CHAPTER SIX

Permutations and Combinations

FUNDAMENTAL PRINCIPLES OF COUNTING

The Sum Rule

Suppose that A and B are two disjoint events (mutually exclusive); that is, they never occur together. Further suppose that A occurs in m ways and B in n ways. Then A or B can occur in $m + n$ ways. This rule can also be applied to more than two mutually exclusive events.

The Product Rule

Suppose that an event X can be decomposed into two stages, A and B. Let stage A occur in m ways and suppose that these stages are unrelated, in the sense that stage B occurs in n ways regardless of the outcome of stage A. Then event X occurs in mn ways. This rule is applicable even if event X can be decomposed in more than two stages.

PERMUTATIONS AND COMBINATIONS

Suppose we have a set of n distinct objects $\{x_1, x_2, \cdots, x_n\}$ from which we have to take a sample of r $(1 \le r \le n)$ objects. In how many ways can this be done? However, the problem is not completely specified; in particular, the phrase 'take a sample' is somewhat ambiguous.

1. Is the order in which we take the sample important? For example, if $r = 2$, do we consider first picking x_1 and then x_2 to be different from picking x_2 and then x_1?
2. Do we allow repetition of objects? For example, do we allow a sample to consists of two x_1?

Since each question has two answers, the product rule says that our original problem is really made up of four separate problems.

1. How many such samples are there if the order is important and we allow repetitions?
2. How many such samples are there if the order is important and we do not allow repetitions?
3. How many such samples are there if the order is not important and we allow repetitions?
4. How many such samples are there if the order is not important and we do not allow repetitions?

We now introduce some terminology. If the order is important, then we say that our sample consists of *arrangements*. If the order is not important then we have *selections*. If repetition is allowed, then we are using *replacement*. An arrangement of r objects with replacement is called an r-*sequence* (or simply sequence). Vehicle numbers are examples of sequences.

An r-*permutation* (or simply *permutation*) is an arrangement without replacement. The different orderings of a deck of cards are examples of permutations. A selection of r objects with replacement is called an r-*multiset*. The letters used to form the word MATHEMATICS can be thought of as a multiset with two Ms, two As, two Ts, one H, one E, one I, one C and one S. An r-*combination* (or simply *combination*) is a selection of r objects without replacement.

In Table 6.1 we illustrate the four different problems and their solutions. We take two objects from $\{x_1, x_2, x_3\}$.

Table 6.1

Repetition		*No repetition*	
$x_1\,x_1$ $x_1\,x_2$ $x_1\,x_3$ $x_2\,x_1$ $x_2\,x_2$ $x_2\,x_3$ $x_3\,x_1$ $x_3\,x_2$ $x_3\,x_3$	r-sequence	$x_1\,x_2$ $x_1\,x_3$ $x_2\,x_1$ $x_2\,x_3$ $x_3\,x_1$ $x_3\,x_2$	r-permutation
$\{x_1, x_1\}$ $\{x_1, x_2\}$ $\{x_1, x_3\}$ $\{x_2, x_2\}$ $\{x_2, x_3\}$ $\{x_3, x_3\}$	r-multiset	$\{x_1, x_2\}$ $\{x_1, x_3\}$ $\{x_2, x_3\}$	r-combination

What happens when $r = 0$? We may view the r-combination as a subset of r elements, and thus a 0-combination may be viewed as the empty set. Since there is exactly one empty set, we may say that there is exactly one 0-combination. Mathematicians also say that there is exactly one 0-sequence, one 0-permutation, and one 0-multiset. By relaxing the definitions we may use the terms sequence and permutation interchangeably and call a multiset, a combination.

SOME IMPORTANT RESULTS

1. The number of sequences (that is, permutations) of n distinct objects taken r at a time, when repetition of objects is allowed, is n^r $(r > 0)$.
2. The number of permutations of n distinct objects taken r $(0 \le r \le n)$ at a time, when repetition is not allowed, is given by

$$^nP_r = \frac{n!}{(n-r)!}$$

Conventionally, $^nP_r = 0$ for $r > n$.

3. The number of permutations of n objects taken all together, when p_1 of the objects are alike and of one kind, p_2 of them are alike and of the second kind, $\cdots$, p_r of them are alike and of the rth kind, where $p_1 + p_2 + \cdots + p_r = n$ is given by

$$\frac{n!}{p_1!\, p_2! \cdots p_r!}$$

4. The number of combinations of n distinct objects taken r $(0 \le r \le n)$ at a time is given by

$$\binom{n}{r} = {}^nC_r = \frac{n!}{(n-r)!\,r!}$$

By definition $^nC_r = 0$ if $r > n$.

5. The number of ways of selecting zero or more objects out of p_1 identical objects of first kind, p_2 identical objects of second kind, …, p_r identical objects of rth kind is $(p_1 + 1)(p_2 + 1) \ldots (p_r + 1)$.
6. The number of combinations of n distinct objects taken r $(\le n)$ at a time, when k $(0 \le k \le r)$ particular objects always occur, is $^{n-k}C_{r-k}$.
7. The number of combinations of n distinct objects taken r at a time, when k $(1 \le k \le n)$ never occur, is $^{n-k}C_r$.
8. If $n > 1$, $^nP_0 < {}^nP_1 < {}^nP_2 < \ldots < {}^nP_{n-1} = {}^nP_r$
9. If $n = 2m + 1$, then

$$^nC_0 < {}^nC_1 < {}^nC_2 \ldots < {}^nC_m = {}^nC_{m+1}$$
$$^nC_{m+1} > {}^nC_{m+2} > {}^nC_{m+3} > \ldots > {}^nC_n$$

10. If $n = 2m$, then

$$^nC_0 < {}^nC_1 < {}^nC_2 \ldots < {}^nC_m$$
$$^nC_m > {}^nC_{m+1} > {}^nC_{m+2} \ldots > {}^nC_n$$

CIRCULAR PERMUTATION

The number of ways of arranging n distinct objects around a circle is $(n-1)!$.

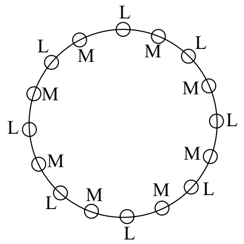
Remark

1. Note that even if one object is lying on the circle, then the circle has to be treated as a row, so far as arranging the objects around the circle is concerned.
2. Note that if no distinction is to be made between clockwise and counter-clockwise arrangements, then the number of arrangements is equal to $(n-1)!/2$.
3. The number of ways of arranging r identical objects and $(n-r)$ distinct objects along a circle is $\dfrac{(n-1)!}{r!}$

Illustration 1

In how many ways can 8 ladies and 5 men can be seated around a round table so that no two men are together.

Solution: 8 ladies can be arranged arround a round table in $(8 - 1)!$ ways. After the ladies have been arranged, there are 8 places for men. We can place 5 men at these places in $^8P_5$ ways. Thus, required number of ways is $(7!)\,(^8P_5)$.



SOME IMPORTANT IDENTITIES

(i) $^nC_0 = 1 = {}^nC_n$.

(ii) $^nC_r = {}^nC_{n-r}$ $\quad (0 \le r \le n)$.

(iii) $^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$.

(iv) $^nC_r = {}^nC_s$ implies $r = s$ or $r + s = n$.

(v) $^nC_r = \dfrac{n-r+1}{r}\, {}^nC_{r-1}$ $\quad (1 \le r \le n)$.

(vi) If n is even, then the greatest value of ^nC_r is ^nC_m, where $m = n/2$. If n is odd, then the greatest value of ^nC_r is ^nC_m, where $m = (n-1)/2$ or $(n+1)/2$.

(vii) $^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$.

(viii) $^nC_0 + {}^nC_2 + \cdots = {}^nC_1 + {}^nC_3 + \cdots = 2^{n-1}$.

(ix) $^{2n+1}C_0 + {}^{2n+1}C_1 + \cdots + {}^{2n+1}C_n = 2^{2n}$
$^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + \ldots + {}^{2n+1}C_{2n+1} = 2^{2n}$

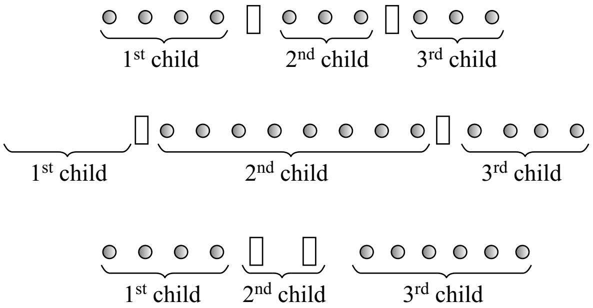
DIVISION OF IDENTICAL OBJECTS

The number of ways of distributing n identical objects among r persons giving zero or more to each is equal to the number of ways of arranging n identical object of one kind and $(r-1)$ identical objects of second kind (separators) in a row

$$= \frac{(n+r-1)!}{n!(r-1)!} = {}^{n+r-1}C_{r-1}$$

Illustration 2

To distribute 10 identical toys among 3 children, we may use two separators as follows:



etc.

The required number of ways

$$= \frac{(10+2)!}{10!2!} = {}^{12}C_2 = 66.$$

Alternative method

1. The number of distributing n identical objects among r persions giving zero or more to each

$= \text{coefficient of } x^n \text{ in } (1-x)^{-r}$

$= \text{coefficient of } x^n \text{ in } (1 + x + x^2 + \ldots)^r$

2. The number of ways of distributing n identical object among r persons giving at least one to each.

= the number of ways of distributing remaining $(n-r)$ identical objects (after giving one to each of r persons) among r persons giving zero or more to each

$$= {}^{(n-r)+(r-1)}C_{r-1} = {}^{n-1}C_{r-1}$$

Illustration 3

(i). To distribute 10 toys among 3 children giving at least one to each, we first give one toy to each of the child and distribute the remaining 7 toys among 3 children giving zero or more to each in ${}^{7+3-1}C_{3-1} = {}^9C_2 = 36$ ways.

(ii). However if we wish to give at least two to first, at least three to second and at least one to the third, then, we first give $(2 + 3 + 1)$ toys to the three children as desired and distribute the remaining 4 toys among three children giving zero or more to each. This can be done in ${}^{4+3-1}C_{3-1} = {}^6C_2 = 15$ ways.

(iii). If wish to distribute n identical object among r persons so that kth person $(1 \le k \le r)$ does not get more than a_k, then the desired number of ways

$= \text{Coeffcient of } x^n \text{ in } (1 + x + \ldots + x^{a_1})(1 + x + \ldots + x^{a_2}) \ldots (1 + x + \ldots + x^{a_r})$

Illustration 4

Find the number of ways of distributing 10 identical toys among 3 children so as give a_i $(1 \le i \le 3)$ to the ith children, where $2 \le a_1 \le 4$, $3 \le a_2 \le 5$, $1 \le a_3 \le 7$.

The required number of ways

$= \text{coefficient of } x^{10} \text{ in } (x^2 + x^3 + x^4)(x^3 + x^4 + x^5)(x + \ldots + x^7)$

$= \text{coefficient of } x^{10} \text{ in } x^6 (1 + x + x^2)^2 (1 + \ldots + x^6)$

$= \text{coefficient of } x^4 \text{ in } \left(\frac{1-x^3}{1-x}\right)^2 \left(\frac{1-x^7}{1-x}\right)$

$= \text{coefficient of } x^4 \text{ in } (1 - 2x^3)(1-x)^{-3}$ [ignore any power that is more than 4]

$= \text{coeffcient of } x^4 \text{ in } (1 - 2x^3)(1 + {}^3C_1 x + {}^4C_2 x^2 + {}^5C_3 x^3 + {}^6C_4 x^4 + \ldots)$

$= {}^6C_4 - 2({}^3C_1) = 15 - 6 = 9$

Finding Number Integral Solutions of a Linear Equation

The number of non-negative integral solution of

$$x_1 + x_2 + \ldots + x_r = n,$$

where $n \in \mathbf{N} \cup \{0\}$, and $r \in \mathbf{N}$

= the number of ways of distributing n identical objects among r persons giving zero or more to each

$= {}^{n+r-1}C_{r-1}$.

MULTIPLICATION OF TWO INFINITE SERIES

The following informal method of multiplying two infinite power series is also useful:

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots)(b_0 + b_1x + b_2x^2 + \ldots)$$
$$= c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$$

where $c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \ldots a_nb_0$ for all $n \ge 0$.

DISTRIBUTION INTO GROUPS

1. The number of ways in which n distinct objects can be split into three groups containing respectively r, s and t objects (where r, s and t are distinct and $r + s + t = n$) is

$$({}^nC_r)({}^{n-r}C_s)({}^{n-r-s}C_t)$$
$$= \frac{n!}{r!(n-r)!} \frac{(n-r)!}{(n-r-s)!\,s!} \frac{(n-r-s)!}{(n-r-s-t)!\,t!}$$
$$= \frac{n!}{r!\,s!\,t!}$$

2. Suppose mn distinct objects are to be divided into m groups, each containing n objects and the order of the groups is not important. Then the number of ways of doing this is given by

$$\frac{(mn)!}{m!(n!)^m}$$

If, however, the order of groups is important, then the number of ways is given by

$$\frac{(mn)!}{(n!)^m}$$

SELECTION

1. The number of ways of selecting one or more objects out of n distinct objects is $2^n - 1$.
2. Suppose we have n distinct objects, and p like objects of one kind, q like objects of a second kind, r like objects of a third kind, etc. Then the number of ways of selecting one or more objects from these objects is
$2^n (p + 1) (q + 1) (r + 1) \cdots - 1$.

EXPONENT OF PRIME p IN n!

Let $E_p(m)$ denote the exponent of the prime p in the prime factorization of positive integer m. We have

$$E_p(n!) = E_p\,(1\cdot 2\cdot 3\cdot 4 \cdots (n-1)\cdot n)$$

The last integer amongst 1, 2, 3, $\cdots$, $(n - 1)$, n which is divisible by p is $[n/p]p$, where $[x]$ denotes greatest integer $\le x$.

Therefore,

$$E_p(n!) = E_p\left(p\cdot 2p\cdot 3p \cdots \left[\frac{n}{p}\right]p\right)$$

because the remaining integers from the set $\{1, 2, 3, \cdots, (n - 1), n\}$ are not divisible by p.

$$E_p(n!) = E_p\left(p\cdot 2p\cdot 3p \cdots \left[\frac{n}{p}\right]p\right)$$

$$= \left[\frac{n}{p}\right] + E_p\left(1\cdot 2\cdot 3 \cdots \left[\frac{n}{p}\right]\right)$$

The last integer amongst 1, 2, $\cdots$, $[n/p]$ which is divisible by p is

$$\left[\frac{[n/p]}{p}\right]p = \left[\frac{n}{p^2}\right]p$$

Since the remaining integers are not divisible by p, we get

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + E_p\left(1\cdot 2 \cdots \left[\frac{n}{p^2}\right]\right)$$

Continuing in this way, we get

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots + \left[\frac{n}{p^s}\right]$$

where $p^s \le n < p^{s+1}$.

Illustration 5

Find exponent of 3 in 32!

As 3 divides 3, 6, 9...., 30 amongst 1, 2, . . . 31, 32,

$E_3(32!) = E_3(3 \times 6 \times 9 \times \ldots \times 30)$

$= E_3(3^{10} \times 1 \times 2 \times \ldots \times 10)$

$= 10 + E_3(3 \times 6 \times 9)$

[Ignore 1, 2, 4, 5, 7, 8, 10 as 3does not divide these.]

$= 10 + E_3(3^3 \times 1 \times 2 \times 3)$

$= 10 + 3 + E_3(3)$

$= 10 + 3 + 1 = 14.$

Note, $10 = \left[\frac{31}{3}\right]$, $3 = \left[\frac{31}{3^2}\right]$ and $1 = \left[\frac{31}{3^3}\right]$

SOLVED EXAMPLES

Concept-based
Straight Objective Type Questions

Example 1: If ${}^{n+5}P_{n+1} = \frac{1}{2}(11)(n-1)\left({}^{n+3}P_n\right)$ then n is equal to:

(a) 10, 11 (b) 9, 10
(c) 8, 7 (d) 6, 7

Ans: (d).

Solution: $\frac{1}{2}\,(11)\,(n-1) = \frac{{}^{n+5}P_{n+1}}{{}^{n+3}P_n}$

$$= \frac{(n+5)!}{4!} \times \frac{3!}{(n+3)!}$$

$\Rightarrow \quad \frac{1}{2}(11)\,(n-1) = \frac{1}{4}(n+5)\,(n+4)$

$\Rightarrow \quad 22n - 22 = n^2 + 9n + 20$

$\Rightarrow \quad n^2 - 13n + 42 = 0 \Rightarrow n = 6, 7$

Example 2: If ${}^{n+2}C_3 = {}^{n+3}P_2 - 20$, then n is equal to:

(a) 6 (b) 5
(c) 4 (d) 3

Ans (d).

Solution: $\frac{1}{6}(n+2)\,(n+1)n = (n+3)\,(n+2) - 20$

$\Rightarrow \quad n^3 + 3n^2 + 2n = 6(n^2 + 5n + 6) - 120$

$\Rightarrow \quad n^3 - 3n^2 - 28n + 84 = 0$

$\Rightarrow \quad (n - 3)(n^2 - 28) = 0$

As n is a positive integer, we get $n = 3$

Example 3: If ${}^{10}C_{r-1} > 2({}^{10}C_r)$ then r is equal to

(a) 6, 7, 8, 9 (b) 7, 8, 9
(c) 8, 9, 10 (d) 5, 6, 9, 10

Ans: (c).

Solution: Note that $0 \le r \le 10$. Now $\dfrac{{}^{10}C_{r-1}}{{}^{10}C_r} > 2$

$$\Rightarrow \quad \frac{10!}{(r-1)!(10-r+1)!} \cdot \frac{r!(10-r)!}{10!} > 2$$

$$\Rightarrow \quad \frac{r}{11-r} > 2 \Rightarrow r > 22 - 2r$$

$$\Rightarrow \quad 3r > 22 \Rightarrow r \ge 8 \Rightarrow r = 8, 9, 10.$$

Example 4: If $\dfrac{1}{a}({}^nP_{r+1}) = \dfrac{1}{b}({}^nP_r) = \dfrac{1}{c}({}^nP_{r-1})$, then $b^2 - (a + b)c$ is equal to.

(a) 0 (b) 1
(c) -1 (d) -2

Ans: (a).

Solution: $\dfrac{{}^nP_{r+1}}{{}^nP_r} = \dfrac{a}{b} \Rightarrow \dfrac{(n-r)!}{(n-r-1)!} = \dfrac{a}{b}$

$$\Rightarrow \quad n - r = \frac{a}{b}$$

Similarly $\quad n - r + 1 = \dfrac{b}{c}$

$$\therefore \quad \frac{b}{c} - \frac{a}{b} = 1$$

$$\Rightarrow \quad b^2 - ac = bc \Rightarrow b^2 - (a + b)c = 0.$$

Exemple 5: Let Q_n be the number of possible quadrilaterals formed by joining vertices of an n sided regular polygon. If $Q_{n+1} - Q_n = 20$, then value of n is:

(a) 8 (b) 7
(c) 6 (d) 5

Ans: (c).

Solution: Q_n = Number of ways of choosing four vertices out of n

$$= {}^nC_4$$

$$\therefore \quad 20 = Q_{n+1} - Q_n = {}^{n+1}C_4 - {}^nC_4$$

$$= {}^nC_4 + {}^nC_3 - {}^nC_4 = {}^nC_3$$

$$[\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]$$

$$\Rightarrow \quad \frac{1}{6}n(n-1)(n-2) = 20 \qquad (1)$$

$\Rightarrow \quad n^3 - 3n^2 + 2n - 120 = 0$

$\Rightarrow \quad n^3 - 6n^2 + 3n^2 - 18n + 20n - 120 = 0$

$\Rightarrow \quad (n - 6)(n^2 + 3n + 20) = 0$

As $n \in \mathbf{N}$, $n = 6$.

TIP

You can directly reach the answer by substituting the given choices in (1).

Example 6: If there are 62 onto mapping from a set X containing n elements to the set $Y = \{-1, 1\}$, then n is equal to:

(a) 4 (b) 5
(c) 6 (d) 7

Ans: (c)

Solution: Let $X = \{x_1, x_2, \ldots, x_n\}$.
Each x_i can have two images *viz* -1 and 1. Thus, there are 2^n mappings from X to Y. But there are exactly two mapping which are not onto. These are when all the elements are mapped to -1 or when all the elements are mapped to 1.
$\therefore$ there are $2^n - 2$ onto mapping from X to Y.

Set $\quad 2^n - 2 = 62 = \Rightarrow 2^n = 64 = 2^6$

$\Rightarrow \quad n = 6$

Example 7: The maximum possible number of points of intersection of 7 straight lines and 5 circles is:

(a) 111 (b) 109
(c) 107 (d) 105

Ans: (a)

Solution: Maximum possible number of points of intersection of 7 lines is ${}^7C_2 = 21$.
Maximum possible number of points of intersection of 5 circles in $2\,({}^5C_2) = 20$
[$\because$ two circles can intersect in two distinct points.]
Maximum possible number of points of intersection of 5 circles and 7 lines is $2(5)(7) = 70$
Thus, maximum possible number of points of intersection is

$$21 + 20 + 70 = 111.$$

Example 8: The number of injective functions from a set X containing m elements to a set Y containing n elements, for $m > n$ is:

(a) ${}^mP_n$ (b) $(m - n)!$
(c) ${}^mC_n$ (d) 0

Ans. (d)

Solution:

TIP

If X and Y are two finite sets and $f : X \to Y$ is an injective function from X to Y then $n \geq m$ where $m = n(X)$ and $n = n(Y)$. The number of injective functions from X to Y is

$n(n-1)(n-2)\ldots(n-m+1) = {}^nP_m$

As $m > n$, there does not exist any injective function from X to Y.

