10. GRAVITATION

1. INTRODUCTION

Have you ever wondered whether we would still be studying about with Gravitation if a stone had fallen on Newton's head instead of an apple? Anyways, the real question is, why does an apple fall down rather than go upward?

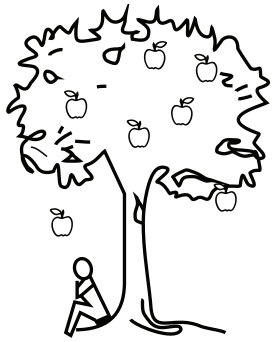


Figure 10.1

2. NEWTON'S LAW OF UNIVERSAL GRAVITATION

"Every particle of matter in the universe attracts every other particle with a force equal to the product of masses of particles and inversely proportional to the square of the distance between them"

If m_1 and m_2 are two point masses separated by a distance r, the gravitational force of attraction F is given by

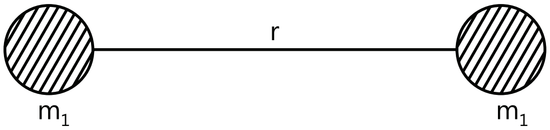


Figure 10.2

$$F \propto \frac{m_1 m_2}{r^2}$$

$$F = \frac{G m_1 m_2}{r^2}$$

Where G is a constant and is called the Universal gravitational constant.

Magnitude (and unit) of G : 6.67×10^{-11} Newton. m^2 / kg^2

Dimension of G : $M^{-1}L^{3}T^{-2}$

CONCEPTS

The direction of force F is independent of the medium, not affected by the presence of the other bodies and acts along the line joining the two particles.

If two persons come very close to each other such that the distance between them is almost 0, the two persons should experience a high force of attraction. Observe keenly the value of G. It's of order -11.

The Universal gravitational constant G is an experimental value calculated by Cavendish 71 years after the law was formulated.

Always remember Gravitational Force is conservative in nature i.e. work done doesn't depend on the path taken and depends only on the end points.

Vaibhav Gupta (JEE 2009, AIR 54)

Illustration 1: Two particles of masses 1.0 kg and 2.0 kg are placed at a separation of 50 cm. Assuming that the only forces acting on the particles are their mutual gravitation, find the initial accelerations of the two particles. **(JEE MAIN)**

Sol: The force of mutual gravitation acting on particles is $F=\frac{Gm_1m_2}{r^2}$. As the particle are accelerating under the force of gravitation, the acceleration is obtained using Newton's laws of motion.

The force of gravitation exerted by one particle on the other is

$$F=\frac{Gm_1m_2}{r^2}=\frac{6.67\times10^{-11}\frac{N-m^2}{kg^2}\times(1.0kg)\times(2.0kg)}{(0.5m)^2}=5.3\times10^{-10}N.$$

The acceleration of 1.0 kg particle is $a_1=\frac{F}{m_1}=\frac{5.3\times10^{-10}N}{1.0kg}=5.3\times10^{-10}ms^{-2}$

This acceleration is towards the 2.0 kg particles. The acceleration of the 2.0 kg particle is

$$a_2=\frac{F}{m_2}=\frac{5.3\times10^{-10}N}{2.0kg}=2.65\times10^{-10}ms^{-2}$$

This acceleration is towards the 1.0 kg particle.

Illustration 2: Spheres of the same material and same radius r are touching each other. Show that gravitational force between them is directly proportional to r^4. **(JEE MAIN)**

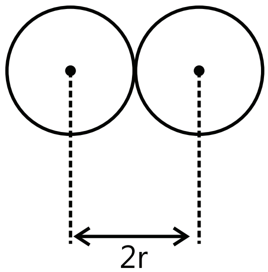


Figure 10.3

Sol: The force of gravitation is directly proportional to the masses of the spheres. As the spheres are having the same masses, and mass $m\propto V\Rightarrow m\propto r^3$ thus the proportionality between the force and distance is easily established.

As the spheres are made of same material, and density so the mass of each sphere is $m_1=m_2=$ (volume) (destiny)

$$=\left(\frac{4}{3}\pi r^3\right)\rho$$

$$F=\frac{Gm_1m_2}{(2r)^2}=\frac{G\left(\frac{4}{3}\pi r^3\right)\left(\frac{4}{3}\pi r^3\right)\rho^2}{4r^2} \quad \text{or} \quad F\propto r^4$$

Illustration 3: Three particles each of mass m, are located at the vertices of an equilateral triangle of side a. At what speed will they move if they all revolve under the influence of their gravitational force of attraction in a circular orbit circumscribing the triangle while still preserving the equilateral triangle? **(JEE MAIN)**

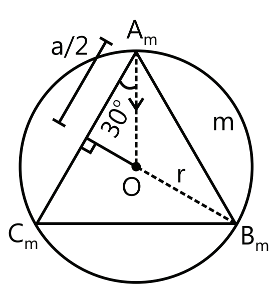


Figure 10.4

Sol: The net force of gravitation on any one particle is due to other two particles. This gravitational force provides the necessary centripetal force to the particles to move in the circular orbit around the equilateral triangle.

$$\vec{F}_A=\vec{F}_{AB}+\vec{F}_{AC}=2\left[\frac{GM^2}{a^2}\right]\cos30^\circ=\left[\frac{GM^2}{a^2}\sqrt{3}\right]$$

$$r=\frac{a}{\sqrt{3}}, \quad \text{Now } \frac{mv^2}{r}=F;\ \text{Or} \quad \frac{mv^2\sqrt{3}}{a}=\frac{GM^2}{a^2}\sqrt{3}; \quad \therefore v=\sqrt{\frac{GM}{a}}$$

3. GRAVITATIONAL FIELD

How would a particle interact with the surrounding or with other particles?

Every particle creates a field and when the other particle comes in to this particle's field, there would be an interaction between the particles.

The intensity of the field i.e. how intensely would it attract another particle in its field is called Gravitational field intensity or Gravitational field strength $\vec{E}$. It is defined as the force experienced by a unit mass placed at a distance r due to mass M, i.e. $\vec{E}=\frac{\vec{F}}{M}$

CONCEPTS

Always remember, it is a vector quantity and should be added vectorially when calculating Gravitational field intensity at a point by one or more masses.

Vaibhav Krishnan (JEE 2009, AIR 22)

4. GRAVITATIONAL FIELD INTENSITY

(a) Due to a point mass M:

$$F=\frac{GMm}{r^2};\qquad E=\frac{F}{m}=\frac{GM}{r^2};\qquad E=\frac{GM}{r^2}$$

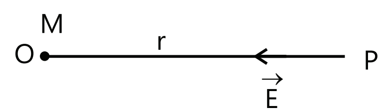


Figure 10.5

(b) Due to uniform ring of Mass M and radius a on its axis.

Consider any particle of mass dm on the ring, say at point A. The distance of this particle from P is $AP=z=\sqrt{a^2+r^2}$. The gravitational field at P is dm is along $\overrightarrow{PA}$ and its magnitude is $dE=\frac{Gdm}{z^2}$

The component along PO is $dE\cos\alpha=\frac{Gdm}{z^2}\cos\alpha$

The net gravitational field at P due to the ring is

$$E=\int\frac{Gdm}{z^2}\cos\alpha=\frac{G\cos\alpha}{z^2}\int dm=\frac{GM\cos\alpha}{z^2}=\frac{GMr}{(a^2+r^2)^{3/2}}$$

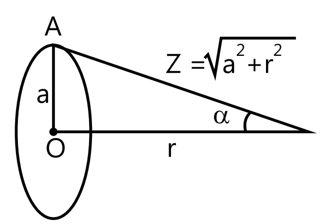


Figure 10.6

The field is directed towards the center of the ring.

(c) Due to uniform disc of mass M and radius a on its axis.

Let us draw a circle of radius x with the center at O. We draw another concentric circle of radius x+dx. The part of the disc enclosed between these two circles can be treated as a uniform ring of radius x. The point P is on its axis at a distance r from the center. The area of this ring is $2\pi x dx$. The area of the whole disc is πa^2. As the disc is uniform, the mass of this ring is

$$dm=\frac{M}{\pi a^2}2\pi x dx=\frac{2Mxdx}{a^2}$$

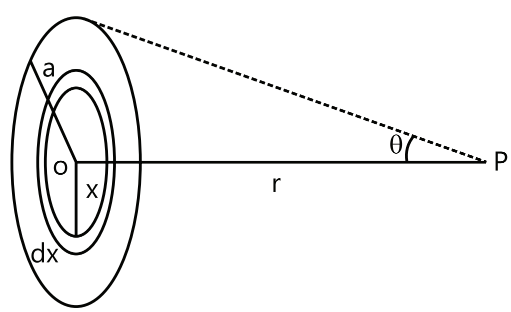


Figure 10.7

The gravitational field at P due to the ring is, by equation,

$$dE=\frac{G\left(\frac{2Mxdx}{a^2}\right)r}{\left(r^2+x^2\right)^{3/2}}=\frac{2GMr}{a^2}\frac{xdx}{\left(r^2+x^2\right)^{3/2}}$$

As x varies from 0 to a, the rings cover up the whole disc. The field due to each of these is in the same direction PO. Thus, the net field due to the whole disc is along PO and its magnitude is

$$E = \int_o^a \frac{2GMr}{a^2} \frac{xdx}{(r^2+x^2)^{3/2}} = \frac{2GMr}{a^2} \int_0^a \frac{xdx}{(r^2+x^2)^{3/2}} \qquad \text{...(i)}$$

Let $r^2 + x^2 = z^2$ then 2x dx=2z dz and

$$\int \frac{xdx}{(r^2+x^2)^{3/2}} = \int \frac{zdz}{z^3} = \int \frac{1}{z^2}dz = -\frac{1}{z} = -\frac{1}{\sqrt{r^2-x^2}}$$

From (i) $E = \frac{2GMr}{a^2}\left[-\frac{1}{\sqrt{r^2+x^2}}\right]_0^a = \frac{2GMr}{a^2}\left[\frac{1}{r} - \frac{1}{\sqrt{r^2+a^2}}\right]$

Equation may be expressed in terms of the angle θ subtended by a radius of the disc at P as,

$$E = \frac{2GM}{a^2}(1-\cos\theta).$$

(d) Due to uniform thin spherical shell of mass M and radius a from the triangle OAP,

$z^2 = a^2 + r^2 - 2ar\cos\theta$ or

$2z\,dz = 2ar\sin\theta d\theta$

or $\sin\theta d\theta = \frac{zdz}{ar}.$ ...(ii)

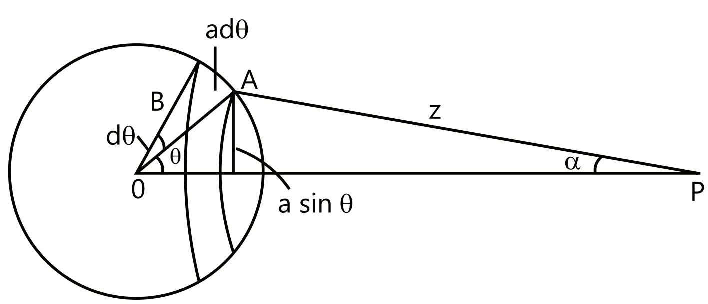


Figure 10.8

Also from the triangle OAP,

$$a^2 = z^2 + r^2 - 2zr\cos\alpha \quad \text{or} \quad \cos\alpha = \frac{z^2+r^2-a^2}{2zr}. \qquad \text{... (iii)}$$

Putting from (ii) and (iii) in (i), $dE = \frac{GM}{4ar^2}\left(1 - \frac{a^2-r^2}{z^2}\right)dz$ or $\int dE = \frac{GM}{4ar^2}\left[z + \frac{a^2-r^2}{z}\right]$

Case I: P is outside the shell (r > a)

In this case, z varies from r – a to r + a. The field due to the whole shell is

$$E = \frac{GM}{4ar^2}\left[z + \frac{a^2-r^2}{z}\right]_{r-a}^{r+a} = \frac{GM}{r^2}$$

We see that the shell may be treated as a point particle of the same mass placed at its center to calculate the gravitational field at an external point.

Case II: P is inside the shell

In this case, z varies from a - r to a + r. The field at P due to the whole shell is $E = \frac{GM}{4ar^2}\left[z + \frac{a^2-r^2}{z}\right]_{a-r}^{a+r} = 0$

Hence the field inside a uniform spherical shell is zero.

(e) Due to uniform solid sphere of mass M and radius a

(i) At an external point r (>a): Let us divide the sphere into thin spherical shells each centered at O. Let the mass of one such shell be dm. To calculate the gravitational field at P, we can replace the shell by a single particle of mass dm placed at the shell that is at O.

The field at P due to this shell is then $dE = \frac{Gdm}{r^2}$

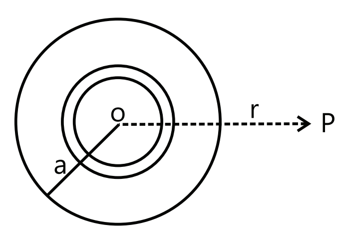


Figure 10.9

Towards PO. The field due to the whole sphere may be obtained by summing the fields of all the shells making the solid sphere.

Thus, $E = \int dE = \int \frac{Gdm}{r^2} = \frac{G}{r^2}\int dm = \frac{GM}{r^2}$

Thus, a uniform sphere may be treated as a single particle of equal mass placed at its center for calculating the gravitational field at an external point.

(ii) At an internal point r (<a):

Suppose the point P is inside the solid sphere (See Fig 10.10). In this case r<a. The sphere may be divided into thin spherical shells all centered at O.

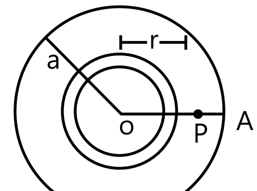


Figure 10.10

Suppose the mass of such a shell is dm. If the radius of the shell is less than r, the point is outside the shell. The field due to the shell is $dE = \frac{Gdm}{r^2}$ along PO.

If the radius of the shell considered is greater than r, the point P is internal and the field due to such a shell is zero. The total field due to the whole sphere is obtained by summing the fields due to all the shells. As all these fields are along the same direction, the net field is

$$E = \int dE = \int \frac{GdM}{r^2} = \frac{G}{r^2}\int dm \qquad \text{... (i)}$$

Only the masses of the shells with radii less than r should be added to get $z = \sqrt{a^2 + r^2}$. These shells form a solid sphere of radius r. The volume of this sphere is $\frac{4}{3}\pi r^3$.The volume of the whole sphere is $\frac{4}{3}\pi a^3$. As the given sphere is uniform, the mass of the sphere of radius r is $\frac{M}{\frac{4}{3}\pi a^3}\left(\frac{4}{3}\pi r^3\right) = \frac{Mr^3}{a^3}$

Thus, $\int dm = \frac{Mr^3}{a^3}$ and by (i) $E = \frac{G}{r^2}\frac{Mr^3}{a^3} = \frac{GM}{a^3}r.$

The gravitational field due to a uniform sphere at an internal point is proportional to the distance of the point from the center of the sphere.

