

6. QUADRATIC EQUATIONS

■ Quadratic Equation

An equation of the form

$$ax^2 + bx + c = 0 \quad \dots\dots(i)$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$ is called a quadratic equation. The numbers a, b, c are called the coefficients of this equation.

● A root of the quadratic Equation

$$\text{Discriminant } D = b^2 - 4ac$$

The roots of Eq (i) are given by the formula

$$x = \frac{-b \pm \sqrt{D}}{2a} \quad \text{or} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

● Properties of Quadratic Equations

- ▶ A quadratic equation has two and only two roots.
- ▶ A quadratic equation cannot have more than two different roots.
- ▶ If α be a root of the quadratic equation $ax^2 + bx + c = 0$, then $(x - \alpha)$ is a factor of $ax^2 + bx + c = 0$.

● Sum and Product of the roots of a Quadratic Equation

Let α, β be the roots of a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, then

$$\alpha + \beta = \frac{-b}{a} = -\left(\frac{\text{coefficient of } x}{\text{coefficient of } x^2}\right)$$

$$\text{and } \alpha\beta = \frac{c}{a} = \left(\frac{\text{constant term}}{\text{coefficient of } x^2}\right)$$

Therefore,

- ▶ If the two roots α and β be reciprocal to each other, then $a = c$.
- ▶ If the two roots α and β be equal in magnitude and opposite in sign $b = 0$.

● Sign of the Roots

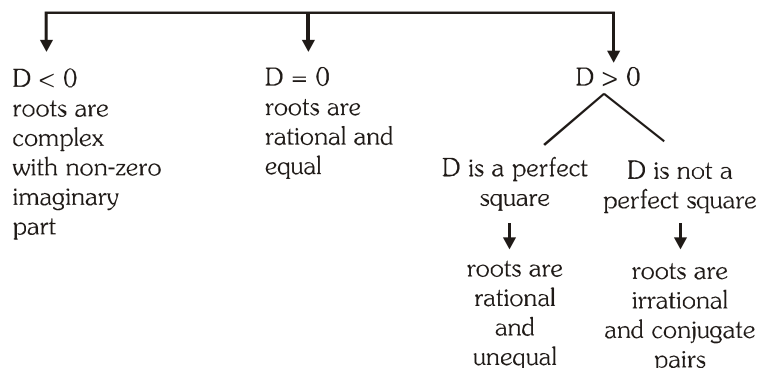
Sign of $(\alpha + \beta)$	Sign of $(\alpha\beta)$	Sign of the α, β
+ve	+ve	α and β are positive
-ve	+ve	α and β are negative
+ve	-ve	α is positive and β is negative if $\alpha > \beta$
-ve	-ve	α is negative and β is positive if $\alpha < \beta$

■ Nature of Roots

For a quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$ and $D = b^2 - 4ac$

(i) If $D < 0$, roots are imaginary

(ii) If $D \geq 0$ roots are real.



- ▶ If $a, b, c \in \mathbb{R}$ and $p + iq$ is one root of quadratic equation (where $q \neq 0$) then the other root must be conjugate $p - iq$ and vice-versa. ($p, q \in \mathbb{R}$ and $i = \sqrt{-1}$)
- ▶ If $a, b, c \in \mathbb{Q}$ and $p + \sqrt{q}$ is one root of the quadratic equation, then the other root must be the conjugate $p - \sqrt{q}$ and vice-versa (where p is a rational and $\sqrt{q}$ is a surd).
- ▶ If $a = 1$ and $b, c \in \mathbb{I}$ and the roots of quadratic equation are rational numbers, then these roots must be integers.

● Condition for Common Roots

Consider two quadratic equations

$$ax^2 + bx + c = 0 \quad \dots\dots\dots(i) \quad a \neq 0$$

$$\text{and } a'x^2 + b'x + c' = 0 \quad \dots\dots\dots(ii) \quad d \neq 0$$

(i) If one root is common then,

$$(ab' - a'b)(bc' - b'c) = (ca' - c'a)^2$$

(ii) If two roots are common then,

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

● Condition that $ax^2 + bx + c = 0$, is Factorizable into two Linear Factors

When $D \geq 0$, then the equation $ax^2 + bx + c = 0$ is factorizable into two linear factors.

i.e., $ax^2 + bx + c \Rightarrow (x - \alpha)(x - \beta) = 0$, where α and β are the roots of quadratic equation.

