6 Sequences and Series

Section-A : JEE Advanced/ IIT-JEE

A **1.** 3050 **2.** 4 **3.** $\frac{n^2(n+1)}{2}$ **4.** 4 : 1 or 1 : 4 **5.** $\frac{1}{4}(n+1)^2(2n-1)$

6. –3, 77

C **1.** (c) **2.** (b) **3.** (c) **4.** (a) **5.** (c) **6.** (c) **7.** (d)

8. (d) **9.** (b) **10.** (d) **11.** (a) **12.** (d) **13.** (c) **14.** (d)

15. (c) **16.** (c) **17.** (c) **18.** (d) **19.** (b)

D **1.** (a, b, d) **2.** (b, c) **3.** (b) **4.** (c) **5.** (b) **6.** (a, d) **7.** (b, d)

8. (a, d) **9.** (a, d)

E **1.** 3 and 6 or 6 and 3 **2.** 9 **3.** yes, infinite **4.** 5, 8, 12 **7.** $\frac{2^{mn}-1}{2^{mn}(2^n-1)}$ **8.** $-\frac{1}{4}$

9. 3 **11.** $\frac{n(2n+1)(4n+1)-3}{3}$ **12.** $\beta \in \left(-\infty, \frac{1}{3}\right], \gamma \in \left[-\frac{1}{27}, \infty\right)$

15. $G = (A_1\ A_2 \ldots\ldots A_n\ H_1\ H_2 \ldots\ldots H_n)^{1/2n}$. **18.** 6

G **1.** (b) **2.** (d) **3.** (b) **4.** (c) **5.** (a) **6.** (b)

H. **1.** (c)

I **1.** 3 **2.** 0 **3.** 9 **4.** 5 **5.** 4 **6.** 9 **7.** 8

Section-B : JEE Main/ AIEEE

1. (b) **2.** (d) **3.** (b) **4.** (b) **5.** (b) **6.** (a) **7.** (a) **8.** (d)

9. (d) **10.** (b) **11.** (b) **12.** (c) **13.** (d) **14.** (d) **15.** (d) **16.** (d)

17. (d) **18.** (b) **19.** (b) **20.** (a) **21.** (a) **22.** (c) **23.** (b) **24.** (c)

25. (a) **26.** (b) **27.** (d) **28.** (d) **29.** (d) **30.** (d)

Section-A JEE Advanced/ IIT-JEE

A. Fill in the Blanks

1. The sum of integers from 1 to 100 that are divisible by 2 or = sum of integers from 1 to 100 divisible by 2 + sum of integers from 1 to 100 divisible by 5 – sum of integers from 1 to 100 divisible by 10

$=(2+4+6+...+100)+(5+10+15+...+100) -(10+20+...+100)$

$=\frac{50}{2}[2\times2+49\times2]+\frac{20}{2}[2\times5+19\times5]$

$=\frac{10}{2}[2\times10+9\times10]=2550+1050-550=3050$

2. The given equation is

$$\log_7 \log_5 (\sqrt{x+5}+\sqrt{x})=0$$

$\Rightarrow \log_5(\sqrt{x+5}+\sqrt{x})=1$

$\Rightarrow \sqrt{x+5}+\sqrt{x}=5 \Rightarrow \sqrt{x+5}=5-\sqrt{x}$

Squaring both sides

$\Rightarrow x+5=25-10\sqrt{x}+x \Rightarrow 10\sqrt{x}=20$

$\Rightarrow \sqrt{x}=2 \Rightarrow x=4$

3. When n is odd, let $n = 2m + 1$

$\therefore$ The req. sum

$=1^2+2.2^2+3^2+2.4^2+...+2(2m)^2+(2m+1)^2$

$=\Sigma(2m+1)^2+4[1^2+2^2+3^2+...+m^2]$

$=\frac{(2m+1)(2m+2)(4m+2+1)}{6}+\frac{4m(m+1)(2m+1)}{6}$

$=\frac{(2m+1)(m+1)}{6}[2(4m+3)+4m]$

$=\frac{(2m+1)(2m+2)(6m+3)}{6}=\frac{(2m+1)^2(2m+2)}{2}$

$=\frac{n^2(n+1)}{2}$ $[\because 2m+1=n)$

4. Let a and b be two positive numbers.

Then, $H.M. = \frac{2ab}{a+b}$ and $G.M. = \sqrt{ab}$

ATQ $HM : GM = 4 : 5$

$\therefore \frac{2ab}{(a+b)\sqrt{ab}} = \frac{4}{5}$

$\Rightarrow \frac{2\sqrt{ab}}{a+b} = \frac{4}{5} \Rightarrow \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{5+4}{5-4}$

$\Rightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{9}{1} \Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = 3, -3$

$\Rightarrow \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{3+1}{3-1}, \frac{-3+1}{-3-1} \Rightarrow \frac{\sqrt{a}}{\sqrt{b}} = 2, \frac{1}{2} \Rightarrow \frac{a}{b} = 4, \frac{1}{4}$

$a : b = 4 : 1$ or $1 : 4$

5. Since n is an odd integer, $(-1)^{n-1} = 1$ and $n-1, n-3, n-5$, ... are even integers.

We have

$n^3 - (n-1)^3 + (n-2)^3 - (n-3)^3 + ... + (-1)^{n-1}1^3$

$= n^3 + (n-1)^3 + (n-2)^3 + ... + 1^3 - 2[(n-1)^3 + (n-3)^3 + ... + 2^3]$

$= [n^3 + (n-1)^3 + (n-2)^3 + ... + 1^3]$

$-2 \times 2^3 \left[\left(\frac{n-1}{2}\right)^3 + \left(\frac{n-3}{2}\right)^3 + ... + 1^3\right]$

[$\because$ n − 1, n − 3......... are even integers]

Here the first square bracket contain the sum of cubes of 1st n natural numbers. Whereas the second square bracket contains the sum of the cubes of natural numbers from 1 to $\left(\frac{n-1}{2}\right)$, where $n-1, n-3$, ... are even integers. Using the formula for sum of cubes of 1st n natural numbers we get the summation

$= \left[\frac{n(n+1)}{2}\right]^2 - 16\left[\left\{\frac{1}{2}\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right)\right\}\right]^2$

$= \frac{1}{4}n^2(n+1)^2 - 16\frac{(n-1)^2(n+1)^2}{16 \times 4}$

$= \frac{1}{4}(n+1)^2[n^2 - (n-1)^2] = \frac{1}{4}(n+1)^2(2n-1)$

6. It is given

$p + q = 2, \quad pq = A$

and $r + s = 18, \quad rs = B$

and it is given that p, q, r, are in A.P.

Therefore, let $p = a - 3s$, $q = a - d$, $r = a + d$ and $s = a + 3d$.

As $p < q < r < s$, we have $d > 0$

Now, $2 = p + q = a - 3d + a - d = 2a - 4d$

$\Rightarrow a - 2d = 1$ ....(1)

Again $18 = r + s = a + d + a + 3d$

$\Rightarrow 18 = 2a + 4d \Rightarrow 9 = a + 2d.$ ....(2)

Subtracting (1) from (2)

$\Rightarrow 8 = 4d$

$\Rightarrow 2 = d$ Putting in (2) we obtain a = 5

$\therefore p = a - 3d = 5 - 6 = -1, \quad q = a - d = 5 - 2 = 3$

$r = a + d = 5 + 2 = 7, \quad s = a + 3d = 5 + 6 = 11$

Therefore, $A = pq = -3$ and $B = rs = 77$.

C. MCQs with ONE Correct Answer

1. (c) $\because x, y, z$ are the p^{th}, q^{th} and r^{th} terms of an AP.

$\therefore x = A + (p-1)D; y = A + (q-1)D;$

$z = A + (r-1)D$

$\Rightarrow x - y = (p-q)D;\ y - z = (q-r)D$

$z - x = (r-p)D$ ....(1)

where A is the first term and D is the common difference.

Also x, y, z are the p^{th}, q^{th}, and r^{th} terms of a GP.

$\therefore x = AR^{p-1}, y = AR^{q-1}, z = AR^{r-1}$

$\therefore x^{y-z}y^{z-x}z^{x-y} = (AR^{p-1})^{y-z}(AR^{q-1})^{z-x}(AR^{r-1})^{x-y}$

$= A^{y-z+z-x+x-y}R^{(p-1)(y-z)+(q-1)(z-x)+(r-1)(x-y)}$

$= A^0 r^{(p-1)(q-r)D+(q-1)(r-p)D+(r-1)(p-q)D} = A^0R^0 = 1$

2. (b) $ar^2 = 4$ ....(1)

$a.\ ar.\ ar^2.ar^3.ar^4 = a^5r^{10} = (ar^2)^5 = 4^5.$

3. (c) $2.\overline{357} = 2 + .357 + 0.000357 + ...\infty$

$= 2 + \frac{357}{10^3} + \frac{357}{10^6} + ...\infty = 2 + \frac{\frac{357}{10^3}}{1 - \frac{1}{10^3}} = 2 + \frac{357}{999} = \frac{2355}{999}$

4. (a) a, b, c are in G.P.

