \cos x & \tan x & \sin^2 x \end{vmatrix}$, then

(a) $\Delta(x)$ is divisible by x (b) $\Delta(x)=0$
(c) $\Delta'(x)=0$ (d) None of these

4. If a, b and c are sides of a triangle and $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}=0$, then

(a) ΔABC is an equilateral triangle
(b) ΔABC is a right angled isosceles triangle
(c) ΔABC is an isosceles triangle
(d) None of the above

5. If $\begin{vmatrix} \alpha & x & x & x \\ x & \beta & x & x \\ x & x & \gamma & x \\ x & x & x & \delta \end{vmatrix}=f(x)-xf'(x)$, then $f(x)$ is equal to

(a) $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$
(b) $(x+\alpha)(x+\beta)(x+\gamma)(x+\delta)$
(c) $2(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$
(d) None of the above

6. If $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix}=0$, the line $ax+by+c=0$ passes through the fixed point which is

(a) (1, 2) (b) (1, 1) (c) $(-2, 1)$ (d) (1, 0)

7. If $f(x)=a+bx+cx^2$ and α, β and γ are the roots of the equation $x^3=1$, then $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is equal to

(a) $f(\alpha)+f(\beta)+f(\gamma)$
(b) $f(\alpha)\,f(\beta)+f(\beta)\,f(\gamma)+f(\gamma)\,f(\alpha)$
(c) $f(\alpha)\,f(\beta)\,f(\gamma)$
(d) $-f(\alpha)\,f(\beta)\,f(\gamma)$

8. When the determinant $\begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$ is expanded in powers of $\sin x$, the constant term in that expression is

(a) 1 (b) 0 (c) -1 (d) 2

9. If [ ] denotes the greatest integer less than or equal to the real number under consideration and $-1 \le x < 0, 0 \le y < 1, 1 \le z < 2$, the value of the determinant $\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$ is

(a) $[x]$ (b) $[y]$
(c) $[z]$ (d) None of these

10. The determinant $\begin{vmatrix} y^2 & -xy & x^2 \\ a & b & c \\ a' & b' & c' \end{vmatrix}$ is equal to

(a) $\begin{vmatrix} bx+ay & cx+by \\ b'x+a'y & c'x+b'y \end{vmatrix}$ (b) $\begin{vmatrix} a'x+b'y & bx+cy \\ ax+by & b'x+c'y \end{vmatrix}$
(c) $\begin{vmatrix} bx+cy & ax+by \\ b'x+c'y & a'x+b'y \end{vmatrix}$ (d) $\begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$

11. If A, B and C are angles of a triangle, the value of $\begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iBA} & e^{2iC} \end{vmatrix}$ is (where $i=\sqrt{-1}$)

(a) 1 (b) -1 (c) -2 (d) -4

12. If $\begin{vmatrix} x^n & x^{n+2} & x^{2n} \\ 1 & x^a & a \\ x^{n+5} & x^{a+6} & x^{2n+5} \end{vmatrix}=0, \forall\, x \in R$, where $n \in N$, the value of a is

(a) n (b) $n-1$
(c) $n+1$ (d) None of these

13. If x, y and z are the integers in AP lying between 1 and 9 and $x51, y41$ and $z31$ are three digits number, the value of $\begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$ is

(a) $x+y+z$
(b) $x-y+z$
(c) 0
(d) None of the above

14. If $a_1 b_1 c_1, a_2 b_2 c_2$ and $a_3 b_3 c_3$ are three digit even natural numbers and $\Delta = \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$, then Δ is

(a) divisible by 2 but not necessarily by 4
(b) divisible by 4 but not necessarily by 8
(c) divisible by 8
(d) None of the above

15. If a, b and c are sides of ΔABC such that

$$\begin{vmatrix} c & b\cos B + c\beta & a\cos A + b\alpha + c\gamma \\ a & c\cos B + a\beta & b\cos A + c\alpha + a\gamma \\ b & a\cos B + b\beta & c\cos A + a\alpha + b\gamma \end{vmatrix} = 0$$

$\left(\text{where } \alpha, \beta, \gamma \in R^+ \text{ and } \angle A, \angle B, \angle C \neq \frac{\pi}{2}\right)$, ΔABC is

(a) an isosceles (b) an equilateral
(c) can't say (d) None of these

16. If x_1, x_2 and y_1, y_2 are the roots of the equations $3x^2 - 18x + 9 = 0$ and $y^2 - 4y + 2 = 0$, the value of the determinant $\begin{vmatrix} x_1x_2 & y_1y_2 & 1 \\ x_1 + x_2 & y_1 + y_2 & 2 \\ \sin(\pi x_1x_2) & \cos(\pi/2y_1y_2) & 1 \end{vmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) None of these

17. If the value of $\Delta = \begin{vmatrix} ^{10}C_4 & ^{10}C_5 & ^{11}C_m \\ ^{11}C_6 & ^{11}C_7 & ^{12}C_{m+2} \\ ^{12}C_8 & ^{12}C_9 & ^{13}C_{m+4} \end{vmatrix}$ is equal to zero, then m is equal to

(a) 6 (b) 4
(c) 5 (d) None of these

18. The value of the determinant $\begin{vmatrix} 1 & \sin(\alpha-\beta)\theta & \cos(\alpha-\beta)\theta \\ a & \sin\alpha\theta & \cos\alpha\theta \\ a^2 & \sin(\alpha-\beta)\theta & \cos(\alpha-\beta)\theta \end{vmatrix}$ is independent of

(a) α (b) β
(c) θ (d) a

19. If $f(x), g(x)$ and $h(x)$ are polynomials of degree 4 and $\begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ p & q & r \end{vmatrix} = mx^4 + nx^3 + rx^2 + sx + t$ be an identity in x, then

$$\begin{vmatrix} f'''(0) - f''(0) & g'''(0) - g''(0) & h'''(0) - h''(0) \\ a & b & c \\ p & q & r \end{vmatrix}$$

is equal to

(a) $2(3n + r)$ (b) $3(2n - r)$
(c) $3(2n + r)$ (d) $2(3n - r)$

20. If $f(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$, then $f(\theta) - 2f(\phi) + f(\psi)$ is equal to

(a) 0 (b) $\alpha - \beta$
(c) $\alpha + \beta + \gamma$ (d) $\alpha + \beta - \gamma$

21. If $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$, where a, b and c are all different, then the determinant $\begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix}$ vanishes when

(a) $a + b + c = 0$ (b) $x = \frac{1}{3}(a + b + c)$
(c) $x = \frac{1}{2}(a + b + c)$ (d) $x = a + b + c$

22. Let $a, b, c \in R$ such that no two of them are equal and satisfy $\begin{vmatrix} 2a & b & c \\ b & c & 2a \\ c & 2a & b \end{vmatrix} = 0$, the equation $24ax^2 + 4bx + c = 0$ has

(a) atleast one root in $\left[0, \frac{1}{2}\right]$
(b) atleast one root in $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(c) atleast one root in $[-1, 0]$
(d) atleast two roots in $[0, 2]$

23. The number of positive integral solution of the equation $\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = 11$ is

(a) 0 (b) 3 (c) 6 (d) 12

24. If $f(x)=ax^2+bx+c$, $a,b,c\in R$ and equation $f(x)-x=0$ has imaginary roots α,β γ and δ be the roots of $f(f(x))-x=0$, then $\begin{vmatrix} 2 & \alpha & \delta \\ \beta & 0 & \alpha \\ \gamma & \beta & 1 \end{vmatrix}$ is

(a) 0 (b) purely real
(c) purely imaginary (d) None of these

25. If the system of equations $2x-y+z=0$, $x-2y+z=0$, $tx-y+2z=0$ has infinitely many solutions and $f(x)$ be a continuous function, such that $f(5+x)+f(x)=2$, then $\int_0^{-2t} f(x)\,dx$ is equal to

(a) 0 (b) $-2t$ (c) 5 (d) t

26. If $(1+ax+bx^2)^4=a_0+a_1x+a_2x^2+\ldots+a_8x^8$, where $a,b,a_0,a_1,\ldots,a_8\in R$ such that $a_0+a_1+a_2\neq 0$ and $\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_0 \\ a_2 & a_0 & a_1 \end{vmatrix}=0$, then

(a) $a=\frac{3}{4}, b=\frac{5}{8}$ (b) $a=\frac{1}{4}, b=\frac{5}{32}$
(c) $a=1, b=\frac{2}{3}$ (d) None of these

27. Given, $f(x)=\log_{10}x$ and $g(x)=e^{\pi ix}$.

If $\phi(x)=\begin{vmatrix} f(x)\cdot g(x) & (f(x))^{g(x)} & 1 \\ f(x^2)\cdot g(x^2) & (f(x^2))^{g(x^2)} & 0 \\ f(x^3)\cdot g(x^3) & (f(x^3))^{g(x^3)} & 1 \end{vmatrix}$, the value of $\phi(10)$, is

(a) 1 (b) 2 (c) 0 (d) None of these

28. The value of the determinant

$$\begin{vmatrix} 1 & (\alpha^{2x}-\alpha^{-2x})^2 & (\alpha^{2x}+\alpha^{-2x})^2 \\ 1 & (\beta^{2x}-\beta^{-2x})^2 & (\beta^{2x}+\beta^{-2x})^2 \\ 1 & (\gamma^{2x}-\gamma^{-2x})^2 & (\gamma^{2x}+\gamma^{-2x})^2 \end{vmatrix}, \text{ is}$$

(a) 0 (b) $(\alpha\beta\gamma)^{2x}$ (c) $(\alpha\beta\gamma)^{-2x}$ (d) None of these

29. If a, b and c are non-zero real numbers and if the equations $(a-1)x=y+z$, $(b-1)y=z+x$, $(c-1)z=x+y$ has a non-trivial solution, then $ab+bc+ca$ equals to

(a) $a+b+c$ (b) abc (c) 1 (d) None of these

30. The set of equations $\lambda x-y+(\cos\theta)z=0$, $3x+y+2z=0$, $(\cos\theta)x+y+2z=0$, $0\le\theta\le 2\pi$, has non-trivial solution(s)

(a) for no value of λ and θ
(b) for all value of λ and θ
(c) for all values of λ and only two values of θ
(d) for only one value of λ and all values of θ

Determinants Exercise 2 :
More than One Correct Option Type Questions

- This section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. The determinant $\Delta=\begin{vmatrix} a^2+x^2 & ab & ac \\ ab & b^2+x^2 & bc \\ ac & bc & c^2+x^2 \end{vmatrix}$ is divisible by

(a) x (b) x^2 (c) x^3 (d) x^4

32. The value of the determinant

$$\begin{vmatrix} \sqrt{6} & 2i & 3+\sqrt{6} \\ \sqrt{12} & \sqrt{3}+\sqrt{8}i & 3\sqrt{2}+\sqrt{6}\,i \\ \sqrt{18} & \sqrt{2}+\sqrt{12}\,i & \sqrt{27}+2i \end{vmatrix}, \text{ is (where } i=\sqrt{-1})$$

(a) complex (b) real (c) irrational (d) rational

33. If $D_k=\begin{vmatrix} 2^{k-1} & \frac{1}{k(k+1)} & \sin k\theta \\ x & y & z \\ 2^n-1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta\sin\frac{n}{2}\theta}{\sin\frac{\theta}{2}} \end{vmatrix}$, then $\sum_{k=1}^{n} D_k$ is equal to

(a) 0 (b) independent of n
(c) independent of θ (d) independent of x, y and z

34. The determinant $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$ is equal to zero, if

(a) a, b and c are in AP
(b) a, b, c, are in GP
(c) a, b and c are in HP
(d) $(x-\alpha)$ is a factor of $ax^2+2bx+c$

35. Let $f(x)=\begin{vmatrix} 2\cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$, then

(a) $f\left(\frac{\pi}{3}\right)=-1$

(b) $f'\left(\frac{\pi}{3}\right)=\sqrt{3}$

(c) $\int_0^{\pi} f(x)dx=0$

(d) $\int_{-\pi}^{\pi} f(x)\,dx=0$

36. If $\Delta(x)=\begin{vmatrix} x^2-5x+3 & 2x-5 & 3 \\ 3x^2+x+4 & 6x+1 & 9 \\ 7x^2-6x+9 & 14x-6 & 21 \end{vmatrix}$

$= ax^3+bx^2+cx+d$, then

(a) $a=0$ (b) $b=0$ (c) $c=0$ (d) $d=47$

37. If a, b and c are the sides of a triangle and A, B and C are the angles opposite to a, b and c respectively, then

$\Delta=\begin{vmatrix} a^2 & b\sin A & C\sin A \\ b\sin A & 1 & \cos A \\ C\sin A & \cos A & 1 \end{vmatrix}$ is independent of

(a) a (b) b (c) c (d) A, B, C

38. Let $f(a,b)=\begin{vmatrix} a & a^2 & 0 \\ 1 & (2a+b) & (a+b)^2 \\ 0 & 1 & (2a+3b) \end{vmatrix}$, then

(a) $(a+b)$ is a factor of $f(a,b)$
(b) $(a+2b)$ is a factor of $f(a,b)$
(c) $(2a+b)$ is a factor of $f(a,b)$
(d) a is a factor of $f(a,b)$

39. If $f(x)=\begin{vmatrix} \sec^2 x & 1 & 1 \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cot^2 x \end{vmatrix}$, then

(a) $\int_{-\pi/4}^{\pi/4} f(x)\,dx=\frac{1}{16}(3\pi+8)$

(b) $f'\left(\frac{\pi}{2}\right)=0$

(c) maximum value of $f(x)$ is 1
(d) minimum value of $f(x)$ is 0

40. If $\begin{vmatrix} a & a+x^2 & a+x^2+x^4 \\ 2a & 3a+2x^2 & 4a+3x^2+2x^4 \\ 3a & 6a+3x^2 & 10a+6x^2+3x^4 \end{vmatrix}$

$= a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7$ and

$f(x)=+a_0x^2+a_3x+a_6$, then

(a) $f(x)\ge 0,\ \forall\ x\in R$ if $a>0$
(b) $f(x)=0$, only if $a=0$
(c) $f(x)=0$, has two equal roots
(d) $f(x)=0$, has more than two root if $a=0$

41. If $\Delta(x)=\begin{vmatrix} 4x-4 & (x-2)^2 & x^3 \\ 8x-4\sqrt{2} & (x-2\sqrt{2})^2 & (x+1)^3 \\ 12x-4\sqrt{3} & (x-2\sqrt{3})^2 & (x-1)^3 \end{vmatrix}$, then

(a) term independent of x in $\Delta(x)=16(5-\sqrt{2}-\sqrt{3})$
(b) coefficient of x in $\Delta(x)=48(1+\sqrt{2}-\sqrt{3})$
(c) coefficient of x in $\Delta(x)=16(5+\sqrt{2}-\sqrt{3})$
(d) coefficient of x in $\Delta(x)$ is divisible by 16

42. If

$f(x)=\begin{vmatrix} 3 & 3x & 3x^2+2a^2 \\ 3x & 3x^2+2a^2 & 3x^3+6a^2x \\ 3x^2+2a^3 & 3x^3+6a^2x & 3x^4+12a^2x^2+2a^4 \end{vmatrix}$,

then

(a) $f'(x)=0$
(b) $y=f(x)$ is a straight line parallel to X-axis
(c) $\int_0^2 f(x)\,dx=32a^4$
(d) None of the above

43. If $a>b>c$ and the system of equations $ax+by+cz=0$, $bx+cy+az=0$, $cx+ay+bz=0$ has a non-trivial solution, then both the roots of the quadratic equation at^2+bt+c, are

(a) real
(b) of opposite sign
(c) positive
(d) complex

44. The values of λ and b for which the equations $x+y+z=3$, $x+3y+2z=6$ and $x+\lambda y+3z=b$ have

(a) a unique solution, if $\lambda\ne 5$, $b\in R$
(b) no solution, if $\lambda\ne 5$, $b=9$
(c) infinite many solution, $\lambda=5$, $b=9$
(d) None of the above

45. Let λ and α be real. Let S denote the set of all values of λ for which the system of linear equations

$\lambda x+(\sin\alpha)y+(\cos\alpha)z=0$

$x+(\cos\alpha)y+(\sin\alpha)z=0$

$-x+(\sin\alpha)y-(\cos\alpha)z=0$

has a non-trivial solution, then S contains

(a) $(-1, 1)$ (b) $[-\sqrt{2}, -1]$
(c) $[1, \sqrt{2}]$ (d) $(-2, 2)$

Determinants Exercise 3 :
Passage Based Questions

- This section contains **7 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Q. Nos. 46 to 48)

Consider the system of equations

$$x + y + z = 5;\ x + 2y + 3z = 9;\ x + 3y + \lambda z = \mu$$

The system is called smart, brilliant, good and lazy according as it has solution, unique solution, infinitely many solutions and no solution, respectively.

46. The system is smart, if

(a) $\lambda \neq 5$ or $\lambda = 5$ and $\mu = 13$ (b) $\lambda \neq 5$ and $\mu = 13$
(c) $\lambda \neq 5$ and $\mu \neq 13$ (d) $\lambda \neq 5$ or $\lambda = 5$ and $\mu \neq 13$

47. The system is good, if

(a) $\lambda \neq 5$ or $\lambda = 5$ and $\mu = 13$ (b) $\lambda = 5$ and $\mu = 13$
(c) $\lambda = 5$ and $\mu \neq 13$
(d) $\lambda \neq 5$, μ is any real number

48. The system is lazy, if

(a) $\lambda \neq 5$ or $\lambda = 5$ and $\mu = 13$ (b) $\lambda = 5$ and $\mu = 13$
(c) $\lambda = 5$ and $\mu \neq 13$ (d) $\lambda \neq 5$ or $\lambda = 5$ and $\mu \neq 13$

Passage II
(Q. Nos. 49 to 51)

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ *and* $C_{ij} = (-1)^{i+j} M_{ij}$, *where* M_{ij} *is a determinant obtained by deleting ith row and jth column, then* $\begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \Delta^2$.

49. If $\begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = 5$ and $\Delta = \begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0 \end{vmatrix}$, then sum of digits of Δ^2, is

(a) 7 (b) 8 (c) 13 (d) 11

50. Suppose $a, b, c \in R$, $a + b + c > 0$, $A = bc - a^2$, $B = ca - b^2$ and $C = ab - c^2$ and $\begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix} = 49$, then the value of $a^3 + b^3 + c^3 - 3abc$, is

(a) −7 (b) 7 (c) −2401 (d) 2401

51. If $a^3 + b^3 + c^3 - 3abc = -3$ and $A = bc - a^2$, $B = ca - b^2$ and $C = ab - c^2$, then the value of $aA + bB + cC$, is

(a) −3 (b) 3 (c) −9 (d) 9

Passage III
(Q. Nos. 52 to 54)

If α, β, γ *are the roots of* $x^3 + 2x^2 - x - 3 = 0$

52. The value of $\begin{vmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{vmatrix}$ is equal to

(a) 14 (b) −2 (c) 10 (d) 14

53. If the absolute value of the expression $\frac{\alpha-1}{\alpha+2} + \frac{\beta-1}{\beta+2} + \frac{\gamma-1}{\gamma+2}$ can be expressed as $\frac{m}{n}$, where m and n are co-prime, the value of $\begin{vmatrix} m & n^2 \\ m-n & m+n \end{vmatrix}$, is

(a) 17 (b) 27 (c) 37 (d) 47

54. If $a = \alpha^2 + \beta^2 + \gamma^2$, $b = \alpha\beta + \beta\gamma + \gamma\alpha$, the value of $\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix}$, is

(a) 14 (b) 49 (c) 98 (d) 196

Passage IV
(Q. Nos. 55 to 57)

Suppose $f(x)$ *is a function satisfying the following conditions:*

(i) $f(0) = 2$, $f(1) = 1$

(ii) $f(x)$ *has a minimum value at* $x = \frac{5}{2}$

(iii) For all x, $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$.

55. The value of $f(2) + f(3)$ is

(a) 1 (b) $\frac{3}{2}$ (c) 2 (d) $\frac{5}{2}$

56. The number of solutions of the equation $f(x) + 1 = 0$ is

(a) 0 (b) 1 (c) 2 (d) infinite

57. Range of $f(x)$ is

(a) $\left(-\infty, \frac{7}{16}\right]$ (b) $\left[\frac{3}{4}, \infty\right)$ (c) $\left[\frac{7}{16}, \infty\right)$ (d) $\left(-\infty, \frac{3}{4}\right]$

Passage V

(Q. Nos. 58 to 60)

If $\begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ x-\ln x & \cos(x-1) & (x-1)^2 \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix} = a_0 + a_1(x-1) + a_2(x-1)^2 + \dots$

58. The value of $\cos^{-1}(a_1)$, is

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

59. The value of $\lim_{x \to a_0} (\sin x)^x$ is

(a) 1 (b) e (c) $e-1$ (d) None of these

60. The equation whose roots are a_0 and a_1, is

(a) $x^2 - x = 0$ (b) $x^2 - 2x = 0$
(c) $x^2 - 3x = 0$ (d) None of these

Passage VI

(Q. Nos. 61 to 63)

Let $\Delta = \begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix}$ and the equation $x^3 - px^2 + qx - r = 0$ has roots a, b, c, where $a, b, c \notin R^+$.

61. The value of Δ is

(a) $\le 9r^3$ (b) $\ge 27r^2$ (c) $\le 27r^2$ (d) $\ge 81r^3$

62. If a, b, c are in GP, then

(a) $r^3 = p^3 q$ (b) $p^3 = r^3 q$ (c) $p^3 = q^3 r$ (d) $q^3 = p^3 r$

63. If $\Delta = 27$ and $a^2 + b^2 + c^2 = 2$, then the value of $\sum a^2 b$, is

(a) $3(2\sqrt{2} - p)$ (b) $3(2\sqrt{2} - r)$
(c) $3(2\sqrt{2} - q)$ (d) $3(2\sqrt{2} - p - q)$

Passage VII

(Q. Nos. 64 to 66)

If $\Delta n = \begin{vmatrix} a^2+n & ab & ac \\ ab & b^2+n & bc \\ ac & bc & c^2+n \end{vmatrix}$, $n \in N$ and the equation $x^3 - \lambda x^2 + 11x - 6 = 0$ has roots a, b, c and a, b, c are in AP.