■ Formation of a Quadratic Equation

Let α, β be the two roots then we can form a quadratic equation as follows

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

$$\text{i.e., } x^2 - (\alpha + \beta)x + (\alpha\beta) = 0$$

$$\text{or } (x - \alpha)(x - \beta) = 0$$

■ Formation of a New Quadratic Equation by Changing the roots of a given Quadratic Equation

Let α, β be the roots of a quadratic equation $ax^2 + bx + c = 0$, then we can form a new quadratic equation as per the following rules.

- A quadratic equation whose roots are p more than the roots of the equation $ax^2 + bx + c = 0$ (i.e., the roots are $\alpha + p$ and $\beta + p$).

The required equation is

$$a(x - p)^2 + b(x - p) + c = 0$$

- A quadratic equation whose roots are less by p than the roots of the equation $ax^2 + bx + c = 0$, (i.e., the roots are $\alpha - p$ and $\beta - p$)

The required equation is

$$a(x + p)^2 + b(x + p) + c = 0$$

- A quadratic equation whose roots are p times the roots of the equation $ax^2 + bx + c = 0$ (i.e., the roots are

$$\frac{\alpha}{p} \text{ and } \frac{\beta}{p}) \text{ The required equation is } a(px)^2 + b(px) + c = 0$$

- A quadratic equation whose roots are the reciprocal of the roots of equation $ax^2 + bx + c = 0$ (i.e. the roots are

$$\frac{1}{\alpha} \text{ and } \frac{1}{\beta}). \text{ The required equation is } a\left(\frac{1}{x}\right)^2 + b\left(\frac{1}{x}\right) + c = 0$$

$$\Rightarrow cx^2 + bx + a = 0$$

- A quadratic equation whose roots are $1/p$ times the roots of the equation $ax^2 + bx + c = 0$ (i.e., the roots are $1/\alpha$ and $1/\beta$).

$$\text{The required equation is } a\left(\frac{x}{p}\right)^2 + b\left(\frac{x}{p}\right) + c = 0$$

- A quadratic equation whose roots are the negative of the roots of the equation $ax^2 + bx + c = 0$ (i.e., the roots are $-\alpha$ and $-\beta$)

$$\text{The required equation is } a(-x)^2 + b(-x) + c = 0$$

$$\Rightarrow ax^2 - bx + c = 0$$

- A quadratic equation whose roots are the square of the roots of the equation $ax^2 + bx + c = 0$ (i.e., the roots are α^2 and β^2)

$$\text{The required equation is } a(\sqrt{x})^2 + b(\sqrt{x}) + c = 0$$

$$\Rightarrow ax + b\sqrt{x} + c = 0$$

- A quadratic equation whose roots are the cubes of the roots of the equation $ax^2 + bx + c = 0$ (i.e., the roots are α^3 and β^3)

$$\text{The required equation is } a(x^{1/3})^2 + b(x^{1/3}) + c = 0$$

$$\Rightarrow ax^{2/3} + bx^{1/3} + c = 0$$

■ Maximum or Minimum value of a Quadratic Equation

At $x = \frac{-b}{2a}$ we get the maximum or minimum value of the quadratic expression.

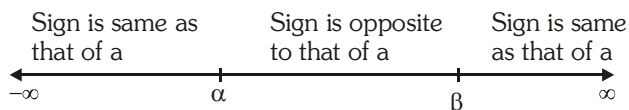
- ▶ When $a > 0$ (in the equation $ax^2 + bx + c$) the expression gives minimum value, $y = \frac{4ac - b^2}{4a}$
- ▶ When $a < 0$ (in the equation $ax^2 + bx + c$) the expression gives maximum value, $y = \frac{4ac - b^2}{4a}$.