$b^2 = ac$ ....(1)

$ax^2 + 2bx + c = 0$

and $dx^2 + 2ex + f = 0$ have a common root

Let it be α, then $a\alpha^2 + 2b\alpha + c = 0$

$d\alpha^2 + 2e\alpha + f = 0$

$\Rightarrow \frac{\alpha^2}{2(bf - ec)} = \frac{\alpha}{cd - af} = \frac{1}{2(ae - bd)}$

$\Rightarrow \alpha^2 = \frac{bf - ce}{ae - bd}; \alpha = \frac{cd - af}{2(ae - bd)}$

Substituting the value of α, we get

$\frac{(cd - af)^2}{4(ae - bd)^2} = \frac{bf - ce}{ae - bd}$

$\Rightarrow (cd - af)^2 = 4(ae - bd)(bf - ce)$

Dividing both sides by a^2c^2 we get

$\left(\frac{d}{a} - \frac{f}{c}\right)^2 = 4\left(\frac{e}{c} - \frac{bd}{ac}\right)\left(\frac{bf}{ac} - \frac{e}{a}\right)$

$\left(\frac{d}{a} - \frac{f}{c}\right)^2 = 4\left(\frac{e}{c} - \frac{d}{b}\right)\left(\frac{f}{b} - \frac{e}{a}\right)$ [Using eq. (1)]

$\Rightarrow \frac{d^2}{a^2} + \frac{f^2}{c^2} - \frac{2df}{ac} = \frac{4ef}{cb} - \frac{4e^2}{ac} - \frac{4df}{b^2} + \frac{4de}{ab}$

$\Rightarrow \frac{d^2}{a^2} + \frac{f^2}{c^2} + \frac{4e^2}{b^2} + 2\frac{d}{a}.\frac{f}{c} - 4\frac{e}{b}.\frac{f}{c} - 4.\frac{d}{a}.\frac{e}{b} = 0$

[Using eq.(1)]

$\Rightarrow \left(\frac{d}{a} + \frac{f}{c} - 2\frac{e}{b}\right)^2 = 0 \Rightarrow \frac{d}{a} + \frac{f}{c} = \frac{2e}{b}$

$\Rightarrow \frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

5. **(c)** Let $S=\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+....n\ terms$

$$=\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{4}\right)+\left(1-\frac{1}{8}\right)+\left(1-\frac{1}{16}\right)....\text{n terms}$$

$$=(1+1+1+....n\text{ terms})-\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+....+\frac{1}{2^n}\right)$$

$$=n-\left[\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}\right]=n-1+2^{-n}$$

6. **(c)** We know that $\log_2 4=2$ and $\log_2 8=3$

$\therefore$ $\log_2 7$ lies between 2 and 3

$\therefore$ $\log_2 7$ is either rational or irrational but not integer or prime number.

If possible let $\log_2 7=\frac{p}{q}$ (a rational number)

$\Rightarrow$ $2^{p/q}=7\Rightarrow 2^p=7^q$

$\Rightarrow$ even number = odd number

$\therefore$ We get a contradiction, so assumption is wrong.

Hence $\log_2 7$ must be an irrational number.

7. **(d)** In $(a+c)$, In $(a-c)$, In $(a-2b+c)$ are in A.P.

$\Rightarrow$ $a+c, a-c, a-2b+c$ are in G.P.

$\Rightarrow$ $(c-a)^2=(a+c)(a-2b+c)$

$\Rightarrow$ $(c-a)^2=(a+c)^2-2b(a+c)$

$\Rightarrow$ $2b(a+c)=(a+c)^2-(c-a)^2$

$\Rightarrow$ $2b(a+c)=4ac\Rightarrow b=\frac{2ac}{a+c}$

$\Rightarrow$ a, b, c are in H.P.

8. **(d)** $a_1=h_1=2, a_{10}=h_{10}=3$

$3=a_{10}=2+9d \Rightarrow d=1/9$

$\therefore$ $a_4=2+3d=7/3$

$$3=h_{10}\Rightarrow\frac{1}{3}=\frac{1}{h_{10}}=\frac{1}{2}+9D \quad\therefore\ D=-\frac{1}{54}$$

$$\frac{1}{h_7}=\frac{1}{2}+6D=\frac{1}{2}-\frac{1}{9}=\frac{7}{18} \qquad \therefore\ a_4h_7=\frac{7}{3}\times\frac{18}{7}=6.$$

9. **(b)** $\frac{1}{H}=\frac{1}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)=\frac{\alpha+\beta}{2\alpha\beta}=\frac{(4+\sqrt{5})}{(5+\sqrt{2})}\cdot\frac{(5+\sqrt{2})}{(8+2\sqrt{5})}\cdot\frac{1}{2}=\frac{1}{4}$

$\therefore$ $H=4.$

10. **(d)** Sum = 4 and second term = 3/4, it is given that first term is a and common ratio r

$$\Rightarrow \frac{a}{1-r}=4 \text{ and } ar=3/4 \Rightarrow r=\frac{3}{4a}$$

$$\text{Therefore, } \frac{a}{1-\frac{3}{4a}}=4 \Rightarrow \frac{4a^2}{4a-3}=4$$

or $a^2-4a+3=0\Rightarrow(a-1)(a-3)=0$

$\Rightarrow$ $a=1$ or 3

When $a=1, r=3/4$ and when $a=3, r=1/4$

11. **(a)** α,β are the roots of $x^2-x+p=0$

$\therefore$ $\alpha+\beta=1$ ....(1)

$\alpha\beta=p$ ...(2)

γ,δ are the roots of $x^2-4x+q=0$

$\therefore$ $\gamma+\delta=4$ ....(3)

$\gamma\delta=q$ ....(4)

$\alpha,\beta,\gamma,\delta$ are in G.P.

$\therefore$ Let $\alpha=a;\ \beta=ar,\ \gamma=ar^2,\ \delta=ar^3$.

Substituting these values in equations (1), (2), (3) and (4), we get

$a+ar=1$ ....(5)

$a^2r=p$ ....(6)

$ar^2+ar^3=4$ ....(7)

$a^2r^5=q$ ....(8)

Dividing (7) by (5) we get

$$\frac{ar^2(1+r)}{a(1+r)}=\frac{4}{1}\Rightarrow r^2=4\Rightarrow r=2,-2$$

$$(5)\ \Rightarrow a=\frac{1}{1+r}=\frac{1}{1+2} \text{ or } \frac{1}{1-2}=\frac{1}{3} \text{ or} -1$$

As p is an integer (given), r is also an integer (2 or – 2).

$\therefore$ (6) $\Rightarrow a\neq\frac{1}{3}$. Hence $a=-1$ and $r=-2$.

$\therefore$ $p=(-1)^2\times(-2)=-2$

$q=(-1)^2\times(-2)^5=-32$

12. **(d)** a, b, c, d are in A.P.

$\therefore$ d, c, b, a are also in A.P.

$\Rightarrow$ $\frac{d}{abcd},\frac{c}{abcd},\frac{b}{abcd},\frac{a}{abcd}$ are also in A.P.

$\Rightarrow$ $\frac{1}{abc},\frac{1}{abd},\frac{1}{acd},\frac{1}{bcd}$ are in A.P.

$\Rightarrow$ abc, abd, acd, bcd are in H.P.

13. **(c)** ATQ $2+5+8+....2n$ terms $=57+59+61+....n$ terms

$$\Rightarrow \frac{2n}{2}[4+(2n-1)3]=\frac{n}{2}[114+(n-1)2]$$

$\Rightarrow$ $6n+1=n+56\Rightarrow 5n=55\Rightarrow n=11$

14. **(d)** Given that a, b, c are in A.P.

$\Rightarrow$ $2b=a+c$

$\Rightarrow$ but given $a+b+c=3/2 \Rightarrow 3b=3/2$

$\Rightarrow$ $b=1/2$ and then $a+c=1$

Again a^2, b^2, c^2, are in G.P. $\Rightarrow b^4=a^2c^2$

$\Rightarrow$ $b^2=\pm ac\Rightarrow ac=\frac{1}{4}$ or $-\frac{1}{4}$

and $a+c=1$ ....(1)

Considering $a+c=1$ and $ac=1/4$

$\Rightarrow$ $(a-c)^2=1-1=0 \Rightarrow a=c$ but $a\neq c$ as given that $a<b<c$

$\therefore$ We consider $a+c=1$ and $ac=-1/4$

$\Rightarrow$ $(a-c)^2=1+1=2\Rightarrow a-c=\pm\sqrt{2}$

but $a<c\Rightarrow a-c=-\sqrt{2}$ ....(2)

Solving (1) and (2) we get $a=\frac{1}{2}-\frac{1}{\sqrt{2}}$

15. **(c)** $\frac{x}{1-r}=5 \Rightarrow r=1-\frac{x}{5}$

Since G.P. contains infinite terms

$\therefore \quad -1<r<1$

$\Rightarrow \quad -1<1-\frac{x}{5}<1 \Rightarrow -2<-\frac{x}{5}<0$

$\Rightarrow \quad -10<x<0. \Rightarrow 0<\frac{x}{5}<2$

$\Rightarrow \quad 0<x<10.$

16. **(c)** In the quadratic equation $ax^2+bx+c=0$

$\Delta=b^2-4ac$ and $\alpha+\beta=-\frac{b}{a}, \alpha\beta=\frac{c}{a}$

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ac}{a^2}$

and $\alpha^3+\beta^3=-\frac{b^3}{a^3}-\frac{3c}{a}\left(-\frac{b}{a}\right)=-\left(\frac{b^3-3abc}{a^3}\right)$

Given $\alpha+\beta, \alpha^2+\beta^2, \alpha^3+\beta^3$ are in G.P.

$\Rightarrow -\frac{b}{a}, \frac{b^2-2ac}{a^2}, -\frac{(b^3-3abc)}{a^3}$ are in G.P.

$\Rightarrow \left(\frac{b^2-2ac}{a^2}\right)^2=\frac{b}{a}\left(\frac{b^3-3abc}{a^3}\right)$

$\Rightarrow b^4+4a^2c^2-4ab^2c=b^4-3ab^2c$

$\Rightarrow 4a^2c^2-ab^2c=0 \Rightarrow ac\,\Delta=0$

$\Rightarrow c\,\Delta=0 \quad (\because$ In quadratic $a\neq 0)$

17. **(c)** Given that for an A.P, $S_n=cn^2$

Then $T_n=S_n-S_{n-1}=cn^2-c(n-1)^2$

$=(2n-1)c$

$\therefore$ Sum of squares of n terms of this A.P

$=\sum T_n^2 == \sum(2n-1)^2.c^2$

$=c^2\left[4\sum n^2-4\sum n+n\right]$

$=c^2\left[\frac{4n(n+1)(2n+1)}{6}-\frac{4n(n+1)}{2}+n\right]$

$=c^2n\left[\frac{2(2n^2+3n+1)-6(n+1)+3}{3}\right]$

$=c^2n\left[\frac{4n^2-1}{3}\right]=\frac{n(4n^2-1)c^2}{3}$

18. **(d)** $\because a_1, a_2, a_3, ...$ are in H.P.

$\therefore \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$ .... are in A.P.

$\therefore \quad \frac{1}{a_1}=\frac{1}{5}$ and $\frac{1}{a_{20}}=\frac{1}{25}$

$\frac{1}{a_1}+19d=\frac{1}{a_{20}} \Rightarrow \frac{1}{5}+19d=\frac{1}{25} \Rightarrow d=\frac{-4}{475}$

Now $\frac{1}{a_n}=\frac{1}{5}+(n-1)\left(\frac{-4}{475}\right)$

Clearly $a_n<0$ if $\frac{1}{a_n}<0 \Rightarrow \frac{1}{5}-\frac{4n}{475}+\frac{4}{475}<0$

$\Rightarrow -4n<-99$ or $n>\frac{99}{4}=24\frac{3}{4} \quad \therefore n\geq 25$

Hence least value of n is 25.