CONCEPTS

One could assume the whole mass is concentrated at the center of mass (now assume it as point mass) for calculating the gravitation field at an external point for spherical shell, sphere nevertheless of mass distribution (uniformly/non-uniformly)

Mass distribution should be a function of radial distance only.

Remember the Gauss theorem in Electricity?

Equivalent Gauss theorem for gravitational field is $\oint \vec{E}.d\vec{S} = -4\pi G(m)$, m=enclosed mass I guess now you could deduce the note above. Can you?

Nivvedan (JEE 2009, AIR 113)

Illustration 4: Three concentric shells of homogenous mass distribution of masses M_1, M_2 and M_3 having radii a, b and c respectively are situated as shown in Fig. 10.11. Find the force on a particle of mass m **(JEE MAIN)**

(a) When the particle is located at Q.

(b) When the particle is located at P.

Sol: For a particle of mass m, lying at a distance r from the center of the spherical shell of mass M and radius r, the gravitational force of attraction is $\left(\frac{GMm}{r^2}\right)$. If the particle is lying inside the spherical shell then the force of gravitation on it is zero.

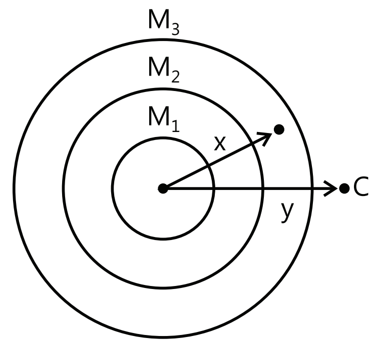


Figure 10.11

Attraction at an external point due to spherical shell of mass M is $\left(\frac{GMm}{r^2}\right)$ while at an internal point is zero.

(a) Point is external to shell M_1, M_2 and M_3,

So, force at Q will be $$F_q = \frac{GM_1m}{y^2} + \frac{GM_2m}{y^2} + \frac{GM_3m}{y^2} = \frac{Gm}{y^2}(M_1+M_2+M_3)$$

(b) Force at P will be $$F_p = \frac{GM_1m}{x^2} + \frac{GM_2m}{x^2} + 0 = \frac{Gm}{x^2}(M_1+M_2)$$

Illustration 5: A uniform ring of mass m and radius a is placed directly above a uniform sphere of mass M and of equal radius. The center of the ring is at a distance $\sqrt{3}a$ from the center of the sphere. Find the gravitational force exerted by the sphere on the ring. **(JEE ADVANCED)**

Sol: The field due to ring at the center of the sphere can be found easily, as the center of the sphere is lying at the axis of the ring. From Newton's third law of motion the force on the sphere due to the ring will be equal in magnitude to the force exerted by the sphere on the ring.

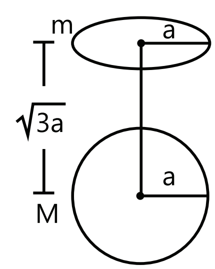


Figure 10.12

The gravitational field at any point on the ring due to the sphere is equal to the field due to a single particle of mass M placed at the center of the sphere. Thus, the force on the ring due to the sphere is also equal to the force on it by a particle of mass M placed at this point. By Newton's third law, it is equal to the force on the particle by the ring.

Now the gravitational field due to the ring at a distance $d = \sqrt{3}a$ on its axis is $$E = \frac{G\,md}{(a^2+d^2)^{3/2}} = \frac{\sqrt{3}\,Gm}{8a^2}$$

The force on a particle of mass M placed here is F=$ME = \frac{\sqrt{3}\,GMm}{8a^2}$. Thus we have used the formula for field due to a ring.

This is also the force due to the sphere on the ring.

5. EARTH'S GRAVITATIONAL FIELD

We have seen what gravitational field is and how an object would interact with other objects. Earth is no different as it creates a gravitational field and interacts with us.

$g = F/m$ (g should be written as g bar and F as F bar. Take care of that)

6. VARIATION IN THE VALUE OF ACCELERATION DUE TO GRAVITY (g)

Variation in the value of g: The value of g varies from place to place on the surface of earth. It also varies as we go above or below the surface of the earth. Thus, value of g depends on the following factors:-

(a) Shape of the earth: The earth is not a perfect sphere. It is somewhat flat at the two poles. The equatorial radius is approximately 21 km more than the polar radius. And since

$$g = \frac{GM}{R^2} \quad \text{Or} \quad g \propto \frac{1}{R^2}$$

The value of g is minimum at the equator and maximum at the poles.

(b) Height above the surface of the earth: The gravitational force on mass m due to Earth of mass M at height h above the surface of earth is

$$F = \frac{GMm}{(R+h)^2}$$

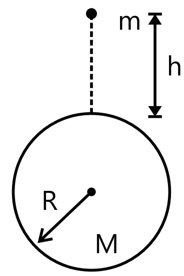


Figure 10.13

So the acceleration due to gravity is $g' = \frac{F}{m} = \frac{GM}{(R+h)^2}$

This can also be written as, $g' = \frac{GM}{R^2\left(1+\frac{h}{R}\right)^2}$ Or $g' = \frac{g}{\left(1+\frac{h}{R}\right)^2}$ as $\frac{GM}{R^2} = g$

Thus, $g' < g$ i.e., the value of acceleration due to gravity g goes on decreasing as we go above the surface of earth. Further,

$$g' = g\left(1+\frac{h}{R}\right)^{-2} \quad \text{or} \quad g' \approx g\left(1-\frac{2h}{R}\right) \text{ if } h << R$$

So on going above the surface of the earth, acceleration due to gravity decreases. Note that mass is always constant.

(c) Depth below the surface of the earth: Let an object of mass m is situated at a depth h below the earth's surface. Its distance from the center of earth is (R - h). This mass is situated at the surface of the inner solid sphere and lies inside the outer spherical shell. The gravitational force of attraction on a mass inside a spherical shell is always zero. Therefore, the object experiences gravitational attraction only due to inner solid sphere.

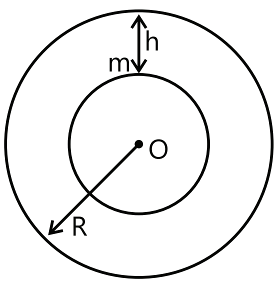


Figure 10.14

The mass of this sphere is $M' = \left(\frac{M}{4/3\,\pi R^3}\right)\frac{4}{3}\pi(R-h)^3$ or $M' = \frac{(R-h)^3}{R^3}M$

$$F = \frac{GM'm}{(R-h)^2} = \frac{GMm(R-h)}{R^3} \quad \text{and} \quad g' = \frac{F}{m}$$

Substituting the values, we get $g' = g\left(1-\frac{h}{R}\right)$ i.e., $g' < g$

(d) Axial rotation of the earth: Let us consider a particle P at rest on the surface of the earth, in latitude ϕ. Then the pseudo force acting on the particles is $mr\omega^2$ in outward direction. The true acceleration g is acting towards the center O of the earth. Thus, the effective accelerating g' is the resultant of g and $r\omega^2$ or

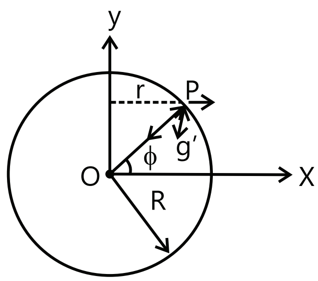


Figure 10.15

$$g' = \sqrt{g^2 + (r\omega^2)^2 + 2g(r\omega^2)\cos(180-\phi)}$$

or $$g' = \sqrt{g^2 + r^2\omega^4 - 2gr\omega^2\cos\phi} \quad \text{... (i)}$$

Here, the term $r^2\omega^4$ comes out to be too small as $\omega = \frac{2\pi}{T} = \frac{2\pi}{24\times 3600}$

rad/s is small. Hence, this term can be ignored. Also, $r = R\cos\phi$. Therefore, Eq. (i) can be written as $g' = (g^2 - 2gR\omega^2\cos^2\phi)^{1/2}$

$$= g\left(1 - \frac{2R\omega^2\cos^2\phi}{g}\right)^{1/2} = g\left(1 - \frac{R\omega^2\cos^2\phi}{g}\right)$$

Thus, $g' = g - R\omega^2\cos^2\phi$ $\quad R\omega^2$ is almost 0.03 m/s^2

CONCEPTS

They is always a decrease in the value of acceleration due to gravity from that of g at the surface irrespective of the condition.

If earth were to rotate faster 'g' would decrease at all points except at the poles. Guessed it? ϕ is 90 at poles. Also remember ϕ is 0 at equator.

Chinmay S Purandare (JEE 2012, AIR 698)

Illustration 6: Suppose the earth increases its speed of rotation. At what new time period will the weight of a body on the equator become zero? Take g = 10 m/s² and radius of earth R = 6400km. **(JEE MAIN)**

Sol: When rotational speed of earth is increased, the centrifugal force acting on the particle at rest at equator also increases. At the equator, the centrifugal force is opposite to the force of gravity. Thus the apparent value of g is $g' = g - R\omega^2$. For mass of body to be zero at the equator, g' = 0 i.e. $\omega = \sqrt{\frac{g}{R}}$. The time period of rotation is $T = \frac{2\pi}{\omega}$.

The weight will become zero, when $g' = 0$ or $g - R\omega^2 = 0$ (on the equator $g' = g - R\omega^2$)

or $\omega = \sqrt{\frac{g}{R}}$; $\quad \therefore \frac{2\pi}{T} = \sqrt{\frac{g}{R}}$ or $T = 2\pi\sqrt{\frac{R}{g}}$

Substituting the values, $\quad T = \dfrac{2\pi\sqrt{\dfrac{6400\times10^3}{10}}}{3600}h$ $\quad$ or $\quad T = 1.4$ h

Thus, the new time period should be 1.4 h instead of 24 h for the weight of a body to be zero on the equator.

Illustration 7: A simple pendulum has a time period exactly 2 s when used in a laboratory at North Pole. What will be the time period if the same pendulum is used in a laboratory at equator? Account for the earth's rotation only. Take $g = \frac{GM}{R^2} = 9.8\,m/s^2$ and radius of earth=6400 km. **(JEE ADVANCED)**

Sol: The time period of simple pendulum is given by $t = 2\pi\sqrt{\frac{\ell}{g}}$ where ℓ is the length of pendulum. At the equator value of acceleration due to gravity 'g' is different than at the pole. The apparent value of g is $g' = g - R\omega^2$. Thus the time periods will be different.

Consider the pendulum in its mean position at the North Pole. As the pole is on the axis of rotation, the bob is in equilibrium. Hence in the mean position, the tension T is balanced by earth's attraction. Thus, $T = \frac{GMm}{R^2} = mg$.

The time period t is $t = 2\pi\sqrt{\frac{\ell}{T/m}} = 2\pi\sqrt{\frac{\ell}{g}}$ ... (i)

At equator, the lab and the pendulum rotate with the earth at angular velocity $\omega = \frac{2\pi \text{ radian}}{24 \text{ hour}}$ in a circle of radius equal to 6400 km. Using Newton's second law,

$$\frac{GMm}{R^2} - T' = \omega^2 R \text{ or, } T' = m(g - \omega^2 R)$$

Where T' is the tension in the string.

The time period will be

$$t' = 2\pi\sqrt{\frac{l}{(T'/m)}} = 2\pi\sqrt{\frac{l}{g - \omega^2 R}} \quad \text{... (ii)}$$

By (i) and (ii)

$$\frac{t'}{t} = \sqrt{\frac{g}{g - \omega^2 R}} = \left(1 - \frac{\omega^2 R}{g}\right)^{-1/2} \quad \text{or,} \quad t' = t\left(1 + \frac{\omega^2 R}{2g}\right)$$

Putting the values, $t' = 2.004$ seconds.

7. GRAVITATIONAL POTENTIAL ENERGY

Suppose I would like to move a particle form another particle's field, work is either done against the gravitational field or extracted from it. This negative work is called as Gravitational Potential energy.

Gravitational force is a conservative in nature. Work done by gravitational field= $U_f - U_i = -\int_i^f \vec{F}.\vec{dr}$.

Let a particle of mass m_1 be kept fixed at a point A (See Fig 10.16) and another particle of mass m_2 is taken from a point B to a point C. Initially, the distance between the particles is $AB = r_1$ and finally it becomes $AC = r_2$. We have to calculate the change in potential energy of the system of the two particles as the distance changes from r_1 to r_2.

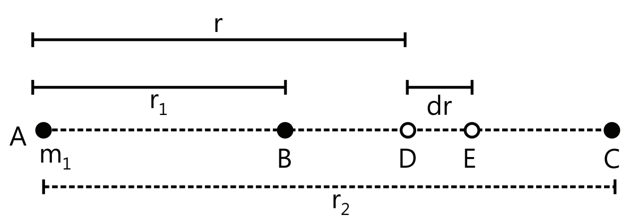


Figure 10.16

Consider a small displacement when the distance between the particles changes from r to r + dr. In the Fig 10.16, this corresponds to the second particle going from D to E.

The force on the second particle is $F = \frac{Gm_1m_2}{r^2}$ along $\overrightarrow{DA}$

The work done by the gravitational force in the displacement is $dW = -\frac{Gm_1m_2}{r^2}dr$.

The change in potential energy of the two-particle system during this displacement is $dU = -dW = \frac{Gm_1m_2}{r^2}dr$.

The change in potential energy as the distance between the particles from r_1 to r_2 is

$$U(r_2) - U(r_1) = \int dU = \int_{r_1}^{r_2} \frac{Gm_1m_2}{r^2}dr = Gm_1m_2\int_{r_1}^{r_2} \frac{1}{r^2}dr = Gm_1m_2\left[-\frac{1}{r}\right]_{r_1}^{r_2} = Gm_1m_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

This is the change in potential energy of the particles when moved from B to C.

Suppose the same particles which are of mass m_1 and m_2 are very far from each other and we need to calculate the change in potential energy when the distance between them becomes r. Then using above formulae,

we get $\quad U(r) - U(\infty) = Gm_1m_2\left[\frac{1}{\infty} - \frac{1}{r}\right] = -\frac{Gm_1m_2}{r}$

We make a standard assumption that the potential energy of the two-particle system to be zero when the distance between them is infinity. This means that we choose $U(\infty) = 0$.

Note: Just as one assumed current to be in opposite direction with the flow of electrons, the potential at infinity is assumed to be zero.

8. GRAVITATIONAL POTENTIAL

The potential at a point may also be defined as the work done per unit mass by an external agent in bringing a particle slowly from the reference point to the given point. Generally the reference point is chosen at infinity so that the potential at infinity is zero.

CONCEPTS

By slowly I mean, the particle is moved in such a way that there is no increase in Kinetic energy.

Since the Kinetic energy of the particle is zero, from the work energy theorem, the total work done is change in potential energy. So, what is the difference between the Potential and Potential energy? Observe it is the work done per unit mass.