■ Sign of Quadratic Expression $ax^2 + bx + c$

- If α, β are the roots of the corresponding quadratic equation, then for $x = \alpha$ and $x = \beta$, the value of the expression is equal to zero. i.e., $f(x) = ax^2 + bx + c = 0$.
- But for other real values of x (i.e., except α and β) the expression is either less than zero or greater than zero, i.e., $f(x) < 0$ or $f(x) > 0$.
- But for other real values of x (i.e., except α and β) the expression is either less than zero or greater than zero, i.e., $f(x) < 0$ or $f(x) > 0$.
- Thus the sign of $ax^2 + bx + c, x \in \mathbb{R}$, is determined by the following rules:
 - ▶ If $D < 0$ i.e., α and β are imaginary, then

$$ax^2 + bx + c > 0, \text{ if } a > 0$$
 and $ax^2 + bx + c < 0, \text{ if } a < 0$
 - ▶ If $D = 0$ i.e., α and β are real and equal, then

$$ax^2 + bx + c \geq 0, \text{ if } a > 0$$
 and $ax^2 + bx + c \leq 0, \text{ if } a < 0$
 - ▶ If $D > 0$ i.e., α and β are real unequal ($\alpha < \beta$), then the sign of the expression $ax^2 + bx + c, x \in \mathbb{R}$ is determined as follows :



■ Relation between roots and coefficients

- For quadratic equation $ax^2 + bx + c = 0$, having the roots α and β , then

$$\alpha + \beta = \frac{-b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

- For cubic equation $ax^3 + bx^2 + cx + d = 0$, having roots α, β and γ , then

$$\alpha + \beta + \gamma = \frac{-b}{a}, \quad \alpha\beta + \beta\gamma + \gamma\alpha = (-1)^2 \frac{c}{a} = \frac{c}{a}$$

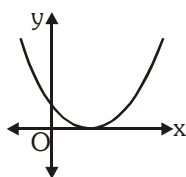
$$\text{and} \quad \alpha\beta\gamma = (-1)^3 \frac{d}{a} = -\frac{d}{a}$$

QUADRATIC EQUATIONS

EXERCISE

1. If the roots, x_1 and x_2 , of the quadratic equation $x^2 - 2x + c = 0$ also satisfy the equation $7x_2 - 4x_1 = 47$, then which of the following is true ?
 (1) $c = -15$ (2) $x_1 = 5, x_2 = 3$
 (3) $x_1 = 4.5, x_2 = -2.5$ (4) None of these
2. The integral values of k for which the equation $(k-2)x^2 + 8x + k + 4 = 0$ has both the roots real, distinct and negative is :
 (1) 0 (2) 2 (3) 3 (4) -4
3. If the roots of the equation $\frac{x^2 - bx}{ax - c} = \frac{m-1}{m+1}$ are equal and of opposite sign, then the value of m will be :
 (1) $\frac{a-b}{a+b}$ (2) $\frac{b-a}{a+b}$ (3) $\frac{a+b}{a-b}$ (4) $\frac{b+a}{b-a}$
4. If α, β are the roots of the equation $x^2 + 2x + 4 = 0$, then $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$ is equal to :
 (1) $-\frac{1}{2}$ (2) $\frac{1}{4}$ (3) 32 (4) $\frac{1}{32}$
5. If α, β are the roots of the equation $x^2 + 7x + 12 = 0$, then the equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$ is :
 (1) $x^2 + 50x + 49 = 0$ (2) $x^2 - 50x + 49 = 0$
 (3) $x^2 - 50x - 49 = 0$ (4) $x^2 + 12x + 7 = 0$
6. The value of k ($k > 0$) for which the equations $x^2 + kx + 64 = 0$ and $x^2 - 8x + k = 0$ both will have real roots is :
 (1) 8 (2) 16 (3) -64 (4) None
7. If α, β are roots of the quadratic equation $x^2 + bx - c = 0$, then the equation whose roots are b and c is
 (1) $x^2 + ax - \beta = 0$
 (2) $x^2 - [(\alpha + \beta) + \alpha\beta]x - \alpha\beta(\alpha + \beta) = 0$
 (3) $x^2 + (\alpha\beta + \alpha + \beta)x + \alpha\beta(\alpha + \beta) = 0$
 (4) $x^2 + (\alpha\beta + \alpha + \beta)x - \alpha\beta(\alpha + \beta) = 0$
8. Solve for x : $x^6 - 26x^3 - 27 = 0$
 (1) -1, 3 (2) 1, 3
 (3) 1, -3 (4) -1, -3
9. Solve : $\sqrt{2x+9} + x = 13$:
 (1) 4, 16 (2) 8, 20
 (3) 2, 8 (4) None of these
10. Solve : $\sqrt{2x+9} - \sqrt{x-4} = 3$
 (1) 4, 16 (2) 8, 20 (3) 2, 8 (4) None
11. Solve for x : $2\left[x^2 + \frac{1}{x^2}\right] - 9\left[x + \frac{1}{x}\right] + 14 = 0$:
 (1) $\frac{1}{2}, 1, 2$ (2) 2, 4, $\frac{1}{3}$ (3) $\frac{1}{3}, 4, 1$ (4) None
12. Solve for x : $\sqrt{x^2 + x - 6} - x + 2 = \sqrt{x^2 - 7x + 10}$, $x \in \mathbb{R}$:
 (1) 2, 6, $-\frac{10}{3}$ (2) 2, 6
 (3) -2, -6 (4) None of these
13. The number of real solutions of $x - \frac{1}{x^2 - 4} = 2 - \frac{1}{x^2 - 4}$ is :
 (1) 0 (2) 1 (3) 2 (4) Infinite
14. The equation $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$ has :
 (1) No solution
 (2) One solution
 (3) Two solutions
 (4) More than two solutions
15. The number of real roots of the equation $(x-1)^2 + (x-2)^2 + (x-3)^2 = 0$:
 (1) 0 (2) 2 (3) 3 (4) 6
16. If the equation $(3x)^2 + (27 \times 3^{1/k} - 15)x + 4 = 0$ has equal roots, then $k =$
 (1) -2 (2) $-\frac{1}{2}$ (3) $\frac{1}{2}$ (4) 0
17. Equation $ax^2 + 2x + 1$ has one double root if :
 (1) $a = 0$ (2) $a = -1$ (3) $a = 1$ (4) $a = 2$
18. Solve for x : $(x+2)(x-5)(x-6)(x+1) = 144$
 (1) -1, -2, -3 (2) 7, -3, 2
 (3) 2, -3, 5 (4) None of these