Example 9: If ${}^nP_r = 2520$ and ${}^nC_r = 21$ then n is equal to:

(a) 6 (b) 7
(c) 8 (d) 10

Ans: (b)

Solution: $(r!)({}^nC_r) = {}^nP_r \Rightarrow (r!)(21) = 2520$

$\Rightarrow \quad r! = 120 \Rightarrow r = 5$

Now $\quad {}^nC_5 = 21 = \dfrac{1}{2!}(7 \times 6) = \dfrac{7!}{5!2!}$

$\Rightarrow \quad n = 7$

Example 10: If ${}^{15}C_{r+1} : {}^{15}C_{3r} = 3 : 11$, then value of r is:

(a) 2 (b) 3
(c) 4 (d) 5

Ans: (b)

Solution: Note that $0 \leq 3r \leq 15$

$\Rightarrow 0 \leq r \leq 5$. Also, $\dfrac{15!}{(14-r)!(r+1)!} \times \dfrac{(3r)!(15-3r)!}{15!} = \dfrac{3}{11}$

$$\Rightarrow \quad \frac{(3r)!(15-3r)!}{(14-r)!(r+1)!} = \frac{3}{11} \qquad (1)$$

For the denominator, to be divisible by 11, $11|(14-r)!$. Setting $14 - r = 11$.

$\Rightarrow \quad r = 3$

Putting $r = 3$, in LHS of (1) we get

LHS of (1) $\dfrac{9!6!}{11!4!} = \dfrac{6 \times 5}{11 \times 10} = \dfrac{3}{11}$ = RHS of (1)

$\therefore \quad r = 3$

LEVEL 1

Straight Objective Type Questions

Example 11: If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P., then a value of n can be

(a) 6 (b) 7
(c) 8 (d) 9

Ans. (b)

Solution: As ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P.,

$2({}^nC_5) = {}^nC_4 + {}^nC_6$

$$\Rightarrow \quad 2 = \frac{{}^nC_4}{{}^nC_5} + \frac{{}^nC_6}{{}^nC_5}$$

$$= \frac{n!}{4!(n-4)!}\,\frac{5!(n-5)!}{n!} + \frac{n!}{6!(n-6)!}\,\frac{5!(n-5)!}{n!}$$

$$\Rightarrow \quad 2 = \frac{5}{n-4} + \frac{n-5}{6} \quad \Rightarrow \quad n^2 - 21n + 98 = 0$$

$\Rightarrow \quad n = 7, 14.$

Example 12: The least positive integer n for which ${}^{n-1}C_5 + {}^{n-1}C_6 < {}^nC_7$ is

(a) 14 (b) 15
(c) 16 (d) 28

Ans. (a)

Solution: Since ${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$ we can write the given inequality as

$${}^nC_6 < {}^nC_7$$

$$\Rightarrow \quad \frac{{}^nC_6}{{}^nC_7} < 1 \quad \Rightarrow \quad \frac{7}{n-6} < 1 \quad \Rightarrow \quad n > 13$$

$\therefore \quad$ the least value of n is 14.

Example 13: If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, then the value of r is equal to

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (c)

Solution: We have

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{36}{84} \Rightarrow \frac{n!}{(r-1)!(n-r+1)!}\,\frac{r!(n-r)!}{n!} = \frac{3}{7}$$

$$\Rightarrow \quad \frac{r}{n-r+1} = \frac{3}{7} \quad \Rightarrow \quad 10r = 3n + 3$$

Similarly, $\quad \dfrac{{}^nC_r}{{}^nC_{r+1}} = \dfrac{84}{126} \quad \Rightarrow \quad \dfrac{r+1}{n-r} = \dfrac{2}{3}$

$\Rightarrow \quad 5r + 3 = 2n \quad \Rightarrow \quad 5r + 3 = 2n$

Solving we obtain $r = 3$.

Example 14: In a group of 8 girls, two girls are sisters. The number of ways in which the girls can sit in a row so that two sisters are not sitting together is

(a) 4820 (b) 1410
(c) 2830 (d) 30240

Ans. (d)

Solution: The required number of ways
= the number of ways in which 8 girls can sit in a row – the number of ways in which two sisters sit together
= 8! – (2) (7!) = 30240.

Example 15: The number of words that can be formed by using the letters of the word MATHEMATICS that start as well as end with T is:

(a) 80720 (b) 90720
(c) 20860 (d) 37528

Ans. (b)

Solution: The word MATHEMATICS contains 11 letters viz. M, M, A, A T, T, H, E, I, C, S. The number of words that begin with T and end with T is

$$\frac{9!}{2!\,2!} = 90720.$$

Example 16: If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ equals:

(a) $\frac{n}{2}a_n$ (b) $\frac{n}{4}a_n$
(c) na_n (d) $(n–1)a_n$

Ans. (a)

Solution: Let $b_n = \sum_{r=0}^{n} \frac{r}{{}^nC_r} = \sum_{r=0}^{n} \frac{n-r}{{}^nC_{n-r}} = \sum_{r=0}^{n} \frac{n-r}{{}^nC_r}$

$$\Rightarrow \quad 2b_n = \sum_{r=0}^{n} \frac{r+(n-r)}{{}^nC_r} = n\sum_{r=0}^{n} \frac{1}{{}^nC_r} = na_n$$

$$\Rightarrow \quad b_n = \frac{n}{2}a_n\,.$$

Example 17: m men and w women are to be seated in a row so that no two women sit together. If $m > w$, then the number of ways in which they can be seated is:

(a) $\frac{m!\,(m+1)!}{(m-w+1)!}$ (b) ${}^mC_{m-w}\,(m-w)!$
(c) ${}^{m+w}C_m\,(m-w)!$ (d) none of these

Ans. (a)

Solution: We first arrange the m men. This can be done in $m!$ ways. After m men have taken their seats, the women must choose w seats out of $(m + 1)$ seats marked with X below.

$$\underset{\text{1st}}{X}\,M\,\underset{\text{2nd}}{X}\,M\,\underset{\text{3rd}}{X}\,M\,X\ldots\underset{m\text{th}}{X}\,M\,\underset{(m+1)\text{th}}{X}$$

They can choose w seats in ${}^{m+1}C_w$ ways and arrange w women in $w!$ ways.
Thus, the required number of arrangements is

$$m!\,({}^{m+1}C_w)\,(w!) = \frac{m!\,(m+1)!\,w!}{w!\,(m+1-w)!} = \frac{m!\,(m+1)!}{(m+1-w)!}$$

Example 18: The number of subsets of the set $A = \{a_1, a_2, \ldots, a_n\}$ which contain even number of elements is

(a) 2^{n-1} (b) $2^n - 1$
(c) 2^{n-2} (d) 2^n

Ans. (a)

Solution: For each of the first $(n – 1)$ elements $a_1, a_2, \ldots, a_{n-1}$ we have two choices: either a_i $(1 \le i \le n–1)$ lies in the subset or a_i doesn't lie in the subset. For the last element we have just one choice. If even number of elements have already been selected, we do not include a_n in the subset, otherwise (when odd number of elements have been selected), we include it in the subset.
Thus, the number of subsets of $A = \{a_1, a_2, \ldots, a_n\}$ which contain even number of elements is equal to 2^{n-1}.

Example 19: The number of positive integers < 1,00,000 which contain exactly one 2, one 5 and one 7 in its decimal representation is

(a) 2940 (b) 7350
(c) 2157 (d) 1582

Ans. (a)

Solution: We may consider a number of up to 5-digits to be a number of the form

$$X\,X\,X\,X\,X$$

where X is a digit from 0 to 9. The digit 2 can occupy any of the five places, 3 can occupy any of the remaining 4 places and 7 any of the 3-remaining places. The remaining 2 places can be filled up by 7 digits. Thus, there are (5) (4) (3) (7) (7) = 2940 positive integers in the desired category.

Example 20: The number of ways of factoring 91,000 into two factors, m and n, such that $m > 1$, $n > 1$ and $gcd\,(m, n) = 1$ is

(a) 7 (b) 15
(c) 32 (d) 37

Ans. (a)

Solution: We have $91,000 = 2^3 \times 5^3 \times 7 \times 13$
Let $A = \{2^3, 5^3, 7, 13\}$ be the set associated with the prime factorization of 91,000. For m, n to be relatively prime, each element of A must appear either in the prime factorization of m or in the prime factorization of n but not in both. Moreover, the 2 prime factorizations must be composed exclusively from the elements of A. Therefore, the number of relatively prime pairs m, n is equal to the number of ways of partitioning A into 2 unordered non-empty subsets. We can partition A as follows:

$\{2^3\} \cup \{5^3, 7, 13\}$, $\{5^3\} \cup \{2^3, 7, 13\}$
$\{7\} \cup \{2^3, 5^3, 13\}$, $\{13\} \cup \{2^3, 5^3, 7\}$
and $\{2^3, 5^3\} \cup \{7, 13\}$, $\{2^3, 7\} \cup \{5^3, 13\}$, $\{2^3, 13\} \cup \{5^3, 7\}$

Therefore, the required number of ways = 4 + 3 = 7.

Example 21: The sum $S = \frac{1}{9!} + \frac{1}{3!7!} + \frac{1}{5!5!} + \frac{1}{7!3!} + \frac{1}{9!}$ equals

(a) $\frac{2^9}{10!}$ (b) $\frac{2^{10}}{8!}$

(c) $\frac{2^{11}}{9!}$ (d) $\frac{2^{10}}{7!}$

Ans. (a)

Solution: We can write S as follows

$$S = \frac{1}{10!}\,[{}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_7 + {}^{10}C_9]$$
$$= \frac{1}{10!}\,(2^9).$$

Example 22: Let $n = 2015$. The least positive integer k for which

$$k(n^2)\,(n^2 - 1^2)\,(n^2 - 2^2)\,(n^3 - 3^2)\,\ldots\,(n^2 - (n-1)^2) = r!$$

for some positive integer r is

(a) 2014 (b) 2013

(c) 1 (d) 2

Ans. (d)

Solution: We can rewrite the given expression as

$$k(n^2)\,(n-1)\,(n+1)\,(n-2)\,(n+2)\,(n-3)\,(n+3)\ldots(n+n-1)\;(n-n+1) = r!$$

$\Rightarrow \quad kn\,(1)\,(2)\ldots(n-1)\,n(n+1)\,(n+2)\ldots(2n-1) = r!$

$\Rightarrow \quad kn\,(2n-1)! = r!$

$\therefore$ To convert L.H.S. to a factorial, we shall require, $k = 2$ which will convert it into $(2n)!$

Example 23: If $0 < r < s \le n$ and ${}^nP_r = {}^nP_s$, then value of $r + s$ is

(a) $2n - 2$ (b) $2n - 1$

(c) 2 (d) 1

Ans. (b)

Solution: ${}^nP_r = {}^nP_s \quad \Rightarrow \quad \frac{n!}{(n-r)!} = \frac{n!}{(n-s)!}$

$\Rightarrow \quad (n-r)! = (n-s)!$

As $r < s$, $n - r > n - s$. But the only two different factorials which are equal are 0! and 1!. Thus $n - r = 1$ and $n - s = 0$

$\Rightarrow r = n–1$ and $s = n$.

$\Rightarrow \quad r + s = 2n - 1.$

Example 24: If $E = \frac{1}{4}.\frac{2}{6}.\frac{3}{8}.\frac{4}{10}\cdots\frac{30}{62}.\frac{31}{64} = 8^x$, then value of x is

(a) – 7 (b) – 9

(c) – 10 (d) – 12

Ans. (d)

Solution: We have

$$E = \frac{31!}{2^{31}(32!)} = \frac{1}{2^{31}(32)} = \frac{1}{2^{36}} = 2^{-36} = (2^3)^{-12} = 8^{-12}$$

Thus, $x = –12$.

Example 25: The value of

$$E = \frac{(1+17)\left(1+\frac{17}{2}\right)\left(1+\frac{17}{3}\right)\ldots\left(1+\frac{17}{19}\right)}{(1+19)\left(1+\frac{19}{2}\right)\left(1+\frac{19}{3}\right)\ldots\left(1+\frac{19}{17}\right)} \text{ is}$$

(a) 1 (b) ${}^{36}C_{17}$

(c) 2/19 (d) ${}^{36}C_{18}$

Ans. (a)

Solution: We have

$$(1+k)\left(1+\frac{k}{2}\right)\left(1+\frac{k}{3}\right)\ldots\left(1+\frac{k}{n}\right)$$
$$= \frac{(1+k)(2+k)(3+k)\ldots(n+k)}{(2)(3)\ldots(n)} = \frac{(n+k)!}{k!n!}$$
$$= {}^{n+k}C_k$$

Thus, both the numerator and the denominator of E equals ${}^{36}C_{17} = {}^{36}C_{19}$.

$\therefore \quad E = 1.$

Example 26: Kunal Gaba has n objects, each of weight w. He weighs them in pairs and finds the sum of the weights of all possible pairs is 120g. When his friend Rakshit weighs them in triplets, the sum of all possible weights is 240g. The value of n is

(a) 7 (b) 6

(c) 5 (d) 10

Ans. (b)

Solution: According to the given condition

$({}^nC_2)\,(2w) = 120 \quad \Rightarrow \quad n(n-1)w = 120$

and $({}^nC_3)\,(3w) = 240 \quad \Rightarrow \quad n(n-1)\,(n-2)w = 480$

Thus, $\frac{n(n-1)(n-2)w}{n(n-1)w} = \frac{480}{120}$

$\Rightarrow \quad n - 2 = 4 \quad \Rightarrow \quad n = 6$

Example 27: If $[y]$ denote the greatest integer $\le y$, and $2\left[\frac{x}{8}\right]^2 + 3\left[\frac{x}{8}\right] = 20$, then x lies in the smallest interval $[a, b)$ where $b– a$ is equal to

(a) 6 (b) 5

(c) 4 (d) 8

Ans. (d)

Solution: $2\left[\frac{x}{8}\right]^2 + 3\left[\frac{x}{8}\right] - 20 = 0$

$$\Rightarrow \quad \left[\frac{x}{8}\right] = \frac{5}{2} \quad \text{or} \quad -4$$

As $\left[\frac{x}{8}\right]$ is an integer, we take

$$\left[\frac{x}{8}\right] = -4 \qquad \Rightarrow \qquad -4 \le x/8 < -3$$

$\Rightarrow -32 \le x < -24$. Thus, $a = -32$, and $b = -24$. Therefore, $b - a = 8$

Example 28: The sum $\sum_{i=0}^{m} \binom{10}{i}\binom{20}{m-i}$ (where $\binom{p}{q} = 0$ if $p < q$) is maximum where m is

(a) 5 (b) 10
(c) 15 (d) 20

Ans. (c)

Solution: We have $\sum_{i=0}^{m} \binom{10}{i}\binom{20}{m-i}$

= the number of ways of choosing m persons out of 10 men and 20 women

= the number of ways of choosing m persons out of 30 persons

= ${}^{30}C_m$

But ${}^{30}C_m$ is maximum for $m = 15$.

Example 29: Let T_n denote the number of triangles which can be formed by using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals

(a) 5 (b) 7
(c) 6 (d) 4

Ans. (b)

Solution: The number of triangles that can be formed by using the vertices of a regular polygon is ${}^nC_3$. That is, $T_n = {}^nC_3$

Now, $\quad T_{n+1} - T_n = 21$

$\Rightarrow \quad {}^{n+1}C_3 - {}^nC_3 = 21$

$\Rightarrow \quad {}^nC_2 + {}^nC_3 - {}^nC_3 = 21 \qquad [\because {}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r]$

$\Rightarrow \quad \frac{1}{2} n(n-1) = 21 \quad \Rightarrow \quad n = -6$ or 7.

As n is a positive integer, $n = 7$.

Example 30: An eight digit number divisible by 9 is to be formed by using 8 digits out of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 without replacement. The number of ways in which this can be done is

(a) 9! (b) 2(7!)
(c) 4(7!) (d) (36) (7!)

Ans. (d)

Solution: We have $0 + 1 + 2 + 3 \ldots + 8 + 9 = 45$

To obtain an eight digit number exactly divisible by 9, we must not use either (0, 9) or (1, 8) or (2, 7) or (3, 6) or (4, 5). [Sum of the remaining eight digits is 36 which is exactly divisible by 9.]

When, we do not use (0, 9), then the number of required 8 digit numbers is 8!.

When one of (1, 8) or (2, 7) or (3, 6) or (4, 5) is not used, the remaining digits can be arranged in 8!– 7! ways.{0 cannot be at extreme left.}

Hence, there are 8! + 4(8! – 7!) = (36) (7!) numbers in the desired category.

Example 31: The number of rational numbers lying in the interval (2015, 2016) all whose digits after the decimal point are non-zero and are in decreasing order is

(a) $\sum_{i=1}^{9} {}^9P_i$ (b) $\sum_{i=1}^{10} {}^9P_i$

(c) $2^9 - 1$ (d) $2^{10} - 1$

Ans. (c)

Solution: A rational number of the desired category is of the form 2015. $x_1 x_2 \ldots x_k$ where $1 \le k \le 9$ and $9 \ge x_1 > x_2 > \ldots > x_k \ge 1$. We can choose k digits out 9 in ${}^9C_k$ ways and arrange them in decreasing order in just one way. Thus, the desired number of rational numbers is ${}^9C_1 + {}^9C_2 + \ldots + {}^9C_9 = 2^9 - 1$.

Example 32: The number of functions f from the set $A = \{0, 1, 2\}$ in to the set $B = \{0, 1, 2, 3, 4, 5, 6, 7\}$ such that $f(i) \le f(j)$ for $i < j$ and $i, j \in A$ is

(a) ${}^8C_3$ (b) ${}^8C_3 + 2({}^8C_2)$
(c) ${}^{10}C_3$ (d) ${}^{10}C_4$

Ans. (c)

Solution: A function $f: A \to B$ such that $f(0) \le f(1) \le f(2)$ falls in one of the following four categories.

Case 1 $\quad f(0) < f(1) < f(2)$

There are ${}^8C_3$ functions in this category.

Case 2 $\quad f(0) = f(1) < f(2)$

There are ${}^8C_2$ functions in this category.

Case 3 $\quad f(0) < f(1) = f(2)$

There are again ${}^8C_2$ functions in this category.

Case 4 $\quad f(0) = f(1) = f(2)$

There are ${}^8C_1$ functions in this category.

Thus, the number of desired functions is

$${}^8C_3 + {}^8C_2 + {}^8C_2 + {}^8C_1 = {}^9C_3 + {}^9C_2 = {}^{10}C_3.$$

Example 33: The number of positive integral solution of the equation $x_1 x_2 x_3 x_4 x_5 = 1050$ is

(a) 1800 (b) 1600
(c) 1400 (d) none of these

Ans. (d)

◎ **Solution:** Using prime factorization of 1050, we can write the given equation as

$$x_1\, x_2\, x_3\, x_4\, x_5 = 2 \times 3 \times 5^2 \times 7$$

We can assign 2, 3 or 7 to any of 5 variables. We can assign entire 5^2 to just one variable in 5 ways or can assign $5^2 = 5 \times 5$ to two variables in ${}^5C_2$ ways. Thus, 5^2 can be assigned in

$${}^5C_1 + {}^5C_2 = 5 + 10 = 15 \text{ ways}$$

Thus, the required number of solutions is $5 \times 5 \times 5 \times 15 = 1875$.

Example 34: The number of ways in which we can arrange the digits 1, 2, 3, …, 9 such that the product of five digits at any of the five consecutive positions is divisible by 7 is

(a) 7! (b) ${}^9P_7$
(c) 8! (d) 5(7!)

Ans. (c)

◎ **Solution:** Let an arrangement of 9 digit number be $x_1\, x_2\, x_3\, x_4\, x_5\, x_6\, x_7\, x_8\, x_9$. Note that we require product of each of $(x_1, x_2, x_3, x_4, x_5)$; $(x_2, x_3, x_4, x_5, x_6)$; …; $(x_5, x_6, x_7, x_8, x_9)$ is divisible by 7.

This is possible if the 5th digit is 7.

Therefore, we can arrange the 9 digits in desired number of ways in 8! ways.

Example 35: Let X be a set containing n elements. The number of reflexive relations that can be defined on X is

(a) 2^{n^2} (b) $2^{n^2 - n}$
(c) $n(2^{n^2 - n})$ (d) $n(2^{n^2})$

Ans. (b)

◎ **Solution:** Let $X = \{x_1, x_2, \ldots x_n\}$. The set $X \times X$ contains n^2 elements. A reflexive relation R on X must contain (x_i, x_i) for $1 \le i \le n$, i.e. R must contain the n elements (x_1, x_1), (x_2, x_2) . . . (x_n, x_n) and any subset of the set containing remaining $n^2 - n$ elements. Therefore, the number of reflexive relations that can be defined on X is $2^{n^2 - n}$

Example 36: Suppose X contains m elements and Y contain n elements. The number of functions from X to Y is

(a) nm (b) n^m
(c) ${}^nP_m$ (d) ${}^nC_m$

Ans. (b)

◎ **Solution:** Let $X = \{x_1, x_2, \ldots x_m\}$ and

$$Y = \{y_1, y_2, \ldots y_n\}$$

For each x_i $(1 \le i \le m)$ we have n possible images. Therefore, the number of functions from X to Y is n^m.

Example 37: Three boys and three girls are to be seated around a circular table. Among them the boy X does not want any girl as a neighbour and girl Y does not want any boy as a neighbour. The number of possible arrangement is:

(a) 9 (b) 4
(c) 8 (d) 6

Ans. (b)

◎ **Solution:** Clearly X should sit between the two remaining boys and Y should sit between the remaining two girls. We have to arrange two groups $\boxed{B_1\ X\ B_2}$ and $\boxed{G_1\ Y\ G_2}$ along a circle. This can be done is (2 – 1)! ways. But group of boys (girls) can be arranged in 2! ways. Thus, the required number of ways is (2) (2) (1) = 4 ways.