Nitin Chandrol (JEE 2012, AIR 134)

We define the "change in potential" $V_B - V_A$ between the two points as $V_B - V_A = \frac{U_B - U_A}{m}$

Calculation of some Gravitational potentials:

(a) Potential due to point mass M at a point P which is at a distance r

(b) (ii) Potential due to Uniform ring of radius "a" and mass M at a point P on its axis.

(c) $V_{(r)} = \frac{U_{(r)} - U_{(\infty)}}{m}$

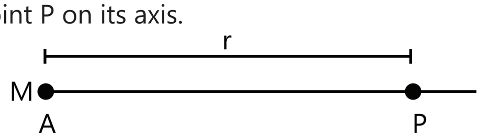


Figure 10.17

But $U(r) - U(\infty) = -\frac{GMm}{r}$ so that $V = -\frac{GM}{r}$

The gravitational potential due to a point mass M at a distance r is $-\frac{GM}{r}$

(d) Consider any small part of the ring of mass dm. The point P is at a distance $z = \sqrt{a^2 + r^2}$ from dm.

$$dV = -\frac{GdM}{r} = -\frac{Gdm}{\sqrt{a^2 + r^2}};$$

$$V = \int dV = \int -\frac{Gdm}{\sqrt{a^2 + r^2}} = -\frac{G}{\sqrt{a^2 + r^2}}\int dm = -\frac{GM}{\sqrt{a^2 + r^2}}$$

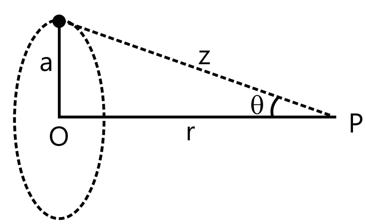


Figure 10.18

CONCEPTS

Remember that potential is a scalar quantity and one can directly add the contributions due to each of the point masses.

Potential due to Uniform Thin spherical shell and due to Uniform sphere can be derived similarly and here is the table of all the results.

CONCEPTS

	Potential	Gravitational Field
Point Mass at a distance r	$\frac{-GM}{r}$	$\frac{-GM}{r^2}\vec{e}_r$
Uniform Ring at a point on its axis	$\frac{-GM}{\sqrt{a^2+r^2}}$	$\frac{GMr}{\left(a^2+r^2\right)^{3/2}}$ towards center of ring
Uniform Thin spherical shell	$\frac{-GM}{a}$ (inside) $\frac{-GM}{r}$ (outside)	0 (inside) $\frac{GM}{r^2}$ (outside)
Uniform Solid Sphere	$\frac{-GMr^2}{a^3}$ (Inside) $\frac{-GM}{2a^3}\left(3a^2-r^2\right)$ (outside)	$\frac{GMr}{a^3}$ (inside) $\frac{GM}{r^2}$ (outside)

Only the magnitudes of gravitational field are written. As the gravitational force is attractive in nature, the direction could be easily found out.

Gravitational force, potential and potential energy all are taken with negative sign because the gravitational force is always attractive in nature.

$$E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y} \text{ and } E_z = -\frac{\partial V}{\partial z}$$

Potential using the field for various cases $V\left(\vec{r}_2\right)-V\left(\vec{r}_1\right) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}.\vec{dr}.$

B Rajiv Reddy (JEE 2012, AIR 11)

Illustration 8: A particle of mass 1 kg is kept on the surface of a uniform sphere of mass 20 kg and radius 1.0 m. Find the work to be done against the gravitational force between them to take the particle away from the sphere.

(JEE MAIN)

Sol: The work done in moving a particle away from the sphere will be equal to the change in gravitational potential energy of the particle in the gravitational field of the sphere.

Potential at the surface of sphere, $V = -\frac{GM}{R} = -\frac{(6.67\times10^{-11})(20)}{1}\,J/kg = -1.334\times10^{-9}\,J/kg$

i.e., 1.334×10^{-9} J work is obtained to bring a mass of 1 kg from infinity to the surface of sphere. Hence, the same amount of work will have to be done to take the particle away from the surface of sphere. Thus, $W = 1.334\times10^{-9}$ J

Illustration 9: A particle is fired vertically upward with a speed of 9.8 km/s. Find the maximum height attained by the particle. Radius of earth = 6400 km and g at the surface=9.8 m/s^2. Consider only earth's gravitation.

(JEE MAIN)

Sol: Particle initially moves with kinetic energy only in upwards direction opposite to the gravitation pull of earth. The loss in its kinetic energy is equal to the gain in the potential energy. At the highest point of its vertical motion, kinetic energy is converted completely into potential energy.

At the surface of the earth, the potential energy of the earth-particle system is $-\frac{GMm}{R}$ with usual symbols. The kinetic energy is $\frac{1}{2}mv_0^2$ where v_0 = 9.8 km/s. At the maximum height the kinetic energy is zero. If the maximum height reached is H, the potential energy of the earth-particle system at this instant is $-\frac{GMm}{R+H}$. Using conservation of energy, $-\frac{GMm}{R}+\frac{1}{2}mv_0^2=-\frac{GMm}{R+H}$

Writing $GM = gR^2$ and dividing by m, $-gR+\frac{v_0^2}{2}=\frac{-gR^2}{R+H}$ or $\frac{R^2}{R+H}=R-\frac{v_0^2}{2g}$ or $R+H=\frac{R^2}{R-\frac{v_0^2}{2g}}$ Putting the values of R, v_0 and g on the right side,

$$R+H=\frac{(6400km)^2}{6400km-\frac{(9.8kms^{-1})^2}{2\times9.8ms^{-2}}}=\frac{(6400km)^2}{1500km}=27300km \quad \text{or} \quad H=(27300-6400)km=20900km.$$

Illustration 10: Two particles of equal masses go round a circle of radius R under the action of their mutual gravitational attraction. Find the speed of each particle. **(JEE MAIN)**

Sol: As the particles go around the circle they always remain diametrically opposite to each other. To sustain their respective circular motion the necessary centripetal acceleration is provided by the gravitation force of attraction between them.

The particles will always remain diametrically opposite so that the force on each particle will be directed along the radius. Consider the motion of one of the particles. The force on the particle is $F=\frac{Gm^2}{4R^2}$. If Thus, by Newton's law,

$\frac{Gm^2}{4R^2}=\frac{mv^2}{R}$ or $v=\sqrt{\frac{Gm}{4R}}$

9. BINDING ENERGY

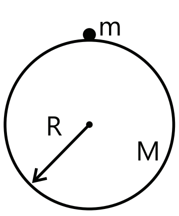


Figure 10.19

It is the energy due to which a system is bound. Suppose the mass m is placed on the surface of earth. The radius of the earth is R and its mass M. Then, the kinetic energy of the particle K=0 and potential energy of the particle is $U=-\frac{GMm}{R}$.

Therefore, the total mechanical energy of the particle is, $E=K+U=0-\frac{GMm}{R}$ or $E=-\frac{GMm}{R}$

It is due to this energy, the particle is attached to the earth. If this amount of energy is supplied to the particle in any form (normally kinetic), the particle no longer remains bound to the earth. It goes out of the gravitational field of earth.

Illustration 11: Assuming the earth to be a sphere of uniform mass destiny, calculate the energy needed to completely disassemble it against the gravitational pull amongst its constituent particles. Given the product of mass and radius of the earth $= 2.5\times10^{31}$kgm, $g = 10 m/s^2$. **(JEE MAIN)**

Sol: The work done to completely disassemble the earth will be equal to change in potential energy of the earth. Initial potential energy is negative and final will be zero.

If M and R are the mass and radius of the earth, then the density ρ of the earth is $\rho = \frac{3M}{4\pi R^3}$

The earth may be supposed to be made up of a large number of thin concentric spherical shells. It can be disassembled by removing such shells one by one. When a sphere of radius x is left, the energy needed to remove a shell of thickness lying between x and $x + dx$ is $dU = \frac{Gm_1m_2}{x}$

Where m_1 = mass of the sphere of radius $x = \frac{4}{3}\pi x^3 p$,

and m_2 = mass of the spherical shell of radius x and thickness dx $= 4\pi x^2 dx\rho$

$$\therefore \quad dw = dU = \frac{G\left(\frac{4}{3}\pi x^3\rho\right)\left(4\pi x^2 dx\rho\right)}{x} = \frac{16}{3}G\pi^2\rho^2x^4dx$$

Total energy required $U = \int dU = \frac{16G\pi^2\rho^2}{3}\int_0^R x^4dx = \frac{16G\pi^2\rho^2}{3}\frac{R^5}{5} = \frac{16}{15}G\pi^2\left(\frac{M}{(4/3)\pi R^3}\right)^2 R^5 = \frac{3}{5}\frac{GM^2}{R}$

$$= \frac{3}{5}gMR = \frac{3}{5}\times10\times2.5\times10^{31} = 1.5\times10^{32}\text{ J.}$$

10. ESCAPE VELOCITY

The minimum velocity needed to take a particle infinitely away from the earth is called the escape velocity. On the surface of earth its value 11.2 km/s.

As we discussed the binding energy of a particle on the surface of earth kept at rest is $\frac{GMm}{R}$. If this much energy in the form of kinetic energy is supplied to the particle, it leaves the gravitational field of the earth. So, if v_e is the escape velocity of the particle, then

$$\frac{1}{2}mv_e^2 = \frac{GMm}{R} \quad \text{or} \quad v_e = \sqrt{\frac{2GM}{R}} \quad \text{or } v_e = \sqrt{2gR} \qquad \text{as} \quad g = \frac{GM}{R^2}$$

CONCEPTS

Escape velocity is independent of angle of projection.

Anand K (JEE 2011, AIR 47)

Illustration 12: Calculate the escape velocity from the surface of moon. The mass of the moon is 7.4×10^{22} kg and radius $= 1.74\times10^6$m **(JEE MAIN)**

Sol: Escape velocity of any object placed on moon is given by $v_e = \sqrt{\frac{2GM_m}{R_m}}$

Escape velocity from the surface of moon is $v_e = \sqrt{\frac{2GM_m}{R_m}}$

Substituting the values, we have $v_e = \sqrt{\frac{2\times 6.67\times 10^{-11}\times 7.4\times 10^{22}}{1.74\times 10^{6}}} = 2.4\times 10^{3}$ m/s or2.4 km/s

11. SATELLITES

Satellites are generally of two types:

Natural Satellites: Moon is a natural satellite of the earth.

Artificial Satellite: These are launched in to space by humans and they help us in weather forecasting, telecommunications etc. The path of these satellites is elliptical with the center of earth at a focus.

Orbital Speed: The necessary centripetal force to the satellite is being provided by the gravitational force exerted by the earth on the satellite. Thus,

$\therefore \quad v_o = \sqrt{\frac{GM}{r}}$ or $v_o \propto \frac{1}{\sqrt{r}}$

Hence, the orbital speed (v_o) of the satellite decreases as the orbital radius (r) of the satellite increases. Further, the orbital speed of a satellite close to the earth's surface $(r \approx R)$ is, $v_o = \sqrt{\frac{GM}{R}} = \sqrt{gR} = \frac{v_e}{\sqrt{2}}$;

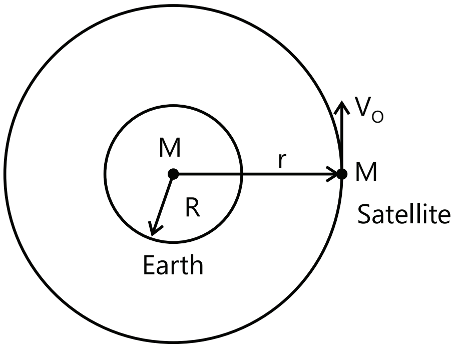


Figure 10.20

Substituting $v_e = 11.2\text{km/s}$; $v_o = 7.9\text{km/s}$

Period of Revolution: The period of revolution (T) is given by $T = \frac{2\pi r}{v_o}$ or $T = \frac{2\pi r}{\sqrt{\frac{GM}{r}}}$ or $T = 2\pi\sqrt{\frac{r^3}{GM}}$

Or $T = 2\pi\sqrt{\frac{r^3}{gR^2}}$ (as $GM = gR^2$)

Energy of Satellite: The potential energy of the system is $U = -\frac{GMm}{r}$

The kinetic energy of the satellite is, $K = \frac{1}{2}mv_0^2 = \frac{1}{2}m\left(\frac{GM}{r}\right)$

or $K = \frac{1}{2}\frac{GMm}{r}$

The total energy is, $E = K + U = -\frac{GMm}{2r}$ or $E = -\frac{GMm}{2r}$

This energy is constant and negative, i.e., the system is closed. The farther the satellite from the earth the greater its total energy.

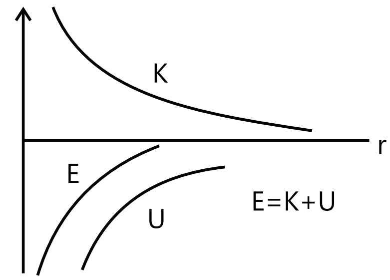


Figure 10.21

CONCEPTS

The velocity of a satellite is independent of its mass. It only depends upon the mass of the planet around which it revolves.

What if the time period of rotation of satellite is exactly 24 hours just as the time period of rotation of earth? Its position w.r.t earth is fixed right! Try calculating the distance from the earth's surface. By the way, these satellites are called Geo-stationary (stationary w.r.t earth) satellites.

Yashwanth Sandupatla (JEE 2012, AIR 821)

Illustration 13: Consider an earth's satellite so positioned that it appears stationary to an observer on earth and serves the purpose of a fixed relay station for international transmission of TV and other communications. What would be the height at which the satellite should be positioned and what would be the direction of its motion? Given that the radius of the earth is 6400 km and acceleration due to gravity on the surface of the earth is 9.8 m/s^2. **(JEE ADVANCED)**

Sol: For any artificial satellite to appear stationary with respect to a point on earth, it must rotate with the same angular speed as that of the earth and in the direction of motion as of the earth. The angular velocity of the satellite at height h above earth surface is given by $\omega = \sqrt{GM/r^3}$ where r=R+h.

For a satellite to remain above a given point on the earth's surface, it must rotate with the same angular velocity as the point on earth's surface. Therefore the satellite must rotate in the equatorial plane from west to east with a time period of 24 hours.

Now as for a satellite orbital velocity is $v_0 = \sqrt{GM/r}$

$$T = \frac{2\pi r}{v_o} = 2\pi r\sqrt{\frac{r}{GM}} = 2\pi r\sqrt{\frac{r}{gR^2}} \;\; (\text{as } g = GM/R^2) \text{ or } r = \left[gR^2 \frac{\frac{1}{3}}{4\pi^2} \right] = 4.23 \times 107\text{m} = 42300\text{km}$$

So the height of the satellite above the surface of earth, $h = r - R = 42300 - 6400 \approx 36000\text{km}$

[The speed of a geostationary satellite $v_o = R\sqrt{g/r} = r\omega = 3.1\,\text{km/s}$]

Illustration 14: Two satellites S_1 and S_2 revolve round a planet in coplanar circular orbits in the same sense. Their periods of revolution are 1 h and 8 h respectively. The radius of the orbit of S_1 is 10^4 km. When S_2 is closest to S_1 find (a) the speed of S_2 relative to S_1 and (b) the angular speed of S_2 as observed by an astronaut in S_1. **(JEE ADVANCED)**

Sol: According to Kepler's laws of planetary motion, $T^2 \propto R^3$. The orbital velocity of the satellite $v_0 = \frac{2\pi R}{T} = R\omega$ where ω is the angular velocity of revolution of satellite.