64. The value of $\sum_{r=1}^{7} \Delta_n$ is

(a) $(12)^3$ (b) $(14)^3$ (c) $(26)^3$ (d) $(28)^3$

65. The value of $\frac{\Delta 2_n}{\Delta_n}$ is

(a) < 8 (b) = 8
(c) > 8 (d) None of these

66. The value of $\sum_{r=1}^{30} \left(\frac{27\Delta_r - \Delta 3_r}{27r^2} \right)$ is

(a) 130 (b) 190 (c) 280 (d) 340

Determinants Exercise 4 :
Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

67. If $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$,

the value of $\sqrt{2^k \sqrt{2^k \sqrt{2^k \dots \infty}}}$ is

68. Let α, β and γ are three distinct roots of

$\begin{vmatrix} x-1 & -6 & 2 \\ -6 & x-2 & -4 \\ 2 & -4 & x-6 \end{vmatrix} = 0$, the value of $\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right)^{-1}$ is

69. If $\begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ x-\ln x & \cos(x-1) & (x-1)^2 \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix} = \sum_{r=0}^{n} a_r (x-1)^r$,

the value of $(2^{a_0} + 3^{a_1})^{a_1+1}$ is

70. If $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$,

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma$ is equal to

71. Let $f(a, b, c) = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$, the greatest integer $n \in N$ such that $(a+b+c)^{n'}$ divides $f(a, b, c)$ is

72. If $0 \le \theta \le \pi$ and the system of equations

$$x = (\sin\theta)\, y + (\cos\theta)\, z$$
$$y = z + (\cos\theta)\, x$$
$$z = (\sin\theta)\, x + y$$

has a non-trivial solution, then $\frac{8\theta}{\pi}$ is equal to

73. The value of the determinant $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$ is

74. If a, b, c and d are the roots of the equation $x^4 + 2x^3 + 4x^2 + 8x + 16 = 0$, the value of the determinant $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$ is

75. If $a \neq 0, b \neq 0, c \neq 0$ and $\begin{vmatrix} 1+a & 1 & 1 \\ 1+b & 1+2b & 1 \\ 1+c & 1+c & 1+3c \end{vmatrix} = 0$, the value of $|a^{-1} + b^{-1} + c^{-1}|$ is equal to

76. If the system of equations

$$ax + hy + g = 0;\ hx + by + f = 0$$

and $ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda = 0$ has a unique solution and

$\dfrac{abc + 2fgh - af^2 - bg^2 - ch^2}{h^2 - ab} = 8$, the value of λ is

Determinants Exercise 5 :
Matching Type Questions

- This section contains **5 questions**. Questions 77 to 81 have three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

77.

	Column I		Column II
(A)	If a,b,c are three complex numbers such that $a^2+b^2+c^2=0$ and $\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \lambda a^2b^2c^2$, then λ is divisible by	(p)	2
(B)	If $a,b,c \in R$ and $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 5a+2b & 7a+5b+2c \\ 3a & 7a+3b & 9a+7b+3c \end{vmatrix} = -1024$, then a is divisible by	(q)	3
(C)	Let $\Delta(x) = \begin{vmatrix} x-1 & 2x^2-5 & x^3-1 \\ 2x^2+5 & 2x+2 & x^3+3 \\ x^3-1 & x+1 & 3x^2-2 \end{vmatrix}$ and $ax+b$ be the remainder, when $\Delta(x)$ is divided by x^2-1, then $4a+2b$ is divisible by	(r)	4
		(s)	5
		(t)	6

78.

	Column I		Column II
(A)	Let $f_1(x) = x + a_1$, $f_2(x) = x^2 + b_1x + b_2$, $x_1 = 2$, $x_2 = 3$ and $x_3 = 5$ and $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ f_1(x_1) & f_1(x_2) & f_1(x_3) \\ f_2(x_1) & f_2(x_2) & f_2(x_3) \end{vmatrix}$, then Δ is	(p)	Even number
(B)	If $\lvert a_1 - b_1 \rvert = 6$ and $f(x) = \begin{vmatrix} 1 & b_1 & a_1 \\ 1 & b_1 & 2a_1 - x \\ 1 & 2b_1 - x & a_1 \end{vmatrix}$, then the minimum value of $f(x)$ is	(q)	Prime number
(C)	If coefficient of x in $f(x) = \begin{vmatrix} x-2 & (x-1)^2 & x^3 \\ x-1 & x^2 & (x+1)^3 \\ x & (x+1)^2 & (x+2)^3 \end{vmatrix}$ is λ, then $\lvert\lambda\rvert$ is	(r)	Odd number
		(s)	Composite number
		(t)	Perfect number

79.

	Column I		Column II
(A)	If $\begin{vmatrix} x^2+3x & x-1 & x+3 \\ x^2+1 & 2+3x & x-3 \\ x^2-3 & x+4 & 3x \end{vmatrix} = ax^4+bx^3+cx^2+dx+e$, then $e+a$ is divisible by	(p)	2
(B)	If $\begin{vmatrix} x-1 & 5x & 7 \\ x^2-1 & x-1 & 8 \\ 2x & 3x & 0 \end{vmatrix} = ax^3+bx^2+cx+d$, then $(e+a-3)$ is divisible by	(q)	3
(C)	If $\begin{vmatrix} x^3+4x & x+3 & x-2 \\ x-2 & 5x & x-1 \\ x-3 & x+2 & 4x \end{vmatrix} = ax^5+bx^4+cx^3+dx^2+ex+f$, then $(f+e)$ is divisible by	(r)	5
		(s)	6
		(t)	7

80.

	Column I		Column II
(A)	If $a^2+b^2+c^2=1$ and $\Delta = \begin{vmatrix} a^2+(b^2+c^2)d & ab(1-d) & ca(1-d) \\ ab(1-d) & b^2+(c^2+a^2)d & bc(1-d) \\ ca(1-d) & bc(1-d) & c^2+(a+b^2)d \end{vmatrix}$, then Δ is	(p)	independent of a
(B)	If $\Delta = \begin{vmatrix} \frac{1}{c} & \frac{1}{c} & \frac{-(a+b)}{c^2} \\ \frac{-(b+c)}{a^2} & \frac{1}{a} & \frac{1}{a} \\ \frac{-bd(b+c)}{a^2c} & \frac{(ad+2bd+cd)}{ac} & \frac{-(a+b)bd}{ac^2} \end{vmatrix}$, then Δ is	(q)	independent of b
(C)	If $\Delta = \begin{vmatrix} \sin a & \cos a & \sin(a+d) \\ \sin b & \cos b & \sin(b+d) \\ \sin c & \cos c & \sin(c+d) \end{vmatrix}$, then Δ is	(r)	independent of c
		(s)	independent of d
		(t)	zero

81.

	Column I		Column II
(A)	If n be the number of distinct values of 2×2 determinant whose entries are from the set $\{-1,0,1\}$, then $(n-1)^2$ is divisible by	(p)	2
(B)	If n be the number of 2×2 determinants with non-negative values whose entries from the set $\{0,1\}$, then $(n-1)$ is divisible by	(q)	3
(C)	If n be the number of 2×2 determinants with negative values whose entries from the set $\{-1, 1\}$, then $n(n+1)$ is divisible by	(r)	4
		(s)	5
		(t)	6

Determinants Exercise 6 :
Statement I and II Type Questions

- **Directions** (Q. Nos. 82 to 87) are Assertion-Reason type questions. Each of these questions contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

82. **Statement-1** If $\Delta(r)=\begin{vmatrix} r & r+1 \\ r+3 & r+4 \end{vmatrix}$ then $\sum_{r=1}^{n}\Delta(r)=-3n$

Statement-2 If $\Delta(r)=\begin{vmatrix} f_1(r) & f_2(r) \\ f_3(r) & f_4(r) \end{vmatrix}$

then $\sum_{r=1}^{n}\Delta(r)=\begin{vmatrix} \sum_{r=1}^{n} f_1(r) & \sum_{r=1}^{n} f_2(r) \\ \sum_{r=1}^{n} f_3(r) & \sum_{r=1}^{n} f_4(r) \end{vmatrix}$

83. Consider the determinant

$$\Delta=\begin{vmatrix} a_1+b_1x^2 & a_1x^2+b_1 & c_1 \\ a_2+b_2x^2 & a_2x^2+b_2 & c_2 \\ a_3+b_3x^2 & a_3x^2+b_3 & c_3 \end{vmatrix}=0,$$

where $a_i, b_i, c_i \in R (i=1,2,3)$ and $x \in R$.

Statement-1 The value of x satisfying $\Delta=0$ are $x=1,-1$

Statement-2 If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$, then $\Delta=0$.

84. **Statement-1** The value of determinant

$$\begin{vmatrix} \sin\pi & \cos\left(x+\frac{\pi}{4}\right) & \tan\left(x-\frac{\pi}{4}\right) \\ \sin\left(x-\frac{\pi}{4}\right) & -\cos\left(\frac{\pi}{2}\right) & \ln\left(\frac{x}{y}\right) \\ \cot\left(\frac{\pi}{4}+x\right) & \ln\left(\frac{y}{x}\right) & \tan\pi \end{vmatrix}$$ is zero.

Statement-2 The value of skew-symmetric determinant of odd order equals zero.

85. **Statement-1** $f(x)=\begin{vmatrix} (1+x)^{11} & (1+x)^{12} & (1+x)^{13} \\ (1+x)^{21} & (1+x)^{22} & (1+x)^{23} \\ (1+x)^{31} & (1+x)^{32} & (1+x)^{33} \end{vmatrix}$,

the coefficient of x in $f(x)=0$

Statement-2 If $P(x)=a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_nx^n$, then $a_1=P'(0)$, where dash denotes the differential coefficient.

86. **Statement-1** If system of equations $2x+3y=a$ and $bx+4y=5$ has infinite solution,

then $a=\frac{15}{4}, b=\frac{8}{5}$

Statement-2 Straight lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are parallel,

if $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$

87. **Statement-1** The value of the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix}\neq 0$

Statement-2 Neither of two rows or columns of $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix}$ is identical.

88. **Statement-1** The digits A, B and C re such that the three digit numbers $A88$, $6B8$, $86C$ are divisible by 72, then the determinant $\begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix}$ is divisible by 288.

Statement-2 $A=B=?$

Determinants Exercise 7 :
Subjective Type Questions

- In this section, there are **20 subjective** questions.

89. Prove that $\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = 4abc.$

90. Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$

91. Find the value of determinant $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}.$

92. Find the value of the determinant $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$, where a, b and c respectively are the pth, qth and rth terms of a harmonic progression.

93. Without expanding the determinant at any stage, prove that $\begin{vmatrix} -5 & 3+5i & \frac{3}{2}-4i \\ 3-5i & 8 & 4+5i \\ \frac{3}{2}+4i & 4-5i & 9 \end{vmatrix}$ has a purely real value.

94. Prove without expanding that $\begin{vmatrix} ah+bg & g & ab+ch \\ bf+ba & f & hb+bc \\ af+bc & c & bg+fc \end{vmatrix} = a \begin{vmatrix} ah+bg & a & h \\ bf+ba & h & b \\ af+bc & g & f \end{vmatrix}.$

95. If A, B and C are the angles of a triangle and

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0,$$

then prove that ΔABC must be isosceles.

96. Prove that $\begin{vmatrix} \beta\gamma & \beta\gamma'+\beta'\gamma & \beta'\gamma' \\ \gamma\alpha & \gamma\alpha'+\gamma'\alpha & \gamma'\alpha' \\ \alpha\beta & \alpha\beta'+\alpha'\beta & \alpha'\beta' \end{vmatrix}$
$= (\alpha\beta'-\alpha'\beta)(\beta\gamma'-\beta'\gamma)(\gamma\alpha'-\gamma'\alpha).$

97. If $y = \frac{u}{v}$, where u and v are functions of x, show that

$$v^3 \frac{d^2y}{dx^2} = \begin{vmatrix} u & v & 0 \\ u' & v & v \\ u'' & v'' & 2v' \end{vmatrix}.$$

98. Show that the determinant $\Delta(x)$ is given by $\Delta(x) =$

$$\begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & a+x\sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & b+x\sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & c+x\sin\gamma \end{vmatrix}$$ is independent of x.

99. Evaluate $\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix}.$

100. (i) Find maximum value of

$$f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}.$$

(ii) Let A, B and C be the angles of a triangle, such that $A \geq B \geq C$.

Find the minimum value of Δ, where

$$\Delta = \begin{vmatrix} \sin^2 A & \sin A\cos A & \cos^2 A \\ \sin^2 B & \sin B\cos B & \cos^2 B \\ \sin^2 C & \sin C\cos C & \cos^2 C \end{vmatrix}.$$

101. If $f(x) = \begin{vmatrix} x^2-4x+6 & 2x^2+4x+10 & 3x^2-2x+16 \\ x-2 & 2x+2 & 3x-1 \\ 1 & 2 & 3 \end{vmatrix}$, then find the value of $\int_{-3}^{3} \frac{x^2 \sin x}{1+x^6} f(x)dx.$

102. If $Y = sX$ and $Z = tX$ all the variables beings functions of x, then prove that $\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$, where suffixes denote the order of differentiation with respect to x.

103. If f, g and h are differentiable functions of x and $\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$, then prove that

$$\Delta' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}.$$

104. If $|a_1| > |a_2| + |a_3|$, $|b_2| > |b_1| + |b_3|$ and $|c_3| > |c_1| + |c_2|$, then show that $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0.$

105. Show that $\begin{vmatrix} (a-a_1)^{-2} & (a-a_1)^{-1} & a_1^{-1} \\ (a-a_2)^{-2} & (a-a_2)^{-1} & a_2^{-1} \\ (a-a_3)^{-2} & (a-a_3)^{-1} & a_3^{-1} \end{vmatrix} =$

$\pm \dfrac{a^2\,\Pi(a_i - a_j)}{\Pi a_i\,\Pi(a-a_i)^2}$. Write out the terms of the product in the numerator and give the resulting expression its correct sign.

106. Show that in general there are three values of t for which the following system of equations has a non-trivial solution $(a-t)x + by + cz = 0$, $bx + (c-t)y + az = 0$ and $cx + ay + (b-t)z = 0$.

Express the product of these values of t in the form of a determinant.

107. Eliminates

(i) a, b and c

(ii) x, y, z from the equations

$$-a + \frac{by}{z} + \frac{cz}{y} = 0, \ -b + \frac{cz}{x} + \frac{ax}{z} = 0$$

and $\quad -c + \dfrac{ax}{y} + \dfrac{by}{x} = 0.$

108. If x, z and y are not all zero and if

$ax + by + cz = 0, \ bx + cy + az = 0$

and $\quad cx + ay + bz = 0$, then

prove that $x : y : z = 1 : 1 : 1$ or $1 : \omega : \omega^2$ or $1 : \omega^2 : \omega$

Determinants Exercise 8 :
Questions Asked in Previous 13 Year's Exam

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

109. If $a^2 + b^2 + c^2 = -2$ and

$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$, then $f(x)$ is a

polynomial of degree **[AIEEE 2005, 3M]**

(a) 3 (b) 2 (c) 1 (d) 0

110. The system of equations

$$\alpha x + y + z = \alpha - 1,$$
$$x + \alpha y + z = \alpha - 1$$

and $\quad x + y + \alpha z = \alpha - 1$

has no solution, if α is **[AIEEE 2005, 3M]**

(a) not -2 (b) 1
(c) -2 (d) Either -2 or 1

111. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in GP, then the determinant

$\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$ is equal to **[AIEEE 2005, 3M]**

(a) 1 (b) 0 (c) 4 (d) 2

112. If $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$ for $x \neq 0, \ y \neq 0$, then D is **[AIEEE 2007, 3M]**

(a) divisible by neither x nor y
(b) divisible by both x and y
(c) divisible by x but not y
(d) divisible by y but not x

113. Consider the system of equations

$$x - 2y + 3z = -1$$
$$-x + y - 2z = k$$
$$x - 3y + 4z = 1$$

Statement-1 The system of equations has no solutions for $k \neq 3$. **[IIT-JEE 2008, 3M]**

and

Statement-2 The determinant $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$, for $k \neq 3$.

(a) Statement-1 is true, Statement-2 is true and Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is true and Statement-2 is not a correct explanation for Statement-1.
(c) Statement-1 is true, Statement-2 is false.
(d) Statement-1 is false, Statement-2 is true.

114. Let a, b, c be any real numbers. Suppose that there are real numbers x, y, z not all zero such that $x = cy + bz, \ y = az + cx$ and $z = bx + ay$. Then, $a^2 + b^2 + c^2 + 2abc$ is equal to **[AIEEE 2008, 3M]**

(a) -1 (b) 0 (c) 1 (d) 2

115. Let a, b, c be such that $b(a+c) \neq 0$. If

$$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0,$$

then the value of n is **[AIEEE 2009, 4M]**

(a) any integer (b) zero
(c) an even integer (d) any odd integer

116. If $f(\theta)=\begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta): 0 \le \theta < \frac{\pi}{2}\right\}$ is **[IIT-JEE 2011, 2M]**

(a) $(-\infty,-1)\cup(1,\infty)$ (b) $[2,\infty)$
(c) $(-\infty,0]\cup[2,\infty)$ (d) $(-\infty,-1]\cup[1,\infty)$

117. The number of values of k for which the linear equations

$$4x+ky+2z=0$$
$$kx+4y+z=0$$
$$2x+2y+z=0$$

Possess a non-zero solution is **[AIEEE 2011,4M]**

(a) zero (b) 3 (c) 2 (d) 1

118. If the trivial solution is the only solution of the system of equations

$$x-ky+z=0$$
$$kx+3y-kz=0$$
$$3x+y-z=0$$

Then, the set of values of k is

(a) $\{2,-3\}$ (b) $R-\{2,-3\}$
(c) $R-\{2\}$ (d) $R-\{-3\}$ **[AIEEE 2011, 4M]**

119. The number of values of k for which the system of equations $(k+1)x+8y=4k$; $kx+(k+3)y=3k-1$ has no solution, is

(a) 1 (b) 2
(c) 3 (d) infinite **[JEE Main 2013, 4M]**

120. If $\alpha,\ \beta \ne 0$ and $f(n)=\alpha^n+\beta^n$ and

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = k(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2,$$

then k is equal to **[JEE Main 2014, 4M]**

(a) 1 (b) -1
(c) $\alpha\beta$ (d) $1/\alpha\beta$

121. The set of all values of λ for which the system of linear equations

$$2x_1-2x_2+x_3=\lambda x_1$$
$$2x_1-3x_2+2x_3=\lambda x_2$$
$$-x_1+2x_2=\lambda x_3$$

has a non-trivial solution **[JEE Main 2015, 4M]**

(a) contains two elements
(b) contains more than two elements
(c) is an empty set
(d) is a singleton

122. Which of the following values of α satisfy the equation

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha\ ?$$

[JEE Advanced 2015, 4M]

(a) -4 (b) 9
(c) -9 (d) 4

123. The system of linear equations

$$x+\lambda y-z=0$$
$$\lambda x-y-z=0$$
$$x+y-\lambda z=0$$

has a non-trivial solution for **[JEE Main 2016, 4M]**

(a) exactly one-value of λ
(b) exactly two values of λ
(c) exactly three values of λ
(d) infinitely many values of λ.

124. The total number of distinct $x \in R$ for which

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10 \text{ is}$$

[JEE Advanced 2016, 3M]

125. Let $a, \lambda, \mu \in R$. Consider the system of linear equations

$$ax+2y=\lambda$$
$$3x-2y=\mu$$

Which of the following statement(s) is (are) correct?
[JEE Advanced 2016, 4M]

(a) If $a=-3$, then the system has infinitely many solutions for all values of λ and μ
(b) If $a \ne -3$, then the system has a unique solution for all values of λ and μ
(c) If $\lambda+\mu=0$, then the system has infinitely many solutions for $a=-3$
(d) If $\lambda+\mu \ne 0$, then the system has no solution for $a=-3$

126. If S is the set of distinct values of 'b' for which the following system of linear equations

$$x+y+z=1$$
$$x+ay+z=1$$
$$ax+by+z=0$$

has no solution, then S is **[JEE Main 2017, 4M]**

(a) an infinite set
(b) a finite set containing two or more elements
(c) a singleton
(d) an empty set

Answers

Chapter Exercises

1. (a) **2.** (b) **3.** (a) **4.** (c) **5.** (a) **6.** (b)
7. (d) **8.** (c) **9.** (c) **10.** (d) **11.** (d) **12.** (c)
13. (c) **14.** (a) **15.** (b) **16.** (a) **17.** (c) **18.** (a)
19. (d) **20.** (a) **21.** (b) **22.** (a) **23.** (b) **24.** (b)
25. (b) **26.** (b) **27.** (c) **28.** (a) **29.** (b) **30.** (a)
31. (a, b, c, d) **32.** (b, d) **33.** (a, b, c, d) **34.**(b, d)
35. (a, c, d) **36.** (a, b, c) **37.** (a, b, c, d) **38.** (a, b, d)
39. (a, b, c, d) **40.** (a, c, d) **41.** (a, b) **42.** (a, b)
43. (a, b) **44.** (a, c) **45.** (a, b, c) **46.** (a)
47. (b) **48.** (c) **49.** (c) **50.** (b) **51.** (b)
52. (c) **53.** (c) **54.** (d) **55.** (a) **56.** (a) **57.** (c)
58. (c) **59.** (a) **60.** (d) **61.** (b) **62.** (d) **63.** (b)
64. (b) **65.** (a) **66.** (c) **67.** (2) **68.** (9) **69.** (2)
70. (1) **71.** (3) **72.** (6) **73.** (1) **74.** (8) **75.** (3)
76.(8) **77.** (A) $\rightarrow$ (p, r); (B) $\rightarrow$ (p, r); (C) $\rightarrow$ (p, q, s, t)
78. (A) $\rightarrow$ (p, s, t); (B) $\rightarrow$ (r, t); (C) $\rightarrow$ (p, q)
79. (A) $\rightarrow$ (r); (B) $\rightarrow$ (r, t); (C) $\rightarrow$ (p, q, s)
80. (A) $\rightarrow$ (p, q, r); (B) $\rightarrow$ (p, q, r, s, t); (C) $\rightarrow$ (p, q, r, s, t)
81. (A) $\rightarrow$ (p, r); (B) $\rightarrow$ (p, q, r, t); (C) $\rightarrow$ (p, r, s)
82. (c) **83.** (b) **84.** (a) **85.** (a) **86.** (b) **87.** (b)
88. (c) **91.** $15\sqrt{2} - 25\sqrt{3}$
92. 0 **99.** $\frac{1}{12}xyz(x-y)(y-z)(z-x)$
100. (i) 6 (ii) 0 **101.** 0
105. $-a^2(a_1-a_2)(a_2-a_3)(a_3-a_1)$
106. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$
107. (i) $\frac{yz}{x^2}+\frac{zx}{y^2}+\frac{xy}{z^2}+1=0$ (ii) $a^3+b^3+c^3=5\,abc$
109. (b) **110.** (c) **111.** (b) **112.** (b) **113.** (a) **114.** (c)
115. (d) **116.** (b) **117.** (c) **118.** (b) **119.** (a) **120.** (a)
121. (a) **122.** (b, c) **123.** (c) **124.** (2) **125.** (b,c,d)
126. (c)