Example 38: If ${}^nC_r : {}^nC_{r+1} : {}^nC_{r+2} = 1 : 2 : 3$, then r is equal to

(a) 5 (b) 4
(c) 3 (d) 0

Ans. (b)

◎ **Solution:** $\dfrac{2}{1} = \dfrac{{}^nC_{r+1}}{{}^nC_r} = \dfrac{n-r}{r+1}$

and $\quad \dfrac{3}{2} = \dfrac{n-r-1}{r+2}$

$\Rightarrow \quad \dfrac{3}{2} = \dfrac{2(r+1)-1}{r+2} \qquad [\because\ n - r = 2(r+1)]$

$\Rightarrow \quad 3r + 6 = 4r + 2 \Rightarrow r = 4.$

Example 39: The number of ways selecting three numbers from 1 to 30, so as to exclude every selection of three consecutive number is:

(a) 4030 (b) 4031
(c) 4032 (d) 4035

Ans. (c)

◎ **Solution:** The numbers of ways of selecting 3 number out of 30 is ${}^{30}C_3$. But this include 28 ways of selecting 3 consecutive numbers. [viz (1, 2, 3), (2, 3, 4), 3, 4, 5), ..., (28, 29, 30)]

Thus, required number of ways is ${}^{30}C_3 - 28 = 4032$.

Example:40: If all permutations of the letters of the word P E N C I L are arranged as in a dictionary, then 413th word is

(a) L I C N E P (b) L I C N P E
(c) L I C P N E (d) L I C P E N

Ans. (d)

◎ **Solution:** Letters in the word PENCIL are C, E, I, L, N, P.

There are 5! words each beginning with C, E and I. That is, there are 3(5!) = 360 words beginning with C, E, I

Words beginning with L have serial numbers 361 to 480. 4! = 24 words beginning with LC have serial numbers 361 to 384. Words beginning with LE are numbered from 385 to 408. Words beginning with LIC are numbered from 409 to 414. Now 414th word is L I C P N E and 413th word is L I C P E N.

Example: 41: Out of 10 consonants and 4 vowels, words with 6 consonants and 3 vowels are formed. The number of such words is

(a) $\frac{7}{11}(12!)$ (b) $\frac{9}{10}(12!)$

(c) $\frac{7}{10}(11!)$ (d) $\frac{4}{5}(12!)$

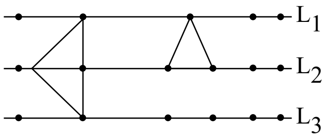
Ans. (a)

Solution: The number of ways of choosing 6 consonants out of 10 is $^{10}C_6$ and 3 vowels out of 4 in $^4C_3$. Thus number of such words is $(^{10}C_4)\,(^4C_3)\,(^9P_9) = \frac{7}{11}(12!)$

Example: 42: Let L_1, L_2, L_3 be three distinct parrallel lines in the XY-plane, p distinct the points are taken on each of the three lines. The maximum number of triangles than can be formed by these $3p$ points is:

(a) $p^2(4p - 3)$ (b) $p^3 + 3(^pC_2)$
(c) $p^2(3p - 4)$ (d) $(p + 1)^3 - 1$

Ans. (a)



Solution: To obtain a triangle we may take one point from each of the three lines. This will give us p^3 triangles. Alternatively, we may choose two lines and take one point on one of them and two points on the other line. This will give us
$(^3C_2)\,(2!)\,(p)\,(^pC_2) = 3p^2\,(p - 1)$ triangles
$\therefore$ required number of triangles is $p^3 + 3p^2\,(p - 1) = p^2(4p - 3)$

Alternative Solution

Choose three points not on the same lines. Number of ways = $^{3p}C_3 - 3(^pC_3) = p^2(4p - 3)$

Example : 43: The number of ways of selecting 4 letters out of the letters of the word MINIMAL is

(a) 16 (b) 17
(c) 18 (d) 20

Ans. (b)

Solution: Letters of the word , M I N I M A L are (M, M) (I, I), A, N, L

We can select 4 letters from letters of the word MINIMAL as follows:

Case 1 All letters are distinct.
This can be done in $^5C_4 = 5$ ways

Case 2 Exactly two letters are identical.
We can choose two identical letters in $^2C_1$ ways and two remaining distinct letters in $^4C_2 = 6$ ways
$\therefore$ In this case the selection can be made in (2) (6) = 12 ways.

Case 3 Two pairs of identical letters are selected.
This can be done in $^2C_2 = 1$ way
$\therefore$ two number of ways
$= 5 + 12 + 1 = 17.$

Example 44: Let $S = \left\{\begin{bmatrix} 18 & a \\ b & 25 \end{bmatrix} : a, b \in \mathbf{N}\right\}$

The number of singular matrices in S is

(a) 18 (b) 25
(c) 450 (d) infinite

Ans: (a)

Solution: det $\begin{bmatrix} 18 & a \\ b & 25 \end{bmatrix} = 2 \times 3^2 \times 5^2 - ab = 0$

$\Leftrightarrow \quad ab = 2 \times 3^2 \times 5^2$

$\Leftrightarrow \quad a \mid 2 \times 3^2 \times 5^2$

That is a is a divisor of $2 \times 3^2 \times 5^2$

Therefore, a is of the form $2^{a}\,3^{b}\,5^{g}$ where
$\alpha \in \{0, 1\}$, $\beta, \gamma \in \{0, 1, 2\}$.

Thus number of elements in S is $2 \times 3 \times 3 = 18$.

Example 45: The number of positive integers n such that 2^n divides $n!$ is

(a) exactly 1 (b) exactly 2
(c) infinite (d) none of these

Ans. (d)

Solution: The exponent of 2 in $n!$ is given by

$$E = \left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right] + \left[\frac{n}{2^3}\right] + \ldots$$

where $[x]$ denotes greatest integer $\le x$, As $[x] \le x \;\; \forall \;\; x$, $\left[\frac{n}{2^m}\right] = 0$ after finite number of terms. Thus we get

$$E < \frac{n}{2} + \frac{n}{2^2} + \frac{n}{2^3} + \ldots = \frac{n/2}{1 - 1/2} = n$$

Thus, there is no positive integer for which 2^n divides $n!$

Example 46: The range of the function $^{7-x}P_{x-3}$ is

(a) $\{1, 2, 3, 4\}$ (b) $\{1, 2, 3, 4, 5, 6\}$
(c) $\{1, 2, 3\}$ (d) $\{1, 2, 3, 4, 5\}$

Ans. (c)

Solution: We must have $7 - x \ge 1$, $x - 3 \ge 0$ and $x - 3 \le 7 - x$

$\Rightarrow \quad x \le 6$, $x \ge 3$ and $x \le 5$

Thus, $\quad 3 \le x \le 5$.

$\therefore \quad$ Range of $^{7-x}P_{x-3}$ is $\{^4P_0, {}^3P_1, {}^2P_2\} = \{1, 3, 2\}$

Example 47: The value of $^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$ is

(a) $^{56}C_3$ (b) $^{56}C_4$
(c) $^{55}C_4$ (d) $^{55}C_3$

Ans. (b)

Solution: $^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$

$$= {}^{50}C_4 + ({}^{55}C_3 + {}^{54}C_3 + \dots + {}^{50}C_3)$$
$$= ({}^{50}C_4 + {}^{50}C_3) + ({}^{51}C_3 + \dots + {}^{55}C_3)$$
$$= ({}^{51}C_4 + {}^{51}C_3) + ({}^{52}C_3 + \dots + {}^{55}C_3)$$
$$= ({}^{52}C_4 + {}^{52}C_3) + ({}^{53}C_3 + \dots + {}^{55}C_3)$$
$$= \dots\dots$$
$$= {}^{56}C_4$$

Example 48: A class consists of 4 boys and g girls. Every Sunday five students, including at least three boys go for a picnic to Appu Ghar, a different group being sent every week. During, the picnic, the class teacher gives each girl in the group a doll. If the total number of dolls distributed was 85, then value of g is

(a) 15 (b) 12
(c) 8 (d) 5

Ans. (d)

Solution: Number of groups having 4 boys and 1 girl

$$= ({}^4C_4)({}^gC_1) = g$$

and number of groups having 3 boys and 2 girls

$$= ({}^4C_3)({}^gC_2) = 2g(g-1)$$

Thus, the number of dolls distributed

$$= g(1) + (2)[2g(g-1)] = 4g^2 - 3g$$

We are given $4g^2 - 3g = 85 \Rightarrow g = 5$.

Example 49: Sum of the factors of 9! which are odd and are of the form $3m + 2$, where m is a natural number is

(a) 40 (b) 45
(c) 51 (d) 54

Ans. (a)

Solution: We have $9! = 2^7 \times 3^4 \times 5 \times 7$

Odd factors of the form $3m + 2$ are neither multiples of 2 nor multiples of 3. So the factors may be 1, 5, 7, 35 of which just 5 and 35 are of the form $3m + 2$. Their sum is 40.

Example 50: Sum of all three digit numbers (no digit being zero) having the property that all digits are perfect squares, is

(a) 3108 (b) 6216
(c) 13986 (d) none of these

Ans. (c)

Solution: The non-zero perfect square digits are 1, 4 and 9. 1 can occur at units place in $3 \times 3 = 9$ ways.

$\therefore$ Sum due to 1 at units place is 1×9. Similarly, sum due to 1 at tens place is $1 \times 10 \times 9$ and sum due to 1 at hundreds place is $1 \times 100 \times 9$. We can deal with the digits 4 and 9 in a similar way.

Thus, sum of the desired number is

$$(1 + 4 + 9)(1 + 10 + 100)(9) = 13986$$

Example 51: The number of ordered pairs (m, n), $m, n \in \{1, 2, \dots, 100\}$ such that $7^m + 7^n$ is divisible by 5 is

(a) 1250 (b) 2000
(c) 2500 (d) 5000

Ans. (c)

Solution: Note that 7^r ($r \in \boldsymbol{N}$) ends in 7, 9, 3 or 1 (corresponding to r = 1, 2, 3 and 4 respectively.)

Thus, $7^m + 7^n$ cannot end in 5 for any values of $m, n \in \boldsymbol{N}$. In other words, for $7^m + 7^n$ to be divisible by 5, it should end in 0.

For $7^m + 7^n$ to end in 0, the forms of m and n should be as follows:

	m	n
1	$4r$	$4s + 2$
2	$4r + 1$	$4s + 3$
3	$4r + 2$	$4s$
4	$4r + 3$	$4s + 1$

Thus, for a given value of m there are just 25 values of n for which $7^m + 7^n$ ends in 0. [For instance, if $m = 4r$, then n = 2, 6, 10, …, 98]

$\therefore$ there are $100 \times 25 = 2500$ ordered pairs (m, n) for which $7^m + 7^n$ is divisible by 5.

Example 52: An n-digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to be formed using only three digits 2, 5 and 7. The smallest value of n for which this is possible is

(a) 6 (b) 7
(c) 8 (d) 9

Ans. (b)

Solution: The number of n-digit distinct numbers that can be formed using digits 2, 5 and 7 is 3^n. We have to find n, so that $3^n \geq 900$

$\Rightarrow \quad 3^{n-2} \geq 100 \quad \Rightarrow \quad n - 2 \geq 5 \quad \Rightarrow \quad n \geq 7$.

Thus, the smallest value of n is 7.

Example 53: The number of ways of arranging p numbers out of 1, 2, 3, …, q so that maximum is q–2 and minimum is 2 (repetition of number is allowed) such that maximum and minimum both occur exactly once, $(p > 5, q > 3)$ is

(a) $^{p-3}C_{q-2}$ (b) $^pC_2\,(q-3)^{q-1}$
(c) $^pC_2 \times {}^qC_3$ (d) $p(p-1)(q-5)^{p-2}$

Ans. (d)

◎ **Solution:** First we take one of the numbers as 2 and one another as $q-2$. We can arrange these two numbers in $p(p-1)$ ways. We have to choose remaining $p-2$ numbers from the numbers 3, 4, …, $q-4$, $q-3$. This can be done in $(q-5)^{p-2}$ ways.
Thus, the total number of ways of arranging the numbers in desired way is

$$p(p-1)\,(q-5)^{p-2}.$$

◉ **Example 54:** The number of ways of selecting 4 cards of an ordinary pack of playing cards so that exactly 3 of them are of the same denomination is

(a) 2496 (b) ${}^{13}C_3 \times {}^4C_3 \times 48$
(c) ${}^{52}C_3 \times 48$ (d) none of these

Ans. (a)

◎ **Solution:** We can choose one denomination in ${}^{13}C_1$ ways, then 3 cards of this denomination can be chosen in ${}^4C_3$ ways and one remaining card can be chosen in ${}^{48}C_1$ ways. Thus, the total number of choices is $({}^{13}C_1)\,({}^4C_3)\,({}^{48}C_1) = 13 \times 4 \times 48 = 2496$.

◉ **Example 55:** The number of integers x, y, z, w such that $x + y + z + w = 20$ and $x, y, z, w \geq -1$, is

(a) ${}^{24}C_3$ (b) ${}^{25}C_3$
(c) ${}^{26}C_3$ (d) ${}^{27}C_3$

Ans. (d)

◎ **Solution:** Put $x = a-1$, $y = b-1$, $z = c-1$, $w = d-1$, then $a, b, c, d \geq 0$ and $(a-1)+(b-1)+(c-1)+(d-1) = 20$

$$\Rightarrow \qquad a + b + c + d = 24$$

The number of non-negative integral solutions of this equation is

$$ {}^{24+4-1}C_{4-1} = {}^{27}C_3$$

◉ **Example 56:** If 20% of three subsets (i.e., subsets containing exactly three elements) of the set $A = \{a_1, a_2, \ldots, a_n\}$ contain a_1, then value of n is

(a) 15 (b) 16
(c) 17 (d) 18

Ans. (a)

◎ **Solution:** The number of subsets of A containing exactly three elements is ${}^nC_3$ whereas the number of three subsets of A that contain a_1, is ${}^{n-1}C_2$. We are given,

$${}^{n-1}C_2 = \frac{20}{100}\left({}^nC_3\right)$$

$$\Rightarrow \frac{(n-1)(n-2)}{2} = \frac{1}{5}\,\frac{n(n-1)\,(n-2)}{(6)} \Rightarrow n = 15.$$

◉ **Example 57:** There are three piles of identical yellow, black and green balls and each pile contains at least 20 balls. The number of ways of selecting 20 balls if the number of black balls to be selected is twice the number of yellow balls, is

(a) 6 (b) 7
(c) 8 (d) 9

Ans. (b)

◎ **Solution:** Let the number of yellow balls be x, that of black be $2x$ and that of green be y. Then

$$x + 2x + y = 20 \text{ or } 3x + y = 20$$

$$\Rightarrow \qquad y = 20 - 3x.$$

As $0 \leq y \leq 20$, we get $0 \leq 20 - 3x \leq 20$

$\Rightarrow \quad 0 \leq 3x \leq 20 \quad$ or $\quad 0 \leq x \leq 6$

$\therefore$ The number of ways of selecting the balls is 7.

◉ **Example 58:** The total number of permutations of n (>1) different things taken not more that r at a time, when a thing may be repeated any number of times, is

(a) $\dfrac{n}{n-1}(n^r - 1)$ (b) $\dfrac{n^{r-1}}{n-1}$

(c) $\dfrac{n^r + 1}{n+1}$ (d) $\dfrac{n^r + n}{n-1}$

Ans. (a)

◎ **Solution:** When k $(1 \leq k \leq r)$ things are arranged, the number of possible arrangements, when repetition is allowed is

$$\underbrace{n \times n \times \ldots\ldots \times n}_{k \text{ times}} = n^k$$

Thus, the total possible arrangements is

$$n + n^2 + n^3 + \ldots + n^r = \frac{n\,(n^r - 1)}{n-1}.$$

◉ **Example 59:** The exponent of 7 in the prime factorization of ${}^{100}C_{50}$ is

(a) 0 (b) 2
(c) 4 (d) none of these

Ans. (a)

◎ **Solution:** We have ${}^{100}C_{50} = \dfrac{100!}{50!50!}$.

The exponent of 7 in 50! is $\left[\dfrac{50}{7}\right] + \left[\dfrac{50}{7^2}\right] = 7 + 1 = 8$,

and the exponent of 7 in 100! is $\left[\dfrac{100}{7}\right] + \left[\dfrac{100}{7^2}\right]$

$= 14 + 2 = 16$

Thus, exponent of 7 in the prime factorization of ${}^{100}C_{50}$ is $16 - 2(8) = 0$.

◉ **Example 60:** In a certain test there are n questions. In this test 2^k students gave wrong answers to at least $(n-k)$ questions, where $k = 0, 1, 2, \ldots, n$. If the total number of wrong answers is 4095, then value of n is

(a) 11 (b) 12
(c) 13 (d) 15

Ans. (b)

◎ **Solution:** The number of students answering at least r questions incorrectly is 2^{n-r}.
$\therefore$ the number of students answering exactly r $(1 \le r \le n-1)$ questions incorrectly is $2^{n-r} - 2^{n-(r+1)}$.
Also, the number of students answering all questions wrongly is $2^0 = 1$.
Thus, the total number of wrong answers is

$$1(2^{n-1} - 2^{n-2}) + 2(2^{n-2} - 2^{n-3}) + 3(2^{n-3} - 2^{n-4}) + \dots + (n-1)(2^1 - 2^0) + n(2^0)$$
$$= 2^{n-1} + 2^{n-2} + \dots + 2^0 = 2^n - 1.$$

Now, $2^n - 1 = 4095 \Rightarrow 2^n = 4096 = 2^{12} \Rightarrow n = 12$.

◉ **Example 61:** The number of ways of permuting letters of the word ENDEANOEL so that none of the letters D, L, N occurs in the last five positions is

(a) $5!$ (b) $2(5!)$
(c) $7(5!)$ (d) $21(5!)$

Ans. (b)

◎ **Solution:** Letters of the word ENDEANOEL are

$$3E\text{'s}, 2N\text{'s}, 1D, 1O, 1L, 1A$$

Letters D, L, N can be permuted at first 4 places in $\dfrac{4!}{2!}$ ways and the remaining letters can be permuted $\dfrac{5!}{3!}$ ways.

$\therefore$ required number of ways $\dfrac{4!}{2!} \times \dfrac{5!}{3!} = 2(5!)$

◉ **Example 62:** Sum of the series $\sum_{r=1}^{n} (r^2 + 1)(r!)$ is

(a) $(n + 1)!$ (b) $(n + 2)! - 1$
(c) $n(n + 1)!$ (d) none of these

Ans. (c)

◎ **Solution:** We can write

$$r^2 + 1 = (r + 2)(r + 1) - 3(r + 1) + 2$$

Thus, $$\sum_{r=1}^{n} (r^2 + 1)(r!) = \sum_{r=1}^{n} [(r+2)(r+1) - (r+1) - 2\{(r+1) - 1\}]\, r!$$

$$= \sum_{r=1}^{n} [(r+2)! - (r+1)!] - 2\sum_{r=1}^{n} \{(r+1)! - r!\}$$
$$= (n + 2)! - 2! - 2\{(n + 1)! - 1\} = n(n + 1)!$$

◉ **Example 63:** If letters of the word SACHIN are arranged in all possible ways and are written out as in a dictionary, then the word SACHIN appears at serial number

(a) 603 (b) 602
(c) 601 (d) 600

Ans. (c)

◎ **Solution:** Letters appearing in the word SACHIN are

$$A, C, H, I, N, S$$

The words beginning with letters A, C, H, I and N appear before the word SACHIN.

There are $5(5!) = 600$ words beginning with A, C, H, I and N.

Word SACHIN appears is the first word beginning with S. Therefore, SACHIN appears at serial number 601.

◉ **Example 64:** The set $S = \{1, 2, 3, \dots 12\}$ is to be partitioned into three sets A, B, C of equal size. Thus $A \cup B \cup C = S$, $A \cap B = B \cap C = C \cap A = \phi$. The number of ways to partition S is

(a) $\dfrac{12!}{3!(4!)^3}$ (b) $\dfrac{12!}{3!(3!)^3}$

(c) $\dfrac{12!}{(4!)^3}$ (d) $\dfrac{12!}{(3!)^4}$

Ans. (a)

◎ **Solution:** Each of the three sets A, B, C contains exactly 4 elements.

Thus, the number of ways of partitioning the set S is

$$\frac{1}{3!}\left({}^{12}C_4\right)\left({}^{8}C_4\right)\left({}^{4}C_4\right) = \frac{1}{3!}\,\frac{12!}{4!\,8!} \times \frac{8!}{4!\,4!}(1) = \frac{12!}{3!\,(4!)^3}$$

◉ **Example 65:** Ten letters of an alphabet are given. Words with five letters are formed by these given letters. Then the number of words which have at least one letter repeated is

(a) 69760 (b) 30240
(c) 99748 (d) none of these

Ans. (a)

◎ **Solution:** Number of five letter words that can be formed from 10 letters

$$= 10 \times 10 \times 10 \times 10 \times 10 = 10^5$$

Number of five letter words that have none of their letter repeated

$$= {}^{10}P_5 = 10 \times 9 \times 8 \times 7 \times 6 = 30240$$

$\therefore$ Number of words which have at least one letter repeated

$$= 10^5 - 30240 = 69760$$

◉ **Example 66:** The number of ways in which six '+' and four '–' signs can be arranged in a line such that no two '–' signs occur together is

(a) 30 (b) 35
(c) 6!5! (d) 10!

Ans. (b)

◎ **Solution:** Six '+' signs can be arranged in just one way. There are seven places for '–' signs as shown in the following figure marked with X.

$$X + X + X + X + X + X + X$$

We can choose 4 places out of 7 in ${}^7C_4 = 35$ ways.

◉ **Example 67:** The remainder when $x = 1! + 2! + 3! + 4! + \ldots + 100!$ is divided by 240, is

(a) 153 (b) 33
(c) 73 (d) 187

Ans. (a)

◎ **Solution:** For $r \ge 6$; $r!$ is divisible by 240. Thus, when x is divided by 240, the remainder is $1! + 2! + \ldots + 5! = 153$.