19. **(b)** $\log_c b_1, \log_c b_2, ----, \log_c b_{101}$ are in A.P.

$\Rightarrow \quad b_1, b_2, ---- b_{101}$ are in G.P.

Also $a_1, a_2, ---- a_{101}$ are in A.P.

where $a_1=b_1$ are $a_{51}=b_{51}$.

$\therefore \quad b_2, b_3, ---, b_{50}$ and GM's and $a_2, a_3, ----, a_{50}$ are AM's between b_1 and b_{51}.

$\because$ GM < AM $\Rightarrow b_2<a_2, b_3<a_3, ----, b_{50}<a_{50}$

$\because \quad b_1+b_2+----+b_{51}<a_1+a_2+----+a_{51}$

$\Rightarrow \quad t<s$

Also a_1, a_{51}, a_{101} are in AP

b_1, b_{51}, b_{101} are in GP

$\because \quad a_1=b_1$ and $a_{51}=b_{51}$

$\therefore \quad b_{101}>a_{101}$

D. MCQs with ONE or MORE THAN ONE Correct

1. **(a,b,d)** Let x be the first term and y the $(2n-1)$th terms of AP, GP and HP whose nth terms are a, b, c respectively.

For AP, $y=x+(2n-2)d$

$\Rightarrow \quad d=\frac{y-x}{2(n-1)}$

$\therefore \quad a=x+(n-1)d=x+\frac{1}{2}(y-x)=\frac{1}{2}(x+y)$ ....(1)

For G.P. $\quad y=xr^{2n-2} \Rightarrow r=\left(\frac{y}{x}\right)^{\frac{1}{2n-2}}$

$\therefore \quad b=xr^{n-1}=x.\left(\frac{y}{x}\right)^{1/2}=\sqrt{xy}$ ....(2)

For H.P. $\quad \frac{1}{y}=\frac{1}{x}+(2n-2)d_1$

$\Rightarrow \quad d_1=\frac{x-y}{2xy\,(n-1)}$

$\therefore \quad \frac{1}{c}=\frac{1}{x}+(n-1)\,d_1=\frac{1}{x}+\frac{x-y}{2xy}$

$\frac{1}{c}=\frac{x+y}{2xy} \Rightarrow c=\frac{2xy}{x+y}$ ....(3)

Thus from (1), (2) and (3), a, b, c are A.M., G.M. and H.M. respectively of x and y.

2. **(b, c)** We have for $0<\phi<\frac{\pi}{2}$

$x=\sum_{n=0}^{\infty}\cos^{2n}\phi=1+\cos^2\phi+\cos^4\phi+....\infty$

$\frac{1}{1-\cos^2\phi}=\frac{1}{\sin^2\phi}$ ....(1)

[Using sum of infinite G.P. $\cos^2\alpha$ being < 1]

$$y=\sum_{n=0}^{\infty}\sin^{2n}\phi=1+\sin^2\phi+\sin^4\phi+....\infty$$

$$=\frac{1}{1-\sin^2\phi}=\frac{1}{\cos^2\phi} \quad ....(2)$$

$$z=\sum_{n=0}^{\infty}\cos^{2n}\phi\sin^{2n}\phi$$

$$=1+\cos^2\phi\sin^2\phi+\cos^4\phi\sin^4\phi+....\infty$$

$$=\frac{1}{1-\cos^2\phi\sin^2\phi} \quad ....(3)$$

Substituting the values of $\cos^2\phi$ and $\sin^2\phi$ in (3), from (1) and (2), we get

$$z=\frac{1}{1-\frac{1}{x}.\frac{1}{y}}\Rightarrow z=\frac{xy}{xy-1}\Rightarrow xyz-z=xy\Rightarrow xyz=xy+z.$$

Also, $x+y+z=\dfrac{1}{\cos^2\phi}+\dfrac{1}{\sin^2\phi}+\dfrac{1}{1-\cos^2\phi\sin^2\phi}$

$$\frac{[\sin^2\phi(1-\cos^2\phi\sin^2\phi)+\cos^2\phi(1-\cos^2\phi\sin^2\phi)+\cos^2\phi\sin^2\phi]}{\cos^2\phi\sin^2\phi\,(1-\cos^2\phi\sin^2\phi)}$$

$$=\frac{(\sin^2\phi+\cos^2\phi)\,(1-\cos^2\phi\sin^2\phi)+\cos^2\phi\sin^2\phi}{\cos^2\phi\sin^2\phi\,(1-\cos^2\phi\sin^2\phi)}$$

$$=\frac{1}{\cos^2\phi\sin^2\phi\,(1-\cos^2\phi\sin^2\phi)}=xyz$$

Thus (*b*) and (*c*) both are correct.

3. **(b)** Putting $\theta=0$, we get $b_0=0$

$$\therefore \quad \sin n\theta=\sum_{r=1}^{n}b_r\sin^r\theta\Rightarrow\frac{\sin n\theta}{\sin\theta}=\sum_{r=1}^{n}b_r(\sin\theta)^{r-1}$$

$$=b_1+b_2\sin\theta+b_3\sin^2\theta+......+b_n\sin^{n-1}\theta$$

Taking limit as $\theta\to0$, we obtain

$$\lim_{\theta\to0}\frac{\sin n\theta}{\sin\theta}=b_1\Rightarrow b_1=n.$$

4. **(c)** $T_m=a+(m-1)\,d=1/n$ and $t_n=a+(n-1)d=1/m$

$\Rightarrow (m-n)\,d=1/n-1/m=(m-n)/mn \Rightarrow d=1/mn$

$\Rightarrow a=\dfrac{1}{mn}$

$$\therefore \quad t_{mn}=a+(mn-1)\,d=\frac{1}{mn}+(mn-1)\frac{1}{mn}$$

$$=\frac{1}{mn}+1-\frac{1}{mn}=1$$

5. **(b)** If x,y,z are in G.P. $(x,\,y,\,z>1)$; $\log x,\log y,\log z$ will be in A.P.

$\Rightarrow$ $1+\log x,\ 1+\log y,\ 1+\log z$ will also be in A.P.

$\Rightarrow$ $\dfrac{1}{1+\log x},\dfrac{1}{1+\log y},\dfrac{1}{1+\log z}$ will be in H.P.

6. **(a, d)** We have

$$a(n)=1+\frac12+\frac13+\frac14+\frac15+.....+\frac{1}{2^n-1}$$

$$=1+\left(\frac12+\frac13\right)+\left(\frac14+\frac15+\frac16+\frac17\right)+\left(\frac18+.....+\frac1{15}\right)+...+\left(\frac{1}{2^{n-1}}+.....+\frac{1}{2^n-1}\right)$$

$$<1+\left(\frac12+\frac12\right)+\left(\frac14+\frac14+.....+\frac14\right)+\left(\frac18+\frac18+...+\frac18\right)+..+\left(\frac{1}{2^{n-1}}+..+\frac{1}{2^{n-1}}\right)$$

$$<1+\frac22+\frac44+\frac88+.....+\frac{2^{n-1}}{2^{n-1}}=1+1+.....+1=n$$

Thus, $a(100)<100$

Also

$$a(n)=1+\frac12+\left(\frac13+\frac14\right)+\left(\frac15+\frac16+\frac17+\frac18\right)+.....+\left(\frac{1}{2^{n-1}+1}+..+\frac{1}{2^n}\right)-\frac{1}{2^n}$$

$$>1+\frac12+\left(\frac14+\frac14\right)+\left(\frac18+\frac18+...+\frac18\right)+...+\left(\frac{1}{2^n}+..+\frac{1}{2^n}\right)-\frac{1}{2^n}$$

$$=1+\frac12+\frac24+\frac48+...+\frac{2^{n-1}}{2^n}-\frac{1}{2^n}$$

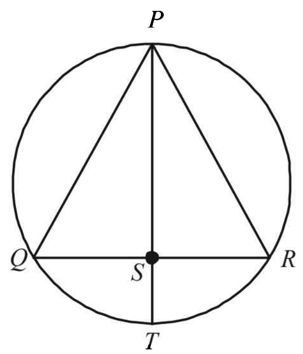
$$=1+\underbrace{\left(\frac12+\frac12+\frac12+.....+\frac12\right)}_{n\text{ times}}-\frac{1}{2^n}$$

$$=1+\frac n2-\frac{1}{2^n}=\left(1-\frac{1}{2^n}\right)+\frac n2$$

Thus, $a(200)>\left(1-\dfrac{1}{2^{200}}\right)+\dfrac{200}{2}>100,$

i.e. $a(200)>100.$

7. **(b, d)**



We know by geometry $PS\times ST=QS\times SR$ ...(1)

$\because$ S is not the centre of circulm circle, GP_3480

$PS \neq ST$

And we know that for two unequal real numbers.

H.M. < G.M.

$$\Rightarrow \frac{2}{\frac{1}{PS}+\frac{1}{ST}} < \sqrt{PS \times ST} \Rightarrow \frac{1}{PS}+\frac{1}{ST} > \frac{2}{\sqrt{PS \times ST}}$$

$$\Rightarrow \frac{1}{PS}+\frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}} \text{ [using eqn (1)] } \quad ...(2)$$

$\therefore$ (b) is the correct option.

Also $\sqrt{QS \times SR} < \frac{QS+SR}{2}$ ($\because$ GM < AM)

$$\Rightarrow \frac{1}{\sqrt{QS \times SR}} > \frac{2}{QR} \Rightarrow \frac{2}{\sqrt{QS \times SR}} > \frac{4}{QR} \quad ...(3)$$

From equations (2) and (3) we get $\frac{1}{PS}+\frac{1}{ST} > \frac{4}{QR}$

$\therefore$ (d) is also the correct option.

8. **(a,d)** We have $S_n = \sum_{k=1}^{n} \frac{n}{n^2+kn+k^2}$

and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2+kn+k^2}$ $n = 1, 2, 3$.........