Solutions

1. $\because f(n) = \alpha^n + \beta^n$

Let $\Delta = \begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$

$$= \begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha-1 & \beta-1 \\ 1 & \alpha^2-1 & \beta^2-1 \end{vmatrix}^2$$

Expanding along R_1, we get

$$\Delta = \begin{vmatrix} \alpha-1 & \beta-1 \\ \alpha^2-1 & \beta^2-1 \end{vmatrix}^2 = (\alpha-1)^2(\beta-1)^2 \begin{vmatrix} 1 & 1 \\ \alpha+1 & \beta+1 \end{vmatrix}^2$$

$$= (\alpha-1)^2(\beta-1)^2(\beta-\alpha)^2 = (1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$$

$$= k(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2 \quad \text{[given]}$$

$\therefore \quad k = 1$

2. $\because a, b, c$ and d are in AP. Let D be the common difference,then

$$b = a + D, c = a + 2D, d = a + 3D \quad \text{...(i)}$$

and $\Delta(x) = \begin{vmatrix} x+a & x+b & x+a-c \\ x+b & x+c & x-1 \\ x+c & x+d & x-b+d \end{vmatrix}$

On putting the values of b, c and d from Eq.(i) in $\Delta(x)$, then

$$\Delta(x) = \begin{vmatrix} x+a & x+a+D & x-2D \\ x+a+D & x+a+2D & x-1 \\ x+a+2D & x+a+3D & x+2D \end{vmatrix}$$

Applying $R_2 \to R_2 - \frac{1}{2}(R_1 + R_3)$, then

$$\Delta(x) = \begin{vmatrix} x+a & x+a+D & x-2D \\ 0 & 0 & -1 \\ x+a+2D & x+a+3D & x+2D \end{vmatrix}$$

Expanding along R_2, then

$$\Delta(x) = \begin{vmatrix} x+a & x+a+D \\ x+a+2D & x+a+3D \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, then

$$\Delta(x) = \begin{vmatrix} x+a & x+a+D \\ 2D & 2D \end{vmatrix}$$

$$= 2D(x + a - x - a - D) = -2D^2$$

Also, $\int_0^2 \Delta(x)\,dx = -16$

$\Rightarrow \quad -2D^2(2) = -16$

$\therefore \quad D^2 = 4$ or $D = \pm 2$

3. Let $\Delta(x) = \begin{vmatrix} x & 1+x^2 & x^3 \\ \log_e(1+x^2) & e^x & \sin x \\ \cos x & \tan x & \sin^2 x \end{vmatrix}$

$$= a + bx + cx^2 + \ldots$$

On putting $x = 0$, we get

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = a$$

$\therefore \quad 0 = a$

or $\quad a = 0$, then

$$\Delta(x) = bx + cx^2 + \ldots$$

Hence, $\Delta(x)$ is divisible by x.

4. Given, $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} a^2 & b^2 & c^2 \\ a^2+2a+1 & b^2+2b+1 & c^2+2c+1 \\ a^2-2a+1 & b^2-2b+1 & c^2-2c+1 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_3$, then

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ 4a & 4b & 4c \\ a^2-2a+1 & b^2-2b+1 & c^2-2c+1 \end{vmatrix} = 0$$

Applying $R_3 \to R_3 - R_1 + \frac{1}{2}R_2$, then

$$4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow -\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0 \quad [\because R_1 \to R_3]$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0$$

$\Rightarrow \quad (a-b)(b-c)(c-a) = 0$

$\therefore \quad a - b = 0$ or $b - c = 0$ or $c - a = 0$

$\Rightarrow \quad a = b$ or $b = c$ or $c = a$

Hence, ΔABC is an isosceles triangle.

5. Let $\Delta = \begin{vmatrix} \alpha & x & x & x \\ x & \beta & x & x \\ x & x & \gamma & x \\ x & x & x & \delta \end{vmatrix}$

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$ and $C_4 \to C_4 - C_1$, then

$$\Delta = \begin{vmatrix} \alpha & x-\alpha & x-\alpha & x-\alpha \\ x & \beta - x & 0 & 0 \\ x & 0 & \gamma - x & 0 \\ x & 0 & 0 & \delta - x \end{vmatrix}$$

Expanding along first column, then

$\Delta = \alpha(\beta - x)(\gamma - x)(\delta - x) - x(x-\alpha)(\gamma - x)(\delta - x)$
$+ x(\delta - x)(x-\alpha)(x-\beta) - x(x-\alpha)(\beta - x)(\gamma - x)$
$= (x-\alpha)(x-\beta)(x-\gamma)(x-\delta) - x[(x-\alpha)(x-\gamma)(x-\delta)$
$+ (x-\beta)(x-\gamma)(x-\delta)$
$+ (x-\alpha)(x-\beta)(x-\delta) + (x-\alpha)(x-\beta)(x-\gamma)]$ [given]
$= f(x) - x f'(x)$

$\therefore \quad f(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

6. Given, $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = 0$

$\Rightarrow \quad \dfrac{1}{a}\begin{vmatrix} a^2 & b-c & c+b \\ a^2+ac & b & c-a \\ a^2-ab & a+b & c \end{vmatrix} = 0$

Applying $C_1 \to C_1 + bC_2 + cC_3$, then

$\Rightarrow \quad \dfrac{1}{a}\begin{vmatrix} a^2+b^2+c^2 & b-c & c+b \\ a^2+b^2+c^2 & b & c-a \\ a^2+b^2+c^2 & a+b & c \end{vmatrix} = 0$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$\Rightarrow \quad \dfrac{1}{a}\begin{vmatrix} a^2+b^2+c^2 & b-c & c-b \\ 0 & c & -b-a \\ 0 & a+c & -b \end{vmatrix} = 0$

Expanding along C_1, then

$\Rightarrow \quad \dfrac{(a^2+b^2+c^2)}{a}\begin{vmatrix} c & -b-a \\ a+c & -b \end{vmatrix} = 0$

$\Rightarrow \quad \dfrac{(a^2+b^2+c^2)}{a}[(-bc + (b+a)(a+c)] = 0$

$\Rightarrow \quad \dfrac{(a^2+b^2+c^2)(-bc+ab+bc+a^2+ac)}{a} = 0$

$\Rightarrow \quad (a^2+b^2+c^2)(a+b+c) = 0$

$\because \quad a^2+b^2+c^2 \neq 0$

$\therefore \quad a+b+c = 0$

Therefore, line $ax + by + c = 0$ passes through the fixed point (1, 1).

7. $\because \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$

$= -(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$

[where ω is cube roots of unity]

$= -f(\alpha)f(\beta)f(\gamma)$ $\quad [\because \alpha = 1, \beta = \omega, \gamma = \omega^2]$

8. Let $\Delta = \begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$

$$= \begin{vmatrix} 1-2\sin^2 x & \sin^2 x & 1-8\sin^2 x(1-\sin^2 x) \\ \sin^2 x & 1-2\sin^2 x & 1-\sin^2 x \\ 1-8\sin^2 x(1-\sin^2 x) & 1-\sin^2 x & 1-2\sin^2 x \end{vmatrix}$$

The required constant term is $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

Applying $C_3 \to C_3 - C_1$, then

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1(0-1) = -1$$

9. $\because \quad -1 \le x < 0 \Rightarrow [x] = -1$

$0 \le y < 1 \Rightarrow [y] = 0$

$1 \le z < 2 \Rightarrow [z] = 1$

Let $\Delta = \begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix}$

Expanding along C_2, then $\Delta = \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} = 1 = [z]$

10. Let $\Delta = \begin{vmatrix} y^2 & -xy & x^2 \\ a & b & c \\ a' & b' & c' \end{vmatrix} = \dfrac{1}{xy}\begin{vmatrix} xy^2 & -xy & x^2y \\ ax & b & cy \\ a'x & b' & c'y \end{vmatrix}$

Applying $C_1 \to C_1 + yC_2$ and $C_3 \to C_3 + xC_2$, then

$$\Delta = \frac{1}{xy}\begin{vmatrix} 0 & -xy & 0 \\ ax+by & b & bx+cy \\ a'x+b'y & b' & b'x+c'y \end{vmatrix}$$

Expanding along R_1, then

$$= \frac{1}{xy}\cdot xy \cdot \begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$$

$$= \begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$$

11. $\because$ In a triangle $A+B+C=\pi$ and $e^{\pi}=\cos\pi+i\sin\pi=-1$

$$e^{i(B+C)}=e^{i(\pi-A)}=e^{i\pi}\cdot e^{iA}=-e^{-iA}$$

$\Rightarrow$ $$e^{-i(B+C)}=-e^{iA}$$

Similarly, $e^{-i(A+B)}=-e^{iC}$ and $e^{-i(C+A)}=-e^{iB}$

Taking e^{iA}, e^{iB}, e^{iC} common from R_1, R_2 and R_3 respectively, we get

$$\Delta=e^{iA}\cdot e^{iB}\cdot e^{iC}\begin{vmatrix} e^{iA} & e^{-i(A+C)} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{iB} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{-i(A+C)} & e^{iC}\end{vmatrix}$$

$$=e^{i\pi}=\begin{vmatrix} e^{iA} & -e^{iB} & -e^{iC} \\ -e^{iA} & e^{iB} & -e^{iC} \\ -e^{iA} & -e^{iB} & e^{iC}\end{vmatrix}$$

Taking e^{iA}, e^{iB}, e^{iC} common from C_1, C_2 and C_3 respectively, we get

$$\Delta=(-1)\,e^{iA}\cdot e^{iB}\cdot e^{iC}\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1\end{vmatrix}$$

$$=(-1)\,e^{i\pi}\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1\end{vmatrix}$$

$$=(-1)(-1)\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1\end{vmatrix}=\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1\end{vmatrix}$$

Applying $C_2\to C_2+C_1$ and $C_3\to C_3+C_1$, then

$$\Delta=\begin{vmatrix} 1 & 0 & 0 \\ -1 & 0 & -2 \\ -1 & -2 & 0\end{vmatrix}=1\,(0-4)=-4$$

12. Taking x^5 common from R_3, then

$$x^5\begin{vmatrix} x^n & x^{n+2} & x^{2n} \\ 1 & x^a & a \\ x^n & x^{a+1} & x^{2n}\end{vmatrix}=0,\ \forall\ x\in R$$

$\Rightarrow$ $$a+1=n+2\ \Rightarrow\ a=n+1$$

13. Since, x, y and z are in AP.

$\therefore$ $$2y=x+z \qquad \text{...(i)}$$

Let $\Delta=\begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z\end{vmatrix}$

$$=\begin{vmatrix} 5 & 4 & 3 \\ 100x+50+1 & 100y+40+1 & 100z+30+1 \\ x & y & z\end{vmatrix}$$

Applying $R_2\to R_2-\frac{1}{2}(R_1+R_3)$, then

$$=\begin{vmatrix} 5 & 0 & 3 \\ 100x+50+1 & 0 & 100z+30+1 \\ x & 0 & z\end{vmatrix} \qquad \text{[from Eq. (i)]}$$

$=0$ [$\because$ all elements of C_2 are zeroes]

14. As $a_1b_1c_1$, $a_2b_2c_2$ and $a_3b_3c_3$ are even natural numbers each of c_1, c_2, c_3 is divisible by 2.

Let $C_i=2\lambda_i$ for $i=1,2,3$ and $\lambda_i\in N$, then

$$\Delta=\begin{vmatrix} 2\lambda_1 & a_1 & b_1 \\ 2\lambda_2 & a_2 & b_2 \\ 2\lambda_3 & a_3 & b_3\end{vmatrix}=2\begin{vmatrix} \lambda_1 & a_1 & b_1 \\ \lambda_2 & a_2 & b_2 \\ \lambda_3 & a_3 & b_3\end{vmatrix}=2m$$

where m is some natural number. Thus, Δ is divisible by 2.
That Δ may not be divisible by 4 can be seen by taking the three numbers as 112, 122 and 134.

$$\Delta=\begin{vmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ 4 & 1 & 3\end{vmatrix}=2(3-2)-1(6-8)+1(2-4)=2$$

which is divisible by 2 but not by 4.

15. Let $\Delta=\begin{vmatrix} c & b\cos B+c\beta & a\cos A+b\alpha+c\gamma \\ a & c\cos B+a\beta & b\cos A+c\alpha+a\gamma \\ b & a\cos B+b\beta & c\cos A+a\alpha+b\gamma\end{vmatrix}$

Applying $C_2\to C_2-\beta C_1$ and $C_3\to C_3-\gamma C_1$, then

$$\Delta=\begin{vmatrix} c & b\cos B & a\cos A+b\alpha \\ a & c\cos B & b\cos A+c\alpha \\ b & a\cos B & c\cos A+a\alpha\end{vmatrix}$$

Applying $C_3\to C_3-\alpha\sec B\,C_2$, then

$$\Delta=\begin{vmatrix} c & b\cos B & a\cos A \\ a & c\cos B & b\cos A \\ b & a\cos B & c\cos A\end{vmatrix}=\cos A\cos B\begin{vmatrix} c & b & a \\ a & c & b \\ b & a & c\end{vmatrix}$$

Applying $C_1\leftrightarrow C_3$, then $\Delta=-\cos A\cos B\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b\end{vmatrix}$

$=-\cos A\cos B\,(a+b+c)\cdot\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$

Given, $\cos A\neq 0$, $\cos B\neq 0$ and $a+b+c\neq 0$

$\because$ $\Delta=0$

$\therefore$ $(a-b)^2+(b-c)^2+(c-a)^2=0$

which is independent, when $a-b=0$, $b-c=0$ and $c-a=0$

i.e., $a=b=c$

Hence, ΔABC is an equilateral.

16. Here, $\left.\begin{array}{l} x_1+x_2=6,\quad x_1x_2=3 \\ \text{and}\quad y_1+y_2=4,\quad y_1y_2=2\end{array}\right\}$ ...(i)

Let $\Delta=\begin{vmatrix} x_1x_2 & y_1y_2 & 1 \\ x_1+x_2 & y_1+y_2 & 2 \\ \sin(\pi x_1x_2) & \cos\left(\frac{\pi}{2}y_1y_2\right) & 1\end{vmatrix}$

$$=\begin{vmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ \sin 3\pi & \cos\left(\frac{\pi}{4}\right) & 1\end{vmatrix} \qquad \text{[from Eq. (i)]}$$

Applying $R_2\to R_2-2R_1$, then $\Delta=\begin{vmatrix} 3 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 1\end{vmatrix}=0$

17. $\because \Delta = \begin{vmatrix} {}^{10}C_4 & {}^{10}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{11}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{12}C_9 & {}^{13}C_{m+4} \end{vmatrix}$

Applying $C_2 \to C_2 + C_1$ and use Pascal's rule $({}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r)$, then

$$\Delta = \begin{vmatrix} {}^{10}C_4 & {}^{11}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{12}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{13}C_9 & {}^{13}C_{m+4} \end{vmatrix} = 0 \qquad \text{[given]}$$

$\therefore \quad m = 5$

18. Let $\Delta = \begin{vmatrix} 1 & \sin(\alpha - \beta)\theta & \cos(\alpha - \beta)\theta \\ a & \sin \alpha\theta & \cos \alpha\theta \\ a^2 & \sin(\alpha - \beta)\theta & \cos(\alpha - \beta)\theta \end{vmatrix}$

Applying $R_1 \to R_1 - R_3$, then

$$\Delta = \begin{vmatrix} 1 - a^2 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ a & & \sin \alpha\theta & & \cos \alpha\theta \\ \vdots & & & & \\ a^2 & & \sin(\alpha - \beta)\theta & & \cos(\alpha - \beta)\theta \end{vmatrix}$$

Expanding along R_1, then

$$\Delta = (1 - a^2) \begin{vmatrix} \sin \alpha\theta & \cos \alpha\theta \\ \sin(\alpha - \beta)\theta & \cos(\alpha - \beta)\theta \end{vmatrix}$$

$$= (1 - a^2)[\sin \alpha\theta \cdot \cos(\alpha - \beta)\theta - \cos \alpha\theta \cdot \sin(\alpha - \beta)\theta]$$

$$= (1 - a^2)\sin(\alpha\theta - \alpha\theta + \beta\theta) = (1 - a^2)\sin \beta\theta$$

19. Let $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ p & q & r \end{vmatrix} = mx^4 + nx^3 + rx^2 + sx + t$ ...(i)

On differentiating twice and thrice of Eq. (i) w.r.t. x, then

$$F''(x) = \begin{vmatrix} f''(x) & g''(x) & h''(x) \\ a & b & c \\ p & q & r \end{vmatrix} = 12mx^2 + 6nx + 2r \qquad \text{...(ii)}$$

$$F'''(x) = \begin{vmatrix} f'''(x) & g'''(x) & h'''(x) \\ a & b & c \\ p & q & r \end{vmatrix} = 24mx + 6n \qquad \text{...(iii)}$$

On putting $x = 0$ in Eqs. (ii) and (iii), we get

$$\begin{vmatrix} f''(0) & g''(0) & h''(0) \\ a & b & c \\ p & q & r \end{vmatrix} = 2r \qquad \text{...(iv)}$$

and $$\begin{vmatrix} f''(0) & g''(0) & h''(0) \\ a & b & c \\ p & q & r \end{vmatrix} = 6n \qquad \text{...(v)}$$

Now, subtracting Eq. (iv) from Eq. (v), we get

$$\begin{vmatrix} f'''(0) - f''(0) & g'''(0) - g''(0) & h'''(0) - h''(0) \\ a & b & c \\ p & q & r \end{vmatrix}$$

$$= 6n - 2r = 2(3n - r)$$

20. $\because f(x) = \begin{vmatrix} \cos(x + \alpha) & \cos(x + \beta) & \cos(x + \gamma) \\ \sin(x + \alpha) & \sin(x + \beta) & \sin(x + \gamma) \\ \sin(\beta - \gamma) & \sin(\gamma - \alpha) & \sin(\alpha - \beta) \end{vmatrix}$

On differentiating both sides w.r.t. x, then

$$f'(x) = \begin{vmatrix} -\sin(x + \alpha) & -\sin(x + \beta) & -\sin(x + \gamma) \\ \sin(x + \alpha) & \sin(x + \beta) & \sin(x + \gamma) \\ \sin(\beta - \gamma) & \sin(\gamma - \alpha) & \sin(\alpha - \beta) \end{vmatrix} + \begin{vmatrix} \cos(x + \alpha) & \cos(x + \beta) & \cos(x + \gamma) \\ \cos(x + \alpha) & \cos(x + \beta) & \cos(x + \gamma) \\ \sin(\beta - \gamma) & \sin(\gamma - \alpha) & \sin(\alpha - \beta) \end{vmatrix}$$

$$= -\begin{vmatrix} \sin(x + \alpha) & \sin(x + \beta) & \sin(x + \gamma) \\ \sin(x + \alpha) & \sin(x + \beta) & \sin(x + \gamma) \\ \sin(\beta - \gamma) & \sin(\gamma - \alpha) & \sin(\alpha - \beta) \end{vmatrix} + \begin{vmatrix} \cos(x + \alpha) & \cos(x + \beta) & \cos(x + \gamma) \\ \cos(x + \alpha) & \cos(x + \beta) & \cos(x + \gamma) \\ \sin(\beta - \gamma) & \sin(\gamma - \alpha) & \sin(\alpha - \beta) \end{vmatrix}$$

$= 0 + 0$ [$\because R_1$ and R_2 are identical]

$= 0$

$\therefore f(x) = c$ [constant]

Now, $f(\theta) - 2f(\phi) + f(\psi) = c - 2c + c = 0$

21. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

Taking a, b, c common from C_1, C_2, C_3, then $= abc \begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ 1 & 1 & 1 \\ a^2 & b^2 & c^2 \end{vmatrix}$

On multiplying in R_1 by abc, then

$$\Delta = \begin{vmatrix} bc & ca & ab \\ 1 & 1 & 1 \\ a^2 & b^2 & c^2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ a^2 & b^2 & c^2 \end{vmatrix} \qquad [R_1 \leftrightarrow R_2]$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} \qquad [R_2 \leftrightarrow R_3]$$

$$= (a - b)(b - c)(c - a)(a + b + c)$$

Now, $D = \begin{vmatrix} 1 & 1 & 1 \\ (x - a)^2 & (x - b)^2 & (x - c)^2 \\ (x - b)(x - c) & (x - c)(x - a) & (x - a)(x - b) \end{vmatrix}$

$$= (b - a)(c - b)(a - c)(3x - a - b - c)$$

Now, given that a, b and c are all different, then $D = 0$

$\therefore \quad 3x - a - b - c = 0$

$\Rightarrow \quad x = \frac{1}{3}(a + b + c)$

22. Given, determinant

$$2a(bc-4a^2)-b(b^2-2ac)+c(2ab-c^2)=0$$

$$\Rightarrow \quad -[(2a)^3+b^3+c^3-3\cdot 2a\cdot b\cdot c]=0$$

$$\Rightarrow \quad \frac{1}{2}(2a+b+c)[(2a-b)^2+(b-c)^2+(c-2a)^2]=0$$

$$\Rightarrow \quad 2a+b+c=0 \quad \ldots\text{(i)} \ [\because b\neq c]$$

Let $f(x)=8ax^3+2bx^2+cx+d$

$\therefore f(0)=d$ and for $\left(\frac{1}{2}\right)=a+\frac{b}{2}+\frac{c}{2}+d=\frac{2a+b+c}{2}+d$

$$=\frac{0}{2}+d=d \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad f(0)=f\left(\frac{1}{2}\right)$$

So, $f(x)$ satisfies Rolle's theorem and hence $f'(x)=0$ has atleast one root in $\left[0,\frac{1}{2}\right]$.