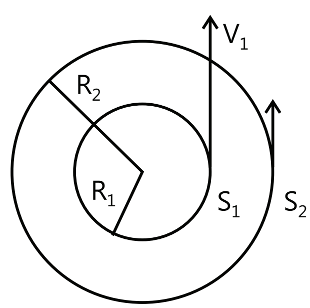


Figure 10.22

Let the mass of the planet be M, that of S_1 be m_1 and of S_2 be m_2.

Let the radius of the orbit of S_1 be $R_1 (=10^4\text{km})$ and so S_2 be R_2.

Let v_1 and v_2 be the linear speeds of S_1 and S_2 with respect to the planet. The given Fig 10.22 shows the situation.

As the square of the time period is proportional to the cube of the radius,

$$\left(\frac{R_2}{R_1}\right)^3 = \left(\frac{T_2}{T_1}\right)^2 = \left(\frac{8h}{1h}\right)^2 = 64 \quad \text{or} \quad \frac{R_2}{R_1} = 4 \text{ or } R_2 = 4R_1 = 4 \times 10^4\text{km}$$

Now the time period of S_1 is 1 h.

So, $\frac{2\pi R_1}{v_1} = 1h$ or $v_1 = \frac{2\pi R_1}{1h} = 2\pi \times 10^4 kmh^{-1}$

Similarly, $v_2 = \frac{2\pi R_2}{8h} = \pi \times 10^4 kmh^{-1}$

(a) At the closest separation, they are moving in the same direction. Hence the speed of S_2 with respect to S_1 is $|v_2 - v_1| = \pi \times 10^4 kmh^{-1}$

(b) As seen from S_1, the satellite S_2 is at a distance $R_2 - R_1 = 3 \times 10^4 km$ at the closest separation. Also, it is moving at $\pi \times 10^4 kmh^{-1}$ in a direction perpendicular to the line joining them.

Thus, the angular speed of S_2 as observed by S_1 is $\omega = \frac{\pi \times 10^4 kmh^{-1}}{3 \times 10^4 km} = \frac{\pi}{3} radh^{-1}$

Illustration 15: A spaceship is launched into a circular orbit close to the earth's surface. What additional velocity is now to be added to the spaceship in the orbit to overcome the gravitational pull? Radius of earth $= 6400$ km, $g = 9.8\ m/s^2$. **(JEE MAIN)**

Sol: The potential energy of the spaceship close to the earth is negative (- mgR). The orbital speed close to the earth is $v = \sqrt{gR}$, so the kinetic energy is mgR/2. The total energy is - mgR/2. We need to provide the additional kinetic energy = mgR/2 such that the spaceship escapes the gravitational pull of the earth.

The extra kinetic energy to be given is $\frac{mv^2}{2} = \frac{mgR}{2}$, so that the extra velocity given is $v' = \sqrt{gR}$.

The velocity is $v' = \sqrt{9.8 \times 6400000} = 7.91 \times 10^3\ m/s = 7.91\ km/s$

Illustration 16: An artificial satellite is moving in a circular orbit around the earth with a speed equal to one fourth the magnitude of escape velocity from the earth.

(i) Determine the height of the satellite above the earth's surface.

(ii) If the satellite is stopped suddenly in its orbit and allowed to fall freely towards the earth, find the speed with which it hits the surface of the earth. **(JEE MAIN)**

Sol: For satellite the escape velocity is $v_e = \sqrt{2Rg}$. According to given data the satellite is moving in the orbit with one fourth the magnitude of this velocity. When satellite stops revolving, it falls freely under action of gravity from the height h above the surface of the earth. The loss in the gravitational potential energy in falling height h is equal to gain in the kinetic energy of the satellite.

(i) Let M and R be the mass and radius of the earth respectively. Let m be the mass of satellite. Here escape velocity from earth $v_e = \sqrt{(2Rg)}$

Velocity of satellite $v_g = \frac{v_e}{4} = \sqrt{(2Rg)}/4$ ...(i)

Further $v_c = \sqrt{\left(\frac{GM}{r}\right)} = \sqrt{\left(\frac{R^2 g}{R+h}\right)}$ $\therefore$ $v_g^2 = \frac{R^2 g}{R+h}$ ...(ii)

From equation (i) and (ii), we get H=7R=44800km

(ii) Now, the total energy at height h=total energy on earth's surface (principle of conservation of energy). Let it reach earth's surface with velocity v.

$\therefore$ $0 - GM\frac{m}{R+h} = \frac{1}{2}mv^2 - GM\frac{m}{R}$ Or $\frac{1}{2}mv^2 = \frac{GMm}{R} - \frac{GMm}{7R}$ $(\because h = 7R)$

Solving we get $v = \sqrt{12Rg/7}$ $\quad\therefore\quad$ $v = \sqrt{(1.714 \times 6400 \times 10^3 \times 9.8)} = 10.368\,km/sec$

12. PLANETS AND THEIR MOTION

12.1 Law of Orbits

All the planets move in elliptical orbits with the sun as one of its focii.

12.2 Law of Areas

The radius vector from the sun at the focus of elliptical orbit to the planet sweeps out equal areas in equal intervals of time.

If the radius vector R sweeps an angle $d\theta$ in time dt, area ASB swept by radius vector in time $dt = dA = \frac{1}{2} \times R \times Rd\theta$

$\therefore\ dA = \frac{1}{2}R^2\frac{d\theta}{dt}dt = \frac{1}{2}\omega R^2 dt$

Areal velocity $= \frac{area}{time} = \frac{dA}{dt} = \frac{1}{2}\omega R^2$

So ωR^2 is constant for area SAB and area SCD. It shows that the angular momentum $mR^2\omega$ is conserved for planetary motion. When R decreases, ω increases so that ωR^2 is constant.

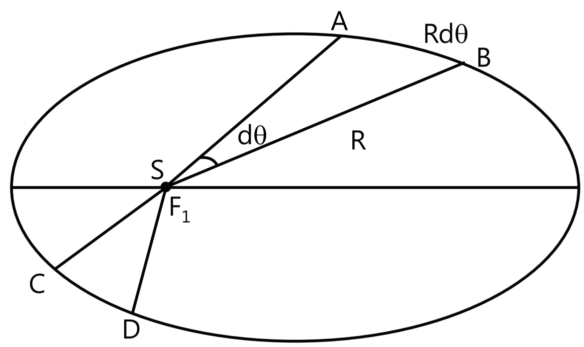


Figure 10.23

12.3 Laws of Periods

The square of the time period of revolution of a planet is proportional to the cube of the mean distance of the planet from the sun.

If a is the mean distance of sun from the planet, T^2 is proportional to a^3 or $T^2 = Ka^3$ where K is a constant.

If a_1 and a_2 are semi-major axis of the orbits of two planets around the sun with respective time periods T_1 and T_2, then $\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$

CONCEPTS

Observe the time period of rotation of satellite. Got it? (It follows Kepler's third law too)

When the planet is farthest from Sun, it is said to be at the Apogee of Aphelion.

When the planet is at nearest to the Sun, it is said to be at Perigee or Perhilion.

GV Abhinav (JEE 2012, AIR 329)

Illustration 17: The minimum and maximum distance of a satellite from the center of the earth are 2R and 4R respectively, where R is the radius of earth and M is the mass of the earth. Find:

(a) Its minimum and maximum speeds,

(b) Radius of curvature at the point of minimum distance. **(JEE ADVANCED)**

Sol: The speed of the satellite is minimum when is at the maximum distance from the earth and vice versa. At the point of minimum or maximum distance from earth the velocity vector is perpendicular to the radius vector from the earth. Apply law of conservation of angular momentum and energy at the two points.

(a) Applying conservation of angular momentum

$$mv_1(2R) = mv_2(4R) \;\; v_1 = 2v_2 \qquad \text{...(i)}$$

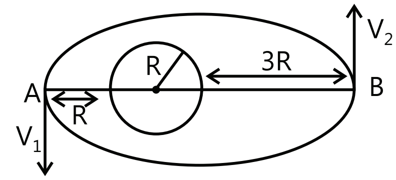


Figure 10.24

From conservation of energy

$$\frac{1}{2}mv_1^2 - \frac{GMm}{2R} = \frac{1}{2}mv_2^2 - \frac{GMm}{4R} \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$v_2 = \sqrt{\frac{GM}{6R}}, \qquad v_1 = \sqrt{\frac{2GM}{3R}}$$

(b) If r is the radius of curvature at point A

$$\frac{mv_1^2}{r} = \frac{GMm}{(2R^2)}; \;\; r = \frac{4v_1^2R^2}{GM} = \frac{8R}{3} \qquad \text{(Putting value of } v_1\text{)}$$

Illustration 18: The planet Neptune travels around the Sun with a period of 165 year. Show that the radius of its orbit is approximately thirty times that of Earth's orbit, both being considered as circular. **(JEE ADVANCED)**

Sol: According to the Kepler's laws of planetary motion $T^2 \propto R^3$ where T is the time period of revolution and R is the radius of the orbit of revolution of planet. Taking the ratio of time periods of revolution of Earth and Neptune, we get the ratio of radius of their orbits.

$T_1 = T_{Earth} = 1$ year; $T_2 = T_{Neptune} = 165$ year $= 165\ T_1$

Let R_1 and R_2 be the radii of the circular orbits of Earth and Neptune respectively.

$$\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3} \quad \therefore \quad R_2^3 = \frac{R_1^3T_2^2}{T_1^2} \quad \text{or} \quad R_2^3 = \frac{R_1^3 \times 165^2}{1^2}$$

$$\therefore \quad R_2^3 = 165^2R_1^3 \quad \text{or} \quad R_2 \approx 30R_1$$

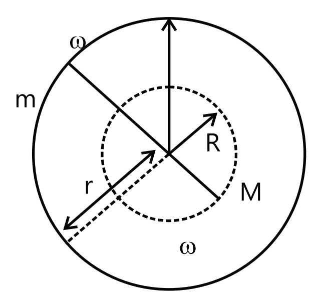
13. MOTION ABOUT THE CENTRE OF MASS

As shown in the Fig 10.25, for the case of circular orbits, two objects are moving about their common center of mass. If we consider the motion of the smaller body,

$$\frac{GMm}{(r+R)^2} = m\omega^2 r$$

The revised law of periods in

$$T^2 = \left(\frac{4\pi^2}{GM}\right)r^3\left(1+\frac{R}{r}\right)^2$$



Two bodies moving in circular orbits under the influences of each other's gravitational attraction

Figure 10.25

Illustration 19: A pair of stars rotate about their common center of mass. One of them has mass m and the other 2m. Their centers are a distance d apart, d being large compared to the size of either star.

(a) Derive an expression for the period or rotation of the stars about their common center of mass in terms of d, m and G

(b) Compare the angular moments of the two stars about their common center of mass.

(c) Compare the kinetic energies of the two stars. **(JEE MAIN)**

Sol: The gravitational pull between two stars provides the necessary centripetal acceleration to make them revolve in a circular orbit. The time period of revolution of each star is $T=\frac{2\pi}{\omega}$. The angular momentum of the revolving body is given by L=I ω = m $r^2\omega$. And the kinetic energy is given by $E=\frac{I\omega^2}{2}$.

The center of mass O is at a distance 2d/3 from the star of mass m and d/3 from the star of mass 2m. Both the stars rotate with the same angular velocity ω.

(a) Since the gravitational force provides the centripetal force, then

$$m\left(\frac{2d}{3}\right)\omega^2=\frac{Gm.2m}{d^2}\Rightarrow \quad \omega=\sqrt{3Gm/d^3} \quad \text{or} \quad T=\frac{2\pi}{\omega}=2\pi\sqrt{d^3/3Gm}$$

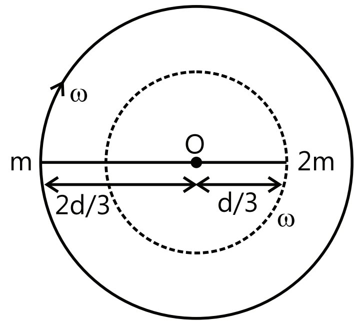


Figure 10.26

(b) Ratio of angular momenta

$$\frac{L_{small}}{L_{large}}=\frac{m\left(2d/3\right)^2\omega}{2m\left(d/3\right)^2\omega}=2$$

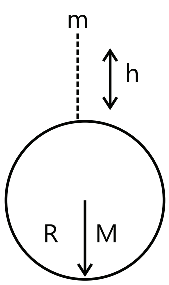
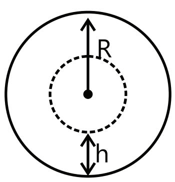
(c) Ratio of kinetic energies

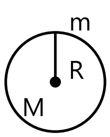
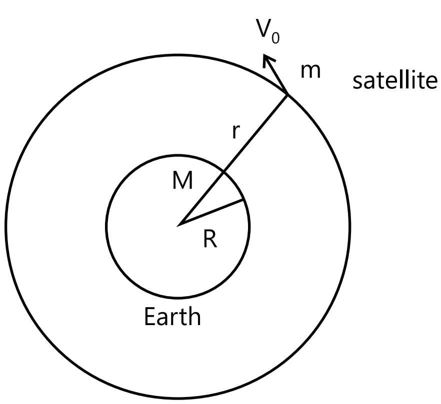
$$\frac{E_{small}}{E_{large}}=\frac{\frac{1}{2}i_{small}\omega^2}{\frac{1}{2}i_{big}\omega^2}=2$$

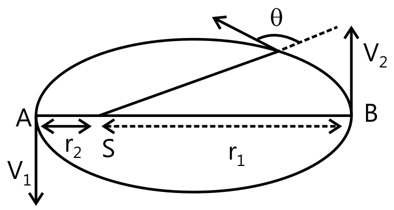
PROBLEM-SOLVING TACTICS

1. Most of the problems are easy, as gravitation and electrostatics are analogous to each other. Just be careful that gravitational force is always attractive, whereas electrostatic force can be attractive as well as repulsive and make changes as necessary.
2. Assumptions are appreciated in real cases of satellites and planetary motion.
3. Ideas and concepts of circular motion must be strong because they are generally applied here.
4. While dealing practical cases on Earth, be careful about Earth's rotation on its own axis.
5. Most questions are solved with ease by using work-energy theorem and laws of motion