◉ **Example 68:** If $x \in \boldsymbol{N}$ and ${}^{x-1}C_4 - {}^{x-1}C_3 < \frac{5}{4}\,({}^{x-2}P_2)$, then x can take

(a) 8 values (b) 7 values
(c) 6 values (d) none of these

Ans. (b)

◎ **Solution:** For $1 \le x \le 3$, each of ${}^{x-1}C_4$, ${}^{x-1}C_3$ and ${}^{x-2}P_2$ is zero. For $x = 4$, ${}^{x-1}C_4 = 0$, ${}^{x-1}C_3 = 1$ and ${}^{x-2}P_2 = 2$ and the inequality is valid trivally. For $x \ge 5$, we have

$$\frac{{}^{x-1}C_4}{{}^{x-2}P_2} - \frac{{}^{x-1}C_3}{{}^{x-2}P_2} < \frac{5}{4}$$

$$\Rightarrow \quad \frac{(x-1)(x-4)}{4!} - \frac{x-1}{3!} < \frac{5}{4}$$

$$\Rightarrow \quad x^2 - 9x - 22 < 0 \Rightarrow (x - 11)(x + 2) < 0$$

$$\Rightarrow \quad x - 11 < 0 \ [\because x + 2 > 0]$$

Thus, $5 \le x < 11$. Hence, x can take 7 values viz. 4, 5, 6, 7, 8, 9 and 10.

◉ **Example 69:** The total number of ways in which 5 balls of different colours can be distributed among three persons so that each person gets at least one ball is:

(a) 75 (b) 150
(c) 210 (d) 243

Ans. (b)

◎ **Solution:** 5 balls can be distributed to 3 persons by giving (2, 2, 1) balls or by giving (3, 1, 1) balls. Each of the above distribution has three such ways. Thus, the required number of ways

$$= (3)({}^5C_2)({}^3C_2)({}^1C_1) + 3({}^5C_3)({}^2C_1)({}^1C_1)$$
$$= (3)(10)(3)(1) + (3)(10)(2)(1) = 150$$

◉ **Example 70:** Let $n \ge 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and rest by red. If the number of red and blue line segments are equal than value of n is

(a) 4 (b) 5
(c) 6 (d) 7

Ans. (b)

◎ **Solution:** Total number of line segments is ${}^nC_2$. The number of line segments joining adjacent points is n and number of lines joining non-adjacent points in ${}^nC_2 - n$. We are given

$${}^nC_2 - n = n \Rightarrow \frac{1}{2}n(n-1) = 2n$$

$$\Rightarrow \quad n = 5.$$

◉ **Example 71:** The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only, is

(a) 55 (b) 66
(c) 77 (d) 88

Ans. (c)

◎ **Solution:** Let x_i $(1 \le i \le 7)$ be the digit at the ith place. As $1 \le x_i \le 3$, and $x_1 + x_2 + \ldots + x_7 = 10$, at most one x_i can be 3.

Two cases arise.

Case 1.

Exactly one of x_i's is 3. In this case exactly one of the remaining x_i's is 2.

In this case, the number of seven digit numbers is

$$\frac{7!}{5!} = 7 \times 6 = 42$$

Case 2.

None of x_i's is 3.

In this case exactly three of x_i's is 2 and the remaining four x_i's are 1.

In this case, the number of seven digit number is

$$\frac{7!}{3!4!} = 35$$

Hence, the required seven digit numbers is 42 + 35 = 77.

◉ **Example 72:** Let $S = \{1, 2, 3, 4\}$. The total number of unordered pairs of disjoint subsets of S is equal to

(a) 25 (b) 34
(c) 42 (d) 41

Ans. (d)

◎ **Solution:** Let A and B be two subsets of S. If $x \in S$, then x will not belong to $A \cap B$ if x belongs to at most one of A, B. This can happen in 3 ways.

Thus, there are $3^4 = 81$ subsets of S for which $A \cap B = \phi$.

Out of these there is just one way for which $A = B = \phi$.

As, we, are interested in unordered pairs of disjoint sets, the number of such subsets is

$$\frac{1}{2}\left(3^4 - 1\right) + 1 = 41.$$

Example 73: How many different words can be formed by jumbling the letters in the word MISSISSIPPI in which no two S are adjacent?

(a) $(8)\left({}^6C_4\right)\left({}^7C_4\right)$ (b) $(6)(7)\left({}^8C_4\right)$

(c) $(6)(8)\left({}^7C_4\right)$ (d) $(7)\left({}^6C_4\right)\left({}^8C_4\right)$

Ans. (d)

Solution: We can permute M, I, I, I, I, P, P in $\frac{7!}{4!2!}$ ways. Corresponding to each arrangement of these seven letters, we have 8 places where S can be arranged as shown below with X.

$$X \square X \square X \square X \square X \square X \square X \square$$

We can choose 4 places out of 8 in ${}^8C_4$ ways.
Thus, the required number of ways

$$= ({}^8C_4)\left(\frac{7!}{4!2!}\right) = (7)\left({}^8C_4\right)\left({}^6C_4\right)$$

Example 74: From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. Then the number of such arrangement is

(a) at least 750 but less than 1000
(b) at least 1000
(c) less than 500
(d) at least 500 but less than 750

Ans. (b)

Solution: We can choose 4 novels out of 6 in ${}^6C_4$ ways and 1 dictionary out of 3 in $3C_1$ ways. We can arrange 4 novels and 1 dictionary in the middle in 4! ways. Thus, required number of ways

$$= ({}^6C_4)\,({}^3C_1)\,(4!) = 1080 > 1000$$

Example 75: There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out and then transferred to the other. The number of ways in which this can be done is

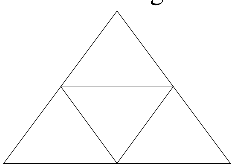
(a) 66 (b) 108
(c) 3 (d) 36

Ans. (b)

Solution: The number of ways of transferring the balls

$$= ({}^3C_2)\,({}^9C_2) = (3)\,(36) = 108$$

Example 76: The number of ways in which the diagram in Figure can be coloured so that each of the smaller triangle is painted with one of the three colours yellow, pink or green and no two adjacent regions are painted with the same colour, is



(a) 24 (b) 12
(c) 36 (d) 16

Ans. (a)

Solution: This can be done in the following ways:

(i) paint the central triangle with one of the three colours;
(ii) paint each of the remaining triangles with any one of the two remaining colours.

Thus, the required number of ways

$$= 3 \times 2 \times 2 \times 2 = 24.$$

Example 77: Four married couples are to be seated in a row having 8 chairs. The number of ways so that spouses are seated next to each other, is

(a) 72 (b) 186
(c) 384 (d) 516

Ans. (c)

Solution: Let us denote the four married couples of C_1, C_2, C_3 and C_4. We consider each couple as one unit. We can permute four units in 4! ways. Each couple can be seated in 2! ways. Thus, the required number of ways is

$$(4!)\,(2)\,(2)\,(2)\,(2) = 384.$$

Example 78: 25 lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent. The number of points in which these lines intersect, is

(a) 300 (b) 315
(c) 325 (d) 450

Ans. (a)

Solution: The number of points in which the lines can intersect = the number of ways of choosing two lines out of 25

$${}^{25}C_2 = \frac{25 \times 24}{2} = 300.$$

Example 79: Letter of the word INDIANOIL are arranged in all possible ways. The number of permutations in which A, I, O occur only at odd places, is

(a) 720 (b) 360
(c) 240 (d) 120

Ans. (c)

Solution: A, I, I, I, O can occur at odd places in $\frac{5!}{3!}$ ways, and the remaining letters N, N, D, L can be arranged at the remaining places in $\frac{4!}{2!}$ ways.

Thus, the required number of ways $= \left(\frac{5!}{3!}\right)\left(\frac{4!}{2!}\right) = 240$ ways

Example 80: Let $a_n = \sum_{r=0}^{n} \frac{1}{\left({}^nC_r\right)^2}$ and $b_n = \sum_{r=0}^{n} \frac{(n-r)}{\left({}^nC_r\right)^2}$, then b_n equals

(a) a_n (b) na_n

(c) $\frac{n}{2}a_n$ (d) 0

Ans. (c)

◎ **Solution:** We have

$$b_n = \sum_{r=0}^{n} \frac{n-r}{\left({}^nC_r\right)^2} = \sum_{r=0}^{n} \frac{n-r}{\left({}^nC_{n-r}\right)^2} = \sum_{r=0}^{n} \frac{r}{\left({}^nC_r\right)^2}$$

Thus,

$$b_n + b_n = \sum_{r=0}^{n} \frac{n-r}{\left({}^nC_r\right)^2} + \sum_{r=0}^{n} \frac{r}{\left({}^nC_r\right)^2} = n\sum_{r=0}^{n} \frac{1}{\left({}^nC_r\right)^2} = n\,a_n$$

$$\Rightarrow \qquad b_n = \frac{n}{2}\,a_n\,.$$

Example 81: A committee of at least three members is to be formed from a group of 6 boys and 6 girls such that it always has a boy and a girl.The number of ways to form such a committee is

(a) $2^{11} - 2^6 - 13$ (b) $2^{12} - 2^7 - 35$
(c) $2^{11} - 2^7 - 35$ (d) $2^{12} - 2^7 - 13$.

Ans. (b)

◎ **Solution:** Required number of ways

$$= ({}^6C_1 + {}^6C_2 + \cdots + {}^6C_6)\,({}^6C_1 + {}^6C_2 + \cdots {}^6C_6) - ({}^6C_1)\,({}^6C_1)$$
$$=(2^6-1)\,(2^6-1) - 36 = 2^{12} - 2^7 - 35$$

Example 82: There are 10 points in a plane, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then

(a) $N \le 100$ (b) $100 < N \le 140$
(c) $140 < N \le 190$ (d) $N > 190$.

Ans. (a)

◎ **Solution:** Number of triangles

$$= {}^{10}C_3 - {}^6C_3 = 120 - 20 = 100$$

Example 83: Assuming the balls to be identical except for difference of colours the number of ways in which one or more ball can be selected from 10 white, 9 green and 7 black balls is:

(a) 629 (b) 630
(c) 879 (d) 880

Ans. (c)

◎ **Solution:** We can select 0 or more balls out of 10 white balls in 11 ways. [select 0 or 1 or 2 ⋯ or 10 white balls.] Thus, the number of ways of making a selection of 0 or more balls

$$= (1+10)\,(1+9)(1+7) = 880$$

Therefore, number of ways of selecting at least one ball = 880 – 1 = 879.

Example 84: Let a_n = number of n-digit positive integers formed by the digits 0, 1 or both such that no consecutive digits in them is 0.

Let b_n = number of such n digit integers ending with digit 1, and c_n = the numner of such n digit integers ending with digit 0, then for $n \ge 3$,

(a) $a_n = b_{n-1} + c_{n-2}$ (b) $a_n = b_{n-1} + c_{n-1}$
(c) $a_n = b_{n-2} + c_{n-1}$ (d) $a_n = b_n + c_n$

Ans. (d)

◎ **Solution:** Note that all such numbers begin with 1.
If number ends in 1, then there $b_n = a_{n-1}$ such numbers. If number ends in 0, then $(n-1)$th digit cannot be 0 and thus, there are $c_n = a_{n-2}$ such numbers.

$$\therefore \qquad a_n = b_n + c_n = a_{n-1} + a_{n-2}$$

Example 85: In Example 84, a_8 equals

(a) 21 (b) 34
(c) 38 (d) 41

Ans. (b)

◎ **Solution:** If # (S) = number of elements of S, then

$a_1 = 1$, $a_2 = \#\,\{10,11\} = 2$,
$a_3 = \#\,\{101, 110, 111\} = 3$

Next, $a_n = b_n + c_n = a_{n-1} + a_{n-2}\ \forall\ n \ge 3$,
we get $a_4 = a_3 + a_2 = 5$,
$a_5 = a_4 + a_3 = 8$, $a_6 = 13$, $a_7 = 21$
and $a_8 = 34$

Example 86: The number of natural numbers less than one million that can be formed by using the digits, 0, 2, and 3 is

(a) 728 (b) 726
(c) 730 (d) 732

Ans. (a)

◎ **Solution:** The number of k digit natural numbers formed by using the digit 0,2 and 3 is $2(3^{k-1})$.

∴ Number of natural numbers less than one million formed by 0,2 and 3 is

$$= 2 + (2)\,(3) + (2)\,(3^2) + 2(3^3) + 2(3^4) + 2(3^5)$$
$$= 728$$

Assertion-Reason Type Questions

Example 87:

Statement-1:
The number of ways of distributing 10 identical balls in 4 distinct boxes such that no box is empty is ${}^9C_3$

Statement-2: The number of ways of choosing 3 places from 9 different place is ${}^9C_3$.

Ans. (b)

◎ **Solution:** Let x_i = number of balls put in ith box, then $x_1 + x_2 + x_3 + x_4 = 10$ where $x_i \geq 1$.

Put $x_i = y_i + 1$, so that equation becomes

$y_1 + y_2 + y_3 + y_4 = 6$. where $y_i \geq 0$

Number of non-negative integral solution of the above equation

= Number of ways of arranging 6 identical balls and 3 identical separators

$= \dfrac{(6+3)!}{6!3!} = {}^9C_3$ number of ways of choosing 3 places out of 9 different places.

Example 88: Statement-1: If n is a natural number then $\dfrac{(n^2)!}{(n!)^{n+1}}$ is a natural number.

Statement-2: The number of ways of dividing mn students into m groups each containing n students is $\dfrac{(mn)!}{m!(n!)^m}$

Ans. (a)

◎ **Solution:** The number of ways of selecting students for the first group is ${}^{mn}C_n$; for the second group is ${}^{mn-n}C_n$ and so on.

$\therefore$ the number of ways of dividing (mn) students into m numbered groups is

$$({}^{mn}C_n)\,({}^{mn-n}C_n)\cdots({}^{n}C_n)$$

$$= \frac{(mn)!}{n!(mn-n)!}\,\frac{(mn-n)!}{n!(mn-2n)!}\cdots\frac{n!}{n!\,0!}$$

$$= \frac{(mn)!}{(n!)^m}$$

As groups are not to be numbered, the desired number of ways is $\dfrac{(mn)!}{m!(n!)^m}$

$\therefore$ Statement-2 is true.

For statement-1, put $m = n$.

Example 89: Let $n \in \boldsymbol{N}$, and

$$f(n) = \begin{vmatrix} n & n+1 & n+2 \\ {}^nP_n & {}^{n+1}P_{n+1} & {}^{n+2}P_{n+2} \\ {}^nC_n & {}^{n+1}C_{n+1} & {}^{n+2}C_{n+2} \end{vmatrix}$$

Statement-1: $f(n)$ is an integer for all $n \in \mathbf{N}$.

Statement-2: If elements of a determinant are integers, then determinant itself is an integer.

Ans. (a)

◎ **Solution:** If each element of a determinant is an integer, then its each cofactor is an integer, and hence determinant itself is an integer.

Example 90: Statement-1: $\displaystyle\sum_{j=1}^{n}\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j} = 3^n - 2^n$

Statement-2: $\displaystyle\sum_{k=1}^{n}\left({}^nC_k\right)^2 = {}^{2n}C_n$

Ans. (c)

◎ **Solution:** $\displaystyle\binom{n}{k}\binom{k}{j} = \frac{n!}{k!(n-k)!}\,\frac{k!}{j!(k-j)!}$

$$= \frac{n!}{j!(n-j)!}\,\frac{(n-j)!}{(k-j)!\,(n-k)!}$$

$$= \binom{n}{j}\binom{n-j}{k-j}$$

Thus, $\displaystyle\sum_{j=1}^{n}\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j} = \sum_{j=1}^{n}\binom{n}{j}\sum_{k=j}^{n}\binom{n-j}{k-j}$

But $\displaystyle\sum_{k=j}^{n}\binom{n-j}{k-j} = \sum_{l=0}^{n-j}\binom{n-j}{l} = 2^{n-j}$

$\therefore$ $\displaystyle\sum_{j=1}^{n}\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j} = \sum_{j=1}^{n}\binom{n}{j}2^{n-j} = (2+1)^n - 2^n$

$= 3^n - 2^n$

Statement-2 is false as $\displaystyle\sum_{k=1}^{n}\left({}^nC_k\right)^2 = {}^{2n}C_n - 1$

Example 91: Statement-1: $\displaystyle T_n = \sum_{k=1}^{n}k\left({}^{2n}C_k\right) = 2^{2n-2}$

Statement-2: $\displaystyle\sum_{k=0}^{n}{}^{2n+1}C_k = 2^{2n}$

Ans. (d)

◎ **Solution:** $\displaystyle 2\sum_{k=0}^{n}{}^{2n+1}C_k = \sum_{k=0}^{n}{}^{2n+1}C_k + \sum_{k=0}^{n}{}^{2n+1}C_k$

$$= \sum_{k=0}^{n}{}^{2n+1}C_k + \sum_{k=0}^{n}{}^{2n+1}C_{2n+1-k}$$

$$= \sum_{k=0}^{2n+1}{}^{2n+1}C_k = 2^{2n+1}$$

$\Rightarrow$ $\displaystyle\sum_{k=0}^{n}{}^{2n+1}C_k = 2^{2n}$

$\therefore$ Statement-2 is true.

Using $k({}^{2n}C_k) = 2n({}^{2n-1}C_{k-1})$, for $k \geq 1$, we get

$$T_n = 2n\sum_{k=1}^{n}\left({}^{2n-1}C_{k-1}\right) = 2n\sum_{j=0}^{n-1}{}^{2n-1}C_j$$

$$= 2n(2^{2n-2}) = n(2^{2n-1})$$

Example 92: **Statement-1:** The number of ways of distributing at most 12 toys to three children A_1, A_2 and A_3 so that A_1 gets at least one, A_2 at least three and A_3 at most five, is 145.

Statement-2: The number of non-negative integral solutions of $x_1 + x_2 + x_3 \le m$ is ${}^{m-1}P_2$.

Ans. (d)

Solution: Suppose A_i gets x_i toys, then

$$x_1 + x_2 + x_3 \le 12.$$

Let $\quad x_4 = 12 - (x_1 + x_2 + x_3)$, then

$$x_1 + x_2 + x_3 + x_4 = 12 \qquad (1)$$

The number of non-negative integral solutions of (1)

$= \text{coefficient of } t^{12} \text{ in } (t + t^2 + \cdots)(t^3 + t^4 + t^5 + \cdots)(1 + t + \cdots + t^5) \times (1 + t + t^2 + \cdots)$

$= \text{coefficient of } t^{12} \text{ in } t^4(1 - t^6)(1 - t)^{-4}$

$= \text{coefficient of } t^8 \text{ in } (1 - t^6)(1 + {}^4C_1 t + {}^5C_2 t^2 + \cdots)$

$= {}^{11}C_8 - {}^5C_2 = 165 - 10 = 155$

Statement-2 is false as the number of non-negative integral solutions of

$$x_1 + x_2 + x_3 \le m$$

equals the number of non-negative integral solution of

$$x_1 + x_2 + x_3 + x_4 = m,$$

which equals ${}^{m+3}C_m$.

LEVEL 2

Straight Objective Type Questions

Example 93: The letters of the word COCHIN are permuted and all the permutations are arranged in an alphabetical order as in an English dictionary. The number of words that appear before COCHIN is

(a) 360 (b) 192
(c) 96 (d) 48

Ans. (c)

Solution: Letters appearing in the word COCHIN are

$$C, C, H, I, N, O$$

Words appearing before COCHIN are of the form

$$C\,X\,\text{- - - -}$$

where X is one of the letters C, H, I, N and the four remaining places can be filled by the remaining four letters.
Thus, the number of words before COCHIN is (4) (4!) = 96

Example 94: If r, s, t are prime numbers and p, q are natural numbers such that LCM of p, q is $r^2 s^4 t^2$, then the number of ordered pairs (p, q) is

(a) 252 (b) 254
(c) 225 (d) 224

Ans. (c)

Solution: Numbers p and q must be of the form

$$p = r^a s^b t^c, \quad q = r^\alpha s^\beta t^\gamma$$

where $\quad 0 \le a, \alpha \le 2$ and at least one of a, α is 2
$\quad 0 \le b, \beta \le 4$ and at least one of b, β is 4
$\quad 0 \le c, \gamma \le 2$ and at least one of c, γ is 2

Possible values of (a, α), and (c, γ) are

(0, 2), (1, 2), (2, 2), (2, 0), (2, 1).

Possible values of (b, β) are

(0, 4), (1, 4), (2, 4), (3, 4), (4, 4), (4, 0), (4, 1), (4, 2), (4, 3).

Thus, number of possible order pairs (p, q) is $5 \times 9 \times 5 = 225$.

Example 95: If $n > 1$ and n divides $(n - 1)! + 1$, then

(a) n must be prime
(b) n must be divisible by exactly two primes
(c) n must be a composite number
(d) none of these.

Ans. (a)

Solution: If n is not prime, then there exists $r \in \mathbf{N}$ such that $2 \le r \le n - 1$ and $r \mid n$.
As $r \mid n$ and $n \mid [(n - 1)! + 1]$, we get

$r \mid [(n - 1)! + 1]$

As $2 \le r \le n - 1$, $r \mid (n - 1)!$, therefore $r \mid 1$. A contradiction.

Example 96: Which of the following statement is false?

(a) There exist 100 consecutive natural numbers none of which is prime
(b) There exist 1000 consecutive natural number none of which is prime.
(c) Given $n \in \mathbf{N}$, there exist n consecutive natural numbers none of which is prime.
(d) none of these.

Ans. (d).

Solution: Given any $n \in \mathbf{N}$, then

$(n + 1)! + 2, (n + 1)! + 3, \ldots\ldots, (n + 1)! + (n + 1)$

are n consecutive natural numbers none of which is prime.

Example 97: The results of 21 football matches (win, lose or draw) are to be predicted. The number of forecasts that contain exactly 18 correct results is

(a) ${}^{21}C_3\, 2^{18}$ (b) ${}^{21}C_{18}\, 2^3$
(c) $3^{21} - 2^{18}$ (d) ${}^{21}C_3\, 3^{21} - 2^{18}$

Ans. (b)

Solution: 18 correct results can be predicted in ${}^{21}C_{18}$ ways and 3 wrong results in 2^3 ways. Thus, required number of ways is ${}^{21}C_{18}\, 2^3$.