23. Given, $\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix}=11$

Taking x, y, z common from C_1, C_2, C_3 respectively, then

$$\Rightarrow \quad xyz\begin{vmatrix} \frac{x^3+1}{x} & x^2 & x^2 \\ y^2 & \frac{y^3+1}{y} & y^2 \\ z^2 & z^2 & \frac{z^3+1}{z} \end{vmatrix}=11$$

On multiplying R_1 by x, R_2 by y and R_3 by z, we get

$$\Rightarrow \quad \begin{vmatrix} x^3+1 & x^3 & x^3 \\ y^3 & y^3+1 & y^3 \\ z^3 & z^3 & z^3+1 \end{vmatrix}=11$$

Applying $R_1\rightarrow R_1+R_2+R_3$, then

$$\begin{vmatrix} x^3+y^3+z^3+1 & x^3+y^3+z^3+1 & x^3+y^3+z^3+1 \\ y^3 & y^3+1 & y^3 \\ z^3 & z^3 & z^3+1 \end{vmatrix}=11$$

Applying $C_2\rightarrow C_2-C_1$ and $C_3\rightarrow C_3-C_1$, then

$$\begin{vmatrix} x^3+y^3+z^3+1 & 0 & 0 \\ y^3 & 1 & 0 \\ z^3 & 0 & 1 \end{vmatrix}=11$$

$$\Rightarrow \quad x^3+y^3+z^3+1=11$$

$$\Rightarrow \quad x^3+y^3+z^3=10$$

Therefore, the ordered triplets are (2, 1, 1), (1, 2, 1) and (1, 1, 2).

24. $\because f(x)-x=0$ has imaginary roots.

Then, $\quad f(x)-x>0$ or $f(x)-x, 0, \forall\ x\in R$

for $\quad f(x)-x>0, \forall\ x\in R$,

then $\quad f[f(x)]-f(x)>0, \forall\ x\in R$

On adding, we get

$$f[f(x)]-x>0, \forall\ x\in R$$

Similarly, $f[f(x)]-x<0, \forall\ x\in R$

Thus, roots of the equation $f[f(x)]-x=0$ are imaginary

Let $\quad z=\begin{vmatrix} 2 & \alpha & \delta \\ \beta & 0 & \alpha \\ \gamma & \beta & 1 \end{vmatrix}$

Then, $\quad \bar{z}=\begin{vmatrix} 2 & \bar{\alpha} & \bar{\delta} \\ \bar{\beta} & 0 & \bar{\alpha} \\ \bar{\gamma} & \bar{\beta} & 1 \end{vmatrix}=\begin{vmatrix} 2 & \beta & \gamma \\ \alpha & 0 & \beta \\ \delta & \alpha & 1 \end{vmatrix}=\begin{vmatrix} 2 & \alpha & \delta \\ \beta & 0 & \alpha \\ \gamma & \beta & 1 \end{vmatrix}=z$

Hence, z is purely real.

25. For infinitely many solutions

$$\Delta=\Delta_1=\Delta_2=\Delta_3=0$$

$$\Delta=0 \Rightarrow \begin{vmatrix} 2 & -1 & 1 \\ 1 & -2 & 1 \\ t & -1 & 2 \end{vmatrix}=0 \Rightarrow t=5$$

For $\quad t=5, \Delta_1=\Delta_2=\Delta_3=0$

Now, $\int_0^{-2t} f(x)\,dx=\int_0^{-10} f(x)\,dx=\int_0^{-5} f(x)\,dx+\int_{-5}^{-10} f(x)\,dx$

$$=\int_{-5}^{-10} f(x+5)\,dx+\int_{-5}^{-10} f(x)\,dx$$

$$=\int_{-5}^{-10}[f(x+5)+f(x)]\,dx$$

$$=\int_{-5}^{-10} 2dx=2(-10+5)$$

$$=-10=-2t$$

26. On putting $x=0$, we get $a_0=1$

On differentiating both sides w.r.t. x and putting $x=0$, we get

$$a_1=4a$$

On differentiating again w.r.t. x and putting $x=0$, we get

$$2a_2=12a^2+8b$$

or $\quad a_2=6a^2+4b$

Also, given $\begin{vmatrix} a_1 & a_1 & a_2 \\ a_0 & a_2 & a_0 \\ a_2 & a_0 & a_1 \end{vmatrix}=0$

$$\Rightarrow \quad -(a_0^3+a_1^3+a_2^3-3a_0a_1a_2)=0$$

$$\Rightarrow \quad \frac{1}{2}(a_0+a_1+a_2)[(a_0-a_1)^2+(a_1-a_2)^2+(a_2-a_0)^2]=0$$

$$\because \quad a_0+a_1+a_2\neq 0$$

$$\therefore \quad (a_0-a_1)^2+(a_1-a_2)^2+(a_2-a_0)^2=0$$

$$\Rightarrow \quad a_0-a_1=0, a_1-a_2=0, a_2-a_0=0$$

$$\therefore \quad a_0=a_1=a_2$$

$$\Rightarrow \quad 1=4a=6a^2+4b$$

$$\Rightarrow \quad a=\frac{1}{4} \text{ and } b=\frac{5}{32}$$

27. $\because f(x)=\log_{10}x$ and $g(x)=e^{\pi i x}$

$$\therefore \quad f(10)=\log_{10}10=1$$

and $\quad g(10)=e^{10\pi i}=(-1)^{10}=1$

$$f(10^2) = \log_{10} 10^2 = 2$$

and $$g(10^2) = e^{100\pi i} = (-1)^{100} = 1$$

$$f(10^3) = \log_{10} 10^3 = 3$$

and $$g(10^3) = e^{1000\pi i} = (-1)^{1000} = 1$$

Given, $\phi(x) = \begin{vmatrix} f(x)\cdot g(x) & [f(x)]^{g(x)} & 1 \\ f(x^2)\cdot g(x^2) & [f(x^2)]^{g(x^2)} & 0 \\ f(x^3)\cdot g(x^3) & [f(x^3)]^{g(x^3)} & 1 \end{vmatrix}$

$\therefore \quad \phi(10) = \begin{vmatrix} f(10)\cdot g(10) & [f(10)]^{g(10)} & 1 \\ f(10^2)\cdot g(10^2) & [f(10^2)]^{g(10^2)} & 0 \\ f(10^3)\cdot g(10^3) & [f(10^3)]^{g(10^3)} & 1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 3 & 1 \end{vmatrix} = 0$$

28. Let $\Delta = \begin{vmatrix} 1 & (\alpha^{2x} - \alpha^{-2x})^2 & (\alpha^{2x} + \alpha^{-2x})^2 \\ 1 & (\beta^{2x} - \beta^{-2x})^2 & (\beta^{2x} + \beta^{-2x})^2 \\ 1 & (\gamma^{2x} - \gamma^{-2x})^2 & (\gamma^{2x} + \gamma^{-2x})^2 \end{vmatrix}$

Applying $C_3 \to C_3 - C_2$, then $\Delta = \begin{vmatrix} 1 & (\alpha^{2x} - \alpha^{-2x})^2 & 4 \\ 1 & (\beta^{2x} - \beta^{-2x})^2 & 4 \\ 1 & (\gamma^{2x} - \gamma^{-2x})^2 & 4 \end{vmatrix} = 0$

29. The given equations can be written as

$$(a-1)x - y - z = 0,$$
$$-x + (b-1)y - z = 0$$

and $$-x - y + (c-1)z = 0$$

For non-trivial solution

$$\begin{vmatrix} a-1 & -1 & -1 \\ -1 & b-1 & -1 \\ -1 & -1 & c-1 \end{vmatrix} = 0$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, then

$$\begin{vmatrix} a & 0 & -1 \\ 0 & b & -1 \\ -c & -c & c-1 \end{vmatrix} = 0$$

Expanding along R_1, then

$\Rightarrow \quad a(bc - b - c) - 0 - 1(0 + bc) = 0$

$\Rightarrow \quad ab + bc + ca = abc$

30. For non-trivial solution $\begin{vmatrix} \lambda & -1 & \cos\theta \\ 3 & 1 & 2 \\ \cos\theta & 1 & 2 \end{vmatrix} = 0$

Applying $R_3 \to R_3 - R_2$, then

$$\begin{vmatrix} \lambda & -1 & \cos\theta \\ \vdots & & \\ 3 & 1 & 2 \\ \vdots & & \\ \cos\theta - 3 \;\cdots & 0 \;\cdots & 0 \end{vmatrix} = 0$$

Expanding along R_3, then

$\Rightarrow \quad (\cos\theta - 3)(-2 - \cos\theta) = 0$

$\Rightarrow \quad (\cos\theta - 3)(2 + \cos\theta) = 0$

$\cos\theta = 3, -2$, where -2 is neglected.

Hence, $\begin{vmatrix} \lambda & -1 & \cos\theta \\ 3 & 1 & 2 \\ \cos\theta & 1 & 2 \end{vmatrix} > 0$ only trivial solution is possible.

31. $\because \Delta = \begin{vmatrix} a^2 + x^2 & ab & ac \\ ab & b^2 + x^2 & bc \\ ac & bc & c^2 + x^2 \end{vmatrix}$

Taking a, b, c common from R_1, R_2, R_3 respectively, then

$$\Delta = abc \begin{vmatrix} \frac{a^2 + x^2}{a} & b & c \\ a & \frac{b^2 + x^2}{b} & c \\ a & b & \frac{c^2 + x^2}{c} \end{vmatrix}$$

On multiplying in C_1, C_2, C_3 by a, b, c respectively, then

$$\Delta = \begin{vmatrix} a^2 + x^2 & b^2 & c^2 \\ a^2 & b^2 + x^2 & c^2 \\ a^2 & b^2 & c^2 + x^2 \end{vmatrix}$$

Now, applying $C_1 \to C_1 + C_2 + C_3$, then

$$\Delta = \begin{vmatrix} x^2 + a^2 + b^2 + c^2 & b^2 & c^2 \\ x^2 + a^2 + b^2 + c^2 & b^2 + x^2 & c^2 \\ x^2 + a^2 + b^2 + c^2 & b^2 & c^2 + x^2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta = \begin{vmatrix} x^2 + a^2 + b^2 + c^2 & b^2 & c^2 \\ 0 & x^2 & 0 \\ 0 & 0 & x^2 \end{vmatrix}$$

$$= x^4 (x^2 + a^2 + b^2 + c^2)$$

32. Let $\Delta = \begin{vmatrix} \sqrt{6} & 2i & 3 + \sqrt{6} \\ \sqrt{12} & \sqrt{3} + \sqrt{8}\,i & 3\sqrt{2} + \sqrt{6}\,i \\ \sqrt{18} & \sqrt{2} + \sqrt{12}\,i & \sqrt{27} + 2i \end{vmatrix}$

Applying $R_2 \to R_2 - \sqrt{2}\,R_1$ and $R_3 \to R_3 - \sqrt{3}\,R_1$, then

$$\Delta = \begin{vmatrix} \sqrt{6} \;\cdots & 2i \;\cdots & 3 + \sqrt{6} \\ \vdots & & \\ 0 & \sqrt{3} & -2\sqrt{3} + 6i \\ \vdots & & \\ 0 & \sqrt{2} & -3\sqrt{2} + 2i \end{vmatrix}$$

Expanding along C_1, we get

$$= \sqrt{6} \begin{vmatrix} \sqrt{3} & -2\sqrt{3} + \sqrt{6}i \\ \sqrt{2} & -3\sqrt{2} + 2i \end{vmatrix}$$

$$= \sqrt{6}\,[-3\sqrt{6} + 2i\sqrt{3} + 2\sqrt{6} - 2i\sqrt{3}]$$

$$= \sqrt{6}\,(-\sqrt{6}) = -6$$ [real and rational]

33. $\sum_{k=1}^{n} 2^{k-1} = 1 + 2 + 2^2 + \ldots + 2^n = 2^n - 1$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$

and $$\sum_{k=1}^{n} \sin k\theta = \frac{\sin\left(\frac{n+1}{2}\right)\theta \sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

Given, $$D_k = \begin{vmatrix} 2^{k-1} & \frac{1}{k(k+1)} & \sin k\theta \\ x & y & z \\ 2^n - 1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta \sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \end{vmatrix}$$

$$\therefore \sum_{k=1}^{n} D_k = \begin{vmatrix} \sum_{k=1}^{n} 2^{k-1} & \sum_{k=1}^{n} \frac{1}{k(k+1)} & \sum_{k=1}^{n} \sin k\theta \\ x & y & z \\ 2^n - 1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta \sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \end{vmatrix}$$

$$= \begin{vmatrix} 2^n - 1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta \sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \\ x & y & z \\ 2^n - 1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta \sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \end{vmatrix} = 0$$

34. We have, $$\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = 0$$

Applying $C_3 \to C_3 - \alpha C_1 - C_2$, then

$$\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ aa + b & ba + c & -(aa^2 + 2ba + c) \end{vmatrix} = 0$$

Expanding along C_3, we get

$$-(a\alpha^2 + 2b\alpha + c)(ac - b^2) = 0$$

$\Rightarrow$ $$(a\alpha^2 + 2b\alpha + c)(b^2 - ac) = 0$$

$\Rightarrow$ $$b^2 - ac = 0 \text{ and } a\alpha^2 + 2b\alpha + c = 0$$

i.e. a, b and c are in GP and $(x - \alpha)$ is a factor of $ax^2 + 2bx + c = 0$.

35. $\because$ $$f(x) = \begin{vmatrix} 2\cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$$

$$= 2\cos x\,(4\cos^2 x - 1) - 1\,(2\cos x - 0) + 0$$
$$= 2\cos x\,(4\cos^2 x - 1 - 1)$$
$$= 4\cos x\,(2\cos^2 x - 1)$$
$$= 4\cos x \cos 2x$$
$$= 2(\cos 3x + \cos x)$$

Option (a)

$$f\left(\frac{\pi}{3}\right) = 2\left(\cos\frac{3\pi}{3} + \cos\frac{\pi}{3}\right) = 2\left(-1 + \frac{1}{2}\right) = -1$$

Option (b)

$$f'(x) = 2(-3\sin 3x - \sin x)$$

$\therefore$ $$f'\left(\frac{\pi}{3}\right) = 2\left(-3\sin\pi - \sin\frac{\pi}{3}\right) = 2\left(0 - \frac{\sqrt{3}}{2}\right) = -\sqrt{3}$$

Option (c)

$$\int_0^{\pi} f(x)\,dx = 2\int_0^{\pi} (\cos 3x + \cos x)\,dx = 2\left[\frac{\sin 3x}{3} + \sin x\right]_0^{\pi}$$

$$= 2[(0 + 0) - (0 + 0)] = 0$$

Option (d)

$$\int_0^{\pi} f(x)\,dx = 2\int_{-\pi}^{\pi} (\cos 3x + \cos x)\,dx = 4\int_0^{\pi} (\cos 3x + \cos x)\,dx$$

$$= 0 \qquad \text{[from option (c)]}$$

36. $\because$ $$\Delta(x) = \begin{vmatrix} x^2 - 5x + 3 & 2x - 5 & 3 \\ 3x^3 + x + 4 & 6x + 1 & 9 \\ 7x^2 - 6x + 9 & 14x - 6 & 21 \end{vmatrix}$$

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 7R_1$, then

$$= \begin{vmatrix} x^2 - 5x + 3 & 2x - 5 & 3 \\ 16x - 5 & 16 & 0 \\ 29x - 12 & 29 & 0 \end{vmatrix}$$

Expanding along C_3, we get $= 3\begin{vmatrix} 16x - 5 & 16 \\ 29x - 12 & 29 \end{vmatrix}$

Applying $C_1 \to C_1 - xC_2$, then

$$\Delta(x) = 3\begin{vmatrix} -5 & 16 \\ -12 & 29 \end{vmatrix} = 3(-145 + 192) = 3 \times 47$$

$$= 141 = ax^3 + bx^2 + cx + d \text{ [given]}$$

$\therefore$ $a = 0, b = 0, c = 0, d = 141$

37. $\because$ $$\Delta = \begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix}$$

Taking common a from each R_1 and C_1, then

$$\Delta = \begin{vmatrix} 1 & \frac{b \sin A}{a} & \frac{c \sin A}{a} \\ \frac{b \sin A}{a} & 1 & \cos A \\ \frac{c \sin A}{a} & \cos A & 1 \end{vmatrix} = \begin{vmatrix} 1 & \sin B & \sin C \\ \sin B & 1 & \cos A \\ \sin C & \cos A & 1 \end{vmatrix}$$

[by sine rule]

Applying $C_2 \to C_2 - \sin B\, C_1$ and $C_3 \to C_3 - \sin C\, C_1$, then

$$\Delta = \begin{vmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ \sin B & & 1 - \sin^2 B & & \cos A - \sin B \sin C \\ \vdots & & & & \\ \sin C & & \cos A - \sin B \sin C & & 1 - \sin^2 C \end{vmatrix}$$

Expanding along R_1, then

$$\Delta = \begin{vmatrix} \cos^2 B & \cos[\pi - (B + C)] - \sin B \sin C \\ \cos[\pi - (B + C)] - \sin B \sin C & \cos^2 C \end{vmatrix}$$

$[\because A + B + C = \pi]$

$$= \begin{vmatrix} \cos^2 B & -\cos(B + C) - \sin B \sin C \\ -\cos(B + C) - \sin B \sin C & \cos^2 C \end{vmatrix}$$

$$= \begin{vmatrix} \cos^2 B & -\cos B \cos C \\ -\cos B \cos C & \cos^2 C \end{vmatrix}$$

$$= \cos^2 B \cos^2 C - \cos^2 B \cos^2 C = 0$$

38. $\because$ $f(a, b) = \begin{vmatrix} a & a^2 & 0 \\ 1 & (2a + b) & (a + b)^2 \\ 0 & 1 & (2a + 3b) \end{vmatrix}$

Applying $C_2 \to C_2 - aC_1$, then

$$f(a, b) = \begin{vmatrix} a & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ 1 & & (a + b) & & (a + b)^2 \\ \vdots & & & & \\ 0 & & 1 & & (2a + 3b) \end{vmatrix}$$

Expanding along R_1, then

$$f(a, b) = a \begin{vmatrix} (a + b) & (a + b)^2 \\ 1 & (2a + 3b) \end{vmatrix}$$

$$= a(a + b) \begin{vmatrix} 1 & (a + b) \\ 1 & (2a + 3b) \end{vmatrix}$$

$$= a(a + b)(2a + 3b - a - b)$$

$$= a(a + b)(a + 2b)$$

39. $\because$ $f(x) = \begin{vmatrix} \sec^2 x & 1 & 1 \\ \cos^2 x & \cos^2 x & \text{cosec}^2 x \\ 1 & \cos^2 x & \cot^2 x \end{vmatrix}$

Applying $C_2 \to C_2 - \cos^2 x\, C_1$, then

$$f(x) = \begin{vmatrix} \sec^2 x & 0 & 1 \\ & \vdots & \\ \cos^2 x & \cdots \cos^2 x - \cos^4 x \cdots & \text{cosec}^2 x \\ & \vdots & \\ 1 & 0 & \cot^2 x \end{vmatrix}$$

Expanding along C_2, then

$$f(x) = \sin^2 x \cos^2 x \begin{vmatrix} \sec^2 x & 1 \\ 1 & \cot^2 x \end{vmatrix}$$

$$= \sin^2 x \cos^2 x (\text{cosec}^2 x - 1)$$

$$= \sin^2 x \cos^2 x \cot^2 x = \cos^4 x$$

option (a)

$$\int_{-\pi/4}^{\pi/4} f(x)\, dx = \int_{-\pi/4}^{\pi/4} \cos^4 x\, dx = 2\int_0^{\pi/4} \cos^4 x\, dx$$

$$= 2\int_0^{\pi/4} \left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= 2 \times \frac{1}{2}\int_0^{\pi/2} \left(\frac{1 + \cos x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int_0^{\pi/2} (1 + 2\cos x + \cos^2 x)\, dx$$

$$= \frac{1}{4}\int_0^{\pi/2} 1 \cdot dx + \frac{1}{2}\int_0^{\pi/2} \cos x\, dx + \frac{1}{4}\int_0^{\pi/2} \cos^2 x\, dx$$

$$= \frac{1}{4}\left(\frac{\pi}{2} - 0\right) + \frac{1}{2}(\sin x)_0^{\pi/2} + \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{8} + \frac{1}{2}(1 - 0) + \frac{\pi}{16} = \frac{1}{16}(2\pi + 8 + \pi) = \frac{1}{16}(3\pi + 8)$$

option (b)

$\because$ $f'(x) = 4\cos^3 x \cdot (-\sin x)$

$\therefore$ $f'\left(\frac{\pi}{2}\right) = 0$

option (c) and (d)

$\because$ $0 \le \cos^4 x \le 1$

$\because$ Maximum value of $f(x)$ is 1.

and minimum value of $f(x)$ is 0.

40. Let $\Delta = \begin{vmatrix} a & a + x^2 & a + x^2 + x^4 \\ 2a & 3a + 2x^2 & 4a + 3x^2 + 2x^4 \\ 3a & 6a + 3x^2 & 10a + 6x^2 + 3x^4 \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, then

$$\Delta = \begin{vmatrix} a & a + x^2 & a + x^2 + x^4 \\ 0 & a & 2a + x^2 \\ 0 & 3a & 7a + 3x^2 \end{vmatrix}$$

Applying $R_3 \to R_3 - 3R_2$, then

$$\Delta = \begin{vmatrix} a & a + x^2 & a + x^2 + x^4 \\ 0 & a & 2a + x^2 \\ 0 & 0 & a \end{vmatrix}$$

$$= a^3 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \quad \text{[given]}$$

$\therefore\ a_0 = a^3, a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0, a_7 = 0$

and $\quad f(x) = a_0 x^2 + a_3 x + a_6 = a^3 x^2$

option (a) $f(x) \ge 0 \Rightarrow a^3 x^2 \ge 0$

If $a^3 > 0$, then $x^2 \ge 0$

$\therefore \quad a > 0, x \in R$

option (b) If $a = 0$, then $f(x) = 0$

and $\quad$ If $x = 0$, then $f(x) = 0$

$\therefore$ Aliter (b) is fail

option (c) $f(x) = 0$

$\Rightarrow \quad a^3 x^2 = 0$ or $x^2 = 0$

$\therefore \quad x = 0, 0$

option (d) For $a = 0$, $f(x) = 0$ is an identity, then it has more than two roots.