19. What does the following graph represent?



- (1) Quadratic polynomial has just one root.
 (2) Quadratic polynomial has equal roots.
 (3) Quadratic polynomial has no root.
 (4) Quadratic polynomial has equal roots and constant term is non-zero.

20. Consider a polynomial $ax^2 + bx + c$ such that zero is one of its roots then

- (1) $c = 0$, $x = \frac{-b}{a}$ satisfies the polynomial equation
 (2) $c \neq 0$, $x = \frac{-a}{b}$ satisfies the polynomial equation
 (3) $x = \frac{-b}{a}$ satisfies the polynomial equation.
 (4) Polynomial has equal roots.

21. Consider a quadratic polynomial $f(x) = ax^2 - x + c$ such that $ac > 1$ and its graph lies below x-axis then:

- (1) $a < 0$, $c > 0$ (2) $a < 0$, $c < 0$
 (3) $a > 0$, $c > 0$ (4) $a > 0$, $c < 0$

22. If α, β are the roots of a quadratic equation $x^2 - 3x + 5 = 0$ then the equation whose roots are $(\alpha^2 - 3\alpha + 7)$ and $(\beta^2 - 3\beta + 7)$ is :

- (1) $x^2 + 4x + 1 = 0$ (2) $x^2 - 4x + 4 = 0$
 (3) $x^2 - 4x - 1 = 0$ (4) $x^2 + 2x + 3 = 0$

23. The expression $a^2x^2 + bx + 1$ will be positive for all $x \in \mathbb{R}$ if :

- (1) $b^2 > 4a^2$ (2) $b^2 < 4a^2$
 (3) $4b^2 > a^2$ (4) $4b^2 < a^2$

24. For what value of a the curve $y = x^2 + ax + 25$ touches the x-axis :

- (1) 0 (2) ± 5
 (3) ± 10 (4) None

25. The value of the expression $x^2 + 2bx + c$ will be positive for all real x if :

- (1) $b^2 - 4c > 0$ (2) $b^2 - 4c < 0$
 (3) $c^2 < b$ (4) $b^2 < c$

26. If the roots of the quadratic equation $ax^2 + bx + c = 0$ are imaginary then for all values of a, b, c and $x \in \mathbb{R}$, the expression $a^2x^2 + abx + ac$ is