Example 98: There are 15 points in a plane, no two of which are in a straight line except 4, all of which are in a straight line. The number of triangles that can be formed by using these 15 points is

(a) 404 (b) 415
(c) 451 (d) 490

Ans. (c)

Solution: The required number of triangle is

$$ {}^{15}C_3 - {}^4C_3 = 455 - 4 = 451 $$

Example 99: If $x, y \in (0, 30)$ such that

$$ \left[\frac{x}{3}\right] + \left[\frac{3x}{2}\right] + \left[\frac{y}{2}\right] + \left[\frac{3y}{4}\right] = \frac{11}{6}x + \frac{5}{4}y $$

(where $[x]$ denotes greatest integer $\le x$), then number of ordered pairs (x, y) is

(a) 0 (b) 2
(c) 4 (d) none of these

Ans. (d)

Solution: Let $\{x\} = x - [x]$ denote the fractional part of x. Note that $0 \le \{x\} < 1$. We can write the given equation as

$$ \frac{x}{3} - \left\{\frac{x}{3}\right\} + \frac{3x}{2} - \left\{\frac{3x}{2}\right\} + \frac{y}{2} - \left\{\frac{y}{2}\right\} + \frac{3y}{4} - \left\{\frac{3y}{4}\right\} = \frac{11}{6}x + \frac{5}{4}y $$

$$ \Rightarrow \quad \left\{\frac{x}{3}\right\} + \left\{\frac{3x}{2}\right\} + \left\{\frac{y}{2}\right\} + \left\{\frac{3y}{4}\right\} = 0 $$

As each number on the L.H.S. lies in the interval $0 \le x < 1$, we must have

$$ \left\{\frac{x}{3}\right\} = \left\{\frac{3x}{2}\right\} = \left\{\frac{y}{2}\right\} = \left\{\frac{3y}{4}\right\} = 0 $$

$$ \Rightarrow \quad \frac{x}{3}, \frac{3x}{2}, \frac{y}{2} \text{ and } \frac{3y}{4} \text{ must be integers.} $$

$\therefore$ $x = 6, 12, 18, 24,$ $y = 4, 8, 12, 16, 20, 24, 28$

$\Rightarrow$ Number of ordered pairs (x, y) equals $4 \times 7 = 28$.

Example 100: The sum $\sum\limits_{0 \le i \le j \le 10}\sum ({}^{10}C_j)({}^jC_i)$ is equal to

(a) $2^{10} - 1$ (b) 2^{10}
(c) $3^{10} - 1$ (d) 3^{10}

Ans. (c)

Solution: We have $\sum\limits_{0 \le}\sum\limits_{i < j \le 10} {}^{10}C_j({}^jC_i)$

$$ = {}^{10}C_1({}^1C_0 + {}^1C_1) + {}^{10}C_2({}^2C_0 + {}^2C_1 + {}^2C_2) + \ldots + {}^{10}C_{10}({}^{10}C_0 + {}^{10}C_1 + \ldots + {}^{10}C_{10}) $$

$$ = {}^{10}C_1(2) + {}^{10}C_2(2^2) + {}^{10}C_3(2^3) + \ldots + {}^{10}C_{10}(2^{10}) $$

$$ = (2 + 1)^{10} - 1 = 3^{10} - 1. $$

Example 101: There are p letters a, q letters b, r letters c. The number of ways of selecting k letters out of these if $p < k < q < r$ is

(a) $\frac{1}{2}(p + 1)^2 - k$

(b) $\frac{1}{2}(p + 1)(2k - p)$

(c) $\frac{1}{3}(p + 1)(q + 1)(r + 1) - k$

(d) none of these

Ans. (b)

Solution: Let x $a's$, y $b's$ and z $c's$ be selected. Then number of selections is equal to the number of non-negative integral solutions of

$$ x + y + z = k \tag{1} $$

If we take $x = l$, $0 \le x \le p$ then $y + z = k - l$ and its number of solutions is $k - l + 1$.

Thus, the desired number of selections

$$ \sum_{l=0}^{p}(k - l + 1) = (k + 1)(p + 1) - \frac{1}{2}(p + 1)(p + 2) $$

$$ = \frac{1}{2}(p + 1)(2k - p). $$

Example 102: The number of ways of choosing n objects out of $(3n + 1)$ objects of which n are identical and $(2n + 1)$ are distinct, is

(a) 2^{2n} (b) 2^{2n+1}
(c) $2^{2n} - 1$ (d) none of these

Ans. (a)

Solution: If we choose k $(0 \le k \le n)$ identical objects, then we must choose $(n - k)$ distinct objects. This can be done in ${}^{2n+1}C_{n-k}$ ways.Thus, the required number of ways

$$ = \sum_{k=0}^{n} {}^{2n+1}C_{n-k} = {}^{2n+1}C_n + {}^{2n+1}C_{n-1} + \ldots + {}^{2n+1}C_0 = 2^{2n}. $$

Example 103: The greatest common divisor ${}^{31}C_3$, ${}^{31}C_5$, …, ${}^{31}C_{29}$ is

(a) 31 (b) 2
(c) 17 (d) none of these

Ans. (a)

Solution: We have, for $3 \le r \le 29$

$$ r!\,(31 - r)!\,({}^{31}C_r) = 31! $$

As the prime 31 divides R.H.S. and 31 does not divide $r!$ and $(31 - r)!$ for $3 \le r \le 29$, we get $31 \mid ({}^{31}C_r)$.

Also, since ${}^{31}C_{29} = (31)(3)(5)$, ${}^{31}C_3 = (31)(29)(5)$ and ${}^{31}C_5 = (31)(29)(7)$

No prime other than 31 can divide all the numbers.

Thus greatest common divisors of the given numbers is 31.

Example 104: For $x \in \boldsymbol{R}$, let $[x]$ denote the greatest integer $\le x$, then value of $\left[-\frac{1}{3}\right] + \left[-\frac{1}{3} - \frac{1}{100}\right] + \left[-\frac{1}{3} - \frac{2}{100}\right] + \ldots + \left[-\frac{1}{3} - \frac{99}{100}\right]$ is

(a) – 100 (b) – 123
(c) – 135 (d) – 153

Ans. (c)

Solution: For $0 \le r \le 66$, $0 \le \frac{r}{100} < \frac{2}{3} \Rightarrow -\frac{2}{3} < -\frac{r}{100} \le 0$

$$\Rightarrow \quad -\frac{1}{3} - \frac{2}{3} < -\frac{1}{3} - \frac{r}{100} \le -\frac{1}{3}$$

$$\therefore \quad \left[-\frac{1}{3} - \frac{r}{100}\right] = -1 \qquad \text{for } 0 \le r \le 66$$

Also, for $67 \le r \le 100$, $\frac{67}{100} \le \frac{r}{100} \le 1 \Rightarrow -1 \le -\frac{r}{100} \le -\frac{67}{100}$

$$\Rightarrow \quad -\frac{1}{3} - 1 \le -\frac{1}{3} - \frac{r}{100} \le -\frac{1}{3} - \frac{67}{100}$$

$$\therefore \quad \left[-\frac{1}{3} - \frac{r}{100}\right] = -2 \text{ for } 67 \le r \le 100$$

Hence, $\sum_{r=0}^{100} \left[-\frac{1}{3} - \frac{r}{100}\right] = 67(-1) + 2(-34) = -135.$

Example 105: If $n \le 8$ and ${}^{n}C_{r+1} = 20$ and ${}^{n-1}C_{r} = 10$, then n is equal to

(a) 5 (b) 6
(c) 7 (d) 8

Ans. (b)

Solution $\frac{{}^{n}C_{r+1}}{{}^{n-1}C_{r}} = \frac{20}{10}$

$$\Rightarrow \quad \frac{n!}{(r+1)!\,(n-r-1)!} \cdot \frac{r!(n-r-1)!}{(n-1)!} = 2$$

$$\Rightarrow \quad \frac{n}{r+1} = 2$$

This shows that n is even and $r + 1 = n/2$ Now. ${}^{n}C_{n/2} = 20$ and $n \le 8$

$\Rightarrow \quad n = 6.$

EXERCISE

Concept-based Straight Objective Type Questions

1. If $x \in \mathbf{N}$ and $\frac{(2x)!}{3!(2x-3)!} : \frac{x!}{2!(x-2)} = 44 : 3$, then x is equal to

(a) 6 (b) 7
(c) 11 (d) 12

2. If $x \in \mathbf{N}$ and $\frac{(x+2)!}{(2x-1)!} \cdot \frac{(2x+1)!}{(x+3)!} = \frac{72}{7}$, then x is equal to

(a) 3 (b) 4
(c) 5 (d) 6

3. If ${}^{m+n}P_2 = 90$ and ${}^{m-n}P_2 = 30$, then (m, n) is equal to

(a) (12, 8) (b) (8, 2)
(c) (6, 4) (d) (7, 4)

4. The number of ways in which you can put five beads of five different colours to form a necklace is:

(a) 12 (b) 24
(c) 60 (d) 120

5. 15 lines are drawn in a plane in such a way that no two of them are parallel and no three are concurrent. The number of points of intersections of these lines is

(a) 455 (b) 465
(c) 475 (d) 485

6. Suppose $p \in \mathbf{N}$, $n = {}^{p}C_2$ and $m = {}^{n}C_2$. If $4m : n = 18 : 1$, then p is equal to

(a) 11 (b) 9
(c) 7 (d) 5

7. If ${}^{2n}C_4$, ${}^{2n}C_5$ and ${}^{2n}C_6$ are in A.P, then n is equal to

(a) 14 (b) 12
(c) 7 (d) 6

8. If ${}^{n}P_{r} = 1680$ and ${}^{n}C_{r} = 70$, then n is equal to

(a) 5 (b) 7
(c) 8 (d) 10

9. If ${}^{n+1}C_{r+1} : {}^{n}C_{r} : {}^{n-1}C_{r-1} = 11 : 6 : 3$, then value of n is

(a) 12 (b) 10
(c) 18 (d) 21

10. Suppose P is a set containing n distinct elements. Let $S = \{(x, y, z.) \mid x, y, z, \in, P$ and at least two of x, y, z are equal$\}$.

The number of elements in S is

(a) $n(3n - 2)$ (b) $n^2 (n - 2)$
(c) $n^3 - {}^{n}C_3$ (d) $n (5n - 4)$

LEVEL 1

Straight Objective Type Questions

11. A man invites 6 non-vegeterian and 5 vegeterian friends for a dinner party. He arrange 6 non-vegeterian friends on one round table and 5 vegeterian friends along another round table. The number of ways this can be done is:
(a) 11! (b) 9!
(c) 2880 (d) 8280

12. In an examination, there are 11 papers. A candidate has to pass in at least 6 papers to pass the examination. The number of ways in which the candidate can pass the examination is:
(a) 2^9 (b) 2^{10}
(c) 2^{11} (d) $2^{11} - {}^{11}C_5$

13. At an election, a voter may vote for any number of candidates not greater than the number to be chosen. There are 10 candidates and 5 members to be selected. The number of ways of in which a voter can vote is:
(a) 630 (b) 635
(c) 637 (d) 639

14. In a class of 10 students there are 3 girls students. The number of ways in which they can be arranged in a row, so that no two girls are consecutive is $k(8!)$, then k is equal to
(a) 12 (b) 24
(c) 36 (d) 42

15. The number of ways of putting 5 identical balls in 10 identical boxes, if not more than one can go in a box is
(a) ${}^{10}P_5$ (b) ${}^{10}C_5$
(c) 10^5 (d) 1

16. The number of four digit numbers that do not contain 4 different digits is:
(a) 4464 (b) 3680
(c) 4120 (d) 7208

17. The number of ways of dividing 10 girls into two groups of 5 each so that two shortest girls are in the different groups is:
(a) 70 (b) 252
(c) 140 (d) 282

18. The number of solutions of $\frac{1}{x}+\frac{1}{y}=\frac{1}{6}$ where $x, y \in \mathbf{N}$ is
(a) 9 (b) 18
(c) 21 (d) 28

19. Let m and n be two digit natural numbers. The number of pairs (m, n) such that n can be subtracted from m without borrowing is:
(a) 2475 (b) 2550
(c) 2675 (d) 2875

20. 5-digit numbers are formed using 2, 3, 5, 7, 9 without repeating the digits. If p is the number of such numbers that exceeds 20000 and q be the number of those that lie between 30000 and 90000 then $p : q$ is:
(a) 5 : 4 (b) 5 : 3
(c) 8 : 3 (d) 15 : 16

21. The number of diagonals of a polygon of 15 sides is
(a) 105 (b) 90
(c) 75 (d) 60

22. A five digit number divisible by 3 is to be formed using the digits 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways, this can be done is
(a) 240 (b) 3125
(c) 600 (d) 216

23. The number of subsets of a set containing n distinct object is
(a) ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_{n-1}$
(b) $2^n - 1$
(c) ${}^nC_0 + {}^nC_1 + \ldots + {}^nC_n$
(d) $2^n + 1$

24. If a, b and r are positive integers, then value of
$${}^aC_r + {}^aC_{r-1}\,{}^bC_1 + {}^aC_{r-2}\,{}^bC_2 + \ldots + {}^bC_r \text{ is}$$
(a) $\frac{a!b!}{r!}$ (b) $\frac{(a+b)!}{r!}$
(c) ${}^{a+b}C_r$ (d) ${}^{ab}C_r$

25. The number of 10 digit numbers that can be written by using the digits 0 and 1 is
(a) ${}^{10}C_2 + {}^9C_2$ (b) 2^{10}
(c) $2^{10}-2$ (d) 10!

26. If P_r stands for ${}^rP_r$, then sum of the series $1 + P_1 + 2P_2 + 3P_3 + \ldots + nP_n$ is
(a) $(n + 1)!$ (b) $P_{n+1} - 1$
(c) $P_{n-1} + 1$ (d) none of these

27. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then value of $\sum_{r=0}^{n} \frac{n-2r}{{}^nC_r}$ is

(a) $\frac{n}{2}a_n$ (b) $\frac{1}{4}a_n$

(c) na_n (d) 0

28. m men and w women are to be seated in a row so that all women sit together. The number of ways in which they can be seated is

(a) $(m+1)!\,w!$ (b) $m!\,w!$

(c) $m!\,(w-1)!$ (d) ${}^{m+w}C_w$

29. A five digit number divisible by 6 is to be formed by using the digits 0, 1, 2, 3, 4 and 8 without repetition. The total number of ways in which this can be done is

(a) 216 (b) 150

(c) 116 (d) 98

30. Rakshit is allowed to select $(n + 1)$ or more books out of $(2n + 1)$ distinct books. If the number of ways in which he may not select all of them is 255, then value of n is

(a) 3 (b) 4

(c) 5 (d) 11

31. The number of ways in which a committee of 3 women and 4 men be chosen from 8 women and 7 men if Mr. X refuses to serve on the committee if Mr. Y is a member of the committee is

(a) 420 (b) 840

(c) 1540 (d) 1400

32. The product of first n odd natural numbers equals

(a) $({}^{2n}C_n)\,({}^{n}P_n)$ (b) $\left(\frac{1}{2}\right)^n ({}^{2n}C_n)\,({}^{n}P_n)$

(c) $\left(\frac{1}{4}\right)^n ({}^{2n}C_n)\,({}^{2n}P_n)$ (d) none of these

33. The number of ways in which two teams A and B of 11 players each can be made up from 22 players so that two particular players are on the opposite sides is

(a) 369512 (b) 184755

(c) 184756 (d) 369514

34. The number of ways in which 20 letters $x_1, x_2, \ldots x_{10}$, $y_1, y_2, \ldots, y_{10}$ be arranged in a line so that suffixes of the letters x and also those of y are respectively in ascending order of magnitude is

(a) 126 (b) 64

(c) 2^{20} (d) 184756

35. The product of n consecutive natural number is always divisible by

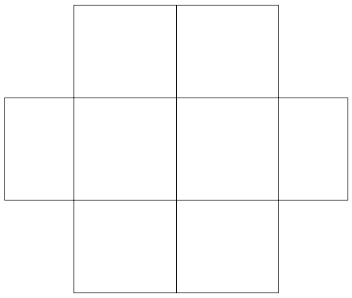
(a) ${}^{n}P_n$ (b) ${}^{2n}C_n$

(c) ${}^{2n}P_n$ (d) ${}^{n+1}P_n$

36. Six X have to be placed in the squares of Figure such that each row contain at least one X. The number of ways in which this can be done is

(a) 25 (b) 26

(c) 27 (d) 30



37. There are three pigeon holes marked M, P, C. The number of ways in which we can put 12 letters so that 6 of them are in M, 4 are in P and 2 are in C is

(a) 2520 (b) 13860

(c) 12530 (d) 25220

38. The greatest number of points of intersection of n circles and m straight lines is

(a) $2mn + {}^{m}C_2$

(b) $\frac{1}{2}\,m(m-1) + n(2m + n - 1)$

(c) ${}^{m}C_2 + 2({}^{n}C_2)$

(d) none of these

39. The number of binary sequences of length n that contain even number of 1's is

(a) 2^{n-1} (b) $2^n - 1$

(c) $2^n - 2$ (d) none of these

40. The number of natural numbers with distinct digits is

(a) $9^{10}-1$ (b) $10^{10} - 9^{10}$

(c) $9^{10} \times 1$ (d) none of these

41. The number of five digit numbers that contain 7 exactly once is

(a) $(41)\,(9^3)$ (b) $(37)\,(9^3)$

(c) $(7)\,(9^4)$ (d) $(41)\,(9^4)$

42. The units digit of $17^{2009} + 11^{2009} - 7^{2009}$ is

(a) 1 (b) 8

(c) 2 (d) 9

43. If $\frac{1}{{}^{4}C_n} = \frac{1}{{}^{5}C_n} + \frac{1}{{}^{6}C_n}$, then value of n is

(a) 3 (b) 4

(c) 1 (d) 2

44. At an election there are five candidates and three members are to be elected, and a voter may vote for any number of candidates not greater than the number

to be elected. The number of ways in which the person can vote is

(a) 25 (b) 30
(c) 35 (d) $2^5–2^3$

45. If n is odd and ${}^nC_0 < {}^nC_1 < {}^nC_2 < \dots < {}^nC_r$, then maximum possible value of r is

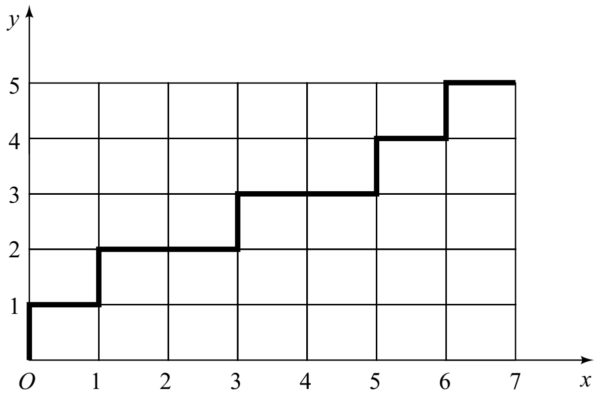
(a) $\frac{n-1}{2}$ (b) $\frac{n+1}{2}$
(c) n (d) none of these

46. The number of arrangements of the letters of the word BANANA in which two N's do not appear adjacently is

(a) 40 (b) 60
(c) 80 (d) 100

47. The number of (staircase) paths in the xy-plane from (0, 0) to (7, 5) where each such path is made up of individual steps going one unit upward (U) or one unit to the right (R). One such path is shown in figure

(a) ${}^{12}C_5$ (b) 12!
(c) 5!7! (d) ${}^{12}P_5$



48. In Question 47, how many such paths are there if each path must pass through the point (3, 4)?

(a) 175 (b) 145
(c) 95 (d) 78

49. The number of ways we can put 5 different balls in 5 different boxes such that at most three boxes is empty, is equal to

(a) $5^5– 5$ (b) $5^5–10$
(c) $2^5–2$ (d) none of these

50. The number of ordered pairs of integers (x, y) satisfying the equation $x^2 + 6x + y^2 = 4$ is

(a) 2 (b) 4
(c) 6 (d) 8

51. The number of the factors of 20! is

(a) 4140 (b) 41040
(c) 4204 (d) 81650

52. The number of five digit numbers that can be formed by using digits 1, 2, 3 only, such that three digits of the formed number are identical, is

(a) 30 (b) 60
(c) 120 (d) 90

53. Let $S_n = \sum_{r=1}^{n} r!$. If $T_n = S_n - 7\left[\frac{S_n}{7}\right]$, where $[x]$ denotes greatest integer $\le x$, then T_{100} is equal to

(a) 7 (b) 5
(c) 11 (d) none of these

54. The number of ordered pairs (m, n), m, $n \in \{1, 2, \dots, 50\}$ such that $6^m + 9^n$ is a multiple of 5 is

(a) 1250 (b) 2500
(c) 500 (d) 625

55. Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. The number of words which have at least one of their letter repeated is

(a) 69760 (b) 30240
(c) 99748 (d) none of these

56. The number of squares that we can find on a chess board is

(a) 64 (b) 160
(c) 224 (d) 204

57. In a football championship, 153 matches were played. Every team played one match with each other, the number of teams participating in the championship is

(a) 18 (b) 16
(c) 9 (d) none of these

58. The total number of ways of dividing mn distinct objects into n equal groups is

(a) $\frac{(mn)!}{(m!)^n(n!)}$ (b) $\frac{(mn)!}{(n!)^m(m!)}$
(c) $\frac{(mn)!}{m!n!}$ (d) none of these

59. The number of ways in which we can get at least 6 successive tails when a coin is tossed 10 times in a row, is

(a) ${}^{10}C_6$ (b) $2^7 - 2^6$
(c) 40 (d) 48

60. Let $R = \{a, b, c, d\}$ and $S = \{1, 2, 3\}$, then the number of functions f, from R to S, which are onto is

(a) 80 (b) 16
(c) 24 (d) 36

61. If ${}^{n+1}C_2 + 2({}^2C_2 + {}^3C_2 + {}^4C_2 + \dots + {}^nC_2) = x$, then x equals

(a) $1 + 2 + 3 \dots + n$ (b) $1^2 + 2^2 + \dots + n^2$
(c) $\frac{1}{2}\, n(n + 1)\,(n + 3)$ (d) none of these

62. The number of ways in which three girls and nine boys can be seated in two cars, each having numbered seats, 3 in the front and 4 at the back, is
(a) ${}^{14}P_{12}$ (b) ${}^{14}P_{12} \times 14 \times 3$
(c) ${}^{11}P_9$ (d) none of these

63. The number of integral solutions of $x + y + z = 0$ with $x \geq -5$, $y \geq -5$, $z \geq -5$ is
(a) 134 (b) 136
(c) 138 (d) 140

64. A library has n different books and has p copies of each of the book. The number of ways of selecting one or more books from the library is
(a) n^p (b) p^n
(c) $(n+1)^p - 1$ (d) $(p+1)^n - 1$

65. The number of ways in which we can get a score of 11 by throwing three dice is
(a) 18 (b) 27
(c) 45 (d) 56

66. Three straight lines l_1, l_2 and l_3 are parallel and lie in the same plane. Five points are taken on each of l_1, l_2 and l_3. The maximum number of triangles which can be obtained with vertices at these points, is
(a) 425 (b) 405
(c) 415 (d) 505

67. The number of 9 digit numbers which have all distinct digits, is
(a) 10! (b) 9!
(c) 8(9!) (d) 10! – 9!

68. A four digit number of distinct digits is formed by using the digits 2, 3, 4, 5, 6, 7, 8. The number of such numbers which are divisible by 25, is
(a) 60 (b) 40
(c) 20 (d) 15

69. The range of the function $f(x) = {}^{15-x}P_{x-8}$ is
(a) $\{1, 6, 20, 24\}$ (b) $\{1, 6, 10, 15\}$
(c) $\{1, 4, 11, 15\}$ (d) $\{1, 8, 11, 19\}$

70. The number of ways of arranging 20 boys so that 3 particular boys are separated is
(a) 9(16!) (b) 15(16!)
(c) 15(17!)/2 (d) none of these

71. A library has n different books and 3 copies of each of the n books. The number of ways of selecting one or more books from the library is
(a) $4^n - 1$ (b) $5^n - 1$
(c) $2^n - 1$ (d) $n^3 - 1$.