41. Let $\Delta(x) = \begin{vmatrix} 4x - 4 & (x-2)^2 & x^3 \\ 8x - 4\sqrt{2} & (x - 2\sqrt{2})^2 & (x+1)^3 \\ 12x - 4\sqrt{3} & (x - 2\sqrt{3})^2 & (x-1)^3 \end{vmatrix}$

$$= a_0 + a_1 x + a_2 x^2 + \dots \quad \dots\text{(i)}$$

On putting $x = 0$ in Eq. (i), then

$$\begin{vmatrix} -4 & 4 & 0 \\ -4\sqrt{2} & 8 & 1 \\ -4\sqrt{3} & 12 & -1 \end{vmatrix} = a_0$$

or $\quad a_0 = -4(-8-12) - 4(4\sqrt{2} + 4\sqrt{3})$

$= 16(5 - \sqrt{2} - \sqrt{3})$ = term independent of x in Δ.

Also, on differentiating Eq. (i) w.r.t. x and then put $x = 0$, we get

$$\begin{vmatrix} 4 & -4 & 0 \\ -4\sqrt{2} & 8 & 1 \\ -4\sqrt{3} & 12 & -1 \end{vmatrix} + \begin{vmatrix} -4 & 4 & 0 \\ 8 & -4\sqrt{2} & 3 \\ -4\sqrt{3} & 12 & -1 \end{vmatrix}$$

$$+ \begin{vmatrix} -4 & 4 & 0 \\ -4\sqrt{2} & 8 & 1 \\ 12 & -4\sqrt{3} & 3 \end{vmatrix} = a_1$$

$\therefore \quad a_1 = 4(-8-12) + 4(4\sqrt{2} + 4\sqrt{3})$

$- 4(4\sqrt{2} - 36) - 4(-8 + 12\sqrt{3})$

$- 4(24 + 4\sqrt{3}) - 4(-12\sqrt{2} - 12)$

$= 48 + 48\sqrt{2} - 48\sqrt{3} = 48(1 + \sqrt{2} - \sqrt{3})$

= Coefficient of x in $\Delta(x)$

42. $\because \quad f(x) = \begin{vmatrix} 3 & 3x & 3x^2 + 2a^2 \\ 3x & 3x^2 + 2a^2 & 3x^3 + 6a^2 x \\ 3x^2 + 2a^2 & 3x^3 + 6a^2 x & 3x^4 + 12a^2 x^2 + 2a^4 \end{vmatrix}$

Applying $C_3 \to C_3 - xC_2$ and $C_2 \to C_2 - x\, C_1$, then

$$f(x) = \begin{vmatrix} 3 & 0 & 2a^2 \\ 3x & 2a^2 & 4a^2 x \\ 3x^2 + 2a^2 & 4a^2 x & 6a^2 x^2 + 2a^4 \end{vmatrix}$$

Applying $C_3 \to C_3 - x\, C_2$, then

$$= \begin{vmatrix} 3 & 0 & 2a^2 \\ 3x & 2a^2 & 2a^2 x \\ 3x^2 + 2a^2 & 4a^2 x & 2a^2 x^2 + 2a^4 \end{vmatrix}$$

$$= 4a^4 \begin{vmatrix} 3 & 0 & 1 \\ 3x & 1 & x \\ 3x^2 + 2a^2 & 2x & x^2 + a^2 \end{vmatrix}$$

Applying $C_1 \to C_1 - 3\, C_3$, then

$$f(x) = 4a^4 \begin{vmatrix} 0 & 0 & 1 \\ \vdots & & \\ 0 & 1 & x \\ \vdots & & \\ -a^2 & \cdots \ 2x \ \cdots & x^2 + a^2 \end{vmatrix}$$

Expanding along C_1, we get

$$= 4a^4 [-a^2 (0-1)] = 4a^6$$

$\therefore \quad f'(x) = 0$

i.e. $y = f(x)$ is a straight line parallel to X-axis.

43. $\because a > b > c$ and given equations are

$$ax + by + cz = 0,$$
$$bx + cy + az = 0$$

and $\quad cx + ay + bz = 0$

For non-trivial solution

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$\Rightarrow \quad 3abc - (a^3 + b^3 + c^3) = 0$

$\therefore \quad a + b + c = 0$

If α and β be the roots of $at^2 + bt + c = 0$

$\therefore \quad \alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$

and $\quad D = b^2 - 4ac = (-a-c)^2 - 4ac = (a-c)^2 > 0$

For opposite sign $|\alpha - \beta| > 0$

$\Rightarrow \quad (\alpha - \beta)^2 > 0 \Rightarrow (\alpha + \beta)^2 - 4\alpha\beta > 0$

$\Rightarrow \quad \dfrac{b^2}{a^2} - \dfrac{4c}{a} > 0 \Rightarrow (-a-c)^2 - 4ac > 0$

$\Rightarrow \quad (a-c)^2 > 0$, true

Hence, the roots are real and have opposite sign.

44. Here, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & \lambda & 3 \end{vmatrix} = 1(9 - 2\lambda) - 1(3-2) + 1(\lambda - 3)$

$= -(\lambda - 5)$

$\Delta_1 = \begin{vmatrix} 3 & 1 & 1 \\ 6 & 3 & 2 \\ b & \lambda & 3 \end{vmatrix} = 3(9 - 2\lambda) - 1(18 - 2b) + 1(6\lambda - 3b)$

$= -(b - 9)$

$$\Delta_2 = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 6 & 2 \\ 1 & b & 3 \end{vmatrix} = 1$$

$$(18 - 2b) - 3(3 - 2) + 1(b - 6) = -(b - 9)$$

and $$\Delta_3 = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & 6 \\ 1 & \lambda & b \end{vmatrix} = 1$$

$$(3b - 6\lambda) - 1(b - 6) + 3(\lambda - 3) = (2b - 3\lambda - 3)$$

Aliter (a) for unique solution $\Delta \neq 0$

i.e. $\lambda \neq 5, b \in R$

Aliter (b) for no solution

$D = 0$ and atleast one of Δ_1, Δ_2, Δ_3 is non-zero

$\therefore \quad \lambda = 5, b \neq 9$

Aliter (c) For infinite many solution

$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$\therefore \quad \lambda = 5, b = 9$

45. For non-trivial solutions

$$\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

Expanding along C_1, we get

$\Rightarrow \quad \lambda(-\cos^2\alpha - \sin^2\alpha) - 1(-\sin\alpha\cos\alpha - \sin\alpha\cos\alpha) - 1(\sin^2\alpha - \cos^2\alpha) = 0$

$\Rightarrow \quad -\lambda + \sin 2\alpha + \cos 2\alpha = 0$

$\Rightarrow \quad \lambda = (\sin 2\alpha + \cos 2\alpha)$

$\because \quad -\sqrt{2} \leq \sin 2\alpha + \cos 2\alpha \leq \sqrt{2}$

$\therefore \quad -\sqrt{2} \leq \lambda \leq \sqrt{2}$

$\Rightarrow \quad S = [-\sqrt{2}, \sqrt{2}]$

Sol. (Q. Nos. 46 to 48)

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & \lambda \end{vmatrix} = (\lambda - 5),$$

$$\Delta_1 = \begin{vmatrix} 5 & 1 & 1 \\ 9 & 2 & 3 \\ \mu & 3 & \lambda \end{vmatrix} = (\lambda + \mu - 18),$$

$$\Delta_2 = \begin{vmatrix} 1 & 5 & 1 \\ 1 & 9 & 3 \\ 1 & \mu & \lambda \end{vmatrix} = (4\lambda - 2\mu + 6)$$

and $$\Delta_3 = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 2 & 9 \\ 1 & 3 & \mu \end{vmatrix} = (\mu - 13)$$

46. The system is smart, if

$\Delta \neq 0 \Rightarrow \lambda \neq 5$

or $\quad \Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$\Rightarrow \quad \lambda = 5$ and $\mu = 13$

47. The system is good, if

$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$\Rightarrow \quad \lambda = 5$ and $\mu = 13$

48. The system is lazy, if

$\Delta = 0$ and atleast one of $\Delta_1, \Delta_2, \Delta_3 \neq 0$

$\Rightarrow \quad \lambda = 5$ and $\mu \neq 13$

Sol. (Q. Nos. 49 to 51)

$$\because \begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ ca - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ca - b^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 \quad \text{...(i)}$$

For $a = 1$, $b = x$ and $c = x^2$

$$\begin{vmatrix} x^3 - 1 & 0 & x - x^4 \\ 0 & x - x^4 & x^3 - 1 \\ x - x^4 & x^3 - 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix}^2$$

$\therefore \quad \Delta = 5^2 = 25$

49. $\because \quad \Delta^2 = (25)^2 = 625$

Sum of digits of $\Delta^2 = 6 + 2 + 5 = 13$

50. From Eq. (i), we get

$$\begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$\Rightarrow \quad 49 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$\Rightarrow \quad q \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \pm 7$$

$\Rightarrow \quad -(a^3 + b^3 + c^3 - 3abc) = \pm 7$

$\Rightarrow \quad a^3 + b^3 + c^3 - 3abc = \mp 7$

$\therefore \quad a^3 + b^3 + c^3 - 3abc = 7 \quad [\because a + b + c > 0]$

51. $$\because aA + bB + cC = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

$= -(-3) = 3$

Sol. (Q. Nos. 52 to 54)

$\because \quad \alpha + \beta + \gamma = -2, \alpha\beta + \beta\gamma + \gamma\alpha = -1$ and $\alpha\beta\gamma = 3$

52. $$\begin{vmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{vmatrix} = -\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$$

$= (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)$

$= (\alpha + \beta + \gamma)[(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)]$

$= (-2)(4 - 9) = 10$

53. Let $x = \dfrac{\alpha - 1}{\alpha + 2} \Rightarrow \alpha = \dfrac{2x + 1}{1 - x}$

$\because \quad \alpha$ is a root of $x^3 + 2x^2 - x - 3 = 0$

$\Rightarrow \quad \alpha^3 + 2\alpha^2 - \alpha - 3 = 0$

$$\Rightarrow \left(\frac{2x+1}{1-x}\right)^3 + 2\left(\frac{2x+1}{1-x}\right)^2 - \left(\frac{2x+1}{1-x}\right) - 3 = 0$$

$$\Rightarrow \quad x^3 + 6x^2 + 21x - 1 = 0 \quad \text{...(i)}$$

Hence, $\frac{\alpha-1}{\alpha+2}, \frac{\beta-1}{\beta+2}$ and $\frac{\gamma-1}{\gamma+2}$ are the roots of Eq. (i), then

$$\frac{\alpha-1}{\alpha+2} + \frac{\beta-1}{\beta+2} + \frac{\gamma-1}{\gamma+2} = -6$$

$$\therefore \left|\frac{\alpha-1}{\alpha+2} + \frac{\beta-1}{\beta+1} + \frac{\gamma-1}{\gamma+2}\right| = \frac{6}{1} = \frac{m}{n}$$

$$\Rightarrow \quad m = 6 \text{ and } n = 1,$$

then $\begin{vmatrix} m & n^2 \\ m-n & m+n \end{vmatrix} = \begin{vmatrix} 6 & 1 \\ 5 & 7 \end{vmatrix} = 42 - 5 = 37$

54. $\because \begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} = \begin{vmatrix} a & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}^2 = (\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma)^2$

$$= (\alpha+\beta+\gamma)^2[(\alpha+\beta+\gamma)^2 - 3(\alpha\beta+\beta\gamma+\gamma\alpha)]^2$$

$$= (-2)^2[(-2)^2 + 3]^2 = 4 \times 49 = 196$$

Sol. (Q. Nos. 55 to 57)

$$\because f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$f'(x) = \begin{vmatrix} 2ax & -1 & b+1 \\ b & 1 & -1-b \\ 2ax+2b & 1 & -b \end{vmatrix}$$

Applying $R_3 \to R_3 - R_1$, then

$$f'(x) = \begin{vmatrix} 2ax & -1 & b+1 \\ b & 1 & -1-b \\ 2b & 2 & -2b-1 \end{vmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, then

$$f'(x) = \begin{vmatrix} 2ax & -1 & b+1 \\ & & \vdots \\ b & 1 & -1-b \\ & & \vdots \\ 0 \cdots & 0 \cdots & 1 \end{vmatrix}$$

$$\Rightarrow \quad f'(x) = (2ax + b)$$

$$\therefore \quad f(x) = ax^2 + bx + c$$

$$f(0) = 2 \Rightarrow c = 2 \quad \text{...(i)}$$

and $f(1) = 1 \Rightarrow a + b + 2 = 1 \Rightarrow a + b = -1$ ...(ii)

Also, $f'\left(\frac{5}{2}\right) = 0 \Rightarrow 5a + b = 0$ ...(iii)

From Eqs. (ii) and (iii), we get

$$a = \frac{1}{4} \text{ and } b = -\frac{5}{4}$$

$$\therefore \quad f(x) = \frac{x^2}{4} - \frac{5x}{4} + 2$$

55. $\because f(2) + f(3) = \left(\frac{4}{4} - \frac{10}{4} + 2\right) + \left(\frac{9}{4} - \frac{15}{4} + 2\right) = 1$

56. $\because f(x) + 1 = 0 \Rightarrow \frac{x^2}{4} - \frac{5x}{4} + 3 = 0$

$$\because \quad D = \frac{25}{16} - 3 = -\frac{23}{16} < 0$$

$\therefore$ Number of solutions = 0

57. Minimum value of $f(x) = -\frac{D}{4a} = -\frac{-\left(\frac{25}{16} - 2\right)}{1} = \frac{7}{16}$

Hence, range of $f(x)$ is $\left[\frac{7}{16}, \infty\right)$

Sol. (Q. Nos. 58 to 60)

Put $x = 1$ on both sides, we get

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \tan 1 & \sin^2 1 & \cos^2 1 \end{vmatrix} = a_0 \Rightarrow 0 = a_0$$

we observe that

$$a_1 = f'(1)$$

where $f(x) = \begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ x - \ln x & \cos(x-1) & (x-1)^2 \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix}$

$$f'(x) = \begin{vmatrix} 1 & e^{x-1} & 3(x-1)^2 \\ x - \ln x & \cos(x-1) & (x-1)^2 \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix} + \begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ 1 - \frac{1}{x} & -\sin(x-1) & 2(x-1) \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix} + \begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ x - \ln x & \cos(x-1) & (x-1)^2 \\ \sec^2 x & \sin 2x & -\sin 2x \end{vmatrix}$$

$$\Rightarrow f'(1) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \tan 1 & \sin^2 1 & \cos^2 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ \tan 1 & \sin^2 1 & \cos^2 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \sec^2 1 & \sin 2 & -\sin 2 \end{vmatrix}$$

$$= 0 + 0 + 0 = 0$$

$$\therefore \quad a_1 = 0$$

58. $\cos^{-1}(a_1) = \cos^{-1}(0) = \frac{\pi}{2}$

59. Let $P = \lim_{x \to a_0} (\sin x)^x = \lim_{x \to 0} (\sin x)^x$

$\therefore \ln p = \lim_{x \to 0} x \ln \sin x$ [form $(0 \times \infty)$]

$$= \lim_{x \to 0} \frac{\ln \sin x}{Yx} = \lim_{x \to 0} \frac{\cot x}{-Yx^2}$$ [by L' Hospital's Rule]

$$= -\lim_{x \to 0} \frac{x^2}{\tan x} = -1 \times 0 = 0$$

$\therefore \quad P = e^0 = 1$

60. Required Equation is

$$(x - a_0)(x - a_1) = 0$$

$$\Rightarrow \quad (x - 0)(x - 0) = 0$$

$$\Rightarrow \quad x^2 = 0$$

Sol. (Q. Nos. 61 to 63)

Multiplying R_1, R_2, R_3 by a, b, c respectively and then taking a, b, c common from C_1, C_2, C_3, we get

$$\Delta = \begin{vmatrix} -bc & ab + ac & ac + ab \\ ab + bc & -ac & bc + ab \\ ac + bc & bc + ac & -ab \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ and then taking $(ab + bc + ca)$ from C_2 and C_3, we get

$$\Delta = (ab + bc + ca)^2 \begin{vmatrix} -bc & 1 & 1 \\ ab + bc & -1 & 0 \\ ac + bc & 0 & -1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= (ab + bc + ca)^2 \begin{vmatrix} ab + bc + ca & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ ab + bc & & -1 & & 0 \\ \vdots & & & & \\ ac + bc & & 0 & & -1 \end{vmatrix}$$

$$= (ab + bc + ca)^3 \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = (ab + bc + ca)^3$$

Also, a, b and c are the roots of

$$x^3 - px^2 + qx - r = 0$$

$\because \quad a + b + c = p, ab + bc + ca = q, abc = r$

$\Rightarrow \quad \Delta = q^3 \quad$ ...(i)

61. $\because$ AM $\geq$ GM

$$\Rightarrow \quad \left(\frac{ab + bc + ca}{3}\right) \geq (ab \cdot bc \cdot ca)^{1/3}$$

$$\Rightarrow \quad \frac{q}{3} \geq (r^2)^{1/3} \Rightarrow q^3 \geq 27r^2$$

or $\quad \Delta \geq 27r^2$

[from Eq. (i)]

62. $\because \quad a, b$ and c are in GP.

$\therefore \quad mb^2 = ac \Rightarrow b^3 = abc = r \Rightarrow b = r^{1/3}$

and b is a root of $x^3 - px^2 + qx - r = 0$

$$\Rightarrow \quad b^3 - pb^2 + qb - r = 0$$

$$\Rightarrow \quad r - pr^{2/3} + qr^{1/3} - r = 0$$

$$\Rightarrow \quad p^3r^2 = q^3r$$

$$\therefore \quad q^3 = p^3r$$

63. $\because \quad \Delta = 27 \Rightarrow q^3 = 27$

$\therefore \quad q = 3$

or $\quad ab + bc + ca = 3$ and $a^2 + b^2 + c^2 = 2$

$$\therefore \quad \sum a^2b = a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$$

$$= (a + b + c)(ab + bc + ca) - 3abc$$

$$= 3p - 3r$$

$$= 6\sqrt{2} - 3r$$

$[\because (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)]$

$$= 3(2\sqrt{2} - r) \qquad [\therefore p^2 = 8 \Rightarrow p = 2\sqrt{2}]$$

Sol. (Q. Nos. 64 to 66)

Taking a, b, c common from R_1, R_2, R_3 respectively and then multiplying by a, b, c is C_1, C_2, C_3 respectively, we get

$$\Delta_n = \begin{vmatrix} a^2 + n & b^2 & c^2 \\ a^2 & b^2 + n & c^2 \\ a^2 & b^2 & c^2 + n \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$\Delta_n = \begin{vmatrix} n + a^2 + b^2 + c^2 & b^2 & c^2 \\ n + a^2 + b^2 + c^2 & b^2 + n & c^2 \\ n + a^2 + b^2 + c^2 & b^2 & c^2 + n \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta_n = \begin{vmatrix} n + a^2 + b^2 + c^2 & b^2 & c^2 \\ & \ddots & \\ 0 & n & 0 \\ & & \ddots \\ 0 & 0 & n \end{vmatrix}$$

$$\Delta_n = n^3 + n^2(a^2 + b^2 + c^2) \quad \text{...(i)}$$

Also, $a + b + c = \lambda$

$$3b = \lambda \qquad [\because a, b, c \text{ are in AP}]$$

$$\therefore \quad b = \frac{\lambda}{3}$$

Also, b is root of $x^3 - \lambda x^2 + 11x - 6 = 0$

$$\Rightarrow \quad b^3 - \lambda b^2 + 11b - 6 = 0$$

$$\Rightarrow \quad \frac{\lambda^3}{27} - \frac{\lambda^3}{9} + \frac{11\lambda}{3} - 6 = 0$$

$$\Rightarrow \quad 2\lambda^3 - 99\lambda + 162 = 0$$

$$\because \quad \lambda = 6$$

Then, equation becomes $x^3 - 6x^2 + 11x - 6 = 0$

$\therefore \quad x = 1, 2, 3$

Let $\quad a = 1, b = 2$ and $c = 3$

From Eq. (i), we get

$$\Delta_n = n^3 + 14n^2$$

$$\therefore \quad \sum_{n=1}^{n} \Delta_n = \frac{n(n+1)(3n^2 + 59n + 28)}{12}$$

64. $\displaystyle\sum_{r=1}^{7} \Delta_r = \frac{7 \cdot 8(147 + 59 \cdot 7 + 28)}{12} = (14)^3$

65. $\displaystyle\frac{\Delta_{2n}}{\Delta_n} = \frac{8(n+7)}{(n+14)} < 8$

$$\therefore \quad \frac{\Delta_{2n}}{\Delta_n} < 8$$

66. $\because \quad \Delta_r = r^3 + 14r^2$

$\therefore \quad \dfrac{27\Delta_r - \Delta_{3r}}{27r^2} = \dfrac{28}{3}$

$\Rightarrow \quad \sum_{r=1}^{30}\left(\dfrac{27\Delta_r - \Delta_{3r}}{27r^2}\right) = \dfrac{28}{3} \times 30 = 280$

67. We have, $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$

Applying $C_3 \to C_3 - C_1$, then

$$\begin{vmatrix} 3^2+k & 4^2 & 3 \\ 4^2+k & 5^2 & 4 \\ 5^2+k & 6^2 & 5 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 16 & 20 & 2 \end{vmatrix} = 0$$

$\Rightarrow (9+k)(18-20) - 16(14-16) + 3(140-144) = 0$

$\Rightarrow \quad -18 - 2k + 32 - 12 = 0 \Rightarrow 2k = 2$

$\therefore \quad k = 1$

Now, $\sqrt{2^k\sqrt{2^k\sqrt{2^k}}\ldots\infty} = (2^k)^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\infty}$

$$= (2^k)^{\frac{\frac{1}{2}}{1-\frac{1}{2}}} = 2^k = 2^1 = 2$$

68. We have, $\begin{vmatrix} x-1 & -6 & 2 \\ -6 & x-2 & -4 \\ 2 & -4 & x-6 \end{vmatrix} = 0$

Applying $C_2 \to C_2 + 3C_3$, then

$$\begin{vmatrix} x-1 & 0 & 2 \\ -6 & x-14 & -4 \\ 2 & 3x-22 & x-6 \end{vmatrix} = 0$$

Expanding along R_1, then

$(x-1)\{(x-14)(x-6) + 4(3x-22)\} - 0 + 2$

$\{-18x + 132 - 2x + 2\} = 0$

$\Rightarrow (x-1)(x^2 - 8x - 4) + 2(-20x + 160) = 0$

$\Rightarrow \quad x^3 - 9x^2 - 36x + 324 = 0$

$\Rightarrow \quad (x-9)(x-6)(x+6) = 0$

$\therefore \quad x = 9 \text{ or } 6 \text{ or } -6$

Now, let $\alpha = 9, \beta = 6, \gamma = -6$

$\therefore \quad \dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{1}{9} + \dfrac{1}{6} - \dfrac{1}{6} = \dfrac{1}{9}$