For $n = 1$ we get

$$S_1 = \frac{1}{1+1+1} = \frac{1}{3} = 0.3 \text{ and } T_1 = \frac{1}{1+0} = 1$$

$$\text{Also } \frac{\pi}{3\sqrt{3}} = \frac{\pi\sqrt{3}}{9} = \frac{3.14 \times 1.73}{9} = 0.34 \times 1.73 = 0.58$$

$$\therefore S_1 < \frac{\pi}{3\sqrt{3}} < T_1, \quad \therefore S_n < \frac{\pi}{3\sqrt{3}} \text{ and } T_n > \frac{\pi}{3\sqrt{3}}$$

9. **(a,d)** $S_n = -1^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 + ...$

$= (3^2 + 7^2 + 11^2 + ...) + (4^2 + 8^2 + 12^2 + ...)$
$-(1^2 + 5^2 + 9^2 + ...) - (2^2 + 6^2 + 10^2 + ...)$

$$= \sum_{r=1}^{n}(4r-1)^2 + \sum_{r=1}^{n}(4r)^2 - \sum_{r=1}^{n}(4r-3)^2 - \sum_{r=1}^{n}(4r-2)^2$$

$$= \left[\sum_{r=1}^{n}(4r-1)^2-(4r-3)^2\right] + 4\left[\sum_{r=1}^{n}(2r)^2-(2r-1)^2\right]$$

$$= 8\sum_{r=1}^{n}(2r-1) + 4\sum_{r=1}^{n}(4r-1)$$

$$= 8\left[2\frac{n(n+1)}{2}-n\right] + 4\left[4\frac{n(n+1)}{2}-n\right]$$

$= 8n^2 + 8n^2 + 4n = 16n^2 + 4n$

For $n = 8$, $16n^2 + 4n = 1056$

and for $n = 9$, $16n^2 + 4n = 1332$

E. Subjective Problems

1. Let the two numbers be a and b, then

$$\frac{2ab}{a+b} = 4...(1); \frac{a+b}{2} = A; \sqrt{ab} = G$$

Also $2A + G^2 = 27 \Rightarrow a + b + ab = 27$ ....(2)

Putting $ab = 27 - (a+b)$ in eqn. (1), we get

$$\frac{54-2(a+b)}{a+b} = 4 \Rightarrow a+b = 9 \text{ then } ab = 27 - 9 = 18$$

Solving the two we get $a = 6, b = 3$ or $a = 3, b = 6$, which are the required numbers.

2. Let there be n sides in the polygon.

Then by geometry, sum of all n interior angles of polygon $= (n-2) \times 180°$

Also the angles are in A.P. with the smallest angle = 120°, common difference = 5°

$\therefore$ Sum of all interior angles of polygon

$$= \frac{n}{2}[2 \times 120 + (n-1) \times 5]$$

Thus we should have

$$\frac{n}{2}[2 \times 120 + (n-1) \times 5] = (n-2) \times 180$$

$$\Rightarrow \frac{n}{2}[5n + 235] = (n-2) \times 180$$

$$\Rightarrow 5n^2 + 235n = 360n - 720$$

$$\Rightarrow 5n^2 - 125n + 720 = 0 \Rightarrow n^2 - 25n + 144 = 0$$

$$\Rightarrow (n-16)(n-9) = 0 \Rightarrow n = 16, 9$$

Also if $n = 16$ then 16th angle $= 120 + 15 \times 5 = 195° > 180°$

$\therefore$ not possible. Hence $n = 9$.

3. If possible let for a G.P.

$T_p = 27 = AR^{p-1}$ ....(1)

$T_q = 8 = AR^{q-1}$ ....(2)

$T_r = 12 = AR^{r-1}$ ....(3)

From (1) and (2)

$$R^{p-q} = \frac{27}{8} \Rightarrow R^{p-q} = (3/2)^3 \quad ....(4)$$

From (2) and (3):

$$R^{q-r} = \frac{8}{12} \Rightarrow R^{q-r} = (3/2)^{-1} \quad ....(5)$$

From (4) and (5):

$R = 3/2; p - q = 3; \ q - r = -1$

$p - 2q + r = 4; \ p, q, r \in N$ ....(6)

As there can be infinite natural numbers for p, q and r to satisfy equation (6)

$\therefore$ There can be infinite G.P's.

4. $2 < a, b, c < 18 \quad a + b + c = 25$ ....(1)

$2, a, b$ are in AP $\Rightarrow 2a = b + 2$

$\Rightarrow 2a - b = 2$ ....(2)

$b, c, 18$ are in GP $\Rightarrow c^2 = 18b$ ....(3)

From (2) $\Rightarrow a = \frac{b+2}{2}$

(1) $\Rightarrow \frac{b+2}{2} + b + c = 25 \Rightarrow 3b = 48 - 2c$

(3) $\Rightarrow c^2 = 6 \ (48 - 2c) \Rightarrow c^2 + 12c - 288 = 0$

$\Rightarrow c = 12, -24$ (rejected) $\Rightarrow a = 5, \ b = 8, \ c = 12$

5. Given that $a, b, c > 0$

We know for +ve numbers A.M. ≥ G.M.

$\therefore$ For +ve numbers a, b, c we get

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \quad ....(1)$$

Also for +ve numbers $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, we get

$$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \ge \sqrt[3]{\frac{1}{abc}} \quad ....(2)$$

Multiplying in eqs (1) and (2) we get

$$\left(\frac{a+b+c}{3}\right)\left(\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\right) \ge \sqrt[3]{abc} \times \frac{1}{\sqrt[3]{abc}}$$

$$\Rightarrow \quad (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 9 \qquad \text{Proved.}$$

6. Given that $n = p_1^{\alpha_1}.\ p_2^{\alpha_2}.\ p_3^{\alpha_3} .... p_k^{\alpha_k}$ ....(1)

Where $n \in N$ and $p_1, p_2, p_3, ...p_k$ are distinct prime numbers.

Taking log on both sides of eq. (1), we get

$\log n = \alpha_1 \log p_1 + \alpha_2 \log p_2 + ... + \alpha_k \log p_k$ ....(2)

Since every prime number is such that

$p_i \ge 2$

$\therefore \quad \log_e p_i \ge \log_e 2$ ....(3)

$\forall \quad i = 1\,(1)\,k$

Using (2) and (3) we get

$\log n \ge \alpha_1 \log 2 + \alpha_2 \log 2 + \alpha_3 \log 2 + ... + \alpha_k \log 2$

$\Rightarrow \quad \log n \ge (\alpha_1 + \alpha_2 + ... + \alpha_k) \log 2$

$\Rightarrow \quad \log n \ge k \log 2$ **Proved.**

7. The given series is

$$\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + ... \text{ up to } m \text{ terms}\right]$$

$$\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left[\left(\frac{1}{2}\right)^r + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + \left(\frac{15}{16}\right)^r + ... \text{to } m \text{ terms}\right]$$

Now, $\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left(\frac{1}{2}\right)^r = 1 - {}^nC_1 . \frac{1}{2} + {}^nC_2 . \frac{1}{2^2} - {}^nC_3 . \frac{1}{2^3} + ...$

$$= \left(1 - \frac{1}{2}\right)^n = \frac{1}{2^n}$$

Similarly, $\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left(\frac{3}{4}\right)^r = \left(1 - \frac{3}{4}\right)^n = \frac{1}{4^n}$ etc.

Hence the given series is,

$$= \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{16^n} + ... \text{to } m \text{ terms}$$

$$= \frac{\frac{1}{2^n}\left(1 - \left(\frac{1}{2^n}\right)^m\right)}{1 - \frac{1}{2^n}} \qquad \text{[Summing the G.P.]}$$

$$= \frac{2^{mn} - 1}{2^{mn}(2^n - 1)}$$

8. The given equation is

$\log_{(2x+3)}(6x^2 + 23 + 21)$
$= 4 - \log_{3x+7}(4x^2 + 12x + 9)$

$\Rightarrow \quad \log_{(2x+3)}(6x^2 + 23x + 21) + \log_{(3x+7)}(4x^2 + 12x + 9) = 4$

$\Rightarrow \quad \log_{(2x+3)}(2x+3)(3x+7) + \log_{(3x+7)}(2x+3)^2 x = 4$

$\Rightarrow \quad 1 + \log_{(2x+3)}(3x+7) + 2\log_{(3x+7)}(2x+3) = 4$

$\Rightarrow \quad \log_{(2x+3)}(3x+7) + \frac{2}{\log_{(2x+3)}(3x+7)} = 3$

Let $\log_{(2x+3)}(3x+7) = y$ ....(1)

$\Rightarrow \quad y + \frac{2}{y} = 3 \Rightarrow y^2 - 3y + 2 = 0$

$\Rightarrow \quad (y-1)(y-2) = 0 \Rightarrow y = 1, 2$

Substituting the values of y in (1), we get

$\Rightarrow \quad \log_{(2x+3)}(3x+7) = 1$ and $\log_{(2x+3)}(3x+7) = 2$

$\Rightarrow \quad 3x + 7 = 2x + 3$ and $3x + 7 = (2x+3)^2$

$\Rightarrow \quad x = -4$ and $4x^2 + 9x + 2 = 0$

$\Rightarrow \quad x = -4$ and $(x+2)(4x+1) = 0$

$\Rightarrow \quad x = -4$ and $x = -2, x = -\frac{1}{4}$

As $\log_a x$ is defined for $x > 0$ and $a > 0\ (a \ne 1)$, the possible value of x should satisfy all of the following inequalities :

$\Rightarrow \quad 2x + 3 > 0$ and $3x + 7 > 0$

Also $2x + 3 \ne 1$ and $3x + 7 \ne 1$

Out of $x = -4, x = -2$ and $x = -\frac{1}{4}$ only $x = -\frac{1}{4}$ satisfies the above inequalities.

So only solution is $x = -\frac{1}{4}$.

9. Given that $\log_3 2, \log_3 (2^x - 5), \log_3 (2^x - 7/2)$ are in A.P.

$\Rightarrow \quad 2\log_3 (2^x - 5) = \log_3 2 + \log_3 (2^x - 7/2)$

$\Rightarrow \quad (2^x - 5)^2 = 2\left(2^x - \frac{7}{2}\right)$

$\Rightarrow \quad (2^x)^2 - 10.2^x + 25 - 2.2^x + 7 = 0$

$\Rightarrow \quad (2^x)^2 - 12.2^x + 32 = 0$

Let $2^x = y$, then we get,

$y^2 - 12y + 32 = 0 \Rightarrow (y-4)(y-8) = 0$

$\Rightarrow \quad y = 4$ or $8 \Rightarrow 2^x = 2^2$ or $2^3 \Rightarrow x = 2$ or 3

But for $\log_3 (2^x - 5)$ and $\log_3 (2^x - 7/2)$ to be defined

$2^x - 5 > 0$ and $2^x - 7/2 > 0$

$\Rightarrow \quad 2^x > 5$ and $2^x > 7/2$

$\Rightarrow \quad 2^x > 5$

$\Rightarrow \quad x \ne 2$ and therefore $x = 3$.