72. Let $a_n = 10^n/n!$ for $n \geq 1$. Then a_n take the greatest value when n equals
(a) 20 (b) 18
(c) 6 (d) 9

Assertion-Reason Type Questions

73. **Statement-1:** The expression
$\binom{40}{r}\binom{60}{0} + \binom{40}{r-1}\binom{60}{1} + \cdots$ attains maximum value when $r = 50$.
Statement-2: $\binom{2n}{r}$ is maximum when $r = n$.

74. **Statement-1:** The number of non-negative integral solution of
$$x_1 + x_2 + \ldots + x_{20} = 100$$
is $\binom{120}{20}$.
Statement-2: The number of ways of distributing n identical objects among r persons giving zero or more objects to a person is $\binom{n+r-1}{r-1}$.

75. **Statement-1:** The sum of divisors of
$n = 2^{10}\, 3^2\, 5^3\, 7^2\, 11^2$ is
$\frac{1}{480}(2^{11} - 1)(3^3 - 1)(5^4 - 1)(7^3 - 1)(11^3 - 1)$
Statement-2: The number of divisors of $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ where $p_1, p_2, \cdots p_r$ are distinct primes and $\alpha_1, \alpha_2, \cdots \alpha_r$ are natural numbers is $(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1)$.

76. **Statement-1:** The number $\binom{1000}{500}$ is not divisible by 11.
Statement-2: If p is a prime, the exponent of p in $n!$ is
$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$$
where $[x]$ denotes the greatest integer $\leq x$.

77. **Statement-1:** A student is allowed to select at most n books from a collection of $(2n + 1)$ books. If the total number of ways in which he can select at least one book is 255, then $n = 3$.
Statement-2: $\binom{2n+1}{0} + \binom{2n+1}{1} + \cdots + \binom{2n+1}{n} = 4^n$

LEVEL 2

Straight Objective Type Questions

78. $2m$ white counters and $2n$ red counters are arranged in a straight line with $(m + n)$ counters on each side of a central mark. The number of ways of arranging the counters, so that the arrangements are symmetrical with respect to the central mark, is
    (a) $\dfrac{(m+n)!}{m!n!}$ (b) ${}^{2m+2n}C_{2m}$
    (c) $\dfrac{1}{2}\dfrac{(m+n)!}{m!n!}$ (d) none of these

79. Let $x = (2n + 1)(2n + 3)(2n + 5) \ldots (4n - 3)(4n - 1)$ and $y = \left(\dfrac{1}{2}\right)^n \dfrac{(4n)!\,n!}{(2n)!\,(2n)!}$, then $x - y + 2^n$ is equal to
    (a) 0 (b) $(2n)!/2^n$
    (c) ${}^{2n}C_n$ (d) 2^n

80. There are 3 set of parallel lines containing respectively p lines, q lines and r lines respectively. The greatest number of parallelograms that can be formed by the system
    (a) $pqr + (p - 1)(q - 1)(r - 1)$
    (b) $\dfrac{1}{4}\{pqr + (p - 1)(q - 1)(r - 1)\}$
    (c) $\dfrac{1}{4} pqr (p + 1)(q + 1)(r + 1)$
    (d) none of these

81. The number of six digit numbers which have sum of their digits as an odd integer, is
    (a) 45000 (b) 450000
    (c) 97000 (d) 970000

82. In the identity $\dfrac{m!}{x(x+1)(x+2)\ldots(x+m)} = \sum_{i=0}^{m} \dfrac{A_i}{x+i}$, then value of A_k is
    (a) ${}^nC_k$ (b) ${}^nC_{k+1}$
    (c) $(-1)^k \cdot {}^nC_k$ (d) none of these

83. The number of integral solutions of $x^2 - y^2 = 352706$, is
    (a) 276 (b) 0
    (c) 720 (d) infinite

84. Let A be a set containing n elements. The number of ways of choosing two subsets P and Q of A such that $P \cap Q = \phi$ is
    (a) 2^n (b) 3^n
    (d) $4^n - 2^n$ (d) $4^n - 3^n$.

85. The last digit of $(1! + 2! + \ldots\ldots + 2009!)^{500}$ is
    (a) 1 (b) 2
    (c) 7 (d) 9

86. If $n = 2^{100}3^2$ and $\{d_1, d_2, \ldots d_k\}$ is the set of all divisors of n, then $\sum_{j=1}^{k} \dfrac{1}{d_j}$ equals
    (a) 2 (b) N
    (c) $2N$ (d) none of these

87. If ${}^nC_{r-1} = (k^2 - 8)({}^{n+1}C_r)$, then k belongs to
    (a) $[-3, -2\sqrt{2})$
    (b) $(-3, 3)$
    (c) $[-3, -2\sqrt{2}) \cup (2\sqrt{2}, 3]$
    (d) $[-2\sqrt{2}, 2\sqrt{2}]$

Previous Years' AIEEE/JEE Main Questions

1. The number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together is given by
    (a) 30 (b) 5! ´ 4!
    (c) $7! \times 5!$ (d) $6! \times 5!$ **[2003]**

2. A student is to answer 10 questions out of 13 questions in an examination such that he must choose at least 4 from the first five questions. The number of choices available to him is
    (a) 196 (b) 280
    (c) 346 (d) 140 **[2003]**

3. If ${}^nC_r$ denotes the number of combinations of n things taken r at a time, then the expression

$${}^nC_{r+1} + {}^nC_{r-1} + 2 \times {}^nC_r$$

equals

(a) ${}^{n+2}C_{r+1}$ (b) ${}^{n+1}C_r$
(c) ${}^{n+1}C_{r+1}$ (d) ${}^{n+2}C_r$ **[2003]**

4. The range of the function ${}^{7-x}P_{x-3}$ is

(a) {1, 2, 3, 4} (b) {1, 2, 3, 4, 5, 6}
(c) {1, 2, 3} (d) {1, 2, 3, 4, 5} **[2004]**

5. How many ways are there to arrange the letters in the word GARDEN with the vowels in alphabetical order?

(a) 360 (b) 240
(c) 120 (d) 480 **[2004]**

6. The number of ways distributing 8 identical balls in 3 distinct boxes so that none of the boxes is empty is

(a) 3^8 (b) 21
(c) 5 (d) ${}^8C_3$ **[2004]**

7. If $S_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$ and $t_n = \sum_{r=0}^{n} \frac{r}{{}^nC_r}$, then $\frac{t_n}{S_n}$ is equal to

(a) $n - 1$ (b) $\frac{1}{2}n - 1$
(c) $\frac{1}{2}n$ (d) $\frac{2n-1}{2}$ **[2004]**

8. The value of ${}^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$ is

(a) ${}^{56}C_3$ (b) ${}^{56}C_4$
(c) ${}^{55}C_4$ (d) ${}^{55}C_3$ **[2005]**

9. If the letters of the word SACHIN are arranged in all possible ways and these are written out as in a dictionary, then the word SACHIN appears at serial number

(a) 603 (b) 602
(c) 601 (d) 600 **[2005]**

10. At an election, a voter may vote for any number of candidates, not greater than the number to be elected. There are 10 candidates and 4 are to be elected. If a voter votes for at least one candidate, then the number of ways in which he can vote is

(a) 1110 (b) 5040
(c) 62010 (d) 385 **[2006]**

11. The set $S = \{1, 2, 3 \ldots 12\}$ is to be partitioned into three sets A, B, C of equal size such that, $A \cup B \cup C = S$, $A \cap B = B \cap C = C \cap A = \phi$. The number of ways to partition S is.

(a) $\frac{12!}{3!(4!)^3}$ (b) $\frac{12!}{3!(3!)^3}$
(c) $\frac{12!}{(4!)^3}$ (d) $\frac{12!}{(3!)^4}$ **[2007]**

12. In a shop there are five types of ice-creams available. A child buys six ice-creams

Statement-1: The number of different ways the child can buy the six ice-creams is ${}^{10}C_5$.

Statement-2: The number of different ways the child can buy the six ice-creams is equal to the number of different ways of arranging 6 A's and 4B's in a row. **[2008]**

13. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. Then the number of such arrangements is

(a) at least 750 but less than 1000
(b) at least 1000
(c) less than 500
(d) at least 500 but less than 750. **[2009]**

14. There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out at random and then transferred to the other. The number of ways in which this can be done is

(a) 66 (b) 108
(c) 3 (d) 36 **[2010]**

15. There are 10 points in a place, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then

(a) $N \le 100$ (b) $100 < N \le 140$
(c) $140 < N \le 190$ (d) $N > 190$ **[2011]**

16. **Statement-1**: The number of ways of distributing 10 identical balls in 4 distinct boxes such that no box is empty is ${}^9C_3$.

Statement-2: The number of choosing 3 places from 9 different places is ${}^9C_3$. **[2011]**

17. Assuming the balls to be identical except for difference in colours, the number of ways in which one or more balls can be selected from 10 white, 9 green and 7 black balls is:

(a) 629 (b) 630
(c) 879 (d) 880 **[2012]**

18. Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1} - T_n = 10$, then value of n is

(a) 5 (b) 10
(c) 8 (d) 7 **[2013]**

19. The number of ways in which an examiner can assign 30 marks to 8 question, giving no less then 2 marks to any question is:
(a) ${}^{30}C_7$ (b) ${}^{21}C_8$
(c) ${}^{21}C_7$ (d) ${}^{30}C_8$ **[2013, online]**

20. 5-digit numbers are formed using 2, 3, 5, 7, 9 without repeating the digits. If p is the number of such numbers that exceeds 20000 and q be the number of those that lie between 30000 and 90000, then $p : q$ is:
(a) 6 : 5 (b) 3 : 2
(c) 4 : 3 (d) 5 : 3 **[2013, online]**

21. A committee of 4 persons is to be formed from 2 ladies, 2 old men and 4 youngmen such that it includes at least 1 lady, at least one old man and at most 2 youngmen. Then the total number of ways in which this committee can be formed is
(a) 40 (b) 41
(c) 16 (d) 32 **[2013, online]**

22. On the sides AB, BC, CA of a triangle ABC, 3, 4, 5 distinct points (excluding vertices A, B, C) are respectively chosen. The number of triangles that can be constructed using these chosen points as vertices are:
(a) 210 (b) 205
(c) 215 (d) 220 **[2013, online]**

23. The sum of the digits in the unit's place of all the 4-digit numbers formed by using the numbers 3, 4, 5 and 6, without repetition, is:
(a) 432 (b) 108
(c) 36 (d) 18 **[2014, online]**

24. An eight digit number divisible by 9 is to be formed using digits from 0 to 9 without repeating the digits. The number of ways in which this can be done is:
(a) 72(7!) (b) 18(7!)
(c) 40(7!) (d) 36(7!) **[2014, online]**

25. 8-digit numbers are formed using the digits 1, 1, 2, 2, 2, 3, 4, 4. The number of such numbers in which the odd digits do not occupy odd places, is:
(a) 160 (b) 120
(c) 60 (d) 48 **[2014, online]**

26. Two women and some men participated in a chess tournament in which every participant played two games with each of the other participants. If the number of games that the men played betweeen themselves exceeds the number of games that the men played with the women by 66, then the number of men who participated in the tournament lies in the interval:
(a) [8, 9] (b) [10, 12)
(c) (11, 13] (d) (14, 1) **[2014, online]**

* None of the given options is correct

27. Let A and B be two sets containing four and two elements respectively. Then the number of subsets of the set $A \times B$, each having at least three elements is:
(a) 219 (b) 256
(c) 275 (d) 510 **[2015]**

28. The number of integers greater than 6,000 that can be formed, using the digits 3, 5, 6, 7 and 8, without repetition, is:
(a) 216 (b) 192
(c) 120 (d) 72 **[2015]**

29. The number of points, having both co-ordinates as integers, that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0), is:
(a) 901 (b) 861
(c) 820 (d) 780 **[2015]**

30*. The number of ways of selecting 15 teams from 15 men and 15 women, such that team consists of a man and a woman, is:
(a) 1120 (b) 1240
(c) 1880 (d) 1960 **[2015 online]**

31. If in a regular polygon the number of diagonals is 54, then the number of sides of this polygon is:
(a) 10 (b) 12
(c) 9 (d) 16 **[2015 online]**

32. If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in a dictionary, then the position of the word SMALL is:
(a) 46^{th} (b) 59^{th}
(c) 52^{nh} (d) 58^{th} **[2016]**

33. The value of $\sum_{r=1}^{15} r^2 \left(\frac{{}^{15}C_r}{{}^{15}C_{r-1}} \right)$ is equal to:
(a) 1240 (b) 560
(c) 1085 (d) 680 **[2016 online]**

34. The sum $\sum_{r=1}^{15} (r^2 + 1) \times (r!)$ is equal to:
(a) $11 \times (11!)$ (b) $10 \times (11!)$
(c) $(11)!$ (d) $101 \times (10!)$
[2016, online]

35. If the four letter words (need not be meaningful) are to be formed using the letters from the word “MEDITERRANEAN” such that the first letter is A and the fourth letter is E, then the total number of all such words is:
(a) 110 (b) 59
(c) $\frac{11!}{(2!)^3}$ (d) 56 **[2016, online]**

36. If $\dfrac{^{n+2}C_6}{^{n-2}P_2} = 11$, then n satisfies the equation:

(a) $n^2 + n - 110 = 0$ (b) $n^2 + 2n - 80 = 0$
(c) $n^2 + 3n - 108 = 0$ (d) $4n^2 + 5n - 84 = 0$ **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. A set B contain 2007 elements. Let C be the set consisting of subsets of B which contain at most 1003 elements. The number of elements in C is:
   (a) 2^{2005} (b) 2^{2006}
   (c) 2^{1003} (d) 2^{2007} **[2006]**
2. A parallelogram is cut by two set of n parallel lines, parallel to the sides of the parallelogram. The number of parallelograms formed is:
   (a) $(^{n+2}C_2)\,(^{n+2}C_2)$ (b) $(^{n}C_2)\,(^{n+1}C_2)$
   (c) $(^{n}C_2)\,(^{n+2}C_2)$ (d) $(^{n}C_2)\,(^{n}C_2)$ **[2007]**
3. If there are 30 onto, mappings from a set containing n elements to the set $\{0, 1\}$, then n equals:
   (a) 3 (b) 5
   (c) 7 (d) 2 **[2007]**
4. Let A be a set containing ten elements. Then the number of subsets of A containing at least four elements is
   (a) 845 (b) 848
   (c) 850 (d) 854 **[2008]**
5. A group of $2n$ students consisting of n boys and n girls are to be arranged in a row such that adjacent members are of opposite sex. The number of ways in which this can be done is:
   (a) $2(n!)$ (b) $(n!)^2$
   (c) $2(n!)^2$ (d) $n!$ **[2009]**
6. The maximum possible number of points of intersection of 8 straight lines and 4 circles is
   (a) 164 (b) 76
   (c) 104 (d) 32 **[2010]**
7. A committee consisting of at least three members is to be formed from a group of 6 boys and 6 girls such that it always has a boy and a girl. The number of ways to form such committee is:
   (a) $2^{12} - 2^7 - 13$ (b) $2^{11} - 2^6 - 13$
   (c) $2^{12} - 2^7 - 35$ (d) $2^{11} - 2^7 - 35$ **[2011]**
8. The total number of injective mappings from a set with m elements to a set with n elements for, $m > n$, is:
   (a) $\dfrac{m!}{n!(m-n)!}$ (b) $\dfrac{m!}{(m-n)!}$
   (c) n^m (d) zero **[2012]**
9. The least positive integral value of x for which $^{10}C_{x-1} > 2(^{10}C_x)$ is:
   (a) 5 (b) 8
   (c) 9 (d) 6 **[2013]**
10. Suppose that six students, including Madhu and Puja, are having six beds arranged in a row. Further, suppose that Madhu does not want a bed adjacent to Puja. Then the number of ways, the beds can be allotted to students is:
    (a) 264 (b) 480
    (c) 600 (d) 384 **[2014]**
11. In a car with seating capacity of exactly five persons, two persons can occupy the front seat and three persons can occupy the back seat. If amongst the seven persons, who wish to travel by this car, only two of them know driving then number of ways in which the car can be fully occupied and driven by them, is:
    (a) 360 (b) 60
    (c) 240 (d) 720 **[2015]**
12. If $S = \sum_{i=1}^{n}\left(\dfrac{^{n}C_{i-1}}{^{n}C_i + {}^{n}C_{i-1}}\right)^3 = \dfrac{36}{13}$, then n is equal to:
    (a) 11 (b) 12
    (c) 13 (d) 10 **[2016]**
13. A code word of length 4 consists two distinct consonants in the English alphabet followed by two digits from 1 to 9, with repetition allowed in digits. If the number of code words so formed ending with an even digit is 432 k, then k is equal to:
    (a) 5 (b) 49
    (c) 35 (d) 7 **[2016]**

Answers

Concept-based

1. (a)	**2.** (b)	**3.** (b)	**4.** (a)
5. (a)	**6.** (d)	**7.** (c)	**8.** (c)
9. (b)	**10.** (a)		

Level 1

11. (c)	**12.** (b)	**13.** (c)	**14.** (d)
15. (d)	**16.** (a)	**17.** (a)	**18.** (a)
19. (a)	**20.** (b)	**21.** (b)	**22.** (d)
23. (c)	**24.** (c)	**25.** (b)	**26.** (a)
27. (d)	**28.** (a)	**29.** (b)	**30.** (b)
31. (d)	**32.** (b)	**33.** (a)	**34.** (d)
35. (a)	**36.** (b)	**37.** (b)	**38.** (b)
39. (a)	**40.** (d)	**41.** (a)	**42.** (a)
43. (d)	**44.** (a)	**45.** (a)	**46.** (a)
47. (a)	**48.** (a)	**49.** (a)	**50.** (d)
51. (b)	**52.** (c)	**53.** (b)	**54.** (a)
55. (a)	**56.** (d)	**57.** (a)	**58.** (a)
59. (d)	**60.** (d)	**61.** (b)	**62.** (a)
63. (b)	**64.** (d)	**65.** (b)	**66.** (a)
67. (d)	**68.** (b)	**69.** (a)	**70.** (d)
71. (a)	**72.** (d)	**73.** (a)	**74.** (d)
75. (b)	**76.** (a)	**77.** (d)	

Level 2

78. (a)	**79.** (d)	**80.** (d)	**81.** (b)
82. (c)	**83.** (b)	**84.** (b)	**85.** (a)
86. (d)	**87.** (c)		

Previous Years' AIEEE/JEE Main Questions

1. (d)	**2.** (a)	**3.** (a)	**4.** (c)
5. (a)	**6.** (b)	**7.** (c)	**8.** (b)
9. (c)	**10.** (d)	**11.** (a)	**12.** (d)
13. (b)	**14.** (b)	**15.** (a)	**16.** (b)
17. (c)	**18.** (a)	**19.** (c)	**20.** (d)
21. (b)	**22.** (b)	**23.** (b)	**24.** (d)
25. (b)	**26.** (b)	**27.** (a)	**28.** (b)
29. (d)	**30.** (**)	**31.** (b)	**32.** (d)
33. (d)	**34.** (b)	**35.** (b)	**36.** (c)

Previous Years' B-Architecture Entrance Examination Questions

1. (b)	**2.** (a)	**3.** (b)	**4.** (b)
5. (c)	**6.** (c)	**7.** (a)	**8.** (d)
9. (b)	**10.** (b)	**11.** (d)	**12.** (b)
13. (c)			

Hints and Solutions

Concept-based

1. $\dfrac{(2x)!}{3!(2x-3)!}\cdot\dfrac{2!(x-2)!}{x!}=\dfrac{44}{3}$

$\Rightarrow \dfrac{(2x)(2x-1)(2x-2)}{3x(x-1)}=\dfrac{44}{3}$

$\Rightarrow 2x - 1 = 11 \Rightarrow x = 6$

2. $\dfrac{(2x+1)(2x)}{x+3}=\dfrac{72}{7}$

$\Rightarrow 7(2x^2 + x) = 36(x + 3)$

$\Rightarrow 14x^2 - 29x - 108 = 0$

$\Rightarrow (14x + 27)(x - 4) = 0 \Rightarrow x = 4$

3. $(m + n)(m + n - 1) = 90$ and $(m - n)(m - n - 1) = 30$

$\Rightarrow m + n = 10,\ m - n = 6$

$\therefore\ m = 8,\ n = 2$

4. Number of ways of putting beads in a necklace is 4! But clockwise arrangement abcde and anticlockwise arrangement aedcb are not really different, for, the arrangement abcde gives rise to arrangement aedcb if the necklace is turned over.