$\therefore \quad \left(\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}\right)^{-1} = 9$

69. We have, $\begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ x - \ln x & \cos(x-1) & (x-1)^2 \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix}$

$$= a_0 + a_1(x-1) + a_2(x-1)^2 + \ldots + a_n(x-1)^n \quad \ldots\text{(i)}$$

On putting $x = 1$ in Eq. (i), we get

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \tan 1 & \sin^2 1 & \cos^2 1 \end{vmatrix} = a_0 + 0 + 0 + \ldots$$

$\therefore \quad a_0 = 0 \quad$ [$\because R_1$ and R_2 an identical]

On differentiating Eq. (i) both sides w.r.t. x, then

$$\begin{vmatrix} 1 & e^{x-1} & 3(x-1)^2 \\ x - \ln x & \cos(x-1) & (x-1)^2 \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix} + \begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ \left(1 - \frac{1}{x}\right) & -\sin(x-1) & 2(x-1) \\ \tan x & \sin^2 x & \cos^2 x \end{vmatrix} + \begin{vmatrix} x & e^{x-1} & (x-1)^3 \\ x - \ln x & \cos(x-1) & (x-1)^2 \\ \sec^2 x & \sin 2x & -\sin 2x \end{vmatrix}$$

$$= 0 + a_1 + 2a_2(x-1) + 3a_3(x-1)^2 + \ldots + na_n(x-1)^{n-1}$$

Now, on putting $x = 1$, we get

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \tan 1 & \sin^2 1 & \cos^2 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ \tan 1 & \sin^2 1 & \cos^2 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \sec^2 1 & \sin 2 & -\sin 2 \end{vmatrix}$$

$= a_1 + 0 + 0 + \ldots + 0$

$\therefore a_1 = 0 + 0 + 0 = 0$

Hence, $(2^{a_0} + 3^{a_1})^{a_1+1} = (2^0 + 3^0)^{0+1} = (1+1)^1 = 2^1 = 2$

70. Given, $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$

$\Rightarrow \quad 1(1 - \cos^2\gamma) - \cos\alpha(\cos\alpha - \cos\beta\cos\gamma)$

$+ \cos\beta(\cos\gamma\cos\alpha - \cos\beta)$

$= 0 - \cos\alpha(0 - \cos\beta\cos\gamma) + \cos\beta(\cos\gamma\cos\alpha - 0)$

$\Rightarrow \; 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma$

$+ 2\cos\alpha\cos\beta\cos\gamma = 2\cos\alpha\cos\beta\cos\gamma$

$\Rightarrow \; 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma = 0$

Hence, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

71. $\because f(a,b,c) = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$f(a, b, c) = \begin{vmatrix} (b+c)^2 & a^2 - (b+c)^2 & a^2 - (b+c)^2 \\ b^2 & (c+a)^2 - b^2 & 0 \\ c^2 & 0 & (a+b)^2 - c^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2 & (a+b+c)(a-b-c) & (a+b+c)(a-b-c) \\ b^2 & (c+a+b)(c+a-b) & 0 \\ c^2 & 0 & (a+b+c)(a+b-c) \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Applying $R_1 \to R_1 - (R_2 + R_3)$, then

$$f(a, b, c) = (a+b+c)^2 \begin{vmatrix} 2bc & -2c & -2b \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Applying $C_2 \to C_2 + \frac{1}{b} C_1$ and $C_3 \to C_3 + \frac{1}{c} C_1$, then

$$f(a, b, c) = (a+b+c)^2 \begin{vmatrix} 2bc & 0 & 0 \\ b^2 & c+a & \frac{b^2}{c} \\ c^2 & \frac{c^2}{b} & a+b \end{vmatrix}$$

Expanding along R_1, then

$$f(a, b, c) = (a+b+c)^2 [2bc \{(c+a)(a+b) - bc\}]$$
$$= (a+b+c)^2 \{2bc(ac + bc + a^2 + ab - bc)\}$$
$$= 2bc(a+b+c)^2 a(a+b+c)$$
$$= 2abc(a+b+c)^3$$

We get, greatest integer $n \in N$ such that $(a+b+c)^n$ divides $f(a, b, c)$ is 3.

72. The system of equations has a non-trivial solution, then

$$\begin{vmatrix} 1 & -\sin\theta & -\cos\theta \\ -\cos\theta & 1 & -1 \\ -\sin\theta & -1 & 1 \end{vmatrix} = 0$$

Applying $C_3 \to C_3 + C_2$, then

$$\begin{vmatrix} 1 & -\sin\theta & -\sin\theta - \cos\theta \\ -\cos\theta & 1 & 0 \\ -\sin\theta & -1 & 0 \end{vmatrix} = 0$$

Expanding along C_3, then

$$(-\sin\theta - \cos\theta)(\cos\theta + \sin\theta) = 0$$
$$\Rightarrow \quad (\sin\theta + \cos\theta)^2 = 0$$
$$\Rightarrow \quad \sin\theta + \cos\theta = 0$$
$$\Rightarrow \quad \sin\theta = -\cos\theta$$
$$\therefore \quad \tan\theta = -1$$
$$\Rightarrow \quad \theta = \frac{3\pi}{4} \qquad [\because \theta \in [0, \pi]]$$

Hence, $\frac{8\theta}{\pi} = 6$

73. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ and $R_4 \to R_4 - R_1$, then

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{vmatrix}$$

Expanding along C_1, then $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 9 \\ 3 & 9 & 19 \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$, then $= \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 3 & 10 \end{vmatrix}$

Expanding along C_1, we get $\Delta = 1 \begin{vmatrix} 1 & 3 \\ 3 & 10 \end{vmatrix} = 10 - 9 = 1$

74. Let $\Delta = \begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$

Taking a, b, c, d common from R_1, R_2, R_3 and R_4 respectively, then

$$\Delta = abcd \begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3 + R_4$ and taking $\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$ common, we get

$$\Delta = abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$ and $C_4 \to C_4 - C_1$, then

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{b} & 1 & 0 & 0 \\ \frac{1}{c} & 0 & 1 & 0 \\ \frac{1}{d} & 0 & 0 & 1 \end{vmatrix}$$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)1\cdot 1\cdot 1$$

$$= abcd + (bcd + acd + abd + abc) = \sigma_4 + \sigma_3$$

$$= \frac{16}{1} + \left(-\frac{8}{1}\right) = 8$$

75. Given, $\begin{vmatrix} 1+a & 1 & 1 \\ 1+b & 1+2b & 1 \\ 1+c & 1+c & 1+3c \end{vmatrix} = 0$

Taking a, b, c common from R_1, R_2 and R_3 respectively, then

$$abc\begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ 1+\frac{1}{b} & 2+\frac{1}{b} & \frac{1}{b} \\ 1+\frac{1}{c} & 1+\frac{1}{c} & 3+\frac{1}{c} \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_2 + R_3$ and taking $\left(3+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common, we get

$$abc\left(3+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ 1+\frac{1}{b} & 2+\frac{1}{b} & \frac{1}{b} \\ 1+\frac{1}{c} & 1+\frac{1}{c} & 3+\frac{1}{c} \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$abc\left(3+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ 1+\frac{1}{2} & 1 & -1 \\ 1+\frac{1}{c} & 0 & 2 \end{vmatrix} = 0$$

Expanding along R_1, we get

$$2\,abc\left(3+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 0$$

$\because \quad a \neq 0, b \neq 0, c \neq 0$

$\therefore \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = -3$ or $|a^{-1}+b^{-1}+c^{-1}| = 3$

76. Given equations

$$ax + hy + g = 0, \quad \text{...(i)}$$

$$hx + by + f = 0 \quad \text{...(ii)}$$

and $ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda = 0$ ...(iii)

Eq. (iii), can be written as

$$x(ax + hy + g) + y(hx + by + f) + gx + fy + c + \lambda = 0$$

$\Rightarrow \quad x \cdot 0 + y \cdot 0 + gx + fy + c + \lambda = 0$ [from Eqs. (i) and (ii)]

$\Rightarrow \quad gx + fy + c + \lambda = 0$ ...(iv)

According to the question Eqs. (i), (ii) and (iii) has unique solution. So, Eqs. (i), (ii) and (iv) has unique solution,

then $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c+\lambda \end{vmatrix} = 0$

$\Rightarrow a(bc + b\lambda - f^2) - h(ch + h\lambda - fg) + g(hf - bg)$

$\Rightarrow (abc + 2fgh - af^2 - bg^2 - ch^2) = \lambda(h^2 - ab)$

or $\quad \dfrac{abc + 2fgh - af^2 - bg^2 - ch^2}{h^2 - ab} = \lambda$

According to the question, $\lambda = 8$

77. (A) $\to$ (p, r); (B) $\to$ (p, r); (C) $\to$ (p, q, s, t)

(A) Using $a^2 + b^2 + c^2 = 0$, we get

$$\Delta = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$$

[taking a, b, c common from C_1, C_2, C_3 respectively]

Applying $C_2 \to C_2 + C_1$ and $C_3 \to C_3 + C_1$, then

$$\Delta = abc\begin{vmatrix} -a & 0 & 0 \\ b & 0 & 2b \\ c & 2c & 0 \end{vmatrix}$$

$$= (abc)(-a)(-4bc) = 4a^2b^2c^2$$

$\therefore \quad \lambda = 4$

(B) Let $\quad \Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 5a+2b & 7a+5b+2c \\ 3a & 7a+3b & 9a+7b+3c \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, then

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & 3a & 5a+3b \\ 0 & 4a & 6a+4b \end{vmatrix}$$

$$= a\begin{vmatrix} 3a & 5a+3b \\ 4a & 6a+4b \end{vmatrix}$$

$$= a(18a^2 + 12ab - 20a^2 - 12ab)$$

$= -2a^3 = -1024$ [given]

$\Rightarrow \quad a^3 = 512 = 8^3$

$\therefore \quad a = 8$

(C) Let $\Delta(x) = \begin{vmatrix} x-1 & 2x^2-5 & x^3-1 \\ 2x^2+5 & 2x+2 & x^3+3 \\ x^3-1 & x+1 & 3x^2-2 \end{vmatrix}$ ...(i)

According to the question,

$\Delta(x) = (x^2-1)\,P(x) + ax + b$

$\therefore\ \Delta(1) = a + b$ and $\Delta(-1) = -a + b$ ...(ii)

From Eq. (i), we get

$$\Delta(1) = \begin{vmatrix} 0 & \cdots & 3 & \cdots & 0 \\ & & \vdots & & \\ 7 & & 4 & & 4 \\ & & \vdots & & \\ 0 & & 2 & & 1 \end{vmatrix} = 3(7-0) = 21$$

and $$\Delta(-1) = \begin{vmatrix} -2 & \cdots & -3 & \cdots & 2 \\ & & \vdots & & \\ 7 & & 0 & & 2 \\ & & \vdots & & \\ -2 & & 0 & & 1 \end{vmatrix} = 3(7+4) = 33$$

From Eq. (ii), $a + b = 21$ and $-a + b = 33$,

we get $a = -6,\ b = 27$

$\therefore\ 4a + 2b = -24 + 54 = 30$

78. (A) $\to$ (p, s, t); (B) $\to$ (r, t); (C) $\to$ (p, q)

(A) $\because\ \Delta = \begin{vmatrix} 1 & 1 & 1 \\ f_1(x_1) & f_1(x_2) & f_1(x_3) \\ f_2(x_1) & f_2(x_2) & f_2(x_3) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ f_1(2) & f_1(3) & f_1(5) \\ f_2(2) & f_2(3) & f_2(5) \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2+a_1 & 3+a_1 & 5+a_1 \\ 4+2b_1+b_2 & 9+3b_1+b_2 & 25+5b_1+b_2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$\Delta = \begin{vmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ 2+a_1 & & 1 & & 3 \\ \vdots & & & & \\ 4+2b_1+b_2 & & 5+b_1 & & 21+3b \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 \\ 5+b_1 & 21+3b_1 \end{vmatrix}$$

$$= 21 + 3b_1 - 15 - 3b_1 = 6$$

(B) $\because\ f(x) = \begin{vmatrix} 1 & b_1 & a_1 \\ 1 & b_1 & 2a_1-x \\ 1 & 2b_1-x & a_1 \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$f(x) = \begin{vmatrix} 1 & \cdots & b_1 & \cdots & a_1 \\ \vdots & & & & \\ 0 & & 0 & & a_1-x \\ \vdots & & & & \\ 0 & & b_1-x & & 0 \end{vmatrix}$$

$$= -(a_1-x)(b_1-x) = -x^2 + (a_1+b_1)x - a_1\,b_1$$

Minimum value of $f(x) = -\dfrac{D}{4a} = -\dfrac{(a_1+b_1)^2 - 4a_1b_1}{4(-1)}$

$$= \frac{(a_1-b_1)^2}{4} = \frac{36}{4} = 9$$

(C) $\because\ f(x)$ is a polynomial of degree atmost 6 in x.

If $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$

$\Rightarrow \lambda = a_1 = f'(0)$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 8 \end{vmatrix} + \begin{vmatrix} -2 & -2 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & 8 \end{vmatrix} + \begin{vmatrix} -2 & 1 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & 12 \end{vmatrix}$$

$$= -8 - 12 + 18 = -2$$

$\therefore\qquad |\lambda| = 2$

79. (A) $\to$ (r); (B) $\to$ (r, t); (C) $\to$ (p, q, s)

(A) Let $f(x) = \begin{vmatrix} x^2+3x & x-1 & x+3 \\ x^2+1 & 2+3x & x-3 \\ x^2-3 & x+4 & 3x \end{vmatrix}$

$f(x) = ax^4 + bx^3 + cx^2 + dx + e$ ...(i)

$\therefore\quad e = f(0) = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 2 & -3 \\ -3 & 4 & 0 \end{vmatrix} = 0 + 1(0-9) + 3(4+6) = 21$

Dividing both sides of Eq. (i) by x^4 i.e., C_1 by x^2, C_2 by x and C_3 by x and then taking $\lim\limits_{x\to\infty}$, we get

$$a = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 1(8) - 1(2) + 1(-2) = 4$$

Hence, $e + a = 25$

(B) Let $f(x) = \begin{vmatrix} x-1 & 5x & 7 \\ x^2-1 & x-1 & 8 \\ 2x & 3x & 0 \end{vmatrix} = ax^3 + bx^2 + cx + d$ ...(i)

$\therefore\quad c = f'(0) = \begin{vmatrix} 1 & 0 & 7 \\ 0 & -1 & 8 \\ 2 & 0 & 0 \end{vmatrix} + \begin{vmatrix} -1 & 5 & 7 \\ -1 & 1 & 8 \\ 0 & 3 & 0 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$

$$= 2(0+7) - 3(-8+7) + 0 = 17$$

Dividing both sides of Eq. (i) by x^3 i.e., c_1 by x^2, c_2 by x and taking $\lim\limits_{x\to\infty}$, we get

$$a = \begin{vmatrix} 0 & & 5 & & 7 \\ \vdots & & & & \\ 1 & \cdots & 1 & \cdots & 8 \\ \vdots & & & & \\ 0 & & 3 & & 0 \end{vmatrix} = -1(0-21) = 21$$

Hence, $c + a - 3 = 35$

(C) Let $g(x) = \begin{vmatrix} x^3+4x & x+3 & x-2 \\ x-2 & 5x & x-1 \\ x-3 & x+2 & 4x \end{vmatrix}$

$$= ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

$$\therefore \quad f = g(0) = \begin{vmatrix} 0 & 3 & -2 \\ -2 & 0 & -1 \\ -3 & 2 & 0 \end{vmatrix} = 0 - 3(0-3) - 2(-4-0) = 17$$

$$\text{and } e = g'(0) = \begin{vmatrix} 4 & 3 & -2 \\ 1 & 0 & -1 \\ 1 & 2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -2 & 5 & -1 \\ -3 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 3 & 1 \\ -2 & 0 & 1 \\ -3 & 2 & 4 \end{vmatrix}$$

$$= 1 - 23 + 11 = -11$$

Hence, $f + e = 17 - 11 = 6$

80. (A) → (p, q, r); (B) → (p, q, r, s, t); (C) → (p, q, r, s, t)

(A) Taking common a, b, c from R_1, R_2 and R_3 respectively and then multiplying in C_1, C_2 and C_3 by a, b, c respectively, then

$$\Delta = \begin{vmatrix} a + (b^2 + c^2)d & b^2(1-d) & c^2(1-d) \\ a^2(1-d) & b^2 + (c^2 + a^2)d & c^2(1-d) \\ a^2(1-d) & b^2(1-d) & c^2 + (a^2 + b^2)d \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$\Delta = \begin{vmatrix} 1 & b^2(1-d) & c^2(1-d) \\ 1 & b^2 + (c^2 + a^2)d & c^2(1-d) \\ 1 & b^2(1-d) & c^2 + (a^2 + b^2)d \end{vmatrix}$$

$[\because a^2 + b^2 + c^2 = 1]$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\Delta = \begin{vmatrix} 1 & b^2(1-d) & c^2(1-d) \\ 0 & d & 0 \\ 0 & 0 & d \end{vmatrix} = d^2$$

$[\because a^2 + b^2 + c^2 = 1]$

(B) Multiplying C_1 by a, C_2 by b and C_3 by c, then

$$\Delta = \frac{1}{abc}\begin{vmatrix} \frac{a}{c} & \frac{b}{c} & -\frac{(a+b)}{c} \\ -\frac{(b+c)}{a} & \frac{b}{a} & \frac{c}{a} \\ -\frac{bd(b+c)}{ac} & \frac{bd(a+2b+c)}{ac} & -\frac{(a+b)bd}{ac} \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$\Delta = \frac{1}{abc}\begin{vmatrix} 0 & \frac{b}{c} & -\frac{(a+b)}{c} \\ 0 & \frac{b}{a} & \frac{c}{a} \\ 0 & \frac{bd(a+2b+c)}{ac} & -\frac{(a+b)bd}{ac} \end{vmatrix} = 0$$

(C) Applying $C_3 \to C_3 - \cos d\, C_1 - \sin d\, C_2$, then

$$\Delta = \begin{vmatrix} \sin a & \cos a & 0 \\ \sin b & \cos b & 0 \\ \sin c & \cos c & 0 \end{vmatrix} = 0$$

81. (A) → (p, r); (B) → (p, q, r, t); (C) → (p, r, s)

(A) Possible values are −2 , −1, 0, 1, 2 and numbering determinant $= 3^4 = 81$

i.e., $\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1, \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = 0, \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1, \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2,$

$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2 \quad \therefore n = 5 \Rightarrow (n-1)^2 = 16$

(B) There are only three determinants of second order with negative value

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$$

Number of possible determinants with elements 0 and 1 are $2^4 = 16$. Therefore, number of determinants with non-negative values is 13.

$\therefore \quad n = 13$

$\Rightarrow \quad (n-1) = 12$

(C) There are only four determinants of second order with negative value

$$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix}$$

$\therefore \quad n = 4 \Rightarrow n(n+1) = 20$

82. **Statement-1**

$$\Delta(r) = \begin{vmatrix} r & r+1 \\ r+3 & r+4 \end{vmatrix} = r(r+4) - (r+1)(r+3)$$

$$= (r^2 + 4r) - (r^2 + 4r + 3) = -3$$

$$\therefore \quad \sum_{r=1}^{n} \Delta(r) = \sum_{r=1}^{n} (-3)$$

$$= \underbrace{(-3) + (-3) + (-3) + \ldots + (-3)}_{n \text{ times}} = -3n$$

⇒ Statement-1 is true.

Statement-2

$$\Delta(r) = \begin{vmatrix} f_1(r) & f_2(r) \\ f_3(r) & f_4(r) \end{vmatrix} = f_1(r)\, f_4(r) - f_2(r)\, f_3(r)$$

$$\therefore \quad \sum_{r=1}^{n} \Delta(r) = \sum_{r=1}^{n} [f_1(r)\, f_4(r) - f_2(r)\, f_3(r)]$$

$$= \sum_{r=1}^{n} [f_1(r)\, f_4(r)] - \sum_{r=1}^{n} [f_2(r)\, f_3(r)] \qquad \text{...(i)}$$

$$\text{and } \begin{vmatrix} \sum_{r=1}^{n} f_1(r) & \sum_{r=1}^{n} f_2(r) \\ \sum_{r=1}^{n} f_3(r) & \sum_{r=1}^{n} f_4(r) \end{vmatrix}$$

$$= \left(\sum_{r=1}^{n} f_1(r)\right)\left(\sum_{r=1}^{n} f_4(r)\right) - \left(\sum_{r=1}^{n} f_2(r)\right)\left(\sum_{r=1}^{n} f_3(r)\right) \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get $\sum_{r=1}^{n} \Delta(r) \neq \begin{vmatrix} \sum_{r=1}^{n} f_1(r) & \sum_{r=1}^{n} f_2(r) \\ \sum_{r=1}^{n} f_3(r) & \sum_{r=1}^{n} f_4(r) \end{vmatrix}$

∴ Statement-2 is false.

Hence, Statement-1 is true and Statement-2 is false.

83. $\because \quad \Delta = \begin{vmatrix} a_1 + b_1x^2 & a_1x^2 + b_1 & c_1 \\ a_2 + b_2x^2 & a_2x^2 + b_2 & c_2 \\ a_3 + b_3x^2 & a_3x^2 + b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} 1 & x^2 & 0 \\ x^2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{...(i)}$$

Statement-1 If $\Delta = 0$, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} 1 & x^2 & 0 \\ x^2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & x^2 & 0 \\ x^2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 \Rightarrow 1 - x^4 = 0 \text{ or } x^4 = 1 \quad [\because x^2 \neq -1]$$

Statement-1 is true

Now, if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, then

$\Delta = 0$ [from Eq. (i)]

Statement-2 is also true.

Hence, both the statements are true but Statement-2 is not a correct explanation of Statement-1.

84. Statement-2 is always true for Statement-1

$$\cos\left(x + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - x\right)\right) = \sin\left(\frac{\pi}{4} - x\right)$$
$$= -\sin\left(x - \frac{\pi}{4}\right)$$
$$\cot\left(\frac{\pi}{4} + x\right) = \cot\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - x\right)\right) = \tan\left(\frac{\pi}{4} - x\right)$$
$$= -\tan\left(x - \frac{\pi}{4}\right)$$

Also, $\ln\left(\frac{y}{x}\right) = -\ln\left(\frac{x}{y}\right)$

Therefore, determinant given in Statement-1 is skew-symmetric and hence its value is zero. Hence, both statements are true and Statement-2 is a correct explanation of Statement-1.