10. Let a and b be two numbers and $A_1, A_2, A_3, ... A_n$ be n A.M's between a and b.

Then a, $A_1, A_2, ... A_n$, b are in A.P.

There are $(n + 2)$ terms in the series, so that

$$a + (n+1)d = b \Rightarrow d = \frac{b-a}{n+1}$$

$$\therefore \quad A_1 = a + \frac{b-a}{n+1} = \frac{an+b}{n+1}$$

$$\therefore \quad p = \frac{an+b}{n+1} \quad ....(1)$$

The first H.M. between a and b, when nHM's are inserted between a and b can be obtained by replacing a by $\frac{1}{a}$ and b by $\frac{1}{b}$ in eq. (1) and then taking its reciptocal.

Therefore, $q = \frac{1}{\frac{\left(\frac{1}{a}\right)n + \frac{1}{b}}{n+1}} = \frac{(n+1)ab}{bn+a}$

$\therefore \quad q = \frac{(n+1)ab}{a+bn}$ ....(2)

We have to prove that q cannot lie between p and $\frac{(n+1)^2}{(n-1)^2}p.$

Now, $n+1 > n-1 \Rightarrow \frac{n+1}{n-1} > 1$

$\Rightarrow \left(\frac{n+1}{n-1}\right)^2 > 1$ or $p\left(\frac{n+1}{n-1}\right)^2 > p$

$\Rightarrow \quad p < p\left(\frac{n+1}{n-1}\right)^2$ ....(3)

Now to prove the given, we have to show that q is less than p.

For this, let, $\frac{p}{q} = \frac{(na+b)\,(nb+a)}{(n+1)^2\,ab}$

$\Rightarrow \quad \frac{p}{q} - 1 = \frac{n(a^2+b^2) + ab(n^2+1) - (n+1)^2\,ab}{(n+1)^2\,ab}$

$\Rightarrow \quad \frac{p}{q} - 1 = \frac{n(a^2+b^2-2ab)}{(n+1)^2\,ab}$

$\Rightarrow \quad \frac{p}{q} - 1 = \frac{n}{(n+1)^2}\left(\frac{a-b}{\sqrt{ab}}\right)^2 = \frac{n}{(n+1)^2}\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2$

$\Rightarrow \quad \frac{p}{q} - 1 > 0$

$\Rightarrow$ (provided a and b and hence p and q are +ve) $p > q$ ....(4)

From 3 and (4), we get, $q < p < \left(\frac{n+1}{n+1}\right)^2 p$

$\therefore$ q can not lie between p and $\left(\frac{n+1}{n+1}\right)^2 p$, if a and b are +ve numbers.

11. We have,

$S_1 = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + ...\infty$

$S_2 = 2 + 2.\frac{1}{3} + 2\left(\frac{1}{3}\right)^2 + ...\infty$

....

....

$S_3 = 3 + 3.\frac{1}{4} + 3\left(\frac{1}{4}\right)^2 + ...\infty$

$S_n = n + n.\frac{1}{n+1} + n\left(\frac{1}{n+1}\right)^2 + ...\infty$

$\Rightarrow \quad S_1 = \frac{1}{1-\frac{1}{2}} = 2 \qquad \left[Using\ S_\infty = \frac{a}{1-r}\right]$

$S_2 = \frac{2}{1-\frac{1}{3}} = 3, \qquad S_3 = \frac{3}{1-\frac{1}{4}} = 4,$

$S_n = \frac{n}{1-\frac{1}{n+1}} = (n+1)$

$\therefore \quad S_1^2 + S_2^2 + S_3^2 + ... + S_{2n-1}^2$

$= 2^2 + 3^2 + 4^2 + ... + (n+1)^2 + ... + (2n)^2$

NOTE THIS STEP:

$\sum_{r=1}^{2n} r^2 - 1 = \frac{2n(2n+1)(4n+1)}{6} - 1^2$

$= \frac{n(2n+1)(4n+1) - 3}{3}$

12. Since x_1, x_2, x_3 are in A.P. Therefore, let $x_1 = a-d, x_2 = a$ and $x_3 = a+d$ and x_1, x_2, x_3 are the roots of $x^3 - x^2 + \beta x + \gamma = 0$

We have $\Sigma\alpha = a - d + a + a + d = 1$ ....(1)

$\Sigma\alpha\beta = (a-d)\,a + a\,(a+d) + (a-d)\,(a+d) = \beta$ ....(2)

$\alpha\,\beta\,\gamma = (a-d)\,a\,(a+d) = -\gamma$ ....(3)

From (1), we get, $3a = 1 \Rightarrow a = 1/3$

From (2), we get, $3a^2 - d^2 = \beta$

$\Rightarrow 3\,(1/3)^2 - d^2 = \beta \Rightarrow 1/3 - \beta = d^2$

We know that $d^2 \geq 0 \ \forall \ d \in R$

$\Rightarrow \quad \frac{1}{3} - \beta \geq 0 \qquad \because d^2 \geq 0$

$\Rightarrow \quad \beta \leq \frac{1}{3} \Rightarrow \beta \in (-\infty, 1/3]$

From (3), $a\,(a^2 - d^2) = -\gamma$

$\Rightarrow \quad \frac{1}{3}\left(\frac{1}{9} - d^2\right) = -\gamma \Rightarrow \frac{1}{27} - \frac{1}{3}d^2 = -\gamma$

$\Rightarrow \quad \gamma + \frac{1}{27} = \frac{1}{3}d^2 \Rightarrow \gamma + \frac{1}{27} \geq 0$

$\Rightarrow \quad \gamma \geq -\frac{1}{27} \Rightarrow \gamma \in \left[-\frac{1}{27}, \infty\right)$

Hence $\beta \in (-\infty, 1/3)$ and $\gamma \in [-1/27, \infty]$

13. Solving the system of equations, $u+2v+3w=6$, $4u+5v+6w=12$ and $6u+9v=4$ we get $u=-1/3$, $v=2/3$, $w=5/3$

$\therefore \quad u+v+w=2,\ \frac{1}{u}+\frac{1}{v}+\frac{1}{w}=-\frac{9}{10}$

Let r be the common ratio of the G.P., a, b, c, d. Then $b=ar$, $c=ar^2$, $d=ar^3$.

Then the first equation

$$\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)x^2+[(b-c)^2+(c-a)^2+(d-b)^2]x+(u+v+w)=0$$

becomes

$$-\frac{9}{10}x^2+[(ar-ar^2)^2+(ar^2-a)^2+(ar^3-ar)^2]\,x+2=0$$

i.e., $9x^2-10a^2(1-r)^2[r^2+(r+1)^2+r^2(r+1)^2]x-20=0$

i.e., $9x^2-10a^2(1-r)^2(r^4+2r^3+3r^2+2r+1)x-20=0$

i.e., $9x^2-10a^2(1-r)^2(1+r+r^2)^2x-20=0$,

i.e., $9x^2-10a^2(1-r^3)^2x-20=0$ ....(1)

The second equation is,

$20x^2+10(a-ar^3)^2x-9=0$

i.e., $20x^2+10a^2(1-r^3)^2x-9=0$ ....(2)

Since (2) can be obtained by the substitution $x \to 1/x$, equations (1) and (2) have reciprocal roots.

14. Let $a-3d$, $a-d$, $a+d$ and $a+3d$ be any four consecutive terms of an A.P. with common difference 2d. $\because$ Terms of A.P. are integers, $2d$ is also an integer.

Hence $P=(2d)^4+(a-3d)(a-d)(a+d)(a+3d)$

$=16d^4+(a^2-9d^2)(a^2-d^2)=(a^2-5d^2)^2$

Now, $a^2-5d^2=a^2-9d^2+4d^2$

$=(a-3d)(a+3d)+(2d)^2=$ some integer

Thus, $P=$ square of an integer.

15. Given that $a_1, a_2, ...a_n$ are +ve real no's in G.P.

$$\therefore \left.\begin{aligned} a_1&=a\\ a_2&=ar\\ a_3&=ar^2\\ &\vdots\\ a_n&=ar^{n-1}\end{aligned}\right\}\quad \text{As } a_1,a_2,....a_n \text{ are +ve} \quad \therefore r>0$$

A_n is A.M. of $a_1, a_2, ..., a_n$

$$\therefore \quad A_n=\frac{a_1+a_2+....+a_n}{n}=\frac{a+ar+....+ar^{n-1}}{n}$$

$$A_n=\frac{a(1-r^n)}{n(1-r)} \quad .... (1)\ (\text{For } r\neq 1)$$

G_n is G.M. of $a_1, a_2, ..., a_n$

$$\therefore \quad G_n=\sqrt[n]{a_1,a_2...a_n}=\sqrt[n]{a.ar.ar^2....ar^{n-1}}$$

$$=\sqrt[n]{a^n, r^{\frac{n(n-1)}{2}}}=ar^{\frac{(n-1)}{2}}$$

$$G_n=ar^{\frac{(n-1)}{2}} \quad .... (2)\ (r\neq 1)$$

H_n is H.M. of $a_1, a_2, ..., a_n$

$$\therefore \quad H_n=\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+....+\frac{1}{a_n}}=\frac{n}{\frac{1}{a}+\frac{1}{ar}+....+\frac{1}{ar^{n-1}}}$$

$$=\frac{n}{\frac{1}{a}\frac{\left(\frac{1}{r^n}-1\right)}{\left(\frac{1}{r}-1\right)}}=\frac{n}{\frac{1}{a}\left(\frac{1-r^n}{r^n}\right).\frac{r}{1-r}}$$

$$H_n=\frac{anr^{n-1}(1-r)}{(1-r^n)} \quad (r\neq 1) \quad ...(3)$$

We also observe that

$$A_nH_n=\frac{a(1-r^n)}{(n(1-r))}\times\frac{anr^{n-1}(1-r)}{(1-r^n)}=a^nr^{n-1}=G_n^2$$

$\therefore \quad A_nH_n=G_n^2$ ....(4)

$\therefore$ Now, G.M. of $G_1, G_2, ... G_n$ is

$G=\sqrt[n]{G_1\ G_2\ ....G_n}$

$G=\sqrt[n]{\sqrt{A_1H_1}\ \sqrt{A_2H_2}...\sqrt{A_nH_n}}$

[Using equation (4)]

$G=(A_1A_2....A_nH_1H_2....H_n)^{1/2n}$ ...(5)

If $r=1$ then

$A_n=G_n=H_n=a$

Also $A_nH_n=G_n^2$

$\therefore$ For $r=1$ also, equation (5) holds.