$\therefore$ required number of ways $= \dfrac{1}{4}(4!) = 12$

5. For a point of intersection, we require two lines, therefor, the number of points of intersection is ${}^{15}C_3$.

6. $m = \dfrac{1}{2}n(n-1) \Rightarrow \dfrac{4m}{n} = 2(n-1) = p(p-1) - 2$

$\therefore\ 18 = p(p-1) - 2 \Rightarrow p = 5$

7. $2({}^{2n}C_5) = {}^{2n}C_4 + {}^{2n}C_6$

$\Rightarrow\ 2=\dfrac{{}^{2n}C_4}{{}^{2n}C_5}+\dfrac{{}^{2n}C_6}{{}^{2n}C_5}=\dfrac{5}{2n-4}+\dfrac{2n-5}{6}$

$\Rightarrow 12(2n-4) = 30 + (2n-4)(2n-5)$

$\Rightarrow 2n^2 - 21n + 49 = 0 \Rightarrow n = 7$

8. ${}^nP_r = (r!)({}^nC_r) \Rightarrow 1680 = (r!)(70)$

$\Rightarrow r! = 24 \Rightarrow r = 4$

Also, $1680 = {}^nP_4 = n(n-1)(n-2)(n-3)$

$\Rightarrow (n^2 - 3n)(n^2 - 3n + 2) = 1680$

$\Rightarrow (n^2 - 3n + 1)^2 = 41^2 \Rightarrow n^2 - 3n + 1 = 41$

$\Rightarrow n^2 - 3n - 40 = 0 \Rightarrow (n + 5)(n - 8) = 0$

$\Rightarrow n = 8.$

9. As $(r + 1)({}^{n+1}C_{r+1}) = (n + 1)({}^nC_r)$,

we get, $\dfrac{n+1}{r+1}=\dfrac{11}{6}$.

Also, $\dfrac{n}{r}=\dfrac{6}{3}\Rightarrow r=\dfrac{1}{2}n$

Thus, $\dfrac{n+1}{n/2+1}=\dfrac{11}{6}\Rightarrow n=10$

10. We can choose x, y, z in n^3 ways out of which ${}^nP_3$ have distinct elements.

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11. Arrange non-vegeterians in $(6 - 1)! = 5!$ ways and vegeterian in $(5 - 1)! = 4!$ways

    $\therefore$ required number of ways

    $= (5!)\,(4!) = (120)\,(24) = 2880$

12. Number of ways passing the examination is

    $S = {}^{11}C_6 + {}^{11}C_7 + {}^{11}C_8 + {}^{11}C_9 + {}^{11}C_{10} + {}^{11}C_{11}$ (1)

    Using ${}^{n}C_{r} = {}^{n}C_{n-r}$, we get

    $S = {}^{11}C_5 + {}^{11}C_4 + {}^{11}C_3 + {}^{11}C_2 + {}^{11}C_1 + {}^{11}C_0$ (2)

    Adding (1) and (2) we get

    $2S = 2^{11} \Rightarrow S = 2^{10}$

13. The voter has to vote for at least one and at most 5 candidates. The number of ways is:

    ${}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 + {}^{10}C_5 = 637$

14. Seven boys can be arranged in 7! ways. If $XB_1\ XB_2\ XB_3\ XB_4\ XB_5\ XB_6\ XB_7\ X$ is one such arrangement, then 3 girls can be arranged at any of the 3 places marked with a X. Thus, the number of arrangements is $({}^{8}P_3)\,(7!) = (6 \times 7)\,(8!) = (42)\,(8!)$. Thus, $k = 42$.

15. There is only one way in which five identical balls can be put in 10 boxes.

16. The number of 4 digit numbers is (9) (10) (10) (10), out of these (9) (9) (8) (7) have distinct digits.

17. Excluding two shortest girls, the remaining eight girls can be divided into two groups in $\frac{1}{2!}({}^{8}C_4)({}^{4}C_4) = 35$ ways. The two shortest girls can be put in two different groups in 2 ways. Therefore, the required number of ways is 70.

18. $\frac{1}{x} + \frac{1}{y} = \frac{1}{6} \Rightarrow 6\,(x + y) = xy$

    $\Rightarrow (x - 6)\,(y - 6) = 36$

    $\Rightarrow (x - 6)|36.$

    $\Rightarrow x - 6 = 1, 2, 3, 4, 6, 9, 12, 18, 36$

    $\Rightarrow (x, y) = (7, 42), (8, 24), (9, 18), (10, 15), (12, 12), (15, 10), (18, 9), (24, 8), (42, 7)$

19. Suppose $m = 10a + b$ and $n = 10c + d$ where $1 \le a, c \le 9$ and $0 \le b, d \le 9$. Note that $0 \le d \le b$. Thus d can take $(b + 1)$ values. Similarly , $1 \le c \le a$, therefore, c can take a values. Hence, the required number of pairs (m, n) is $(1 + 2 + \ldots + 10)\,(1 + 2 + .. + 9) = 2475$

20. Any number 5-digit formed by 2, 3, 5, 7, 9, exceeds 20000. Therefore, $p = 5! = 120$

    A number will lie between 30000 and 90000 if it begins with 3, 5 or 7

    $\therefore\ q = (3)\,(4!)$. Thus $p : q = 5 : 3$.

21. Number of diagonals $= {}^{15}C_2 - 15 = 90$.

22. As $0 + 1 + 2 + 3 + 4 + 5 = 15$, for a five digit number to be divisible by 3 either do not use 0 or 3.

    When 0 is not used, the number of such numbers is 5! and when 3 is not used, the number of such numbers is $5! - 4!$

    $\therefore$ Required number of ways $= 5! + (5! - 4!) = 216$

23. Use: The number of subsets containing exactly r elements is ${}^{n}C_r$.

24. The given expression = number of ways of selecting r persons out of a men and b women $= {}^{a+b}C_r$

25. For each place we have two choices: either use 0 or use 1.

26. $1 + P_1 + 2P_2 + 3P_3 + \ldots + nP_n$

    $= 1 + 1! + 2(2!) + 3(3!) + \ldots + n(n!)$

    $= 1 + \sum_{r=1}^{n} [(r+1) - 1]r! = 1 + \sum_{r=1}^{n} \{(r+1)! - r!\}$

    $= 1 + (n + 1)! - 1! = (n + 1)!$

27. $$\sum_{r=0}^{n} \frac{n-2r}{{}^{n}C_r} = \sum_{r=0}^{n} \frac{n-r}{{}^{n}C_r} - \sum_{r=0}^{n} \frac{r}{{}^{n}C_r}$$

    $$= \sum_{r=0}^{n} \frac{n-r}{{}^{n}C_{n-r}} - \sum_{r=0}^{n} \frac{r}{{}^{n}C_r} = 0$$

28. Treating w women as one block, we can permute m men and one block in $(m + 1)!$ ways and women in the block in $w!$ ways. Thus, the required number of ways is $(m + 1)!\ w!$

29. Since $0 + 1 + 2 + 3 + 4 + 8 = 18$, the 5 digit number will be divisible by 3 if either 0 or 3 is not used.

    When 0 is not used. For the unit place we have 3 choices (2, 4 or 8) and for the remaining place we have 4! Choices.

    When 3 is not used

    In this case, 0 is used at the unit's place, the number of choices is 4! If 0 is not used at the unit's place, then unit's place can be filled up in 3 ways and the remaining places in $(4! - 3!)$ ways.

    Thus, required number of numbers

    $= 3(4!) + 4! + 3(4! - 3!) = 150$

30. Use ${}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + \ldots + {}^{2n+1}C_{2n} = 2^{2n} - 1$
31. Women can be chosen in ${}^{8}C_3$ ways and men in ${}^{7}C_4 - {}^{5}C_2$
32. $1.3.5 \ldots (2n - 1) = \dfrac{(2n)!}{2^n (n!)} = \left(\dfrac{1}{2}\right)^n \left({}^{2n}C_n\right)\left({}^{n}P_n\right)$
33. We can choose 10 players from (22 – 2) players in ${}^{20}C_{10}$ ways and one player from 2 players in ${}^{2}C_1$ ways.

   $\therefore$ required number of ways

   $= \dfrac{20!}{10!\,10!} \times 2 = 369512$

34. Choose 10 places (out of 20) for $x_1, x_2, \ldots, x_{10}$ in ${}^{20}C_{10}$ ways.
35. Let n consecutive natural numbers be $m + 1$, $m + 2$, …, $m + n$ where $m \geq 0$.

   $$P = (m + 1)(m + 2) \ldots (m + n)$$
   $$= \left(\frac{m!\,(m+1)\cdots(m+n)}{m!\,n!}\right) n!$$
   $$= ({}^{m+n}C_n)({}^{n}P_n)$$

   $\therefore$ P is divisible by ${}^{n}P_n$

36. This can be done in four mutually exclusive ways as follows:

	Row R_1	*Row R_2*	*Row R_3*	*Number of ways*
I	1	3	2	$({}^{2}C_1)({}^{4}C_3)({}^{2}C_2) = 8$
II	1	4	1	$({}^{2}C_1)({}^{4}C_4)({}^{2}C_1) = 4$
III	2	2	2	$({}^{2}C_2)({}^{4}C_2)({}^{2}C_2) = 6$
IV	2	3	1	$({}^{2}C_2)({}^{4}C_3)({}^{2}C_1) = 8$
			Total	26

37. Number of ways is ${}^{12}C_6$ for M, ${}^{6}C_4$ for P and ${}^{2}C_2$ for C.

   Thus, the required number of ways

   $= ({}^{12}C_6)({}^{6}C_4)({}^{2}C_2)$

38. The number of points of intersections

   $= ({}^{n}C_2)(2) + ({}^{m}C_2)(1) + ({}^{n}C_1)({}^{m}C_1)(2)$

   $= n(n - 1) + \dfrac{1}{2} m(m - 1) + 2mn$

39. Imitate Example 18.
40. Required number of natural numbers

   $= {}^{9}P_1 + 9 \times {}^{9}P_1 + 9 \times {}^{9}P_2 + \ldots + 9 \times {}^{9}P_9$

41. When 7 is right in the beginning, the number of numbers $= 9^4$ and when 7 is not in the beginning, the number of numbers is $({}^{4}C_1)(8)(9^3)$

   $\therefore$ Required number of numbers

   $= 9^4 + ({}^{4}C_1)(8)\,9^3 = (41)(9^3)$

42. Use unit's digit of 17^{2009} is same as that of the unit's digit of 7^{2009}.
43. $\dfrac{{}^{5}C_n}{{}^{4}C_n} = 1 + \dfrac{{}^{5}C_n}{{}^{6}C_n}$

   $\Rightarrow \dfrac{5}{5-n} = 1 + \dfrac{6-n}{6}$

   $\Rightarrow n^2 - 17n + 30 = 0 \quad \Rightarrow \quad n = 2.$

44. Number of ways $= {}^{5}C_1 + {}^{5}C_2 + {}^{5}C_3 = 25$
45. Use ${}^{2m+1}C_r$ is maximum when $r = m$ or $r = m + 1$.
46. Required number of ways

   = Total number of ways – Number of ways when two N's are together

   $= \dfrac{6!}{3!\,2!} - \dfrac{5!}{3!} = 40$

47. The number of paths

   = the number of ways of arranging 5 $U'S$ and 7 R's.

   $= \dfrac{12!}{5!\,7!} = {}^{12}C_5$

48. Required number of paths

   = (Number of paths upto (3, 4)) × (Number of paths from (3, 4) to (5, 7))

   $= \dfrac{7!}{3!\,4!} \times \dfrac{5!}{4!} = 175$

49. The number of ways of putting all the balls in exactly one box = 5

   As for each ball there are 5 choices for the box, the number of ways of putting balls in the boxes $= 5^5$

   Thus, the required number of ways of putting the balls in the boxes $= 5^5 - 5$

50. $x^2 + 6x + y^2 = 4 \quad \Rightarrow \quad (x + 3)^2 + y^2 = 13$

   $\Rightarrow (x + 3, y) = (\pm 3, \pm 2)$ or $(\pm 2, \pm 3)$

   $\Rightarrow x = 0, y = \pm 2, x = -6, y = \pm 2,$

   $x = -1, y = \pm 3, y = -5, y = \pm 3$

51. Use $20! = 2^{18}\, 3^8\, 5^4\, 7^2\, (11)(13)(17)(19)$
52. When exactly three digits are identical and the remaining two are different, then the number of such numbers

   $$\left({}^{3}C_1\right)\left(\frac{5!}{3!}\right) = 60$$

   When three digits are identical and the remaining two are also identical, then the number of ways

   $$= \left({}^{3}C_2\right)(2)\left(\frac{5!}{3!\,2!}\right) = 60$$

   Thus, the number of such numbers = 120.

53. Use the fact that $r!$ is divisible by 7 whenever $r \geq 7$.

54. As the last digit of 6^m, $m \in \mathbf{N}$ is 6, $6^m + 9^n$ will be divisible by 5 if the unit's digit of 9^n is 4 or 9. This is possible when n is odd.

    $\therefore$ required number of ordered pairs = 50×25 = 1250.

55. The required number of words = $10^5 - {}^{10}P_5$

56. To get a square of size $r \times r$, we must choose $(r + 1)$ consecutive horizontal and $(r + 1)$ consecutive vertical lines. This can be done in $(9 - r)(9 - r) = (9 - r)^2$ ways.

57. ${}^nC_2 = 153 \Rightarrow n = 18$

58. As groups are indistinguishable, the number of ways

$$= \frac{1}{n!}({}^{mn}C_m)({}^{mn-m}C_m) \ldots ({}^mC_m)$$

$$= \frac{1}{n!}\frac{(mn)!}{m!(mn-m)!}\frac{(mn-m)!}{m!(mn-2m)!}\cdots\frac{m!}{m!\,0!}$$

$$= \frac{1}{n!}\frac{(mn)!}{(m!)^n}$$

59. The sequence of six tails may begin at with the ith toss where $i = 1, 2, 3, 4, 5$. Thus, the number of ways is

$$2^4 + 1(2^3) + 2(1)(2^2) + 2^2(1)2 + 2^3(1) = 48$$

60. Total number of functions = 3^4.

    All the four elements can be mapped to exactly one element in 3 ways, and exactly two elements in $3(2^4 - 2)$.

    Thus, the number of onto functions = $3^4 - 3 - 3(2^4 - 2) = 36$.

61. We have

$$ {}^2C_2 + {}^3C_2 + {}^4C_2 + \ldots + {}^nC_2$$
$$= ({}^3C_3 + {}^3C_2) + {}^4C_2 + \ldots + {}^nC_2$$
$$= {}^4C_3 + {}^4C_2 + \ldots + {}^nC_2$$
$$= {}^5C_3 + \ldots + {}^nC_2 = \ldots = {}^{n+1}C_3$$

Thus, $x = {}^{n+1}C_2 + 2({}^{n+1}C_3)$

$$= {}^{n+1}C_2 + {}^{n+1}C_3 + {}^{n+1}C_3 = {}^{n+2}C_3 + {}^{n+1}C_3$$
$$= \frac{1}{6}\,n(n+1)(2n+1)$$

62. We have to arrange 12 persons at 14 seats.

63. Let $x_1 = x + 5$, $y_1 = y + 5$, $z_1 = z + 5$ and find the number of non-negative integral solutions of

$$x_1 + y_1 + z_1 = 15$$

64. For each book we have $p + 1$ choices.

65. Find the number of positive integral solutions of

$$x + y + z = 11$$

where $1 \leq x, y, z \leq 6$.

66. The total number of points is 15. From these 15 points we can obtain ${}^{15}C_3$ triangles. However, if all the 3 points are chosen on the same straight line, we do not get a triangle. Therefore, the required number of triangles

$$= {}^{15}C_3 - 3({}^5C_3) = 425.$$

67. The required number of ways

    = The number of ways of permuting nine digits out of ten digits 0, 1, 2, …, 9

    – The number of ways of permuting nine digits out of nine digits 1, 2, …, 9

$$= {}^{10}P_9 - {}^9P_9 = \frac{10!}{1!} - 9! = 10! - 9!$$

68. The number will be divisible by 25 if it ends in 25 or 75. Therefore, the required number of numbers is

$$({}^5P_2)(2) = 40$$

69. We must have $15 - x \geq 1$ and $x - 8 \geq 0$ and $15 - x \geq x - 8 \Rightarrow 8 \leq x \leq 11$. Now,

$$f(8) = {}^7P_0 = 1, \quad f(9) = {}^6P_1 = 6,$$
$$f(10) = {}^5P_2 = 20, \quad f(11) = {}^4P_3 = 24,$$

Thus, range of f is $\{1, 6, 20, 24\}$.

70. We can arrange remaining 17 boys in 17! ways. For 3 particular boys we have 18 positions. We can arrange 3 particulars boys at these places in ${}^{18}P_3$ ways. Thus, the required number of ways

$$= (17!)\left({}^{18}P_3\right) = 17!\,\frac{18!}{15!}$$

71. For each book we have four choices. We can choose 0, 1, 2 or 3 volumes of the book.

72. We have $\dfrac{a_n}{a_{n+1}} = \dfrac{10^n}{n!} \times \dfrac{(n+1)!}{10^{n+1}} = \dfrac{n+1}{10}$.

Note that $\dfrac{a_1}{a_2} < 1$, $\dfrac{a_2}{a_3} < 1$, $\dfrac{a_3}{a_4} < 1$, $\dfrac{a_4}{a_5} < 1$,

$$\ldots \frac{a_8}{a_9} < 1, \frac{a_9}{a_{10}} = 1, \frac{a_{10}}{a_{11}} > 1, \frac{a_{11}}{a_{12}} > 1, \ldots$$

Thus, $a_1 < a_2 < a_3 < \ldots < a_9 = a_{10} > a_{11} > a_{12}$

$\therefore$ a_n is greatest if $n = 9$ or 10.

73. We have $\dfrac{\binom{2n}{r}}{\binom{2n}{r+1}} = \dfrac{(2n)!}{r!(2n-r)!}\dfrac{(r+1)!(2n-r-1)!}{(2n)!}$

$$= \frac{r+1}{2n-r}$$

Since for $0 \le r \le n-1$, $\dfrac{r+1}{2n-r} < 1$, we get

$$\binom{2n}{0} < \binom{2n}{1} < \cdots < \binom{2n}{n-1} < \binom{2n}{n}$$

Also, as $\binom{2n}{r} = \binom{2n}{2n-r}$

$$\binom{2n}{2n} < \binom{2n}{2n-1} \cdots < \binom{2n}{2n+1} < \binom{2n}{n}$$

Thus, $\binom{2n}{r}$ is maximum when $r = n$.

Next,

$$\binom{40}{r}\binom{60}{0} + \binom{40}{r-1}\binom{60}{1} + \cdots$$

= the number ways of selecting r persons out of 40 men and 60 women

$$= \binom{100}{r}$$

which is maximum when $r = 50$.

74. The number of ways of distributing n identical objects among r persons giving zero of more objects to a person is equivalent to arranging n identical objects of one kind and $(r-1)$ identical objects of second kind in a row, which is equal to $\dfrac{(n+r-1)!}{n!(r-1)!} = \binom{n+r-1}{r-1}$.

Next, the number of non-negative integral solutions of $x_1 + x_2 + \cdots + x_{20} = 100$ equals the number of ways of distributing 100 identical objects among 20 persons giving zero or more objects to a person, which equal $\binom{100+20-1}{20-1} = \binom{109}{19}$.

75. Sum of the divisors of n

$= (1 + 2 + \cdots - 2^{10})\ (1 + 3 + 3^2)\ (1 + 5 + 5^2 + 5^3)\ (1 + 7 + 7^2)\ (1 + 11 + 11^2)$

$$= (2^{11} - 1)\left(\frac{3^3-1}{3-1}\right)\left(\frac{5^4-1}{5-1}\right)\left(\frac{7^3-1}{7-1}\right)\left(\frac{11^3-1}{11-1}\right)$$

$$= \frac{1}{480}(2^{11} - 1)\ (3^3 - 1)\ (5^4 - 1)\ (7^3 - 1)\ (11^3 - 1)$$

A divisors of m is of the form $p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$ where $0 \le \beta_i \le \alpha_i$ for $i = 1, 2, \ldots, r$.

That is, β_i can take $\alpha_i + 1$ values. Thus, the number of divisors of m is $(\alpha_1 + 1)\ (\alpha_2 + 1) \ldots\ (\alpha_r + 1)$

76. For the truth of statement-2, see theory of this chapter.

We have

$$\binom{1000}{500} = \frac{1000!}{500!500!}$$

Exponent of 11 in the prime factorization of 1000! is

$$\left[\frac{1000}{11}\right] + \left[\frac{1000}{11^2}\right] = 90 + 8 = 98$$

Exponent of 11 in the prime factorization of 500! is

$$\left[\frac{500}{11}\right] + \left[\frac{500}{11^2}\right] = 45 + 4 = 49.$$

Thus, exponent of 11 in the prime factorization of $\binom{1000}{500}$ is 0.

$\Rightarrow\ \binom{1000}{500}$ is not divisible by 11.

77. Let

$$S = \binom{2n+1}{0} + \binom{2n+1}{1} + \cdots + \binom{2n+1}{n} \qquad (1)$$

Using $\binom{n}{r} = \binom{n}{n-r}$, we can write (1) as

$$S = \binom{2n+1}{2n+1} + \binom{2n+1}{2n} + \cdots + \binom{2n+1}{n+1} \qquad (2)$$

Adding (1) and (2), we get

$$2S = \binom{2n+1}{0} + \cdots + \binom{2n+1}{2n+1} = 2^{2n+1}$$

$\Rightarrow\quad S = 2^{2n} = 4^n$

The number of ways of choosing r books out of $2n + 1$ is $\binom{2n+1}{r}$. We are given

$$\Rightarrow \quad \binom{2n+1}{1} + \binom{2n+1}{2} + \cdots + \binom{2n+1}{n} = 255$$

$$\Rightarrow \quad 4^n - 1 = 255 \ \Rightarrow\ 4^n = 256 = 4^4 \ \Rightarrow\ n = 4.$$

Level 2

78. Arrange m white and n red counters on one side of the central mark. This can be done in $\frac{(m+n)!}{m!n!}$

79. $x = \frac{(2n+1)(2n+2)\cdots(4n-1)(4n)}{(2n+2)(2n+4)\cdots(4n)}$

$= \frac{1}{(2n)!}\frac{(4n)!n!}{2^n(2n)!} = y$

Thus, $x - y + 2^n = 2^n$

80. Number of parallelograms

$= ({}^pC_2)({}^qC_2) + ({}^qC_2)({}^rC_2) + ({}^rC_2)({}^pC_2)$

81. For the first place we have nine choices. For each of the next four places, we have 10 choices. At this stage we add the numbers already selected and choose the digit at unit's place in 5 ways.