- (1) Positive
 (2) Non-negative
 (3) Negative
 (4) May be positive, zero or negative

27. The value of k , so that the equations $2x^2 + kx - 5 = 0$ and $x^2 - 3x - 4 = 0$ have one root in common is :

- (1) $-2, -3$ (2) $-3, -\frac{27}{4}$
 (3) $-5, -6$ (4) None of these

28. If the expression $x^2 - 11x + a$ and $x^2 - 14x + 2a$ must have a common factor and $a \neq 0$, then the common factor is :

- (1) $(x - 3)$ (2) $(x - 6)$ (3) $(x - 8)$ (4) None

29. The value of m for which one of the roots of $x^2 - 3x + 2m = 0$ is double of one of the roots of $x^2 - x + m = 0$ is :

- (1) 0, 2 (2) 0, -2 (3) 2, -2 (4) None

30. If the equations $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$, ($b \neq c$) have a common root then :

- (1) $b + c = 0$ (2) $b + c = 1$
 (3) $b + c + 1 = 0$ (4) None of these

31. If both the roots of the equations

$$k(6x^2 + 3) + rx + 2x^2 - 1 = 0 \text{ and}$$

$$6k(2x^2 + 1) + px + 4x^2 - 2 = 0 \text{ are common, then } 2r - p \text{ is equal to :}$$

- (1) 1 (2) -1
 (3) 2 (4) 0

32. If $x^2 - ax - 21 = 0$ and $x^2 - 3ax + 35 = 0$; $a > 0$ have a common root, then a is equal to :

- (1) 1 (2) 2
 (3) 4 (4) 5

33. The values of a for which the quadratic equation $(1 - 2a)x^2 - 6ax - 1 = 0$ and $ax^2 - x + 1 = 0$ have at least one root in common are :

- (1) $\frac{1}{2}, \frac{2}{9}$ (2) 0, $\frac{1}{2}$
 (3) $\frac{2}{9}$ (4) 0, $\frac{1}{2}, \frac{2}{9}$

34. If the quadratic equation $2x^2 + ax + b = 0$ and $2x^2 + bx + a = 0$ ($a \neq b$) have a common root, the value of $a + b$ is :

(1) -3 (2) -2 (3) -1 (4) 0

35. If the equation $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ have a common root and $b \neq c$, then their other roots will satisfy the equation :

(1) $x^2 - (b + c)x + bc = 0$

(2) $x^2 - ax + bc = 0$

(3) $x^2 + ax + bc = 0$

(4) None of these

36. If both the roots of the equations $x^2 + mx + 1 = 0$ and $(b - c)x^2 + (c - a)x + (a - b) = 0$ are common, then :

(1) $m = -2$ (2) $m = -1$ (3) $m = 0$ (4) $m = 1$

37. For the equation $3x^2 + px + 3 = 0$, $p > 0$, if one of the roots is square of the other, then $p =$

(1) $\frac{1}{3}$

(2) 1

(3) 3

(4) $\frac{2}{3}$

38. The roots of the equation $|x^2 - x - 6| = x + 2$ are

(1) -2, 1, 4

(2) 0, 2, 4

(3) 0, 1, 4

(4) -2, 2, 4

39. The equation $x - \frac{2}{x-1} = 1 - \frac{2}{x-1}$ has

(1) Two roots

(2) Infinitely many roots

(3) Only one root

(4) No root

40. The value of x which satisfy the expression :

$(5 + 2\sqrt{6})^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10$

(1) $\pm 2, \pm \sqrt{3}$

(2) $\pm \sqrt{2}, \pm 4$

(3) $\pm 2, \pm \sqrt{2}$

(4) $2, \sqrt{2}, \sqrt{3}$

41. Find all the integral values of a for which the quadratic equation $(x - a)(x - 10) + 1 = 0$ has integral roots :

(1) 12, 8

(2) 4, 6

(3) 2, 0

(4) None

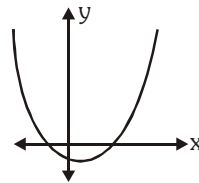
42. Graph of $y = ax^2 + bx + c$ is given adjacently. What conclusions can be drawn from the graph?