Hence we get,

$G=(A_1A_2....A_nH_1H_2....H_n)^{1/2n}$

16. Clearly $A_1+A_2=a+b$

$$\frac{1}{H_1}+\frac{1}{H_2}=\frac{1}{a}+\frac{1}{b}$$

$$\Rightarrow \frac{H_1+H_2}{H_1H_2}=\frac{a+b}{ab}=\frac{A_1+A_2}{G_1G_2} \Rightarrow \frac{G_1G_2}{H_1H_2}=\frac{A_1+A_2}{H_1+H_2}$$

Also $\frac{1}{H_1}=\frac{1}{a}+\frac{1}{3}\left(\frac{1}{b}-\frac{1}{a}\right) \Rightarrow H_1=\frac{3ab}{2b+a}$

$$\frac{1}{H_2}=\frac{1}{a}+\frac{2}{3}\left(\frac{1}{b}-\frac{1}{a}\right) \Rightarrow H_2=\frac{3ab}{2a+b}$$

$$\Rightarrow \frac{A_1+A_2}{H_1+H_2}=\frac{a+b}{3ab\left(\frac{1}{2b+a}+\frac{1}{2a+b}\right)}$$

$$=\frac{(2b+a)(2a+b)}{9ab}$$

17. Given that a, b, c are in A.P.

$\Rightarrow \quad 2b=a+c$ ....(1)

and a^2, b^2, c^2 are in H.P.

$$\Rightarrow \quad \frac{1}{b^2}-\frac{1}{a^2}=\frac{1}{c^2}-\frac{1}{b^2}$$

$$\Rightarrow \quad \frac{(a-b)(a+b)}{b^2a^2}=\frac{(b-c)(b+c)}{b^2c^2}$$

$\Rightarrow \quad ac^2+bc^2=a^2b+a^2c \quad [\because a-b=b-c]$

$\Rightarrow \quad ac(c-a)+b(c-a)(c+a)=0$

$\Rightarrow \quad (c-a)(ab+bc+ca)=0$

$\Rightarrow \quad$ either $c-a=0$ or $ab+bc+ca=0$

$\Rightarrow$ either $c = a$ or $(a+c)\,b + ca = 0$
and then from (i) $2b^2 + ca = 0$

Either $a = b = c$ or $b^2 = a\left(\frac{-c}{2}\right)$

i.e. a, b, – c/2 are in G.P. **Hence Proved.**

18. $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + ... + (-1)^{n-1}\left(\frac{3}{4}\right)^n$

$$= \frac{\frac{3}{4}\left(1-\left(-\frac{3}{4}\right)^n\right)}{1+\frac{3}{4}} = \frac{3}{7}\left(1-\left(-\frac{3}{4}\right)^n\right)$$

$b_n = 1 - a_n$ and $b_n > a_n \ \forall\ n \ge n_0$

$\therefore\ 1 - a_n > a_n \Rightarrow 2a_n < 1$

$$\Rightarrow \frac{6}{7}\left[1-\left(-\frac{3}{4}\right)^n\right] < 1 \Rightarrow -\left(-\frac{3}{4}\right)^n < \frac{1}{6}$$

$\Rightarrow (-3)^{n+1} < 2^{2n-1}$

For n to be even, inequality always holds. For n to be odd, it holds for n ≥ 7.

$\therefore$ The least natural no., for which it holds is 6
($\because$ it holds for every even natural no.)

G. Comprehension Based Questions

1. **(b)** $V_1 + V_2 + ... + V_n = \sum_{r=1}^{n} V_r = \sum_{r=1}^{n}\left(r^3 - \frac{r^2}{2} + \frac{r}{2}\right)$

$$= \Sigma n^3 - \frac{\Sigma n^2}{2} + \frac{\Sigma n}{2}$$

$$= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)\,(2n+1)}{12} + \frac{n(n+1)}{4}$$

$$= \frac{n(n+1)}{4}\left[n(n+1) - \frac{2n+1}{3} + 1\right]$$

$$= \frac{n(n+1)\,(3n^2+n+2)}{12}$$

2. **(d)** $T_r = V_{r+1} - V_r - 2$

$$= \left[(r+1)^3 - \frac{(r+1)^2}{2} + \frac{r+1}{2}\right] - \left[r^3 - \frac{r^2}{2} + \frac{r}{2}\right] - 2$$

$= 3r^2 + 2r + 1$

$T_r = (r+1)(3r-1)$

For each r, T_r has two different factors other than 1 and itself.

$\therefore$ T_r is always a composite number.

3. **(b)** $\therefore Q_{r+1} - Q_r = T_{r+2} - T_{r+1} - (T_{r+1} - T_r)$

$= T_{r+2} - 2T_{r+1} + T_r$

$= (r+3)(3r+5) - 2(r+2)(3r+2) + (r+1)(3r-1)$

$\because Q_{r+1} - Q_r = 6(r+1) + 5 - 6r - 5 = 6$ (constant)

$\therefore Q_1, Q_2, Q_3, ...$ are in AP with common difference 6.

4. **(c)** Given $A_1 = \frac{a+b}{2}, G_1 = \sqrt{ab},\ H_1 = \frac{2ab}{a+b}$

also $A_n = \frac{A_{n-1}+H_{n-1}}{2}, G_n = \sqrt{A_{n-1}H_{n-1}}$

$$H_n = \frac{2A_{n-1}H_{n-1}}{A_{n-1}+H_{n-1}}$$

$\Rightarrow G_n^2 = A_n H_n \Rightarrow A_n H_n = A_{n-1}H_{n-1}$

Similarly we can prove

$A_n H_n = A_{n-1}H_{n-1} = A_{n-2}H_{n-2} = ... = A_1 H_1$

$\Rightarrow A_n H_n = ab\ \ \therefore$

$\therefore\ G_1^2 = G_2^2 = G_3^2\ .... = ab$

$\Rightarrow\ G_1 = G_2 = G_3\ .... = \sqrt{ab}$

5. **(a)** We have $A_n = \frac{A_{n-1}+H_{n-1}}{2}$

$$\therefore\ A_n - A_{n-1} = \frac{A_{n-1}+H_{n-1}}{2} - A_{n-1}$$

$$= \frac{H_{n-1} - A_{n-1}}{2} < 0\ (\because A_{n-1} > H_{n-1})$$

$\Rightarrow A_n < A_{n-1}$ or $A_{n-1} > A_n$

$\therefore$ We can conclude that $A_1 > A_2 > A_3 > ....$

6. **(b)** We have $A_n H_n = ab \Rightarrow H_n = \frac{ab}{A_n}$

$\because\ \frac{1}{A_{n-1}} < \frac{1}{A_n} \Rightarrow H_{n-1} < H_n\ \ \therefore H_1 < H_2 < H_3 < ....$

H. Assertion & Reason Type Questions

1. **(c)** Given a_1, a_2, a_3, a_4 are in GP.

Then b_1, b_2, b_3, b_4 are the numbers

$a_1, a_1+a_2, a_1+a_2+a_3,\ a_1+a_2+a_3+a_4$

or $a, a+ar,\ a+ar+ar^2,\ a+ar+ar^2+ar^3$

Clearly above numbers are neither in AP nor in G.P. and hence statement 1 is true.

Also $\frac{1}{a}, \frac{1}{a+ar}, \frac{1}{a+ar+ar^2}, \frac{1}{a+ar+ar^2+ar^3}$ are not in A.P. $\therefore b_1, b_2, b_3, b_4$ are not in H.P.

$\therefore$ Statement 2 is false.

I. Integer Value Correct Type

1. **(3)** Using $S_\infty = \frac{a}{1-r}$, we get

$$S_k = \begin{cases} \dfrac{\frac{k-1}{k!}}{1-\frac{1}{k!}}, & k \ne 1 \\ 0, & k = 1 \\ \dfrac{1}{(k-1)!}, & k \ge 2 \end{cases}$$

Now $\sum_{k=1}^{100}|(k^2-3k+1)S_k| = \sum_{k=2}^{100}|(k^2-3k+1)|\frac{1}{(k-1)!}$

$= |-1| + \sum_{k=3}^{100}\frac{(k^2-1)+1-3(k-1)-2}{(k-1)!}$ as $k^2-3k+1>0\,\forall k\ge 3$

$= 1+\sum_{k=3}^{100}\left(\frac{1}{(k-3)!}-\frac{1}{(k-1)!}\right)$

$= 1+\left(1-\frac{1}{2!}\right)+\left(\frac{1}{1!}-\frac{1}{3!}\right)+\left(\frac{1}{2!}-\frac{1}{4!}\right)+....$

$....+\left(\frac{1}{96!}-\frac{1}{98!}\right)+\left(\frac{1}{97!}-\frac{1}{99!}\right)$

$= 3-\frac{1}{98!}-\frac{1}{99!} = 3-\frac{9900}{100!}-\frac{100}{100!} = 3-\frac{10000}{100!} = 3-\frac{(100)^2}{100!}$

$\therefore\ \frac{100^2}{100!}+3\sum_{k=1}^{100}\left|\left(k^2-3k+1\right)S_k\right| = 3.$

2. **(0)** Given that $a_k = 2a_{k-1}-a_{k-2}$

$\Rightarrow \frac{a_{k-2}+a_k}{2} = a_{k-1},\ 3\le k\le 11$

$\Rightarrow a_1, a_2, a_3, ..., a_{11}$ are in AP.