FORMULAE SHEET

S. No.	Description	Formulae	
1	Magnitude of gravitational force between two particles of mass m_1 & m_2 placed at a distance r is	$F = \frac{Gm_1m_2}{r^2}$ $G = 6.67 \times 10^{-11} N - m^2 / kg^2$ Note: It acts along the line joining two particles.	
2	Acceleration due to gravity (g)	$g = \frac{GM}{R^2}$ SI units:- m/s^2 M is the mass of the earth and its radius R.	
3		Gravitational force $= \frac{GMm}{(R+h)^2}$ Acceleration due to gravity $= g' = F/m = \frac{GM}{(R+h)^2}$ If $h << R$ $g' = g\left(1 - \frac{2h}{R}\right)$	
4		At a certain, Depth H, acceleration due to gravity g' is $g' = g\left(1 - \frac{h}{R}\right)$ g is acceleration due to gravity at surface of earth.	
5	Effect of g due to axial rotation of earth	$g' = g - R\omega^2 \cos^2 \phi$ g' is the acceleration due to gravity on the particle on the earth surface in latitude ϕ.	
6	Gravitational field strength	$\vec{E} = \frac{\vec{F}}{m}$ SI unit is N/kg.	
		Gravitational Field	**Gravitational Potential**
7	Point Mass	$\frac{GM}{r^2}$	$-\frac{GM}{r}$
8	Uniform ring at point on its axis	$\frac{GMr}{(a^2 + r^2)^{3/2}}$ (towards center of ring)	$-\frac{GM}{\sqrt{a^2 + r^2}}$
9	Uniform thin spherical shell	Inside θ Outside $\frac{GM}{r^2}$	Inside $-\frac{GM}{a}$ Outside $-\frac{GM}{r}$

10	Uniform solid sphere	Inside $\frac{GMr}{a^3}$ Outside $\frac{GM}{r^2}$	Inside $-GMr^2/a^3$ Outside $-\frac{GM}{2a^2}\left(3a^2-r^2\right)$ Here, a is the radius and r is the location of point mass.
11	Gravitational potential	Note: It is a scalar; SI unit is J/kg.	
12		$\vec{E}=-\left[\frac{\partial y}{\partial x}\hat{i}+\frac{\partial V}{\partial y}\hat{j}+\frac{\partial V}{\partial z}\hat{k}\right]$ Note: It is partial derivative $dV=-\vec{E}.\vec{dr}$.	
13	Gravitational potential energy	$U=-\frac{Gm_1m_2}{r}$ System of particle $\left(m_1\,m_2\,m_3\,m_4\right)$ $U=-G\left[\frac{m_4m_3}{r_{43}}+\frac{m_4m_2}{r_{42}}+\frac{m_4m_1}{r_{41}}+\frac{m_3m_2}{r_{32}}+\frac{m_3m_1}{r_{31}}+\frac{m_2m_1}{r_{21}}\right]$ They are $\frac{4\left(4-1\right)}{2}=6$ Pairs	
14	For an n particle system, no. of pairs would be	$\frac{n\left(n-1\right)}{2}$ Pairs	
15	Binding Energy  	$E=\frac{GMm}{R}$ It is due to this energy particle is bound to earth.	
16	Escape Velocity	$v_e=\sqrt{2gR}$	
17	Motion of Satellites  	Orbital Speed $v_o=\sqrt{\frac{GM}{r}}$ Time period: $T=\frac{2\pi r}{v_o}=2\pi\sqrt{\frac{r^3}{GM}}$ Energy of satellite: $U=-\frac{GMm}{r};K=\frac{GMm}{2r}$ U is The potential energy Total Energy "E" K is The kinetic energy $E=K+U=-\frac{GMm}{2r}=-K$	

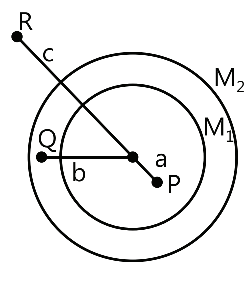
18	Kepler's Laws 	1^{st} Law:- Law of elliptical orbits 2^{nd} Law:- Law of conservation of angular momentum 3^{rd} Law:- Harmonic law $\left(T^2 \propto r^3\right)$ $v_1 r_1 = v_2 r_2$ $v_1 = a(1+e)\;\; r_2 = a(1-e)$ $V_{min} = V_1 = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)}$ $V_{max} = V_2 = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}$

Solved Examples

JEE Main/Boards

Example 1: Two concentric shells of mass M_1 and M_2 are as shown. Calculate the gravitational force on m due to M_1 at points P, Q and R.

Sol: For a particle of mass m, lying at a distance r from the center of the spherical shell of mass M, the gravitational force of attraction is $\left(\frac{GMm}{r^2}\right)$. If the particle is lying inside the spherical shell then the force of gravitation on it is zero.



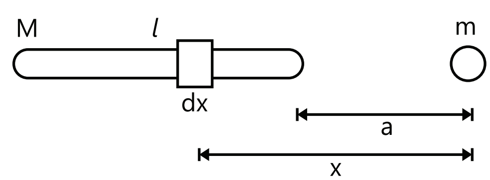
At P, $F = 0$ At Q, $F = \frac{GM_1 m}{b^2}$

At R, $F = \frac{G(M_1 + M_2)m}{c^2}$

Example 2: Find the potential energy of gravitational interaction of a point mass m and a thin uniform rod of mass M and length l, if they are located along a straight line at a distance a from each other.

Sol: The gravitational potential energy is given by $U = \frac{Gm_1 m_2}{r}$ where m_1 and m_2 are point masses. Consider the gravitational potential energy of interaction between the point mass m and an infinitesimal element of the rod of mass dm. The total potential energy will be the summation of energy of interaction of all the small elements.

Consider small element dx of the rod whose mass $dm = \frac{M}{l}dx$



$$\Rightarrow dU = -\frac{Gm\left(\frac{M}{l}dx\right)}{x}$$

$$\Rightarrow U = \int dU = -\frac{GmM}{l}\int_a^{a+l}\frac{dx}{x} = -\frac{GmM}{l}\left[\ln x\right]_a^{a+l}$$

$$\Rightarrow U = -\frac{GmM}{l}\log_e\left(\frac{a+l}{a}\right)$$

Example 3: If the radius of the earth contracts to half of its present value without change in its mass, what will be the new duration of the day?

Sol: The angular momentum of the earth is given by $L = I\omega = \frac{2}{5}MR^2\,\omega$ since earth is considered to be sphere of uniform mass density. As there is no external force is acting on the earth, the angular momentum of the earth must remain constant after the radius of earth reduces to half of its original size. The time period of revolution is $T = \frac{2\pi}{\omega}$.

Present angular momentum of earth $L_1 = I\omega = \frac{2}{5}MR^2\omega$

New angular momentum because of change in radius

$$L_2 = \frac{2}{5}M\left(\frac{R}{2}\right)^2\omega'$$

If external torque is zero then angular momentum must be conserved

$$L_1 = L_2$$

$$\frac{2}{5}MR^2\omega = \frac{1}{4}\times\frac{2}{5}MR^2\omega' \text{ i.e., } \omega' = 4\omega$$

$$T' = \frac{1}{4}T = \frac{1}{4}\times 24 = 6h$$

Example 4: Two particles of equal mass go round a circle of radius R under the action of their mutual gravitational attraction. Find the speed of each particle.

Sol: As the particles go around the circle they always remain diametrically opposite to each other. To sustain their respective circular motion the necessary centripetal acceleration is provided by the gravitation force of attraction between them.

The particles will always remain diametrically opposite so that the force on each particle will be directed along the radius. Consider the motion of one of the particles.

The force on the particle is $F = \frac{Gm^2}{4R^2}$. If the speed is v, its acceleration is v^2/R.

Thus, by Newton's law,

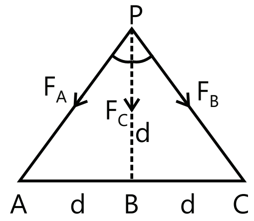
$$\frac{Gm^2}{4R^2} = \frac{mv^2}{R} \qquad \text{Or, } v = \sqrt{\frac{Gm}{4R}}$$

Example 5: Three particles A, B and C, each of mass m, are placed in a line with AB=BC=d. Find the gravitational force on a fourth particle P of same mass, placed at a distance d from the particle B on the perpendicular bisector of the line AC.

Sol: The gravitational force acting on the particle P due to each of other particles is given by $F = \frac{Gm^2}{(r)^2}$ where r is the separation between P and the other particle. As the force is vector quantity the resultant force on particle P has to be found by vector addition.

The force at P due to A is

$F_A = \frac{Gm^2}{(AP)^2} = \frac{Gm^2}{2d^2}$ along PA. The force at P due to C is



$F_C = \frac{Gm^2}{(CP)^2} = \frac{Gm^2}{2d^2}$ along PC. The force at P due to B is

$F_B = \frac{Gm^2}{d^2}$ along PB

The resultant of F_A, F_B and F_C will be along PB. Clearly $\angle APB = \angle BPC = 45°$

Components of F_A along $PB = F_A\cos 45° = \frac{Gm^2}{2\sqrt{2}d^2}$

Component of F_c along $PB = F_c\cos 45° = \frac{Gm^2}{2\sqrt{2}d^2}$

Component of F_B along $PB = \frac{Gm^2}{d^2}$

Hence, the resultant of the three forces is

$$\frac{Gm^2}{d^2}\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + 1\right) = \frac{Gm^2}{d^2}\left(1 + \frac{1}{\sqrt{2}}\right) \text{ along PB.}$$

Example 6: What is the fractional decrease in the value of free-fall acceleration g for a particle when it is lifted from the surface to an elevation h? (h<<R)

Sol: The gravitational acceleration g at height h is given by $g = \frac{GM}{(R+h)^2}$. As here R>>h then $g \approx \frac{GM}{R^2}$. The fractional decrease in g at height h above the surface of the earth is given by $\frac{\Delta g}{g}$.

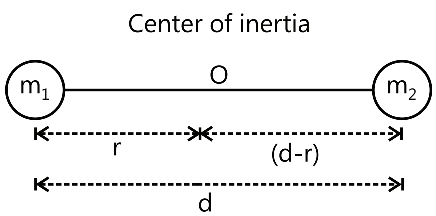
The acceleration due to gravity is $g = \frac{GM}{R^2}$

$$\therefore \frac{\Delta g}{\Delta R} = \frac{-2GM}{R^3} \text{ (Differentiating)}$$

$$\Rightarrow \frac{dg}{h} = \frac{-2GM}{R^2}\frac{1}{R} \Rightarrow \frac{dg}{g} = -2\left(\frac{h}{R}\right)$$

Example 7: A double star is a system of two stars moving around the center of inertia of the system due to gravitation. Find the distance between the components of the double star, if its total mass equals M and period of revolution is T.

Sol: Each star is moving in circular orbit whose center is at the combined center of inertia. Find the radius of orbit of one of the stars in terms of the separation between them and find the orbital velocity of the star in terms of d.



The situation is shown in the above figure

Here $m_1 r = m_2(d-r)$

$$\therefore (m_1 + m_2)r = m_2 d$$

$$r = \frac{m_2 d}{(m_1 + m_2)}$$

Also $M = (m_1 + m_2)$

As gravitational force provides the necessary centripetal force for rotation, we have

$$G\frac{m_1 m_2}{d^2} = \frac{m_1 v_1^2}{r} = \frac{m_1 v_1^2 (m_1 + m_2)}{m_2 d}$$

$$\therefore v_1 = \left[\frac{Gm_2^2}{(m_1 + m_2)d}\right]^{1/2} = m_2\left[\frac{G}{Md}\right]^{1/2}$$

$$\text{Now } T = \frac{2\pi}{\omega} = \frac{2\pi r}{v_1} = \frac{2\pi r}{m_2\sqrt{G/Md}} = \frac{2\pi d^{3/2}}{\sqrt{GM}} \left(\text{as } r = \frac{m_2 d}{M}\right)$$

$$\therefore \frac{T}{2\pi} = \frac{d^{3/2}}{\sqrt{GM}} \quad \text{or } d = \sqrt[3]{\left(\frac{T}{2\pi}\right)^2 GM}.$$

Example 8: Find the distance of a point from the earth's center where the resultant gravitational field due to the earth and the moon is zero. The mass of the earth is 6.0×10^{24} kg and that of the moon is 7.4×10^{22} kg. The distance between the earth and the moon is 4.0×10^5 km.

Sol: If a body is placed between moon and the earth then it is under action of gravitational force due to earth and moon simultaneously. When the gravitational field due to earth is equal in magnitude but opposite in direction to gravitational field due to moon then the net field is zero.

The point must be on the line joining the centers of the earth and the moon and in between them. If the distance of the point from the earth is x, the distance from the moon is (4.0×10^5 km – x). The magnitude of the gravitational field due to the earth is

$$E_1 = \frac{GM_e}{x^2} = \frac{G \times 6 \times 10^{24}\,kg}{x^2}$$

and the magnitude of the gravitational field due to the moon is

$$E_2 = \frac{GM_m}{(4.0 \times 10^5 km - x)^2} = \frac{G \times 7.4 \times 10^{22}\,kg}{(4.0 \times 10^5 km - x)^2}$$

These fields are in opposite directions. For the resultant field to be zero $E_1 = E_2$,

$$\text{Or, } \frac{6 \times 10^{24}\,kg}{x^2} = \frac{7.4 \times 10^{22}\,kg}{(4.0 \times 10^5 km - x)^2}$$

$$\text{Or, } \frac{x}{4.0 \times 10^5 km - x} = \sqrt{\frac{6 \times 10^{24}}{7.4 \times 10^{22}}} = 9$$

Or, $x = 3.6 \times 10^5$ km

Example 9: A planet of mass m_1 revolves around the sun of mass m_2. The distance between the sun and the planet is r. Taking into consideration the motion of the sun, find the total energy of the system assuming the orbits to be circular.

Sol: The gravitational pull between sun and planet provides the necessary centripetal acceleration to make them revolve in circular orbits with same angular velocities. The center of each circular orbit will be at the combined center of mass but their radii will be different.

Both the planet and the sun revolve around their center of mass with same angular velocity (say ω)

$$r = r_1 + r_2 \quad \text{... (i)}$$

$$m_1 r_1 \omega^2 = m_2 r_2 \omega^2 = \frac{Gm_1m_2}{r^2} \quad \text{... (ii)}$$

Solving Eqs. (i) and (ii), we get

$$r_1 = r\left(\frac{m_2}{m_1+m_2}\right)$$

$$r_2 = r\left(\frac{m_1}{m_1+m_2}\right)$$

$$\omega^2 = \frac{G(m_1+m_2)}{r^3}$$

And now, total energy of the system is E= P.E. + K.E.

or $E = -\frac{Gm_1m_2}{r} + \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2$

Substituting the values of r_1, r_2 and ω^2, we get

$$E = -\frac{Gm_1m_2}{2r}.$$

Example 10: Two particles A and B of masses 1 kg and 2 kg respectively are kept 1 m apart and are released to move under mutual attraction. Find the speed of A when that of B is 3.6 cm/hour. What is the separation between the particles at this instant?

Sol: As the particles A and B are initially at rest, the system has potential energy only, but as they move towards each other the loss in potential energy is equal to gain in kinetic energy. As particle is moving under their mutual interaction, the linear momentum system must be conserved.