85. $\begin{vmatrix} (1+x)^{11} & (1+x)^{12} & (1+x)^{13} \\ (1+x)^{21} & (1+x)^{22} & (1+x)^{23} \\ (1+x)^{31} & (1+x)^{32} & (1+x)^{33} \end{vmatrix} = A_0 + A_1x + A_2x^2 + \ldots$ [let]

On differentiating both sides w.r.t. x and then put $x = 0$, we get

$$\begin{vmatrix} 11 & 12 & 13 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 21 & 22 & 23 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 31 & 32 & 33 \end{vmatrix} = 0 + A_1 + 0 + 0 + \ldots$$

$\Rightarrow \quad 0 + 0 + 0 = A_1 \quad \therefore \quad A_1 = 0$

$\therefore$ Coefficient of x in $f(x) = 0$

Both statements are true, Statement-2 is a correct explanation of Statement-1.

86. Here, $\Delta = \begin{vmatrix} 2 & 3 \\ b & 4 \end{vmatrix} = 8 - 3b,$

$$\Delta_1 = \begin{vmatrix} a & 3 \\ 5 & 4 \end{vmatrix} = 4a - 15$$

and $\Delta_2 = \begin{vmatrix} 2 & a \\ b & 5 \end{vmatrix} = 10 - ab$

For infinite solutions, $\Delta = \Delta_1 = \Delta_2 = 0$

We get, $a = \frac{15}{4}$ and $b = \frac{8}{3}$

$\therefore$ Statement-1 is true and if lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel, then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

$\therefore$ Statement-2 is true, but in Statement-1

$$\frac{2}{b} = \frac{3}{4} = \frac{a}{5}$$
$$\Rightarrow \quad \frac{3}{4} = \frac{3}{4} = \frac{3}{4}$$

[both equation are identical]

$\therefore$ Statement-2 is not a correct explanation for Statement-1.

87. $\because \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = 1(0 - 48) - 2(0 - 42) + 3(32 - 35)$

$$= -48 + 84 - 9$$
$$= 84 - 57 = 27 \neq 0$$

$\therefore$ Statement-1 is true.

Also, in given determinant neither two rows or columns are identical, Statement-2 is true, Statement-2 is not a correct explanation for Statement-1.

88. $\because$ $A88, 6B8, 86C$ are divisible by 72, then $A88 = 72\lambda$, $6B8 = 72\mu$ and $86C = 72\upsilon$, where $\lambda, \mu, \upsilon \in N$.

$$\because \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix}$$

Applying $R_3 \to R_3 + 10R_2 + 100R_1$, then

$$\begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 100A + 80 + 8 & 600 + 10B + 8 & 800 + 60 + c \end{vmatrix}$$

$$= \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 72\lambda & 72\mu & 72\upsilon \end{vmatrix} = 72 \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ \lambda & \mu & \upsilon \end{vmatrix} \quad \text{...(i)}$$

Now, $A88$ is also divisible by 9, then

$A + 8 + 8 = A + 16$ is divisible by 9

$\therefore \quad A = 2$

and $6B8$ is also divisible by 9, then $6 + B + 8 = B + 14$ is divisible by 9

$\therefore \quad B = 4$

From Eq. (i), we get

$$= 72 \begin{vmatrix} 2 & 6 & 8 \\ 8 & 2 & 6 \\ \lambda & \mu & \upsilon \end{vmatrix} = 288 \begin{vmatrix} 1 & 3 & 4 \\ 4 & 1 & 3 \\ \lambda & \mu & \upsilon \end{vmatrix} = 288 \qquad \text{[integer]}$$

Statement-1 is true and Statement-2 is false.

89. Let $\Delta = \begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$

Applying $R_1 \to R_1 - (R_2 + R_3)$, then

$\therefore \quad \Delta = \begin{vmatrix} 0 & -2a & -2a \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$

Taking $(-2a)$ common from R_1, then

$$\Delta = (-2a) \begin{vmatrix} 0 & 1 & 1 \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$$

Applying $C_2 \to C_2 - C_3$, then

$\therefore \quad \Delta = (-2a) \begin{vmatrix} 0 & 0 & 1 \\ c & c & a \\ b & -b & a+b \end{vmatrix}$

Expanding along R_1, we get

$$\Delta = (-2a) \cdot 1 \cdot \begin{vmatrix} c & c \\ b & -b \end{vmatrix} = (-2a)(-2bc)$$

Hence, $\Delta = 4abc$

90. Let $\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Since, the answer is $(a+b+c)^3$, we shall try to get $(a+b+c)$.

Applying $R_1 \to R_1 + R_2 + R_3$, then

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking $(a+b+c)$ common from R_1, we get

$$\Delta = (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_1 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$\therefore \quad \Delta = (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -a-b-c & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$

[by property, since all elements above leading diagonal are zero]

$$= (a+b+c) \cdot 1 \cdot (-a-b-c) \cdot (-c-a-b)$$

Hence, $\Delta = (a+b+c)^3$

91. Let $\Delta = \begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$

$$= \begin{vmatrix} \sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15} & 5 & \sqrt{10} \\ 3 & \sqrt{15} & 5 \end{vmatrix} + \begin{vmatrix} \sqrt{13} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{26} & 5 & \sqrt{10} \\ \sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$$

Taking common from 1st determinant $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{5}$ from C_1, C_2 and C_3 respectively and taking common from 2nd determinant $\sqrt{13}$, $\sqrt{5}$ and $\sqrt{5}$ from C_1, C_3 and C_3 respectively, we get

$$= \sqrt{3} \times \sqrt{5} \times \sqrt{5} \begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix} + \sqrt{13} \times \sqrt{5} \times \sqrt{5} \begin{vmatrix} 1 & 2 & 1 \\ \sqrt{2} & \sqrt{5} & \sqrt{2} \\ \sqrt{5} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

$$= \sqrt{3} \times 5 \begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix} + 0 \qquad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

$$= 5\sqrt{3} \begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$,

then $\Delta = 5\sqrt{3} \begin{vmatrix} 1 & 1 & 1 \\ \sqrt{5} & 0 & \sqrt{2} \\ \sqrt{3} & 0 & \sqrt{5} \end{vmatrix}$

Expanding along C_2, then

$$\Delta = 5\sqrt{3} \cdot (-1) \begin{vmatrix} \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{5} \end{vmatrix} = -5\sqrt{3}(5 - \sqrt{6})$$

$$= -25\sqrt{3} + 15\sqrt{2}$$

$$= 15\sqrt{2} - 25\sqrt{3}$$

92. Given that, a, b and c are p th, q th and r th terms of HP $\Rightarrow \frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are p th, q th and r th terms of an AP. Let A and D are the first term and common difference of AP, then

$$\frac{1}{a} = A + (p-1)D \qquad \text{...(i)}$$

$$\frac{1}{b} = A + (q-1)D \qquad \text{...(ii)}$$

$$\frac{1}{c} = A + (r-1)D \qquad \text{...(iii)}$$

Now, given determinant is

$$\Delta = \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = abc \begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

On substituting the values of $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ from Eqs. (i), (ii) and (iii) in Δ, then

$$\Delta = abc\begin{vmatrix} A+(p-1)D & A+(q-1)D & A+(r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - (A-D)R_3 - DR_2$, then

$$\Delta = abc\begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

93. Let $$z = \begin{vmatrix} -5 & 3+5i & \frac{3}{2}-4i \\ 3-5i & 8 & 4+5i \\ \frac{3}{2}+4i & 4-5i & 9 \end{vmatrix}$$

Then, $$\bar{z} = \begin{vmatrix} -5 & 3-5i & \frac{3}{2}+4i \\ 3+5i & 8 & 4-5i \\ \frac{3}{2}-4i & 4+5i & 9 \end{vmatrix} \quad \text{[i.e., conjugate of } z]$$

$$= \begin{vmatrix} -5 & 3+5i & \frac{3}{2}-4i \\ 3-5i & 8 & 4+5i \\ \frac{3}{2}+4i & 4-5i & 9 \end{vmatrix}$$

[interchanging rows into columns]

$\Rightarrow \quad \bar{z} = z$

Hence, z is purely real.

94. $$\text{LHS} = \begin{vmatrix} ah+bg & g & ab+ch \\ bf+ba & f & hb+bc \\ af+bc & c & bg+fc \end{vmatrix}$$

$$= b\begin{vmatrix} ah+bg & g & a \\ bf+ba & f & h \\ af+bc & c & g \end{vmatrix} + c\begin{vmatrix} ah+bg & g & h \\ bf+ba & f & b \\ af+bc & c & f \end{vmatrix}$$

In second determinant, applying $C_1 \to C_1 - bC_2 - aC_3$, then

$$= \begin{vmatrix} ah+bg & bg & a \\ bf+ba & bf & h \\ af+bc & bc & g \end{vmatrix} + c\begin{vmatrix} 0 & g & h \\ 0 & f & b \\ 0 & c & f \end{vmatrix}$$

In first determinant, applying $C_2 \to C_2 - C_1$, then

$$= \begin{vmatrix} ah+bg & -ah & a \\ bf+ba & -ba & h \\ af+bc & -af & q \end{vmatrix} + 0 = a\begin{vmatrix} ah+bg & a & h \\ bf+ba & h & b \\ af+bc & g & f \end{vmatrix} = \text{RHS}$$

95. Let $$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B-\sin A & \sin C-\sin A \\ \sin A+\sin^2 A & (\sin B-\sin A)(\sin B+\sin A+1) & (\sin C-\sin A)(\sin C+\sin A+1) \end{vmatrix}$$

Expanding along R_1, then

$$\Delta = \begin{vmatrix} \sin B-\sin A & \sin C-\sin A \\ (\sin B-\sin A)(\sin B+\sin A+1) & (\sin C-\sin A)(\sin C+\sin A+1) \end{vmatrix}$$

$$= (\sin B-\sin A)(\sin C-\sin A)\begin{vmatrix} 1 & 1 \\ \sin B+\sin A+1 & \sin C+\sin A+1 \end{vmatrix}$$

$$= (\sin B-\sin A)(\sin C-\sin A)(\sin C-\sin B)$$

But, given $\Delta = 0$

$\therefore \quad (\sin B-\sin A)(\sin C-\sin A)(\sin C-\sin B) = 0$

$\therefore \quad \sin B-\sin A = 0$ or $\sin C-\sin A = 0$

or $\quad \sin C-\sin B = 0$

$\Rightarrow \quad \sin B = \sin A$ or $\sin C = \sin A$ or $\sin C = \sin B$

$B = A$ or $C = A$ or $C = B$

In all the three cases, we will have an isosceles triangle.

96. Let $$\Delta = \begin{vmatrix} \beta\gamma & \beta\gamma' + \beta'\gamma & \beta'\gamma' \\ \gamma'\alpha & \gamma'\alpha' + \gamma'\alpha & \gamma'\alpha' \\ \alpha\beta & \alpha\beta' + \alpha'\beta & \alpha'\beta' \end{vmatrix}$$

Taking $\beta'\gamma'$, $\gamma'\alpha'$ and $\alpha'\beta'$ common from R_1, R_2 and R_3 respectively, then

$$\Delta = (\beta'\gamma')(\gamma'\alpha')(\alpha'\beta')\begin{vmatrix} \frac{\beta}{\beta'}\frac{\gamma}{\gamma'} & \frac{\beta}{\beta'}+\frac{\gamma}{\gamma'} & 1 \\ \frac{\gamma}{\gamma'}\frac{\alpha}{\alpha'} & \frac{\gamma}{\gamma'}+\frac{\alpha}{\alpha'} & 1 \\ \frac{\alpha}{\alpha'}\frac{\beta}{\beta'} & \frac{\alpha}{\alpha'}+\frac{\beta}{\beta'} & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

Then, $$\Delta = (\alpha'\beta'\gamma')^2\begin{vmatrix} \frac{\beta}{\beta'}\frac{\gamma}{\gamma'} & \frac{\beta}{\beta'}+\frac{\gamma}{\gamma'} & 1 \\ \frac{\gamma}{\gamma'}\left(\frac{\alpha}{\alpha'}-\frac{\beta}{\beta'}\right) & \left(\frac{\alpha}{\alpha'}-\frac{\beta}{\beta'}\right) & 0 \\ \frac{\beta}{\beta'}\left(\frac{\alpha}{\alpha'}-\frac{\gamma}{\gamma'}\right) & \left(\frac{\alpha}{\alpha'}-\frac{\gamma}{\gamma'}\right) & 0 \end{vmatrix}$$

$$= (\alpha'\beta'\gamma')^2\left(\frac{\alpha}{\alpha'}-\frac{\beta}{\beta'}\right)\left(\frac{\alpha}{\alpha'}-\frac{\gamma}{\gamma'}\right)\begin{vmatrix} \frac{\beta}{\beta'}\frac{\gamma}{\gamma'} & \frac{\beta}{\beta'}+\frac{\gamma}{\gamma'} & 1 \\ \frac{\gamma}{\gamma'} & 1 & 0 \\ \frac{\beta}{\beta'} & 1 & 0 \end{vmatrix}$$

Expanding along C_3, then

$$\Delta = (\alpha'\beta'\gamma')^2\left(\frac{\alpha}{\alpha'}-\frac{\beta}{\beta'}\right)\left(\frac{\alpha}{\alpha'}-\frac{\gamma}{\gamma'}\right)\left(\frac{\gamma}{\gamma'}-\frac{\beta}{\beta'}\right)$$

$$= (\alpha'\beta'\gamma')^2\left(\frac{\alpha}{\alpha'}-\frac{\beta}{\beta'}\right)\left(\frac{\beta}{\beta'}-\frac{\gamma}{\gamma'}\right)\left(\frac{\gamma}{\gamma'}-\frac{\alpha}{\alpha'}\right)$$

$$= (\alpha'\beta'\gamma')^2\frac{(\alpha\beta'-\alpha'\beta)(\beta\gamma'-\beta'\gamma)(\gamma\alpha'-\gamma'\alpha)}{(\alpha'\beta'\gamma')^2}$$

Hence, $\Delta = (\alpha\beta'-\alpha'\beta)(\beta\gamma'-\beta'\gamma)(\gamma\alpha'-\gamma'\alpha)$

97. Since, $y = \frac{u}{v}$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{vu' - uv'}{v^2}$$

$$\Rightarrow \quad v^2\frac{dy}{dx} = vu' - uv' \qquad \text{...(i)}$$

On differentiating both sides w.r.t. x, we get

$$v^2\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 2vv' = (vu'' + u'v') - (uv'' + v'u')$$

$$\Rightarrow \quad v^2\frac{d^2y}{dx^2} + 2vv'\frac{dy}{dx} = vu'' - uv''$$

On multiplying both sides by v, then

$$v^3\frac{d^2y}{dx^2} + 2v'\left(v^2\frac{dy}{dx}\right) = v^2u'' - uvv''$$

$$\Rightarrow \quad v^3\frac{d^2y}{dx^2} + 2v'(vu' - uv') = v^2u'' - uvv'' \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad v^3\frac{d^2y}{dx^2} = 2uv^2 - uvv'' - 2vu'v' + v^2u'' \qquad \text{...(ii)}$$

and $\begin{vmatrix} v & v & 0 \\ v' & v' & v \\ v'' & v'' & 2v'' \end{vmatrix} = u(2v'^2 - vu'') - v(2u'v' - u''v)$

$$= 2uv'^2 - 4vv'' - 2vu'v' + v^2u'' \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$v^3\frac{d^2y}{dx^2} = \begin{vmatrix} u & v & 0 \\ u' & v' & v \\ u' & v'' & 2v' \end{vmatrix}$$

98. Here, we have to prove that $\Delta(x)$ is independent of x. So, it is sufficient to prove that $\Delta'(x) = 0$

Now, $\Delta(x) = \begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & a + x\sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & b + x\sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & c + x\sin\gamma \end{vmatrix}$

On differentiating w.r.t. x, we get

$$\Delta'(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\alpha) & a + x\sin\alpha \\ \cos(x+\beta) & \cos(x+\beta) & b + x\sin\beta \\ \cos(x+\gamma) & \cos(x+\gamma) & c + x\sin\gamma \end{vmatrix}$$

$$+ \begin{vmatrix} \sin(x+\alpha) & -\sin(x+\alpha) & a + x\sin\alpha \\ \sin(x+\beta) & -\sin(x+\beta) & b + x\sin\beta \\ \sin(x+\gamma) & -\sin(x+\gamma) & c + x\sin\gamma \end{vmatrix}$$

$$+ \begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & \sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & \sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & \sin\gamma \end{vmatrix}$$

$$= 0 - 0 + \begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & \sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & \sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & \sin\gamma \end{vmatrix}$$

$$= \begin{vmatrix} \sin(x+\alpha) & \cos(x+\alpha) & \sin\alpha \\ \sin(x+\beta) & \cos(x+\beta) & \sin\beta \\ \sin(x+\gamma) & \cos(x+\gamma) & \sin\gamma \end{vmatrix}$$

Applying $C_1 \to C_1 - (\cos x)C_3$ and $C_2 \to C_2 + (\sin x)C_3$, we get

$$\Delta'(x) = \begin{vmatrix} \sin x\cos\alpha & \cos x\cos\alpha & \sin\alpha \\ \sin x\cos\beta & \cos x\cos\beta & \sin\beta \\ \sin x\cos\gamma & \cos x\cos\gamma & \sin\gamma \end{vmatrix}$$

$$= \sin x\cdot\cos x \begin{vmatrix} \cos\alpha & \cos\alpha & \sin\alpha \\ \cos\beta & \cos\beta & \sin\beta \\ \cos\gamma & \cos\gamma & \sin\gamma \end{vmatrix}$$

$$= \sin x\cdot\cos x \times 0 \qquad [\because C_1 \text{ and } C_2 \text{ are identical}]$$

$$= 0$$

Thus, $\Delta(x)$ is independent of x.

99. Let $\Delta = \begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix} = \begin{vmatrix} x & \frac{x(x-1)}{1\cdot2} & \frac{x(x-1)(x-2)}{1\cdot2\cdot3} \\ y & \frac{y(y-1)}{1\cdot2} & \frac{y(y-1)(y-2)}{1\cdot2\cdot3} \\ z & \frac{z(z-1)}{1\cdot2} & \frac{z(z-1)(z-2)}{1\cdot2\cdot3} \end{vmatrix}$

$$= \frac{xyz}{12}\begin{vmatrix} 1 & x-1 & x^2 - 3x + 2 \\ 1 & y-1 & y^2 - 3y + 2 \\ 1 & z-1 & z^2 - 3z + 2 \end{vmatrix}$$

Applying $C_2 \to C_2 + C_1$, then

$$\Delta = \frac{xyz}{12}\begin{vmatrix} 1 & x & x^2 - 3x + 2 \\ 1 & y & y^2 - 3y + 2 \\ 1 & z & z^2 - 3z + 2 \end{vmatrix}$$

Applying $C_3 \to C_3 + 3C_2 - 2C_1$, then

$$\Delta = \frac{xyz}{12}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \frac{1}{12}xyz(x-y)(y-z)(z-x)$$

100. (i) $\because \quad f(x) = \begin{vmatrix} 1 + \sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1 + \cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1 + 4\sin 2x \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$f(x) = \begin{vmatrix} 1 + \sin^2 x & \cos^2 x & 4\sin 2x \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 + C_1$, then

$$f(x) = \begin{vmatrix} 1 + \sin^2 x & 2 & 4\sin 2x \\ \vdots & & \\ -1 \;\dots & 0 & \dots\; 0 \\ \vdots & & \\ -1 & -1 & 1 \end{vmatrix}$$

Expanding along R_2, then

$$f(x) = \begin{vmatrix} 2 & 4\sin 2x \\ -1 & 1 \end{vmatrix} = 2 + 4\sin 2x$$

$\therefore$ Maximum value of

$$f(x) = 2 + 4(1) = 6$$

(ii) $\because$ $\Delta = \begin{vmatrix} \sin^2 A & \sin A \cos A & \cos^2 A \\ \sin^2 B & \sin B \cos B & \cos^2 B \\ \sin^2 C & \sin C \cos C & \cos^2 C \end{vmatrix}$

$$= \cos^2 A \cos^2 B \cos^2 C \begin{vmatrix} \tan^2 A & \tan A & 1 \\ \tan^2 B & \tan B & 1 \\ \tan^2 C & \tan C & 1 \end{vmatrix}$$

$$= -\cos^2 A \cos^2 B \cos^2 C \begin{vmatrix} 1 & \tan A & \tan^2 A \\ 1 & \tan B & \tan^2 B \\ 1 & \tan C & \tan^2 C \end{vmatrix}$$

$$= -\cos^2 A \cos^2 B \cos^2 C (\tan A - \tan B)(\tan B - \tan C)(\tan C - \tan A)$$

$$= -\sin (A - B) \sin(B - C) \sin (C - A)$$

$$= \sin (A - B) \sin (B - C) \sin (A - C) \geq 0 \quad [\because A \geq B \geq C]$$

$\therefore$ $\Delta \geq 0$

Hence, minimum value of Δ is 0.

101. Let $f(x) = \begin{vmatrix} x^2 - 4x + 6 & 2x^2 + 4x + 10 & 3x^2 - 2x + 16 \\ x - 2 & 2x + 2 & 3x - 1 \\ 1 & 2 & 3 \end{vmatrix}$

On differentiating w.r.t. x, we get

$$f'(x) = \begin{vmatrix} 2x - 4 & 4x + 4 & 6x - 2 \\ x - 2 & 2x + 2 & 3x - 1 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} x^2 - 4x + 6 & 2x^2 + 4x + 10 & 3x^2 - 2x + 16 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} x^2 - 4x + 6 & 2x^2 + 4x + 10 & 3x^2 - 2x + 16 \\ x - 2 & 2x + 2 & 3x - 1 \\ 0 & 0 & 0 \end{vmatrix}$$

$f'(x) = 0, \forall x \in R$ and $f(x) =$ Constant

As, $f(0) = \begin{vmatrix} 6 & 10 & 16 \\ -2 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = 2 \therefore f(x) = 2$

Now, $I = \int_{-3}^{3} \frac{x^2 \sin x}{1 + x^6} f(x)\, dx = 2 \int_{-3}^{3} \frac{x^2 \sin x}{1 + x^6}\, dx$

Let $g(x) = \frac{x^2 \sin x}{1 + x^6}$

$\therefore$ $g(-x) = \frac{-x^2 \sin x}{1 + x^6} = -g(x)$

Hence, g is an odd function.