(i) $a > 0$

(ii) $b < 0$

(iii) $c < 0$

(iv) $b^2 - 4ac > 0$



(1) (i) and (iv)

(2) (ii) and (iii)

(3) (i), (ii) & (iv)

(4) (i), (ii), (iii) & (iv)

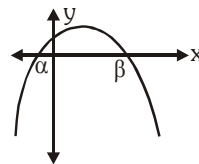
43. The adjoining figure shows the graph of $y = ax^2 + bx + c$. Then which of the following is correct :

(i) $a > 0$

(ii) $b > 0$

(iii) $c > 0$

(iv) $b^2 < 4ac$



(1) (i) and (iv)

(2) (ii) and (iii)

(3) (iii) & (iv)

(4) None of these

44. If $x^2 - (a + b)x + ab = 0$, then the value of $(x - a)^2 + (x - b)^2$ is

(1) $a^2 + b^2$

(2) $(a + b)^2$

(3) $(a - b)^2$

(4) $a^2 - b^2$

45. The sum of the roots of $\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$ is zero.

The product of the roots is

(1) 0

(2) $\frac{1}{2}(a + b)$

(3) $-\frac{1}{2}(a^2 + b^2)$

(4) $2(a^2 + b^2)$

46. If the roots of the equations

$(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$

for $a \neq 0$ are real and equal, then the value of $a^3 + b^3 + c^3$ is

(1) abc

(2) $3abc$

(3) zero

(4) None of these

47. If α, β are the roots of $X^2 - 8X + P = 0$ and $\alpha^2 + \beta^2 = 40$. then the value of P is

(1) 8

(2) 10

(3) 12

(4) 14

48. If, ℓ , m , n are real and $\ell=m$, then the roots of the equations

$$(\ell-m)x^2-5(\ell+m)x-2(\ell-m)=0 \text{ are}$$

- (1) Real and Equal
(2) Complex
(3) Real and Unequal
(4) None of these

49. In a family, eleven times the number of children is greater than twice the square of the number of children by 12. How many children are there ?

- (1) 3 (2) 4
(3) 2 (4) 5

50. The sum of all the real roots of the equation

$$|x-2|^2 + |x-2| - 2 = 0 \text{ is}$$

- (1) 2 (2) 3
(3) 4 (4) None of these

51. If the ratio between the roots of the equation $\ell x^2 + mx + n = 0$ is $p:q$, then the value of

$$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{\ell}} \text{ is}$$

- (1) 4 (2) 3
(3) 0 (4) -1

52. Find the root of the quadratic equation $bx^2 - 2ax + a = 0$

(1) $\frac{\sqrt{b}}{\sqrt{b} \pm \sqrt{a-b}}$ (2) $\frac{\sqrt{a}}{\sqrt{b} \pm \sqrt{a-b}}$

(3) $\frac{\sqrt{a}}{\sqrt{a} \pm \sqrt{a-b}}$ (4) $\frac{\sqrt{a}}{\sqrt{a} \pm \sqrt{a+b}}$

53. If 4 is a solution of the equation $x^2 + 3x + k = 10$, where k is a constant, what is the other solution ?

- (1) -18 (2) -7
(3) -28 (4) None of these

54. The coefficient of x in the equation $x^2 + px + p = 0$ was wrongly written as 17 in place of 13 and the roots thus found were -2 and -15. The roots of the correct equation would be

- (1) -4, -9 (2) -3, -10
(3) -3, -9 (4) -4, -10

55. If α and β are the roots of the quadratic equation

$$ax^2 + bx + c = 0, \text{ then the value of } \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} \text{ is}$$

(1) $\frac{2bc - a^3}{b^2c}$ (2) $\frac{3abc - b^3}{a^2c}$

(3) $\frac{3abc - b^2}{a^3c}$ (4) $\frac{ab - b^2c}{2b^2c}$

ANSWER KEY

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	1
Ans.	1	3	1	2	2	2	3	1	2	2	1	2	1	1	
Que.	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Ans.	2	3	2	4	1	2	2	2	3	4	1	2	3	2	
Que.	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
Ans.	4	3	3	2	1	1	3	4	4	3	1	4	2	3	
Que.	46	47	48	49	50	51	52	53	54	55					
Ans.	2	3	3	2	3	1	3	2	2	2					