The linear momentum of the pair A+B is initially zero. As only mutual attraction is taken into account – which is internal when A+B is taken as the system – the linear momentum will remain zero. The particles move in opposite directions. If the speed of A is v when the speed of B is $3.6\ \text{cm/hour} = 10^{-5}\ \text{m/s}$,

$$(1\text{kg})v = (2\text{kg})(10^{-5}\text{ms}^{-1})$$

or, $v = 2\times10^{-5}\text{ms}^{-1}$

The potential energy of the pair is $-\frac{Gm_Am_B}{R}$ with usual symbols. Initial potential energy

$$= -\frac{6.67\times10^{-11}\text{N}-\text{m}^2/\text{kg}^2\times2\text{kg}\times1\text{kg}}{1\text{m}}$$

$$= -13.34\times10^{-11}\text{J}.$$

If the separation at the given instant is d, using conservation of energy,

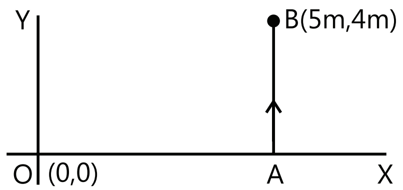
$$-13.34\times10^{-11}\text{J}+0$$

$$= -\frac{13.34\times10^{-11}\text{J}-\text{m}}{d} + \frac{1}{2}(2\ \text{kg})(10^{-5}\text{m/s})^2$$

$$+\frac{1}{2}(1\,\text{kg})(2\times10^{-5}\text{m/s})^2$$

Solving this, $d = 0.31\text{m}$.

Example 11: The gravitational field in a region is given by $\vec{E} = (10\text{Nkg}^{-1})(\vec{i}+\vec{j})$. Find the work done by an external agent to slowly shift a particle of mass of 2 kg from the point (0,0) to a point (5m, 4m).



Sol: As the particle is moving slowly, the kinetic energy of the particle remains zero during its motion. The work done by the external agent to move the particle is given by $W = -\Delta U = \int_i^f \vec{F}.\vec{dr}$

As the particle is slowly shifted, its kinetic energy remains zero. The total work done on the particle is thus zero. The work done by the external agent should be negative of the work done by the gravitational field.

The work done by the field is $dW = -dU = \int_i^f \vec{F}.\vec{dr}$

Consider the figure. Suppose the particle is taken from O to A and then from A to B. The force on the particle is

$$\vec{F} = m\vec{E} = (2\text{kg})(10\text{Nkg}^{-1})(\vec{i}+\vec{j}) = (20\text{N})(\vec{i}+\vec{j})$$

The work done by the field during the displacement OA is

$$W_1 = \int_0^{5\text{m}} F_x dx = \int_0^{5\text{m}} (20\text{N})dx = 20\text{N}\times5\text{m} = 100\text{J}.$$

Similarly, the work done in displacement AB is

$$W_2 = \int_0^{4\text{m}} F_y dy = \int_0^{4\text{m}} (20\text{N})dy = (20\text{N})(4\text{m}) = 80\text{J}$$

Thus, the total work done by the field, as the particle is shifted from O to B, is 180 J.

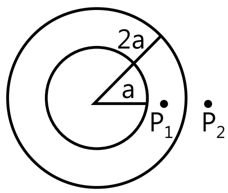
The work done by the external agent is -180 J.

Note that the work is independent of the path so that we can choose any path convenient to us from O to B.

Example 12: A uniform solid sphere of mass M and radius 'a' is surrounded symmetrically by a uniform thin and spherical shell of equal mass and radius 2a. Find the gravitational field at a distance

(a) $\frac{3}{2}$ a from the center, (b) $\frac{5}{2}$a from the center.

Sol: If the particle is inside the spherical shell then the gravitation field due to the shell is zero. The gravitational field at distance r from the center of the sphere is given by $E=\frac{GM}{r^2}$.



Given figure shows the situation. The point p_1 is at a distance $\frac{3}{2}$ a from the center and p_2 is at a distance $\frac{5}{2}$ a from the center. As p_1 is inside the cavity of the thin spherical shell, the field here due to the shell is zero. The field due to the solid sphere is

$$E=\frac{GM}{\left(\frac{3}{2}a\right)^2}=\frac{4GM}{9a^2}$$

This is also the resultant field. The direction is towards the center. The point p_2 is outside the sphere as well as the shell. Both may be replaced by single particles of the same mass at the center. The field due to each of them is

$$E'=\frac{GM}{\left(\frac{5}{2}a\right)^2}=\frac{4GM}{25a^2}$$

The resultant field is $E=2E'=\frac{8GM}{25a^2}$ towards the center.

Example 13: A planet of mass m revolves in an elliptical orbit around the sun so that its maximum and minimum distance from the sun are equal to r_a and r_p respectively. Find the angular momentum of this planet relative to the sun.

Sol: At the apogee and perigee the radius vector is perpendicular to the velocity vector of the plane. Use the law of conservation of angular momentum and energy at these two points.

Using conservation of angular momentum $mv_pr_p=mv_ar_a$

As velocities are perpendicular to the radius, vectors at apogee and perigee, $v_pr_p=v_ar_a$

Using conservation of energy,

$$-\frac{GMm}{r_p}+\frac{1}{2}mv_p^2=\frac{-GMm}{r_a}+\frac{1}{2}mv_a^2$$

By solving, the above equations,

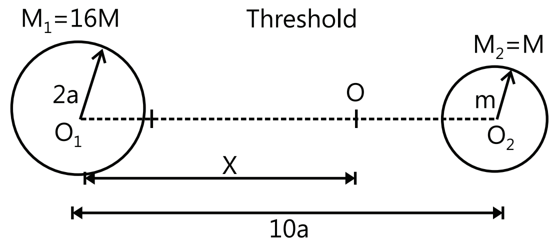
$$v_p=\sqrt{\frac{2GMr_a}{r_p\left(r_p+r_a\right)}};\ L=mv_pr_p=m\sqrt{\frac{2GMr_pr_a}{\left(r_p+r_a\right)}}$$

JEE Advanced/Boards

Example 1: The distance between the centers of two stars is 10 a. The masses of these stars are M and 16 M and their radii, `a' and `2a' respectively. A body of mass m is fired straight from the surface of the larger star towards the smaller star. What should be its minimum initial speed to reach the surface of the smaller star? Obtain the expression in terms of G, M and a.

Sol: At a certain distance from the centers of the stars, the gravitational fields due to the stars are equal in magnitude but opposite in direction. As the body of mass m is projected from the surface of larger star towards the surface of smaller star, the kinetic energy lost by the body is equal to gain of its potential energy when it reaches at the point of zero field.

Let O be the point along O_1O_2 where gravitational intensities due to both the stars balance each other.



Let $O_1O=x$

$$\therefore\ \frac{GM_1}{x^2}=\frac{GM_2}{\left(10a-x\right)^2}$$

Or $16\left(10-x\right)^2=x^2$ or $x=8a$

Potential energy of the body on surface of larger star,

$$U_1 = -\frac{Gm(16M)}{2a} - \frac{GmM}{8a} = -\frac{65GMm}{8a}$$

Potential energy at

$$O = -\frac{GMm}{2a} - \frac{G(16M)m}{8a} = -\frac{5GMm}{2a} = U_O$$

As $U_1 + \frac{1}{2}mv_{min}^2 = U_O$

$$\therefore -\frac{65GMm}{8a} + \frac{1}{2}mv_{min}^2 = -\frac{5GMm}{2a}$$

or $\frac{1}{2}mv_{min}^2 = \frac{65GMm}{8a} - \frac{5GMm}{2a} = \frac{45}{8}\frac{GMm}{a}$

$$v_{min}^2 \frac{45}{4}\frac{GM}{a} = \frac{9 \times 5}{4} \times \frac{GM}{a}$$

$$v_{min} = \frac{3}{2} \times \sqrt{\frac{5GM}{a}}$$

Example 2: Two masses m_1 and m_2, at an infinite distance from each other are initially at rest, start interacting gravitationally. Find their velocity of approach when they are at a distance r apart.

Sol: As the masses move towards each other gain in kinetic energy is equal to loss in gravitational potential energy. This problem is best solved in center of mass frame where the total kinetic energy of masses depends on the square of velocity of approach.

Let v_r be their velocity of approach. From conservation of energy:

Increase in kinetic energy=decrease in gravitational potential energy

Or $\frac{1}{2}\mu v_r^2 = \frac{Gm_1m_2}{r}$ ... (i)

Here, μ = reduced mass $= \frac{m_1m_2}{m_1 + m_2}$

Substituting in Eq. (i), we get

$$v_r = \sqrt{\frac{2G(m_1 + m_2)}{r}}$$

Example 3: A satellite is revolving round the earth in a circular orbit of radius e and velocity v_o. A particle is projected from the satellite in forward direction with relative velocity $v = \left(\sqrt{5/4} - 1\right)v_o$. Calculate its minimum and maximum distance from earth's center during subsequent motion of the particle.

Sol: As the particle is projected from the satellite while the satellite is still in circular motion, the net velocity of the particle is sum of velocity relative to satellite and the velocity of the satellite. As the particle is still bound to the gravitational attraction of the earth, the orbit of the particle will be ellipse. The point of projection is perigee. Conserve the angular momentum at the apogee and perigee.

The orbital speed of satellite is

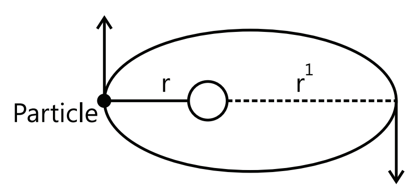
$$v_o = \sqrt{\frac{GM}{r}}$$ ... (i)

Where M=mass of earth

Absolute velocity of particle would be:

$$v_p = v + v_o = \sqrt{\frac{5}{4}}v_o = \sqrt{1.25}\,v_o$$ ... (ii)

Since, v_p lies between orbital velocity and escape velocity, path of the particle would be an ellipse with r being the minimum distance.



Let r′ be the maximum distance and v_p' its velocity at the moment. $v_p = \sqrt{\frac{5}{4}}v_o$

Then, from the conservation of angular momentum and conservation of mechanical energy, we get

$$mv_pr = mv_p'r'$$ ... (iii)

$$\frac{1}{2}mv_p^2 - \frac{GMm}{r} = \frac{1}{2}mv_p'^2 - \frac{GMm}{r'}$$ ... (iv)

Solving the above Eqs. (i), (ii), (iii) and (iv), we get $r' = \frac{5r}{3}$ and r.

Hence, the maximum and minimum distance are $\frac{5r}{3}$ and r respectively.

Example 4: An earth satellite is revolving in a circular orbit of radius 'a' with velocity v_o. A gun is in the satellite and is aimed towards the earth. A bullet is fired from the gun with muzzle velocity $\frac{v_o}{2}$. Neglecting resistance offered by cosmic dust and recoil of gun, calculate maximum and minimum distance of bullet from the center of earth during its subsequent motion.

Sol: Conserve the angular momentum and energy of the particle between the points, the point of projection and at perigee. At perigee velocity is perpendicular to radius.

The orbital speed of the satellite is

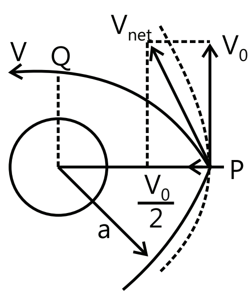
$$v_o = \sqrt{\frac{GM}{a}} \qquad \text{... (i)}$$

From conservation of angular momentum at P and Q we have

$$mav_o = mvr$$

$$\text{Or } v = \frac{av_o}{r} \qquad \text{...(ii)}$$

From conservation of mechanical energy at P and Q. we have



$$\frac{1}{2}m\left(v_o^2 + \frac{v_o^2}{4}\right) - \frac{GMm}{a} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$\text{or } \frac{5}{8}v_o^2 - \frac{GM}{a} = \frac{v^2}{2} - \frac{GMm}{r}$$

Substituting values of v and v_o from Eqs. (i) and (ii), we get

$$\frac{5}{8}\frac{GM}{a} - \frac{GM}{a} = \frac{a^2}{r^2}\left(\frac{GM}{2a}\right) - \frac{GM}{r}$$

$$\text{or } -\frac{3}{8a} = \frac{a}{2r^2} - \frac{1}{r} \text{ or } -3r^2 = 4a^2 - 8ar$$

$$\text{or } 3r^2 - 8ar + 4a^2 = 0$$

$$\text{or } r = \frac{8a \pm \sqrt{64a^2 - 48a^2}}{6}$$

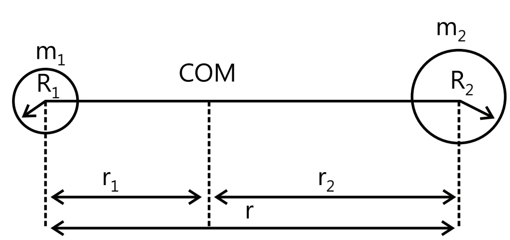
$$\text{or } r = \frac{8a \pm 4a}{6} \text{ or r=2a and } \frac{2a}{3}$$

Hence, the maximum and minimum distance are 2a and $\frac{2a}{3}$ respectively.

Example 5: Binary stars of comparable masses m_1 and m_2 rotate under the influence of each other's gravity with a time period T. If they are stopped suddenly in their motion, find their relative velocity when they collide with each other. The radii of the stars are R_1 and R_2 respectively. G is the universal constant of gravitation.

Sol: They rotate about center of mass, such that the necessary centripetal acceleration for the rotational motion is provided by the gravitational force of attraction. As the stars start approaching each other and collide, the loss in the gravitational energy of system is equal to the gain in the kinetic energy of the system. Find the initial separation in terms of the time period.

Both the stars rotate about their center of mass (COM).



For the position of COM

$$\frac{r_1}{m_2} = \frac{r_2}{m_1} = \frac{r_1 + r_2}{m_1 + m_2} = \frac{r}{m_1 + m_2} \quad (r = r_1 + r_2)$$

$$\text{Also, } m_1 r_1 \omega^2 = \frac{Gm_1 m_2}{r^2} \text{ or } \omega^2 = \frac{Gm_2}{r_1 r^2} \quad \left(\omega = \frac{2\pi}{T}\right)$$

$$\text{But, } r_1 = \frac{m_2 r}{m_1 + m_2}$$

$$\therefore \omega^2 = \frac{G(m_1 + m_2)}{r^3}$$

$$\text{Or } r = \left\{\frac{G(m_1 + m_2)}{\omega^2}\right\}^{1/3} \qquad \text{...(i)}$$

Applying conservation of mechanical energy we have

$$-\frac{Gm_1 m_2}{r} = -\frac{Gm_1 m_2}{(R_1 + R_2)} + \frac{1}{2}\mu v_r^2 \qquad \text{...(ii)}$$

Here, μ = reduced mass $= \frac{m_1 m_2}{m_1 + m_2}$ and

v_r = relative velocity between the two stars.