$\therefore$ $I = 0$

102. Since, $Y = X$ and $Z = tX$

$Y_1 = sX_1 + Xs_1$ ...(i)

$Y_2 = sX_2 + Xs_2 + 2X_1 s_1$ ...(ii)

$Z_1 = tX_1 + Xt_1$ ...(iii)

and $Z_2 = tX_2 + Xt_2 + 2X_1 t_1$ ...(iv)

$$\text{LHS} = \begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$$

$$= \begin{vmatrix} X & sX & tX \\ X_1 & sX_1 + Xs_1 & tX_1 + Xt_1 \\ X_2 & sX_2 + Xs_2 + 2X_1 s_1 & tX_2 + Xt_2 + 2X_1 t_1 \end{vmatrix}$$

[using Eqs. (i), (ii), (iii) and (iv)]

Applying $C_2 \to C_2 - sC_1$ and $C_3 \to C_3 - tC_1$, then

$$= \begin{vmatrix} X & 0 & 0 \\ X_1 & Xs_1 & Xt_1 \\ X_2 & Xs_2 + 2X_1 s_1 & Xt_2 + 2X_1 t_1 \end{vmatrix}$$

Expanding w.r.t. R_1, then

$$= X^2 \begin{vmatrix} s_1 & t_1 \\ Xs_2 + 2X_1 s_1 & Xt_2 + 2X_1 t_1 \end{vmatrix}$$

Applying $R_2 \to R_2 - 2X_1 R_1$, then

$$= X^2 \begin{vmatrix} s_1 & t_1 \\ Xs_2 & Xt_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix} = \text{RHS}$$

103. Given determinant may be expressed as

$$\Delta = \begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ (x^2 f'' + 4xf' + 2f) & (x^2 g'' + 4xg' + 2g) & (x^2 h'' + 4xh' + 2h) \end{vmatrix}$$

Now, applying $R_3 \to R_3 - 4R_2 + 2R_1$, then

$$\Delta = \begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, then $\Delta = \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix}$

$\Rightarrow$ $\Delta = x \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix}$

$\Rightarrow$ $\Delta = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix}$

$\therefore$ $\Delta' = \begin{vmatrix} f' & g & h' \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$

$$= 0 + 0 + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

Hence, $\Delta' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$

104. Let the given determinant be equal to zero. Then, there exist x, y and z not all zero, such that

$$a_1 x + a_2 y + a_3 z = 0,\ b_1 x + b_2 y + b_3 z = 0$$

and $c_1 x + c_2 y + c_3 z = 0$

Assume that, $|x| \ge |y| \ge |z|$ and $x \ne 0$. Then, from $a_1 x = (-a_2 y) + (-a_3 z)$

$\therefore \quad |a_1 x| = |-a_2 y - a_3 z| \le |a_2 y| + |a_3 z|$

$\Rightarrow \quad |a_1|\,|x| \le |a_2|\,|y| + |a_3|\,|z|$

But $\quad x \ne 0$ i.e. $|a_1| \le |a_2| + |a_3|$

Similarly, $\quad |b_2| \le |b_1| + |b_3|$

$\quad |c_3| \le |c_1| + |c_2|$

which is contradiction. Hence, the assumption that the determinant is zero must be wrong.

105. LHS $= \begin{vmatrix} (a-a_1)^{-2} & (a-a_1)^{-1} & a_1^{-1} \\ (a-a_2)^{-2} & (a-a_2)^{-1} & a_2^{-1} \\ (a-a_3)^{-2} & (a-a_3)^{-1} & a_3^{-1} \end{vmatrix}$

$$= (a-a_1)^{-2}(a-a_2)^{-2}(a-a_3)^{-2} \begin{vmatrix} 1 & (a-a_1) & a_1^{-1}(a-a_1)^2 \\ 1 & (a-a_2) & a_2^{-1}(a-a_2)^2 \\ 1 & (a-a_3) & a_3^{-1}(a-a_3)^2 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to -R_3 - R_1$, then

$$\text{LHS} = \frac{1}{\prod (a-a_i)^2} \begin{vmatrix} 1 & (a-a_1) & a_1^{-1}(a-a_1)^2 \\ 0 & (a_1-a_2) & \dfrac{(a^2-a_1a_2)(a_1-a_2)}{a_1a_2} \\ 0 & (a_1-a_3) & \dfrac{(a^2-a_1a_3)(a_1-a_3)}{a_1a_3} \end{vmatrix}$$

Expanding w.r.t. 1st column, then

$$\text{LHS} = \frac{1}{\prod (a-a_i)^2} \begin{vmatrix} (a_1-a_2) & \dfrac{(a^2-a_1a_2)(a_1-a_2)}{a_1a_2} \\ (a_1-a_3) & \dfrac{(a^2-a_1a_3)(a_1-a_3)}{a_1a_3} \end{vmatrix}$$

$$= \frac{(a_1-a_2)(a_1-a_3)}{\prod (a-a_i)^2} \begin{vmatrix} 1 & \dfrac{a^2-a_1a_2}{a_1a_2} \\ 1 & \dfrac{a^2-a_1a_3}{a_1a_3} \end{vmatrix}$$

$$= \frac{(a_1-a_2)(a_1-a_3)\,a^2(a_2-a_3)}{a_1a_2a_3 \prod (a-a_i)^2} = \frac{-a^2 \prod (a_i - a_j)}{\prod a_i \prod (a-a_i)^2}$$

Numerator $= -a^2(a_1-a_2)(a_2-a_3)(a_3-a_1)$

The resulting expression has negative sign.

106. The given system of equation will have a non-trivial solution in the determinant of coefficients.

$$\therefore \quad \Delta = \begin{vmatrix} a-t & b & c \\ b & c-t & a \\ c & a & b-t \end{vmatrix}$$

$\Delta = 0$ is a cubic equation in t.

So, it has in general three solutions t_1, t_2 and t_3.

Let $\quad \Delta = a_0 t^3 + a_1 t^2 + a_2 t + a_3$

Clearly, $a_0 =$ Coefficient of $t^3 = -1$,

so $t_1 t_2 t_3 = -\dfrac{a_3}{a_0} = -\dfrac{a_3}{-1} = a_3 =$ Constant term in the expansion of Δ. i.e. Δ (at $t = 0$)

$$\therefore \quad t_1 t_2 t_3 = a_3 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

107. (i) Eliminating a, b and c from given equations, we obtain

$$\begin{vmatrix} -1 & \dfrac{y}{z} & \dfrac{z}{y} \\ -1 & \dfrac{z}{x} & \dfrac{x}{z} \\ -1 & \dfrac{x}{y} & \dfrac{y}{z} \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\begin{vmatrix} -1 & \dfrac{y}{z} & \dfrac{z}{y} \\ 0 & \dfrac{z}{x} - \dfrac{y}{z} & \dfrac{x}{z} - \dfrac{z}{y} \\ 0 & \dfrac{x}{y} - \dfrac{y}{z} & \dfrac{y}{x} - \dfrac{z}{y} \end{vmatrix} = 0$$

Expanding along C_1, then

$$-\left(\frac{z}{x} - \frac{y}{x}\right)\left(\frac{y}{x} - \frac{z}{y}\right) + \left(\frac{x}{y} - \frac{y}{z}\right)\left(\frac{x}{z} - \frac{z}{y}\right) = 0$$

$$\Rightarrow \quad \frac{yz}{x^2} + \frac{zx}{y^2} + \frac{xy}{z^2} + 1 = 0$$

(ii) To eliminate x, y and z.

Let $\alpha = \dfrac{y}{z}$, $\beta = \dfrac{z}{x}$ and $\gamma = \dfrac{x}{y}$ in the given equations,

$$b\alpha + \frac{c}{\alpha} = a, \quad \text{...(i)}$$

$$c\beta + \frac{a}{\beta} = b \quad \text{...(ii)}$$

and $$a\gamma + \frac{b}{\gamma} = c \quad \text{...(iii)}$$

Also, $\quad \alpha\beta\gamma = 1$

From Eqs. (i), (ii) and (iii), we get

$$\left(b\alpha + \frac{c}{\alpha}\right)\left(c\beta + \frac{a}{\beta}\right)\left(a\gamma + \frac{b}{\gamma}\right) = abc$$

$$\Rightarrow \quad 2abc + ac^2 \frac{\beta\gamma}{\alpha} + a^2 b \frac{\alpha\gamma}{\beta} + b^2 c \frac{\alpha\beta}{\gamma} + a^2 c \frac{\gamma}{\alpha\beta} + bc^2 \frac{\beta}{\gamma\alpha} + ab^2 \frac{\alpha}{\beta\gamma} = abc$$

$[\because \alpha\beta\gamma = 1]$

$$\Rightarrow \quad ac^2 \frac{1}{\alpha^2} + a^2 b \frac{1}{\beta^2} + b^2 c \frac{1}{\gamma^2} + a^2 c\gamma^2 + bc^2\beta^2 + ab^2\alpha^2 = -abc$$

$$\Rightarrow a\left(\frac{c^2}{\alpha^2} + b^2\alpha^2\right) + b\left(\frac{a^2}{\beta^2} + \beta^2 c^2\right) + c\left(\frac{b^2}{\gamma^2} + a^2\gamma^2\right) = -abc \quad \text{...(iv)}$$

On squaring Eqs. (i), (ii) and (iii), we get

$$b^2\alpha^2 + \frac{c^2}{\alpha^2} = a^2 - 2bc,\ c^2\beta^2 + \frac{a^2}{\beta^2} = b^2 - 2ca \text{ and}$$

$$a^2\gamma^2 + \frac{b^2}{\gamma^2} = c^2 - 2ab$$

On putting these values in Eq. (iv), we get

$$a(a^2 - 2bc) + b(b^2 - 2ca) + c(c^2 - 2ab) = -abc$$

$$a^3 + b^3 + c^3 = 5abc$$

108. Here, $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. According to the question, x, y and z not all zero. Hence, the given system of equations has non-trivial solution.

$$\Delta = 0$$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2] = 0$$

$\therefore$ $a + b + c = 0$

or $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

***Case* I** If $a + b + c = 0$

From first two equations,

$$ax + by - (a + b)z = 0$$

$$bx - (a + b)y + ax = 0$$

[by cross-multiplication law]

$$\therefore \frac{x}{ab - (a + b)^2} = \frac{y}{-b(a + b) - a^2} = \frac{z}{-a(a + b) - b^2}$$

$$\Rightarrow \frac{x}{-(a^2 + ab + b^2)} = \frac{y}{-(a^2 + ab + b^2)} = \frac{z}{-(a^2 + ab + b^2)}$$

$\therefore$ $x : y : z = 1 : 1 : 1$

***Case* II** If $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

It is possible only, when

$$a - b = 0,\ b - c = 0 \text{ and } c - a = 0$$

Then, $a = b = c$

In this case all the three equations reduce in the forms

$$x + y + z = 0 \quad \text{...(i)}$$

Then, Eq. (i) will be satisfied, if

$$x = k,\ y = k\omega,\ z = k\omega^2$$

or $x = k,\ y = k\omega^2,\ z = k\omega$

where ω is the cube root of unity.

Then, $x : y : z = 1 : \omega : \omega^2$ or $1 : \omega^2 : \omega$

Hence, combined both cases, we get

$x : y : z = 1 : 1 : 1$

or $1 : \omega : \omega^2$

or $1 : \omega^2 : \omega$

109. Applying $C_1 \to C_1 + C_2 + C_3$, then

$$f(x) = \begin{vmatrix} 1 & (1 + b^2)x & (1 + c^2)x \\ 1 & 1 + b^2x & (1 + c^2)x \\ 1 & (1 + b^2)x & 1 + c^2x \end{vmatrix} \quad [\because a^2 + b^2 + c^2 + 2 = 0]$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$= \begin{vmatrix} 1 & (1 + b^2)x & (1 + c^2)x \\ 0 & 1 - x & 0 \\ 0 & 0 & 1 - x \end{vmatrix} = (1 - x^2)$$

Hence, degree of $f(x) = 2$

110. For no solution or infinitely many solutions

$$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$\begin{vmatrix} \alpha + 2 & 1 & 1 \\ \alpha + 2 & \alpha & 1 \\ \alpha + 2 & 1 & \alpha \end{vmatrix} = 0$$

Applying $R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\begin{vmatrix} \alpha + 2 & 1 & 1 \\ 0 & \alpha - 1 & 0 \\ 0 & 0 & \alpha - 1 \end{vmatrix} = 0 \Rightarrow (\alpha - 1)^2(\alpha + 2) = 0$$

$\therefore$ $\alpha = 1, -2$

For $\alpha = 1$, clearly there an infinitely many solutions and when we put $\alpha = -2$ in given system of equations and adding them together LHS $\neq$ RHS. i.e., no solution.

111. $\because$ $a_1, a_2, a_3, \ldots$ are in GP.

$\therefore$ Using $a_n = a_1 r^{n-1}$, we get the given determinant, as

$$\begin{vmatrix} \log(a_1 r^{n-1}) & \log(a_1 r^n) & \log(a_1 r^{n+1}) \\ \log(a_1 r^{n+2}) & \log(a_1 r^{n+3}) & \log(a_1 r^{n+4}) \\ \log(a_1 r^{n+5}) & \log(a_1 r^{n+6}) & \log(a_1 r^{n+7}) \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ and using $\log m - \log n = \log\left(\frac{m}{n}\right)$, we get

$$\begin{vmatrix} \log(a_1 r^{n-1}) & \log r & 2\log r \\ \log(a_1 r^{n+2}) & \log r & 2\log r \\ \log(a_1 r^{n+5}) & \log r & 2\log r \end{vmatrix} = 0$$

[$\because$ C_2 and C_3 are proportional]

112. Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 0 & x & 0 \\ 0 & 0 & y \end{vmatrix} = xy$$

113. $\because$

$$D = \begin{vmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = 0$$

and $$D_1 = \begin{vmatrix} -1 & -2 & 3 \\ k & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = (3 - k) = 0, \text{ if } k = 3$$

$$D_2 = \begin{vmatrix} 1 & -1 & 3 \\ -1 & k & -2 \\ 1 & -3 & 4 \end{vmatrix} = (k - 3) = 0, \text{ if } k = 3$$

$$D_3 = \begin{vmatrix} -1 & -2 & 3 \\ k & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = (k = 3) = 0, \text{ if } k = 3$$

$\therefore$ System of equations has no solution for $k \neq 3$.

114. The system of equations

$x - cy - bz = 0,\ -cx + y - az = 0$ and $-bx - ay + z = 0$

have non-trivial solution, if

$$\begin{vmatrix} 1 & -c & -b \\ -c & 1 & -a \\ -b & -a & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 1 + 2(-a)(-b)(-c) - a^2 - b^2 - c^2 = 0$$

or $\quad a^2 + b^2 + c^2 + 2abc = 1$

115. $$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ a & -b & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & a-1 & a \\ b+1 & b-1 & -b \\ c-1 & c+1 & c \end{vmatrix} = 0$$

[by property]

$$\Rightarrow \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+2} \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} = 0$$

116. Applying $R_1 \to R_1 + R_3$, then

$$f(\theta) = \begin{vmatrix} \theta & \cdots & 0 & \cdots & 2 \\ & & & & \vdots \\ -\tan\theta & & 1 & & \tan\theta \\ & & & & \vdots \\ -1 & & -\tan\theta & & 1 \end{vmatrix}$$

$$= 2(1 + \tan^2\theta) = 2\sec^2\theta \geq 2$$

$\therefore f(\theta) \in [2, \infty)$

117. Non-zero solution means non-trivial solution.

For non-trivial solution of the given system of linear equations

$$\begin{vmatrix} 4 & k & 2 \\ k & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 4(4-2) - k(k-2) + (2k-8) = 0$

$\Rightarrow \quad -k^2 + 6k - 8 = 0$

$\Rightarrow \quad k^2 - 6k + 8 = 0$

$\Rightarrow \quad (k-2)(k-4) = 0$

$\therefore \quad k = 2, 4$

Clearly, there exist values of k.

118. For trivial solution $\begin{vmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{vmatrix} \neq 0$

$\Rightarrow 1(-3+k) + k(-k+3k) + 1(k-9) \neq 0$

$\Rightarrow \quad 2k^2 + 2k - 12 \neq 0$

$\Rightarrow \quad k^2 + k - 6 \neq 0$

$\Rightarrow \quad (k+3)(k-2) \neq 0$

$\Rightarrow \quad k \neq 2, -3$

or $\quad k \in R - \{2, -3\}$

119. $$\Delta = \begin{vmatrix} k+1 & 8 \\ k & k=3 \end{vmatrix} = (k+1)(k+3) - 8k = k^2 - 4k + 3$$

$\therefore \quad \Delta = (k-1)(k-3)$

$$\Delta_1 = \begin{vmatrix} 4k & 8 \\ 3k-1 & k+3 \end{vmatrix} = 4k^2 + 12k - 24k + 8 = 4k^2 - 12k + 8$$

$$\Delta_1 = 4(k-1)(k-2)$$

and $$\Delta_2 = \begin{vmatrix} k+1 & 4k \\ k & 3k-1 \end{vmatrix} = (k+1)(3k-1) - 4k^2 = -k^2 + 2k + 1$$

$\therefore \quad \Delta_2 = -(k-1)^2$

As given no solutions

$\Rightarrow \quad \Delta_1$ and $\Delta_2 \neq 0$

but $\quad \Delta = 0$

$k = 3$

120. $$\because \begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$$

$$= \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

$$= \{(1-\alpha)(1-\beta)(\alpha-\beta)\}^2$$

$$= (1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$$

So, $k = 1$.

121. The given system can be written as

$$(2-\lambda)x_1 - 2x_2 + x_3 = 0$$

$$2x_1 - (3+\lambda)x_2 + 2x_3 = 0$$

$$-x_1 + 2x_2 - \lambda x_3 = 0$$

For non-trivial solutions, $\Delta = 0$

$$\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \ (2-\lambda)(\lambda^2 + 3\lambda - 4) + 2(-2\lambda + 2) + 1(4 - 3 - \lambda) = 0$

$\Rightarrow \quad \lambda^3 + \lambda^2 - 5\lambda + 3 = 0$

$\Rightarrow \quad \lambda = 1, 1, -3$

Hence, λ has two values.

122. Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 2\alpha+3 & 4\alpha+3 & 6\alpha+3 \\ 4\alpha+8 & 8\alpha+8 & 12\alpha+8 \end{vmatrix} = -648\alpha$$

Applying $R_3 \to R_3 - 2R_2$, then

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 2\alpha+3 & 4\alpha+3 & 6\alpha+3 \\ 2 & 2 & 2 \end{vmatrix} = -648\alpha$$

Applying $C_2 \to C_2 - \frac{1}{2}(C_1 + C_3)$, then

$$\begin{vmatrix} (1+\alpha)^2 & -\alpha^2 & (1+3\alpha)^2 \\ 2\alpha+3 & 4\alpha+3 & 6\alpha+3 \\ 2 & 0 & 2 \end{vmatrix} = -648\alpha$$

$\Rightarrow \quad \alpha^2(4\alpha + 6 - 12\alpha - 6) = -648\alpha$

$\Rightarrow \quad -8\alpha^3 = -648\alpha$

$\Rightarrow \quad \alpha^3 - 81\alpha = 0$

$\therefore \quad \alpha = 0, 9, -9$

123. For non-trivial solution

$$\begin{bmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{bmatrix} = 0$$

$\Rightarrow \quad 1(\lambda+1) - \lambda(-\lambda^2+1) - 1(\lambda+1) = 0$

$\Rightarrow \quad \lambda(\lambda^2 - 1) = 0$

$\Rightarrow \quad \lambda = 0, \ \pm 1$

124.

$$x^3\begin{bmatrix} 1 & 1 & 1+x^3 \\ 2 & 4 & 1+8x^3 \\ 3 & 9 & 1+27x^3 \end{bmatrix} = 10$$

$$\Rightarrow \quad x^3\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 3 & 9 & 1 \end{bmatrix} + x^6\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{bmatrix} = 10$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, then

$$x^3\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 3 & 6 & -2 \end{bmatrix} + x^6\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 6 \\ 3 & 6 & 24 \end{bmatrix} = 10$$

$\Rightarrow \quad 2x^3 + 12x^6 = 10$ or $6x^6 + x^3 - 5 = 0$

or $\quad (6x^3 - 5)(x^3 + 1) = 0$

$\Rightarrow \quad x^3 = \frac{5}{6}$ or $x^3 = -1$

$\therefore \quad x = \left(\frac{5}{6}\right)^{1/3}, -1$

i.e. Two distinct values of x.

125.

$$\Delta = \begin{vmatrix} a & 2 \\ 3 & -2 \end{vmatrix} = -2a - 6,$$

$$\Delta_1 = \begin{vmatrix} \lambda & 2 \\ \mu & -2 \end{vmatrix} = -2(\lambda + \mu)$$

or
$$\Delta_2 = \begin{vmatrix} a & \lambda \\ 3 & \mu \end{vmatrix} = \alpha\mu - 3\lambda$$

System has unique solution for $\Delta \neq 0$

$\therefore \ a \neq -3$ for all values λ and μ

System has infinitely many solution for

$$\Delta = \Delta_1 = \Delta_2 = 0$$

$\therefore \ a = -3, \ \lambda + \mu = 0, \ a\mu - 3\lambda = 0$

and system has no solution

$$\Delta = 0 \Rightarrow a = -3$$

and $\quad \lambda + \mu \neq 0$

126. $\because \ \Delta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{bmatrix} = 1(a-b) - 1(1-a) + 1(b-a^2) = -(a-1)^2$

$$\Delta_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 0 & b & 1 \end{bmatrix} = 1(a-b) - 1(1) + 1(b) = (a-1)$$

$$\Delta_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a & 0 & 1 \end{bmatrix} = 1(1) - 1(1-a) + 1(0-a) = 0$$

and $\Delta_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 0 \end{bmatrix} = 1(-b) - 1(-a) + 1(b-a^2) = -a(a-1)$

For $a = 1$, $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

and for $b = 1$ only

$$x + y + z = 1$$
$$x + y + z = 1$$
$$x + y + z = 0$$

i.e. no solution ($\because$ RHS are not equal)

Hence, for no solution $b = 1$ only.