If a is the first term and D the commen difference then

$a_1^2+a_2^2+...+a_{11}^2 = 990$

$\Rightarrow 11a^2+d^2\left(1^2+2^2+...+10^2\right)+2ad\left(1+2+...+10\right)$

$= 990$

$\Rightarrow 11a^2+\frac{10\times 11\times 21}{6}d^2+2ad\times\frac{10\times 11}{2} = 990$

$\Rightarrow a^2+35d^2+150d = 90$

Using $a = 15$, we get

$35d^2+150d+135 = 0$ or $7d^2+30d+27 = 0$

$\Rightarrow (d+3)(7d+9) = 0 \Rightarrow d = -3$ or $-9/7$

then $a_2 = 15-3 = 12$ or $15-\frac{9}{7} = \frac{96}{7} > \frac{27}{2}$

$\therefore$ d $\ne -9/7$

Hence $\frac{a_1+a_2+...+a_{11}}{11} = \frac{\frac{11}{2}\left[2\times 15+10(-3)\right]}{11} = 0$

3. **(9)**

We have $\frac{S_m}{S_n} = \frac{S_{5n}}{S_n} = \frac{\frac{5n}{2}\left[2\times 3+(5n-1)d\right]}{\frac{n}{2}\left[6+(n-1)d\right]}$

$= \frac{5[(6-d)+5nd]}{(6-d)+nd}$

which will be independent of n if $d = 6$ or $d = 0$

For a proper A.P. we take $d = 6$

then $a_2 = 3+6 = 9$

4. **(5)** Let k, $k+1$ be removed from pack.

$\therefore (1+2+3+...+n)-(k+k+1) = 1224$

$\frac{n(n+1)}{2}-2k = 1225$

$k = \frac{n(n+1)-2450}{4}$

for $n = 50$, $k = 25$

$\therefore\ k-20 = 5$

5. **(4)** $\because a, b, c$ are in G.P

$\therefore b = ar$ and $c = ar^2$

Also $\frac{b}{a}$ is an integer

$\Rightarrow r$ is an integer

$\because$ A.M. of a, b, c is $b+2$

$\Rightarrow \frac{a+b+c}{3} = b+2$

$\Rightarrow a+ar+ar^2 = 3ar+6$

$\Rightarrow a\left(r^2-2r+1\right) = 6$

$\Rightarrow a\left(r-1\right)^2 = 6$

$\because a$ and r are integers

$\therefore$ The only possible values of a and r can be 6 and 2 respectively.

Then $\frac{a^2+a-14}{a+1} = \frac{36+6-14}{6+1} = \frac{28}{7} = 4$

6. **(9)** $\frac{\frac{7}{2}[2a+6d]}{\frac{11}{2}[2a+10d]} = \frac{6}{11} \Rightarrow a = 9d$

$a_7 = a+6d = 15d$

$\because\ 130 < 15d < 140 \Rightarrow d = 9$

($\because$ All terms are natural numbers $\therefore\ d\in N$)

7. **(8)** In expansion of $(1+x)(1+x^2)(1+x^3)....(1+x^{100})$

x^9 can be found in the following ways

$x^9, x^{1+8}, x^{2+7}, x^{3+6}, x^{4+5}, x^{1+2+6}, x^{1+3+5}, x^{2+3+4}$

The coefficient of x^9 in each of the above 8 cases is 1.

$\therefore$ Required coefficient = 8.

Section-B JEE Main/ AIEEE

1. **(b)** $1, \log_9 (3^{1-x}+2), \log_3 (4.3^x - 1)$ are in A.P.

$\Rightarrow 2\log_9 (3^{1-x}+2) = 1 + \log_3 (4.3^x - 1)$

$\Rightarrow \log_3 (3^{1-x}+2) = \log_3 3 + \log_3 (4.3^x - 1)$

$\Rightarrow \log_3 (3^{1-x}+2) = \log_3 [3(4.3^x - 1)]$

$\Rightarrow 3^{1-x}+2 = 3(4.3^x - 1)$

$\Rightarrow 3.3^{-x}+2 = 12.3^x - 3.$

Put $3^x = t$

$\Rightarrow \frac{3}{t}+2 = 12t - 3$ or $12t^2 - 5t - 3 = 0;$

Hence $t = -\frac{1}{3}, \frac{3}{4} \Rightarrow 3^x = \frac{3}{4}$ (as $3^x \neq -ve$)

$\Rightarrow x = \log_3\left(\frac{3}{4}\right)$ or $x = \log_3 3 - \log_3 4$

$\Rightarrow x = 1 - \log_3 4$

2. **(d)** $l = AR^{p-1} \Rightarrow \log l = \log A + (p-1)\log R$

$m = AR^{q-1} \Rightarrow \log m = \log A + (q-1)\log R$

$n = AR^{r-1} \Rightarrow \log n = \log A + (r-1)\log R$

Now,

$$\begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} = \begin{vmatrix} \log A+(p-1)\log R & p & 1 \\ \log A+(q-1)\log R & q & 1 \\ \log A+(r-1)\log R & r & 1 \end{vmatrix}$$

Operating $C_1 - (\log R)C_2 + (\log R - \log A)C_3$

$$= \begin{vmatrix} 0 & p & 1 \\ 0 & q & 1 \\ 0 & r & 1 \end{vmatrix} = 0$$

3. **(b)** The product is $P = 2^{1/4}.2^{2/8}.2^{3/16}.........$

$= 2^{1/4+2/8+3/16+.........\infty}$

Now let $S = \frac{1}{4}+\frac{2}{8}+\frac{3}{16}+........\infty$ ......(1)

$\frac{1}{2}S = \frac{1}{8}+\frac{2}{16}+........\infty$ ......(2)

Subtracting (2) from (1)

$\Rightarrow \frac{1}{2}S = \frac{1}{4}+\frac{1}{8}+\frac{1}{16}+........\infty$

or $\frac{1}{2}S = \frac{1/4}{1-1/2} = \frac{1}{2} \Rightarrow S = 1$

$\therefore P = 2^S = 2$

4. **(b)** $ar^4 = 2$

$a \times ar \times ar^2 \times ar^3 \times ar^4 \times ar^5 \times ar^6 \times ar^7 \times ar^8$

$= a^9 r^{36} = (ar^4)^9 = 2^9 = 512$

5. **(b)** Let a = first term of G.P. and r = common ratio of G.P.; Then G.P. is a, ar, ar^2

Given $S_\infty = 20 \Rightarrow \frac{a}{1-r} = 20 \Rightarrow a = 20(1-r)$... (i)

Also $a^2 + a^2r^2 + a^2r^4 + ...$ to $\infty = 100$

$\Rightarrow \frac{a^2}{1-r^2} = 100 \Rightarrow a^2 = 100(1-r)(1+r)$... (ii)

From (i), $a^2 = 400(1-r)^2$;

From (ii), we get $100(1-r)(1+r) = 400(1-r)^2$

$\Rightarrow 1 + r = 4 - 4r \Rightarrow 5r = 3 \Rightarrow r = 3/5.$

6. **(a)** $1^3 - 2^3 + 3^3 - 4^3 + ...+ 9^3$

$= 1^3 + 2^3 + 3^3 + ...+ 9^3 - 2(2^3 + 4^3 + 6^3 + 8^3)$

$= \left[\frac{9\times10}{2}\right]^2 - 2.2^3\left[1^3+2^3+3^3+4^3\right]$

$= (45)^2 - 16.\left[\frac{4\times5}{2}\right]^2 = 2025 - 1600 = 425$

7. **(a)** $\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4}...............\infty$

$|T_n| = \frac{1}{n(n+1)} = \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$S = T_1 - T_2 + T_3 - T_4 + T_5............\infty$

$= \left(\frac{1}{1}-\frac{1}{2}\right) - \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) - \left(\frac{1}{4}-\frac{1}{5}\right).....$

$= 1 - 2\left[\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}................\infty\right]$

$= 1 - 2[-\log(1+1)+1] = 2\log 2 - 1 = \log\left(\frac{4}{e}\right).$

8. **(d)** $S_n = \frac{1}{^nC_0} + \frac{1}{^nC_1} + \frac{1}{^nC_2} + ... + \frac{1}{^nC_n}$

$t_n = \frac{0}{^nC_0} + \frac{1}{^nC_1} + \frac{2}{^nC_2} + ... + \frac{n}{^nC_n}$

$t_n = \frac{n}{^nC_n} + \frac{n-1}{^nC_{n-1}} + \frac{n-2}{^nC_{n-2}} + ... + \frac{0}{^nC_0}$

Add, $2t_n = (n)\left[\frac{1}{^nC_0} + \frac{1}{^nC_1} + ... \frac{1}{^nC_n}\right] = nS_n$

$\therefore \frac{t_n}{S_n} = \frac{n}{2}$

9. **(d)** $T_m = a + (m-1)d = \frac{1}{n}$ ....(1)

$T_n = a + (n-1)d = \frac{1}{m}$ ....(2)

$(1) - (2) \Rightarrow (m-n)d = \frac{1}{n} - \frac{1}{m} \Rightarrow d = \frac{1}{mn}$

From (1) $a = \frac{1}{mn} \Rightarrow a - d = 0$

10. **(b)** If n is odd, the required sum is

$1^2 + 2.2^2 + 3^2 + 2.4^2 + ... + 2.(n-1)^2 + n^2$

$= \frac{(n-1)(n-1+1)^2}{2} + n^2$

[$\because$ $(n–1)$ is even

$\therefore$ using given formula for the sum of $(n–1)$ terms.]

$= \left(\frac{n-1}{2} + 1\right) n^2 = \frac{n^2(n+1)}{2}$

11. **(b)** We know that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...$

and $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ...$

$\therefore e + e^{-1} = 2\left[1 + \frac{1}{2!} + \frac{1}{4!} + ...\right]$

$\therefore \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ... = \frac{e + e^{-1}}{2} - 1$

$= \frac{e^2 + 1 - 2e}{2e} = \frac{(e-1)^2}{2e}$

12. **(c)** Given ${}^mC_{r-1}$, ${}^mC_r$, ${}^mC_{r+1}$ are in A.P.

$2{}^mC_r = {}^mC_{r-1} + {}^mC_{r+1}$

$\Rightarrow 2 = \frac{{}^mC_{r-1}}{{}^mC_r} + \frac{{}^mC_{r+1}}{{}^mC_r} = \frac{r}{m-r+1} + \frac{m-r}{r+1}$

$\Rightarrow m^2 - m(4r+1) + 4r^2 - 2 = 0$.