From Eq. (ii), we find that

$$v_r^2 = \frac{2Gm_1m_2}{\mu}\left(\frac{1}{R_1+R_2} - \frac{1}{r}\right)$$

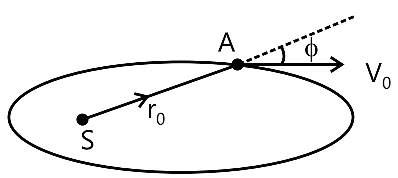
$$= \frac{2Gm_1m_2}{\frac{m_1m_2}{m_1+m_2}}\left(\frac{1}{R_1+R_2} - \frac{1}{r}\right)$$

$$= 2G\,(m_1+m_2)\left(\frac{1}{R_1+R_2} - \frac{1}{r}\right)$$

Substituting the value of r from Eq. (i), we get

$$v_r = \sqrt{2G(m_1+m_2)\left[\frac{1}{R_1+R_2} - \left\{\frac{4\pi^2}{G(m_1+m_2)T^2}\right\}^{1/3}\right]}$$

Example 6: Find the maximum and minimum distance of the planet A from the sun S, if at a certain moment of time it was a distance r_o and travelling with the velocity v_o with the angle between the radius vector and velocity vector being equal to ϕ.



Sol: As the planet revolves around the sun, the mechanical energy of the system is conserved. Conserve the angular momentum between the given point and apogee.

At minimum and maximum distance velocity vector $(\vec{v})$ makes an angle of $90°$ with radius vector. Hence, from conservation of angular momentum,

$$mv_or_o \sin\phi = mrv \qquad ...(i)$$

Here, m is the mass of the planet.

From energy conservation law it follows that.

$$\frac{mv_o^2}{2} - \frac{GMm}{r_o} = \frac{mv^2}{2} - \frac{GMm}{r} \qquad ...(ii)$$

Here, M is the mass of the sun.

Solving Eqs. (i) and (ii) for r, we get we values of r, one is r_{max} and another is r_{min}. So,

$$r_{max} = \frac{r_o}{2-K}\left(1+\sqrt{1-K(2-K)\sin^2\phi}\right)$$

and $$r_{min} = \frac{r_o}{2-K}\left(1-\sqrt{1-K(2-K)\sin^2\phi}\right)$$

Here, $K = \frac{r_0^2 v_0^2}{GM}$

Example 7: The density inside a solid sphere of radius `a' is given by $\rho = \rho_o a/r$, where $\rho_o$ is the destiny at the surface and r denotes the distance from the center. Find the gravitational field due to this sphere at a distance of `2a' from its center.

Sol: The given mass distribution is having spherical symmetry. Any spherically symmetrical body can be replaced by a point particle of the same mass situated at the center of the spherical body. The gravitational field due to the sphere at the point 2a from the center of the sphere is given by $E = \frac{GM}{(2a)^2}$

The field is required at a point outside the sphere. Dividing the sphere in concentric shells, each shell can be replaced by a point particle at its center having mass equal to the mass of the shell. Thus, the whole sphere can be replaced by a point particle at its center having mass equal to the mass of the given sphere. If the mass of the sphere is M, the gravitational field at the given point is

$$E = \frac{GM}{(2a)^2} = \frac{GM}{4a^2} \qquad ... (i)$$

The mass M may be calculated as follows: Consider a concentric shell of radius r and thickness dr. Its volume is

$dV = (4\pi^2)dr$ and its mass is

$$dM = \rho dV = \left(\rho_o \frac{a}{r}\right)(4\pi^2 dr) = 4\rho_o ar\,dr.$$

The mass of the whole sphere is

$$M = \int_0^a 4\rho_0 ar\,dr = 2\pi\rho_0 a^3$$

Thus, by (i) the gravitational field is

$$E = \frac{2\pi G\rho_0 a^3}{4a^2} = \frac{1}{2} \times G\rho_0 a.$$

Example 8: Two satellites of same mass are launched in the same orbit round the earth so as to rotate opposite to each other. They collide solidly and stick together as wreckage. Obtain the total energy of the system before and just after the collision. Describe the subsequent motion of the wreckage.

Sol: Both the satellites are moving in the same orbit so their orbital velocity will be same. As the masses of the satellites are equal, and they are moving in the opposite direction their total momentum before and after the collision is zero.

The two satellites round the earth are shown in figure

Potential energy of the satellite in its orbit $= -GMm/r$

Kinetic energy of satellite in its orbit is

$K = GMm/2r$

Where m is mass of satellite, M is the mass of the earth and r is the orbital radius.

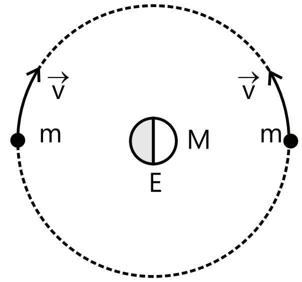
Total energy $= \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$

When there are two satellites, the total energy would be

$$\left(-\frac{GMm}{2r}\right) + \left(-\frac{GMm}{2r}\right) = \left(-\frac{GMm}{r}\right)$$

Let after collision, v' be the velocity of wreckage by the law of conservation of momentum $mv - mv = (m+m)v'$

$\therefore v' = 0$



The wreckage of mass (2m) has no kinetic energy, but it has only potential energy,

So, energy after collision $= -\frac{GM(2m)}{r}$

Now the combined mass has zero velocity just after collision and therefore, the wreckage stops rotating and falls down under gravity.

JEE Main/Boards

Exercise 1

Q.1 Why Newton's law of gravitation is called a universal law?

Q.2 On earth value of $G = 6.67 \times 10^{-11} Nm^2 kg^2$. What is its value on moon, where g is nearly one-sixth than that of earth?

Q.3 An artificial satellite is revolving around the earth at a height 200 km from the earth's surface. If a packet is released from the satellite, what will happen to it? Will it reach the earth?

Q.4 A spring balance is suspended inside an artificial satellite revolving around the earth. If a boy of mass 2 kg is suspended from it, what would be its reading?

Q.5 The escape velocity from earth for a piece of 10 gram is $11.2\,kms^{-1}$. What would it be for a piece of mass 100 gram?

Q.6 Where will the true weight of the body be zero?

Q.7 If the force of gravity acts on all bodies in proportion to their masses, why does not a heavy body fall correspondingly faster than a light body.

Q.8 The gravitational potential energy of a body at a distance from the center of earth is U. What is the weight of the body at that point?

Q.9 The distance of two planets from the sun are 10^{11} and 10^{10} meters respectively. What is the ratio of time periods of these two planets?

Q.10 For a satellite, escape speed is $11 kms^{-1}$. If the satellite is launched at an angle of 60° with the vertical, what will be the escape speed?

Q.11 Prove that the value of acceleration due to gravity at a point above the surface of the earth is inversely proportional to the square of the distance of that point from the center of the earth.

Q.12 Gravitational force between two bodies is 1 newton. If the distance between them is made twice, what will be the force?

Q.13 If a person goes to a height equal to radius of earth from its surface, what would be his weight relative to that on the earth?

Q.14 If the change in the value of g at a height h above the surface of the earth is the same as at a depth x below it, both x and h being much smaller than the radius of the earth, find the relation between x and h.

Q.15 The gravitational force acting on a rocket at a height h from the surface of earth is 1/3 of the force acting on a body at sea level. What is the ratio of h and R (radius of earth)?

Q.16 Does the gravitational force of attraction of the earth becomes zero at some height above the surface of earth? Explain.

Q.17 What do you understand by gravity and acceleration due to gravity. Establish a relation between g and G.

Q.18 Explain how the knowledge of g helps us to find (i) mass of earth and (ii) mean density of earth?

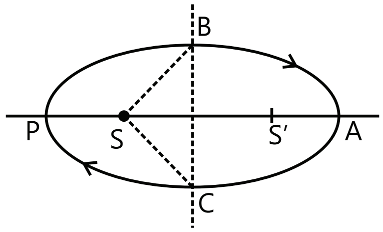
Q.19 What do you understand by 'Escape velocity'? Derive an expression for it in terms of parameters of given planet.

Q.20 What do you understand by Gravitational field, Intensity of gravitational field. Prove that gravitational intensity at a point is equal to the acceleration due to gravity at that point.

Q.21 Explain Kepler's laws of planetary motion and deduce Newton's law of gravitation from them.

Q.22 Explain Newton's law of gravitation. Define gravitational constant, and give its dimensional formula. Give the evidences in support of the Newton's law of gravitation.

Q.23 Let the speed of the planet at the perihelion P in the given figure be v_P and the sun-planet distance SP be r_P. Relate r_P, v_P to the corresponding quantities at the aphelion $\left(r_A, v_A\right)$. Will the planet take equal take equal times to traverse BAC and CPB?



Q.24 Two satellites of a planet have period 32 days and 256 days. If the radius of the orbit of former is R, find the orbital radius of the latter.

Q.25 If the distance of the earth from the sun were half the present value, how many days will make one year? Given, 1 year= 365 days.

Q.26 Estimate the mass of the sun, assuming the orbit of earth round the sun to be a circle. The distance between the sun and the earth is 1.49×10^{11}m, and $G = 6.66\times10^{-11}Nm^2kg^{-2}$.

Q.27 If the mass of the sun is 2×10^{30}kg, the distance of the earth from sun is 1.5×10^{11}m and period of revolution of the earth around sun is one year (=365.3 days), calculate the value of gravitational constant.

Q.28 Calculate the mass and mean density of earth from the following data:

Radius of earth $= 6.37\times10^6$m, acceleration due to gravity $= 9.8ms^{-2}$ and

Gravitational constant $= 6.6\times10^{-11}$ Nm^2kg^{-2}

Q.29 If the radius of the earth shrinks by 2.5%, mass remaining constant, then how would the value of acceleration due to gravity change?

Q.30 At what altitude the acceleration due to gravity above the earth's surface would be half of its value on the surface of the earth? Radius of earth is 6400 km.

Q.31 The radius of earth is approximately 6000 km. What will be your weight at 600 km above the surface of earth? At 12000 km above? At 18000 km above? Your weight on earth is 80 kg wt.

Q.32 At what height from the surface of earth, the acceleration due to gravity is the same at a depth 160 km below the surface of earth. Radius of earth is 6400 km.

Q.33 What is the minimum energy required to launch a satellite of mass m from the surface of earth of mass M, radius R in a circular orbit at an attitude 2 R.

Q.34 A rocket is launched vertically from the surface of the earth with an initial velocity $10 kms^{-1}$. How far above the surface of the earth would it go? Radius of the earth $= 6400 km; g = 9.8 ms^{-2}$.

Q.35 A remote sensing satellite of the earth revolves in a circular orbit at a height of 250km above the earth's surface. What is the (a) orbital speed, and (b) period of revolution of satellite? Radius of the earth $= 6.38 \times 10^6$ m, and acceleration due to gravity at the surface of earth $= 9.8 ms^{-2}$

Q.36 A satellite revolves round a planet in an orbit just above the surface of planet. Taking $G = 6.67 \times 10^{-11} Nm^2kg^{-2}$ and the means density of the planet $= 5.51 \times 10^3 kgm^{-3}$, find the period of satellite.

Q.37 Find the speed of escape at the moon given that its radius 1.7×10^6 m and the value of g at its surface is $1.63 ms^{-2}$.

Q.38 If the earth has a mass nine times and radius twice that of the planet Mars, calculate the maximum speed required by a rocket to pull out of the gravitational force of Mars. Given escape speed on the surface of earth is $11.2 kms^{-1}$

Exercise 2

Single Correct Choice Type

Q. 1 At what altitude will the acceleration due to gravity be 25% of that at the earth's surface (given radius of earth is R)?

(A) R/4 (B) R (C) 3R/8 (D) R/2

Q.2 Let ω be the angular velocity of the earth's rotation about its axis. Assume that the acceleration due to gravity on the earth's surface has the same value at the equator and the poles. An object weighed at the equator gives the same reading as a reading taken at a depth d below earth's surface at a pole (d<<R). The value of d is

(A) $\frac{\omega^2R^2}{g}$ (B) $\frac{\omega^2R^2}{2g}$ (C) $\frac{2\omega^2R^2}{g}$ (D) $\frac{\sqrt{Rg}}{g}$

Q.3 If the radius of the earth be increased by a factor of 5, by what factor its density be changed to keep the value of g the same?

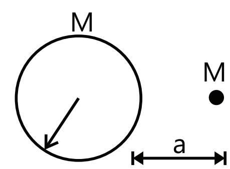
(A) 1/25 (B) 1/5 (C) $1/\sqrt{5}$ (D) 5

Q.4 The mass and diameter of a planet are twice those of earth. What will be the period of oscillation of a pendulum on this planet if it is a second's pendulum on earth?

(A) $\sqrt{2}$ second (B) $2\sqrt{2}$ second

(C) $\frac{1}{\sqrt{2}}$ second (D) $\frac{1}{2\sqrt{2}}$ second

Q.5 A particle of mass M is at a distance a form surface of a thin spherical shell of equal mass and having radius a.



(A) Gravitational field and potential both are zero at center of the shell.

(B) Gravitational field is zero not only inside the shell but at a point outside the shell also.

(C) Inside the shell, gravitational field alone is zero.

(D) Neither gravitational field nor gravitational potential is zero inside the shell.

Q.6 A spherical uniform planet is rotating about its axis. The velocity of a point on its equator is v. Due to the rotation of planet about its axis the acceleration due to gravity g at equator is $\frac{1}{2}$ of g at poles. The escape velocity of a particle on the pole of planet in terms of V.

(A) $v_e = 2v$ (B) $v_e = v$

(C) $v_e = v\sqrt{2}$ (D) $v_e = \sqrt{3}v$

Q.7 Two planets A and B have the same material density. If the radius of A is twice that of B, then the ratio of the escape velocity $\frac{v_A}{v_B}$ is.

(A) 2 (B) $\sqrt{2}$ (C) $1/\sqrt{2}$ (D) $1/2$

Q.8. The escape velocity for a planet is v_e. A tunnel is dug along a diameter of the planet and a small body is dropped into it at the surface. When the body reaches the center of the planet, its speed will be

(A) v_e (B) $\frac{v_e}{\sqrt{2}}$ (C) $\frac{v_e}{2}$ (D) Zero

Q.9 A hollow spherical shell is compressed to half its radius. The gravitational potential at the center

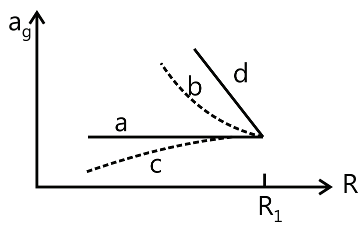
(A) Increases

(B) Decreases

(C) Remains same

(D) During the compression increases then returns at the previous value.

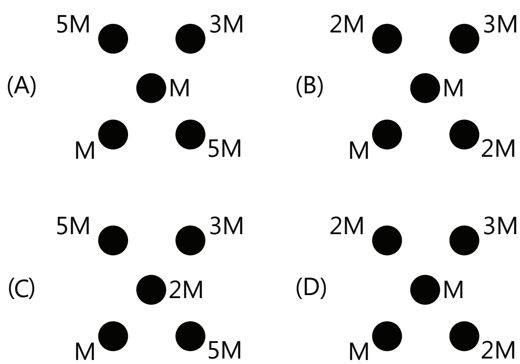
Q.10 A (nonrotating) star collapses onto itself from an initial radius R_i with its mass remaining unchanged. Which curve in the following figure best gives the gravitational acceleration a_g on the surface of the star as function of the radius of the star during the collapse?