82. A_k is obtained by putting $x = -k$ everywhere in the expression except in $x + k$. Therfore,

$A_k = \frac{m!}{(-k)\cdots(-k+k-1)(-k+k+1)\cdots(-k+m)}$

$= \frac{(-1)^k m!}{k!(m-k)!} = (-1)^k\left({}^mC_k\right)$

83. As 352706 is even, either both x and y are even or both x and y are odd.

If x, y are both even, $x^2 - y^2$ is divisble by 4, but 352706 is not divisible by 4.

If x, y are both odd, $x^2 - y^2$ is divisible by 4, but 352706 is not divisible by 4.

84. For each $x \in A$, we have four choices;

(i) $x \in P$, $x \in Q$

(ii) $x \in P$, $x \notin Q$

(iii) $x \notin P$, $x \in Q$

(iv) $x \notin P$, $x \notin Q$

Out of these four choices, last 3 choices imply $x \notin P \cap Q$.

Thus, P and Q can be chosen in 3^n ways, so that $P \cap Q = \phi$.

85. As $1! + 2! + 3! + 4! = 33$ and last digit of $n!$, for $n \geq 5$ is 0, last digit of $(1! + 2! + \cdots + 2009!)^{500}$ is same as that of last digit of 3^{500}

But $3^{500} = 9^{250} = (10 - 1)^{250}$, and last digit of $(10 - 1)^{250}$ is 1.

86. Note that $d \Big| n \;\Leftrightarrow\; \frac{n}{d}\Big| n$

Now, $\sum_{j=1}^{k}\frac{1}{d_j} = \sum_{j=1}^{k}\frac{d_j}{n} = \frac{1}{n}\sum_{j=1}^{k} d_j$

But divisors of $n = 2^{100}\,3^2$ are

$1, 2, \cdots, 2^{100}, 3, (3)(2), \cdots, 3(2^{100}), 3^2\,(3^2)(2), \cdots (3^2)(2^{100})$

Thus, $\sum_{j=1}^{k} d_j = (1 + 2 + \cdots + 2^{100})(1 + 3 + 3^2)$

$= \left(\frac{2^{101}-1}{2-1}\right)(13) = 13(2^{101}-1)$

87. We have, $r - 1 \geq 0$, $r \leq n + 1$

$\Rightarrow 1 \leq r \leq n + 1 \Rightarrow \frac{1}{n+1} \leq \frac{r}{n+1} \leq 1$

Also, $k^2 - 8 = \frac{n!}{(r-1)!\,(n-r+1)!}\cdot\frac{r!(n+1-r)!}{(n+1)!}$

$= \frac{r}{n+1}$

Thus, $\frac{1}{n+1} \leq k^2 - 8 \leq 1$

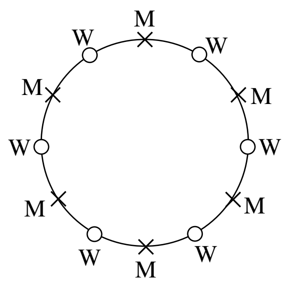
$\Rightarrow 8 < \frac{1}{n+1} + 8 \leq k^2 \leq 9$

$\Rightarrow 8 < k^2 \leq 9 \Rightarrow -3 \leq k < -2\sqrt{2}$ or $2\sqrt{2} < k \leq 3$

Hence, $k \in [-3, -2\sqrt{2})$

Previous Years' AIEEE/JEE Main Questions

1. We can arrange 6 men around the round table in 5! ways. There are 6 places for women to sit. Out of these we choose 5 and arrange women there. This can be done in $(5!)\,({}^6C_5)\,(5!) = (5!)\,(6!)$ ways.



2. Required number of ways

$= ({}^5C_4)({}^8C_6) + ({}^5C_5)({}^8C_5) = 196$

3. ${}^nC_{r+1} + {}^nC_{r-1} + 2({}^nC_r)$

$= ({}^nC_{r-1} + {}^nC_r) + ({}^nC_r + {}^nC_{r+1})$

$= {}^{n+1}C_r + {}^{n+1}C_{r+1} = {}^{n+2}C_{r+1}$

4. We must have

$7 - x \geq 1$, $x - 3 \geq 0$ and $7 - x \geq x - 3$

$\Rightarrow x \leq 6$, $x \geq 3$ and $x \leq 5$

Thus, $3 \leq x \leq 5$

$\therefore$ Range $= \{{}^4P_0, {}^3P_1, {}^2P_2\}$

$= \{1, 3, 2\}$

5. We can arrange the letters of the word GARDEN in 6! ways. Exactly half of these will be with vowels in alphabetical order.

So the required number of ways is (1/2) 6! = 360.

6. If x_1 is the number of balls in the ith box, then we must have

$$x_1 + x_2 + x_3 = 8 \qquad (1)$$

where $x_i \geq 1$.

The number of solutions of (1) is ${}^7C_2 = 21$.

7. We have $t_n = \sum_{r=0}^{n} \frac{r}{{}^nC_r} = \sum_{r=0}^{n} \frac{n-r}{{}^nC_{n-r}}$

$$= \sum_{r=0}^{n} \frac{n-r}{{}^nC_r} = n\, s_n - t_n$$

$$\Rightarrow 2t_n = n\, s_n \Rightarrow \frac{t_n}{s_n} = \frac{n}{2}.$$

8. ${}^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$

$= {}^{50}C_4 + ({}^{50}C_3 + {}^{51}C_3 + \ldots + {}^{55}C_3)$

$= ({}^{50}C_4 + {}^{50}C_3) + {}^{51}C_3 + \ldots + {}^{55}C_3$

$= ({}^{51}C_4 + {}^{51}C_3) + {}^{52}C_3 + \ldots + {}^{55}C_3$

[using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$]

$= {}^{52}C_4 + {}^{52}C_3 + \ldots + {}^{55}C_3$

$= \ldots = {}^{56}C_4$

9. Number of words which begin with A, C, H, I or N is 5(5!) = 600. These have ranks from 1 to 600. Then words beginning S are listed and SACHIN is the first word beginning with S, therefore, its rank is 601.

10. Required number of ways

$= {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4$

$= 10 + 45 + 120 + 210 = 385$

11. Number of ways

$= ({}^{12}C_4)\,({}^8C_4)\,({}^4C_4)$

$$= \frac{12!}{(41)^3}$$

12. Let x_i = number of ice creams of ith type bought by the child.

Then, $x_1 + x_2 + x_3 + x_4 + x_5 = 6 \qquad (1)$

The number of non-negative integral solution of (1) ${}^{6+5-1}C_{5-1} = {}^{10}C_4 = N$ (say)

$\therefore$ Statement-1 is false.

The number of ways of arranging 6 A's and 4 B's in a row

$$= \frac{10!}{4!6!} = {}^{10}C_4 = N.$$

Thus, statement-2 is true.

13. We can choose 4 novels out of 6 in ${}^6C_4$ ways and 1 dictionary out of 3 in ${}^3C_1$ ways. We can arrange 4 novels and 1 dictionary in the middle in 4! Ways. Thus, required number of ways

$= ({}^6C_4)\,({}^3C_1)\,(4!) = 1080 > 1000$

14. Number of ways

$= ({}^3C_2)\,({}^9C_2) = (3)\,(36) = 108$

15. Number of triangles

$N = {}^{10}C_3 - {}^6C_3$

$= \frac{1}{6}(10 \times 9 \times 8) - \frac{1}{6}(6 \times 5 \times 4)$

$= 100$

16. Let x_i = number of balls put in the ith box, then

$$x_1 + x_2 + x_3 + x_4 = 10 \qquad (1)$$

where $x_i \geq 1$.

Let $x_i = y_i + 1$ where $y_i \geq 0$

Thus, (1) becomes

$$y_1 + y_2 + y_3 + y_4 = 6 \qquad (2)$$

Number of non-negative integral solutions of (2) = number of ways of arranging 6 identical balls and 3 identical separators in a row.

$$= \frac{(6+3)!}{6!3!} = {}^9C_3$$

= number of ways of choosing 3 places out of 9 different places.

17. We can choose 0 or more balls out of 10 white balls in 11 ways.

[Select 0 or 1 or 2, … or 10 balls.]

Thus, the number ways of selecting 0 or more balls is (1 + 10) (1 + 9) (1 + 7) = 880.

Therefore, the number of ways selecting one or more balls is 880 – 1 = 879.

18. As $T_n = {}^nC_3$,

$T_{n+1} - T_n = 10$

$\Rightarrow {}^{n+1}C_3 - {}^nC_3 = 10$

$\Rightarrow {}^nC_2 + {}^nC_3 - {}^nC_3 = 10$

$\Rightarrow \frac{1}{2}\, n(n-1) = 10 \Rightarrow n = 5$

19. Let x_i = marks assigned to ith question, then

$x_1 + x_2 + \ldots + x_8 = 30,$

$x_i \geq 2$ for $i = 1, 2, \ldots, 8.$

Put $x_i = y_i + 2$, then

$y_1 + y_2 + \ldots + y_8 = 14$ (1)

The number of solutions of (1) is

${}^{14+8-1}C_{8-1} = {}^{21}C_7$

20. Any number that is formed by 2, 3, 5, 7, 9 exceeds 20000.

$\therefore\ p = 5!$

A number that begins with 3, 5 or 7 lies between 3000 and 90000. Therefore,

$q = (3)\ (4!)$

Thus, $p : q = 5 : 3$

21. Different number of ways of forming committee are as under.

	Ladies	Old men	Young men
OPTION 1	1	1	2
OPTION 2	1	2	1
OPTION 3	2	1	1

Thus, the required number of ways

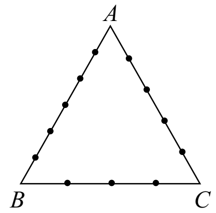
$= ({}^2C_1)\ ({}^2C_1)\ ({}^4C_2) + ({}^2C_1)\ ({}^2C_2)\ ({}^4C_1) + ({}^2C_2)\ ({}^2C_1)\ ({}^4C_1)$

= 24 + 8 + 8 = 40.

22. Take one vertex on each side.

This can be done in $({}^3C_1)\ ({}^4C_1)\ ({}^5C_1)$

= 60 ways.



Choose two vertices on one side and one from the remaining two sides. This can be done in

$({}^3C_2)\ ({}^9C_1) + ({}^4C_2)\ ({}^8C_1) + ({}^5C_2)\ ({}^7C_1)$

= 27 + 48 + 70 = 145.

Thus, number of required triangle is 60 + 145 = 205.

23. Each digit occurs at the unit's place exactly 3! = 6 times. This, sum of the digits at the units place

= 6[3 + 4 + 5 + 6] = 108

24. As 0 + 1 + 2 + … + 9 = 45, a number will be divisible by 9, if we do not use (0, 9), (1, 8), (2, 7), (3, 6), (4, 5).

Thus, the number of 8 digit number that are divisible by 9 is 8! + (8! – 7!) (4) = (36) (7!)

25. We can choose places for 1, 1, 3 in ${}^4C_3 = 4$ ways.

$\therefore$ We can arrange the digits

1, 1, 2, 2, 2, 3, 4, 4 is

$(4)\ \frac{3!}{2!}\frac{5!}{3!2!} = 120$

26. Number of games played by men among themselves $= 2({}^nC_2) = n(n-1)$

Number of games played by men with women = $2(2n) = 4n$

$\therefore\ n(n-1) - 4n = 66 \Rightarrow n(n-5) = 66$

$\Rightarrow (n-11)\ (n+6) = 0 \Rightarrow n = 11 \in [10, 12)$

27. $n(A \times B) = n(A)\ n(B) = (4)\ (2) = 8$

$\therefore$ number of subset of $A \times B$ which contain at least three elements

$= {}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6 + {}^8C_7$

$= 2^8 - ({}^8C_0 + {}^8C_1 + {}^8C_2)$

= 256 – (1 + 8 + 28) = 219

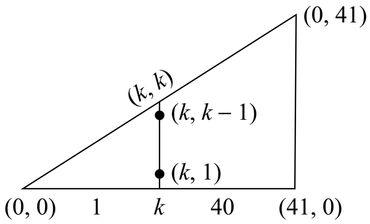
28. Any four digit number beginning with 6, 7 or 8 is greater than 6000. Also, any five digit number is greater than 6000. Thus, required number of numbers is

$({}^3P_1)\ ({}^4P_3) + {}^5P_5 = 192$

29. For $1 \le k \le 40$, the number of integral points on the line $x = k$ and lying in the interior of triangles is $k - 1$.

Thus, required number of points $= \sum_{k=1}^{40} (k-1)$

$\frac{1}{2}$ (39) (40) = 780



30. If we allow same man (woman) to be in two distinct teams (for instance if (M_1, W_1) and (M_1, W_2) are two distinct teams), then the answer is ${}^{225}C_{15}$.

If we do not allow same man (woman) to be in two distinct teams, then first team can be chosen in (15) (15) ways, second team can be chosen in (14) (14) ways and so on.

As all the teams 15 are to be chosen, the number of ways

$= (15)^2\ (14)^2\ (13)^2 \ldots (2^2)\ (1^2)$

$= (15!)^2$

It seems that answer 1240 is arrived by the following way

$15^2 + 14^2 + \ldots + 2^2 + 1^2$

$= \frac{1}{6}\ (15)\ (16)\ (2 \times 15 + 1) = 120$

This is an incorrect way as we have to select all the 15 teams.

31. Let number of sides of the regular polygon be n. The number of diagonals of the polygon

$= {}^nC_2 - n = \frac{1}{2} n(n-1) - n = \frac{1}{2} n(n-3)$

Thus, $\frac{1}{2} n(n-3) = 54$

$\Rightarrow n^2 - 3n - 108 = 0$

$\Rightarrow (n - 12)\ (n + 9) = 0$

As $n \ge 3$, $n = 12$.

32. Number of 5 letter words that can be formed by using letters of word SMALL is $\frac{5!}{2!} = 60$.

Last word (60^{th} word) in the list is SMLLA, 59^{th} word is SMLAL, and 58^{th} word is SMALL.

Alternate solution

Number of words beginning with

(i) A is $\frac{4!}{2!} = 12$	(Ranked 1 to 12)
(ii) L is $4! = 24$	(Ranked 13 to 36)
(iii) M is $\frac{4!}{2!} = 12$	(Ranked 37 to 48)
(iv) SA is $\frac{3!}{2!} = 3$	(Ranked 49 to 51)
(v) SL is $3! = 6$	(Ranked 52 to 57)
(vi) SM (ALL) is 1	(Ranked 58^{th})

33. $\frac{{}^{15}C_r}{{}^{15}C_{r-1}} = \frac{15!}{r!(15-r)!} \cdot \frac{(r-1)!(16-r)!}{15!}$

$= \frac{15-r}{r}$

$\therefore\ a_r = r^2 \left(\frac{{}^{15}C_r}{{}^{15}C_{r-1}}\right) = r(15 - r)$

$\Rightarrow \sum_{r=1}^{15} a_r = \sum_{r=1}^{15} r(16-r)$

$= 16\left\{\frac{1}{2}(15)(16)\right\} - \frac{1}{6}\ (15)(16)(31)$

$= \frac{1}{2}(15)(16)\left[16 - \frac{31}{3}\right]$

$= \frac{1}{2}(15)(16)\left(\frac{17}{3}\right) = 680$

34. As $r^2 + 1 = (r + 2)\ (r + 1) - 3(r + 1) + 2$, we get

$(r^2 + 1)r! = (r + 2)! - 3(r + 1)! + 2(r!)$

$= [(r + 2)! - (r + 1)!] - 2[(r + 1)! - r!]$

$\Rightarrow \sum_{r=1}^{10} (r^2+1)r! = (12! - 2!) - 2(11! - 1!)$

$= 11!(12 - 2) = (10)(11!)$

35. Letters in the word M E D I T E R R A N E A N are A, A, D, E, E, E, I, M, N, N, R, R, T. Two middle letters must be chosen from

A, A, D, E, E, I, M, N, N, R, T. If the chosen letters are distinct, then these can be arranged in ${}^8P_2 = 56$ ways.

If the chosen letters are indentical, then these can be chosen and arranged in ${}^3C_2 = 3$ ways.

Thus, total number of such letters is 59.

36. $\dfrac{(n+2)!}{6!(n-4)!} \cdot \dfrac{(n-4)!}{(n-2)!} = 11$

$\Rightarrow \dfrac{(n+2)(n+1)(n)(n-1)}{(6)(5)(4)(3)(2)} = 11$

$\Rightarrow (n + 2)(n + 1)(n)(n - 1) = (11)(10)(9)(8)$

$\Rightarrow n + 2 = 11 \Rightarrow n = 9$

$\Rightarrow n^2 + 3n - 108 = 0$

Previous Years' B-Architecture Entrance Examination Questions

1. The number of required subsets is

$N = {}^{2007}C_0 + {}^{2007}C_1 + {}^{2007}C_2 + \ldots + {}^{2007}C_{1003}$

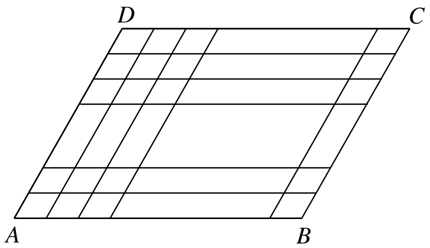
$N = {}^{2007}C_{2007} + {}^{2007}C_{2006} + \ldots + {}^{2007}C_{1004}$

Adding above two equations we get

$2N = {}^{2007}C_0 + {}^{2007}C_1 + \ldots + {}^{2007}C_{2007}$

$\Rightarrow N = 2^{2006}$

2. There are $(n + 2)$ line segments from A to B and $(n + 2)$ line segments from A to D.



To get a parallelogram, choose two line segments from A to B and two line segments from A to D.

$\therefore$ number of required ways

$= ({}^{n+2}C_2)({}^{n+2}C_2)$

3. Suppose the set contains n elements.

Then number of onto mappings

$= 2^n - 1 - 1 = 2^n - 2$

Thus, $2^n - 2 = 30 \Rightarrow n = 5$.

4. Required number of subsets

$= {}^{10}C_4 + {}^{10}C_5 + \ldots + {}^{10}C_{10}$

$= 2^{10} - ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3)$

$= 1024 - (1 + 10 + 45 + 120) = 848$

5. There are following ways for the photograph.

$M_1\ W_1\ M_2\ W_2 \ldots M_n\ W_n$

or

$W_1\ M_1\ W_2\ M_2 \ldots W_n\ M_n$

This can be done in $(2)(n!)(n!) = 2(n!)^2$ ways.

6. Number of points of intersections

$= ({}^{8}C_2) + 2({}^{4}C_2) + 2({}^{8}C_1)({}^{4}C_1)$

$= 104$

7. Required number of ways

$= ({}^{6}C_1 + {}^{6}C_2 + \ldots + {}^{6}C_6)({}^{6}C_1 + {}^{6}C_2 + \ldots + {}^{6}C_6) - ({}^{6}C_1)({}^{6}C_1)$

$= (2^6 - 1)(2^6 - 1) - 36 = 2^{12} - 2^7 - 35$

8. If A and B are two finite sets, and $f : A \to B$ is an injective mapping, then $n(A) \le n(B)$.

As $m > n$, there does not exist any mapping from set with m elements to a set with n elements.

9. ${}^{10}C_{x-1} > 2({}^{10}C_x)$

$\Rightarrow \dfrac{10!}{(x-1)!(11-x)!} > 2\dfrac{10!}{x!(10-x)!}$

$\Rightarrow x > 2(11 - x) \Rightarrow 3x > 22$

$\Rightarrow x > 7 \Rightarrow x \ge 8$.

Thus, least positive integral value of x is 8.

10. Number of ways of arranging beds

$= {}^{6}P_6 = 6! = 720$

Number of ways in which Madhu and Puja have their beds together is $(2)({}^{5}P_5) = 2(5!) = 240$.

Thus, required number of ways

$= 720 - 240 = 480$.

11. Two cases arise:

Case 1: Both the drivers are occupying the car. In this case we have to choose 3 persons out of 5 for the remaining seats and car can be driven by any of the two drivers. Thus, in this case car can be occupied and driven in $({}^{5}C_3)({}^{2}C_1)(4!) = 480$ ways.

Case 2: Exactly one drive is occupying the car. In this case we have to choose 4 persons out of the remaining 5 people. In this case car can be occupied and driven in $({}^{5}C_4)({}^{2}C_1)(4!) = 240$ ways.

Thus, the required number of ways

$= 480 + 240 = 720$

12. ${}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_n$

$\therefore \dfrac{{}^{n}C_{r-1}}{{}^{n}C_{r-1} + {}^{n}C_r} = \dfrac{n!}{(r-1)!(n-r+1)!} \cdot \dfrac{r!(n+1-r)!}{(n+1)!}$

$$= \frac{r}{n+1}$$

$$\Rightarrow S = \sum_{r=1}^{n}\left(\frac{r}{n+1}\right)^3 = \frac{1}{(n+1)^3}\cdot\frac{n^2(n+1)^2}{4}$$

$$= \frac{n^2}{4(n+1)} = \frac{36}{13}$$

$\Rightarrow 13n^2 = 144(n + 1) \Rightarrow n = 12, -12/13$

Thus, $n = 12$

13. Number of code words ending with even digit

$= (21)\ (20)\ (9)\ (4)$

$\therefore\ 432k = (21)\ (20)\ (9)\ (4)$

$\Rightarrow k = 35.$