13. **(d)** $x = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ $\qquad a = 1 - \frac{1}{x}$

$y = \sum_{n=0}^{\infty} b^n = \frac{1}{1-b}$ $\qquad b = 1 - \frac{1}{y}$

$z = \sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$ $\qquad c = 1 - \frac{1}{z}$

a, b, c are in A.P. OR $2b = a + c$

$2\left(1 - \frac{1}{y}\right) = 1 - \frac{1}{x} + 1 - \frac{1}{y}$

$\frac{2}{y} = \frac{1}{x} + \frac{1}{z} \Rightarrow x, y, z$ are in H.P.

14. **(d)** $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}$.............

Putting $x = \frac{1}{2}$ we get

$1 + \frac{1}{4.2!} + \frac{1}{16.4!} + \frac{1}{64.6!} + ...$

$\infty = \frac{e^{\frac{1}{2}} + e^{\frac{-1}{2}}}{2} = \frac{\sqrt{e} + \frac{1}{\sqrt{e}}}{2} = \frac{e+1}{2\sqrt{e}}$

15. **(d)** $\frac{\frac{p}{2}[2a_1 + (p-1)d]}{\frac{q}{2}[2a_1 + (q-1)d]} = \frac{p^2}{q^2} \Rightarrow \frac{2a_1 + (p-1)d}{2a_1 + (q-1)d} = \frac{p}{q}$

$\frac{a_1 + \left(\frac{p-1}{2}\right)d}{a_1 + \left(\frac{q-1}{2}\right)d} = \frac{p}{q}$

For $\frac{a_6}{a_{21}}$, $p = 11, q = 41 \Rightarrow \frac{a_6}{a_{21}} = \frac{11}{41}$

16. **(d)** $\frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = ... = \frac{1}{a_n} - \frac{1}{a_{n-1}} = d$ (say)

Then $a_1a_2 = \frac{a_1 - a_2}{d}$, $a_2a_3 = \frac{a_2 - a_3}{d}$,

..........., $a_{n-1}a_n = \frac{a_{n-1} - a_n}{d}$

$\therefore \quad a_1a_2 + a_2a_3 + ... + a_{n-1}a_n$

$= \frac{a_1 - a_2}{d} + \frac{a_2 - a_3}{d} + ... + \frac{a_{n-1} - a_n}{d}$

$= \frac{1}{d}[a_1 - a_2 + a_2 - a_3 + ... + a_{n-1} - a_n] = \frac{a_1 - a_n}{d}$

Also, $\frac{1}{a_n} = \frac{1}{a_1} + (n-1)d$

$\Rightarrow \frac{a_1 - a_n}{a_1a_n} = (n-1)d \Rightarrow \frac{a_1 - a_n}{d} = (n-1)a_1a_n$

Which is the required result.

17. (d) We know that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty$

Put x = – 1

$\therefore \quad e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}........\infty$

$\therefore \quad e^{-1} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}........\infty$

18. (b) Let the series a, ar, ar^2, ... are in geometric progression.
given, $a = ar + ar^2$

$\Rightarrow \quad 1 = r + r^2 \quad \Rightarrow r^2 + r - 1 = 0$

$\Rightarrow \quad r = \frac{-1 \pm \sqrt{1 - 4 \times -1}}{2} \Rightarrow \quad r = \frac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow \quad r = \frac{\sqrt{5} - 1}{2}$ [$\because$ terms of G.P. are positive

$\therefore$ r should be positive]

19. (b) As per question,

$a + ar = 12 \quad ...(1)$

$ar^2 + ar^3 = 48 \quad ...(2)$

$\Rightarrow \quad \frac{ar^2(1+r)}{a(1+r)} = \frac{48}{12} \Rightarrow r^2 = 4, \Rightarrow r = -2$

($\because$ terms are = + ve and –ve alternately)

$\Rightarrow \quad a = -12$

20. (a) We have

$S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + ...\infty \quad ....(1)$

Multiplying both sides by $\frac{1}{3}$ we get

$\frac{1}{3} S = \frac{1}{3} + \frac{2}{3^2} + \frac{6}{3^3} + \frac{10}{3^4} + ...\infty \quad ....(2)$

Subtracting eqn. (2) from eqn. (1) we get

$\frac{2}{3} S = 1 + \frac{1}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + ...\infty$

$\Rightarrow \frac{2}{3} S = \frac{4}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + ...\infty$

$\Rightarrow \frac{2}{3} S = \frac{\frac{4}{3}}{1 - \frac{1}{3}} = \frac{4}{3} \times \frac{3}{2} \Rightarrow S = 3$

21. (a) Till 10th minute number of counted notes = 1500

$3000 = \frac{n}{2}[2 \times 148 + (n-1)(-2)] = n[148 - n + 1]$

$n^2 - 149n + 3000 = 0$

$\Rightarrow n = 125, 24$

But $n = 125$ is not possible

$\therefore$ total time = 24 + 10 = 34 minutes.

22. (c) Let required number of months = n

$\therefore 200 \times 3 + (240 + 280 + 320 + ... + (n-3)^{th}\text{ term}) = 11040$

$\Rightarrow \frac{n-3}{2}[2 \times 240 + (n-4) \times 40] = 11040 - 600$

$\Rightarrow (n-3)[240 + 20n - 80] = 10440$

$\Rightarrow (n-3)(20n + 160) = 10440$

$\Rightarrow (n-3)(n+8) = 522$

$\Rightarrow n^2 + 5n - 546 = 0 \Rightarrow (n+26)(n-21) = 0$

$\therefore n = 21$

23. (b) nth term of the given series

$= T_n = (n-1)^2 + (n-1)n + n^2$

$= \frac{((n-1)^3 - n^3)}{(n-1) - n} = n^3 - (n-1)^3$

$\Rightarrow S_n = \sum_{k=1}^{n}\left[k^3 - (k-1)^3\right] \Rightarrow 8000 = n^3$

$\Rightarrow \quad n = 20$ which is a natural number.

Now, put n = 1,2,3,....20

$T_1 = 1^3 - 0^3$

$T_2 = 2^3 - 1^3$

$\vdots$

$T_{20} = 20^3 - 19^3$

Now, $T_1 + T_2 + --- + T_{20} = S_{20}$

$\Rightarrow \quad S_{20} = 20^3 - 0^3 = 8000$

Hence, both the given statements are true and statement 2 supports statement 1.

24. (c) Given sequence can be written as

$\frac{7}{10} + \frac{77}{100} + \frac{777}{10^3} + ...+$ up to 20 terms

$= 7\left[\frac{1}{10} + \frac{11}{100} + \frac{111}{10^3} + ... + \text{up to 20 terms}\right]$

Multiply and divide by 9

$= \frac{7}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + ...\text{up to 20 terms}\right]$

$= \frac{7}{9}\left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + ...\text{up to 20 terms}\right]$

$= \frac{7}{9}\left[20 - \frac{\frac{1}{10}\left(1 - \left(\frac{1}{10}\right)^{20}\right)}{1 - \frac{1}{10}}\right]$

$= \frac{7}{9}\left[\frac{179}{9} + \frac{1}{9}\left(\frac{1}{10}\right)^{20}\right] = \frac{7}{81}[179 + (10)^{-20}]$

25. (a) Let $10^9+2.(11)(10)^8+3(11)^2(10)^7+...+10(11)^9=k(10)^9$

Let $x=10^9+2.(11)(10)^8+3(11)^2(10)^7+...+10(11)^9$

Multiplied by $\frac{11}{10}$ on both the sides

$$\frac{11}{10}x=11.10^8+2.(11)^2.(10)^7+...+9(11)^9+11^{10}$$

$$x\left(1-\frac{11}{10}\right)=10^9+11(10)^8+11^2\times(10)^7+...+11^9-11^{10}$$

$$\Rightarrow -\frac{x}{10}=10^9\left[\frac{\left(\frac{11}{10}\right)^{10}-1}{\frac{11}{10}-1}\right]-11^{10}$$

$$\Rightarrow -\frac{x}{10}=(11^{10}-10^{10})-11^{10}=-10^{10}$$

$\Rightarrow x=10^{11}=k.10^9$ Given $\Rightarrow k=100$

26. (b) Let a, ar, ar^2 are in G.P.
According to the question
$a, 2ar, ar^2$ are in A.P.
$\Rightarrow 2\times 2ar=a+ar^2$
$\Rightarrow 4r=1+r^2 \Rightarrow r^2-4r+1=0$

$$r=\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}$$

Since $r>1$

$\therefore$ $r=2-\sqrt{3}$ is rejected

Hence, $r=2+\sqrt{3}$

27. (d) n^{th} term of series $=\frac{\left[\frac{n(n+1)}{2}\right]^2}{n^2}=\frac{1}{4}(n+1)^2$

Sum of n term $=\Sigma\frac{1}{4}(n+1)^2$

$$=\frac{1}{4}\left[\Sigma n^2+2\Sigma n+n\right]$$

$$=\frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6}+\frac{2n(n+1)}{2}+n\right]$$

Sum of 9 terms

$$=\frac{1}{4}\left[\frac{9\times10\times19}{6}+\frac{18\times10}{2}+9\right]=\frac{384}{4}=96$$

28. (d) $m=\frac{l+n}{2}$ and common ratio of G.P. $=r=\left(\frac{n}{l}\right)^{\frac{1}{4}}$

$\therefore$ $G_1=l^{3/4}n^{1/4}, G_2=l^{1/2}n^{1/2}, G_3=l^{1/4}n^{3/4}$

$G_1^4+2G_2^4+G_3^4=l^3n+2l^2n^2+ln^3$
$=ln(l+n)^2$
$=ln\times 2m^2$
$=4lm^2n$

29. (d) Let the GP be a, ar and ar^2 then $a=A+d$; $ar=A+4d$; $ar^2=A+8d$

$$\Rightarrow \frac{ar^2-ar}{ar-a}=\frac{(A+8d)-(A+4d)}{(A+4d)-(A+d)}$$

$$r=\frac{4}{3}$$

30. (d) $\left(\frac{8}{5}\right)^2+\left(\frac{12}{5}\right)^2+\left(\frac{16}{5}\right)^2+\left(\frac{20}{5}\right)^2...+\left(\frac{44}{5}\right)^2$

$$S=\frac{16}{25}\left(2^2+3^2+4^2+...+11^2\right)$$

$$=\frac{16}{25}\left(\frac{11(11+1)(22+1)}{6}-1\right)$$

$$=\frac{16}{25}\times505=\frac{16}{5}\times101$$

$$\Rightarrow\frac{16}{5}m=\frac{16}{5}\times101$$

$\Rightarrow m=101.$