(A) a (B) b (C) c (D) d

Q.11 A mass is at the center of a square, with four masses at the corners as shown

Rank the choices according to the magnitude of the gravitational force on the center mass.



(A) $F_A = F_B < F_C = F_D$ (B) $F_A > F_B < F_D < F_C$

(C) $F_A = F_B > F_C = F_D$ (D) None

Q.12 A satellite of the earth is revolving in circular orbit with a uniform velocity V. If the gravitational force suddenly disappears, the satellite will

(A) Continue to move with the same velocity in the same orbit.

(B) Move tangentially to the original orbit with velocity V.

(C) Fall down with increasing velocity.

(D) Come to a stop somewhere in its original orbit.

Q.13 A satellite revolves in the geostationary orbit but in a direction east to west. The time interval between its successive passing about a point on the equator is:

(A) 48 hrs (B) 24 hrs

(C) 12 hrs (D) Never

Q.14 Two point masses of mass 4m and m respectively separated by d distance are revolving under mutual force of attraction. Ratio of their kinetic energies will be:

(A) 1:4 (B) 1:5 (C) 1:1 (D) 1:2

Q.15 Select the correct choice(s)

(A) The gravitational field inside a spherical cavity, within a spherical planet must be non-zero and uniform.

(B) When a body is projected horizontally at an appreciable large height above the earth, with a velocity less than for a circular orbit, it will fall to the earth along a parabolic path

(C) A body of zero total mechanical energy placed in a gravitational field if it is travelling away from source of field will escape the field.

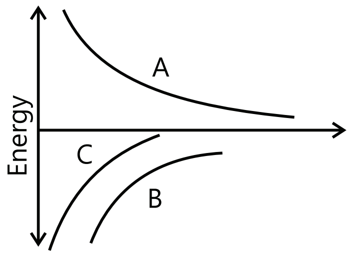
(D) Earth's satellite must be in equatorial plane.

Q.16 A satellite of mass m, initially at rest on the earth, is launched into a circular orbit at a height equal to the radius of the earth. The minimum energy required is.

(A) $\frac{\sqrt{3}}{4}mgR$ (B) $\frac{1}{2}mgR$

(C) $\frac{1}{4}mgR$ (D) $\frac{3}{4}mgR$

Q.17 The following figure shows the variation of energy with the orbit radius of a circular planetary motion. Find the correct statement about the curves A, B and C



(A) A shows the kinetic energy, B the total energy and C the potential energy of the system.

(B) C shows the total energy, V the kinetic energy and A the potential energy of the system.

(C) C and A are kinetic and potential energies respectively and B is the total energy of the system.

(D) A and B are kinetic and potential energies and C is the total energy of the system.

Q.18 When a satellite moves around the earth in a certain orbit, the quantity which remains constant is:

(A) Angular velocity (B) Kinetic energy

(C) Aerial velocity (D) Potential energy

Q.19 A satellite of mass 5 M orbits the earth in a circular orbit. At one point in its orbit, the satellite explodes into two pieces, one of mass M and the other of mass 4M. After the explosion the mass M ends up travelling in the same circular orbit, but in opposite direction. After explosion the mass 4M is.

(A) In a circular orbit

(B) Unbound

(C) Elliptical orbit

(D) Data is insufficient to determine the nature of the orbit.

Q.20 A satellite can be in a geostationary orbit around earth at a distance r from the center. If the angular velocity of earth about its axis doubles, a satellite can now be in a geostationary orbit around earth if its distance from the center is

(A) $\frac{r}{2}$ (B) $\frac{r}{2\sqrt{2}}$ (C) $\frac{r}{(4)^{1/3}}$ (D) $\frac{r}{(2)^{1/3}}$

Q.21 A planet of mass m is in an elliptical orbit around the sun $(m << M_{sun})$ with an orbital period T. If A be the area of orbit, then its angular momentum would be:

(A) $\frac{2mA}{T}$ (B) mAT (C) $\frac{mA}{2T}$ (D) $2mAT$

Q.22 Satellite A and B are orbiting around the in orbits of ratio R and 4R respectively. The ratio of their aerial velocities is:

(A) 1:2 (B) 1:4 (C) 1:8 (D) 1:16

Q.23 In older times, people used to think that the Earth was flat. Imagine that the earth is indeed not a sphere of radius R, but an infinite plate of thickness H. What value of H is needed to allow the same gravitational acceleration to be experienced as on the surface of the actual Earth? (Assume that the Earth's density is uniform and equal in the two models.)

(A) $\frac{2R}{3}$ (B) $\frac{4R}{3}$ (C) $\frac{8R}{3}$ (D) $\frac{R}{3}$

Q.24 A planet revolves about the sun in elliptical orbit. The aerial velocity $\left(\frac{dA}{dt}\right)$ of the planet is $4.0\times10^{16}\ m^2/s$. The least distance between planet and the sun is $2\times10^{12}m$. Then the maximum speed of the planet in km/s is:

(A) 10 (B) 20 (C) 40 (D) None of these

Previous Years' Questions

Q.1 If the radius of x the earth were to shrink by one per cent, its mass remaining the same, the acceleration due to gravity on the earth's surface would ***(1981)***

(A) Decrease (B) Remain unchanged

(C) Increase (D) Be zero

Q.2 If g is the acceleration due to gravity on the earth's surface, the gain in the potential energy of an object of mass m raised from the surface of the earth to a height equal to the radius R of the earth, is ***(1983)***

(A) $\frac{1}{2}mgR$ (B) $2mgR$

(C) mgR (D) $\frac{1}{4}mgR$

Q.3 Imagine a light planet revolving around a very massive star in a circular orbit of radius R with a period of revolution T. If the gravitational force of attraction between the planet and the star is proportional to $R^{-5/2}$, then ***(1989)***

(A) T^2 is proportional to R^2

(B) T^2 is proportional to $R^{7/2}$

(C) T^2 is proportional to $R^{3/2}$

(D) T^2 is proportional to $R^{3.75}$

Q.4 If the distance between the earth and the sun were half its present value, the number of days in a year would have been ***(1996)***

(A) 64.5 (B) 129 (C) 182.5 (D) 730)

Q.5 A simple pendulum has a time period T_1 when on the earth's surface and T_2 when taken to a height R above the earth's surface, where R is the radius of the earth. The value of T_2/T_1 is ***(2001)***

(A) 1 (B) $\sqrt{2}$ (C) 4 (D) 2

Q.6 A geostationary satellite orbits around the earth in a circular orbit of radius 36,000km. Then, the time period of a spy satellite orbiting a few hundred km above the earth's surface $(R_e = 6400\text{km})$ will approximately be ***(2002)***

(A) $1/2$h (B) 1 h (C) 2 h (D) 4 h

Q.7 A double star system consists of two stars A and B which have time periods T_A and T_B. Radius R_A and R_B and mass M_A and M_B. Choose the correct option. **(2006)**

(A) $T_A > T_B$ then $R_A > R_B$

(B) if $T_A > T_B$ then $M_A > M_B$

(C) $\left(\frac{T_A}{T_B}\right)^2 = \left(\frac{R_A}{R_B}\right)^3$

(D) $T_A = T_B$

Q.8 A satellite is moving with a constant speed v in a circular orbit about the earth. An object of mass m is ejected from the satellite such that it just escapes from the gravitational pull of the earth. At the time of its ejection, the kinetic energy of the object is ***(2011)***

(A) $\frac{1}{2}mv^2$ (B) mv^2 (C) $\frac{3}{2}mv^2$ (D) $2mv^2$

Q.9 A planet in a distant solar system is 10 times more massive than the earth and its radius is 10 times smaller. Given that the escape velocity from the earth is 11 kms^{-1}, the escape velocity from the surface of the planet would be ***(2008)***

(A) 1.1 kms^{-1} (B) 11 kms^{-1}

(C) 110 kms^{-1} (D) 0.11 kms^{-1}

Q. 10 Statement-I: For a mass M kept at the centre of a cube of side 'a', the flux of gravitational field passing through its sides is 4π GM.

Statement-II: If the direction of a field due to a point source is radial and its dependence on the distance 'r' for the source is given as $1/r^2$, its flux through a closed surface depends only on the strength of the source enclosed by the surface and not on the size or shape of the surface **(2008)**

(A) Statement-I is false, statement-II is true.

(B) Statement-I is true, statement-II is true; statement-II is correct explanation for statement-I.

(C) Statement-I is true, statement-II is true; statement-II is not a correct explanation for statement-I.

(D) Statement-I is true, statement-II is false.

Q.11 The height at which the acceleration due to gravity becomes $\frac{g}{9}$ (where g = the acceleration due to gravity on the surface of the earth) in terms of R, the radius of the earth is **(2009)**

(A) 2R (B) $\frac{R}{\sqrt{2}}$ (C) $\frac{R}{2}$ (D) $\sqrt{2}R$

Q.12 Two bodies of masses m and 4 m are placed at a distance r. The gravitational potential at a point on the line joining them where the gravitational field is zero is: ***(2011)***

(A) $-\frac{4Gm}{r}$ (B) $-\frac{6Gm}{r}$

(C) $-\frac{9Gm}{r}$ (D) Zero

Q.13 The mass of a spaceship is 1000 kg. It is to be launched from the earth's surface out into free space. The value of 'g' and 'R' (radius of earth) are 10 m/s^2 and 6400km respectively. The required energy for this work will be: ***(2012)***

(A) 6.4 x 10^{11} Joules (B) 6.4 x 10^8 Joules

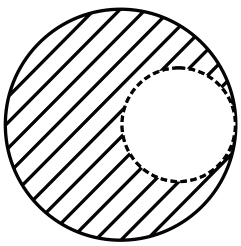
(C) 6.4 x 10^9 Joules (D) 6.4 x 10^{10} Joules

Q.14 What is the minimum energy required to launch a satellite of mass m from the surface of a planet of mass M and radius R in a circular orbit at an altitude of 2R? ***(2013)***

(A) $\frac{5GmM}{6R}$ (B) $\frac{2GmM}{3R}$

(C) $\frac{GmM}{2R}$ (D) $\frac{GmM}{3R}$

Q.15 From a solid sphere of mass M and radius R, a spherical portion of radius R/2 is removed, as shown in the figure. Taking gravitational potential V = 0 at r = ∞, the potential at the centre of the cavity thus formed is: (G = gravitational constant) ***(2015)***



(A) $\frac{-GM}{2R}$ (B) $\frac{-GM}{R}$

(C) $\frac{-2GM}{3R}$ (D) $\frac{-2GM}{R}$

Q.16 A satellite is revolving in a circular orbit at a height 'h' from the earth's surface (radius or earth R; h<<R). The minimum increase in its orbital velocity required, so that the satellite could escape from the earth's gravitational field, is close to: (Neglect the effect of atmosphere.) ***(2016)***

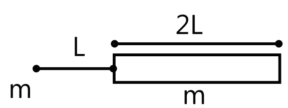
(A) $\sqrt{gR}$ (B) $\sqrt{gR/2}$

(C) $\sqrt{gR}(\sqrt{2}-1)$ (D) $\sqrt{2gR}$

JEE Advanced/Boards

Exercise 1

Q.1 A small mass and a thin uniform rod each of mass 'm' are positioned along the same straight line as shown. Find the force of gravitational attraction exerted by the rod on the small mass.



Q.2 A particle is forced vertically form the surface of the earth with a velocity kv_e where v_e is the escape velocity and k<1. Neglecting air resistance and assuming earth's radius as R_e, calculate the height to which it will rise from the surface of the earth.

Q.3 A point P lies on the axis of a fixed ring of mass M and radius a, at a distance a from its center C. A small particle starts from P and reaches C under gravitational attraction only. Its speed at C will be______.

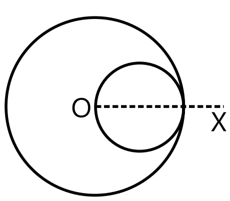
Q.4 Calculate the distance from the surface of the earth at which above and below the surface acceleration due to gravity is the same.

Q.5 An object is projected vertically upward from the surface of the earth of mass M with a velocity such that the maximum height reached is eight times the radius R of the earth. Calculate:

(i) The initial speed of projection

(ii) The speed at half the maximum height.

Q.6 A sphere of radius R has it center at the origin. It has a uniform mass density ρ_0 except that there is a spherical hole of radius $r = R/2$ whose center is at $x = R/2$ as in the given figure. (a) Find gravitational field at points on the axis for x>R

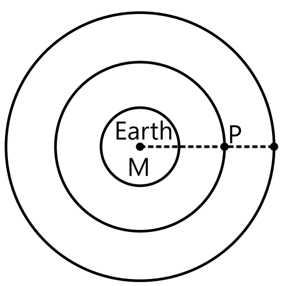


(b) Show that the gravitational field inside the hole is uniform. Find its magnitude and direction.

Q.7 A small body of mass is projected with a velocity just sufficient to make it reach from the surface of a planet (of radius 2R and mass 3M) to the surface of another planet (of radius R and mass M). The distance between the centers of the two spherical planet is 6R. The distance of the body from the center of bigger planet is 'x' at any moment. During the journey, find the distance x where the speed of the body is (a) maximum (b) minimum. Assume motion of body along the line joining centers of planets.

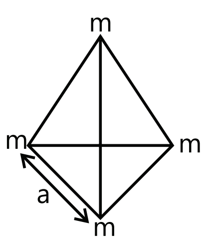
Q.8 A man can jump over b=4m wide trench on earth. If mean density of an imaginary planet is twice that of the earth, calculate its maximum possible radius so that he may escape from it by jumping. Given radius of earth=6400km.

Q.9 A satellite P is revolving around the earth at a height h = radius of earth (R) above equator. Another satellite Q is at a height 2h revolving in opposite direction. At an instant the two are at same vertical line passing through center of sphere. Find the least time after which again they are in this situation.



Q.10 Two small dense stars rotate about their common center of mass as a binary system with the period 1 year for each. One star is of double the mass of the other and the mass of the lighter one is 1/3 of the mass of the sun. Find the distance between the stars if distance between the earth & the sun is R.

Q.11 Four masses (each of m) are placed at the vertices of a regular pyramid (triangular base) of side 'a'. Find the work done by the system while taking them apart so that they form the pyramid of side '2a'



Q.12 A thin spherical shell of total mass M and radius R is held fixed. There is a small hole in the shell. A mass m is released from rest at a distance R from the hole along a line that passes through the hole and also through the center of the shell. This mass subsequently moves under the gravitational force of the shell. How long does the mass take to travel from the hole to the point diametrically opposite?

Q.13 A satellite close to the earth is in orbit above the equator with a period of rotation of 1.5 hours. If it is above a point P on the equator at some time, it will be above P again after time______.

Q.14 A satellite is moving in a circular orbit around the earth. The total energy of the satellite is $E = -2\times10^{5}\,J$. The amount of energy to be imparted to the satellite to transfer it to a circular orbit where its potential energy is equal to________.

Q.15 A satellite of mass m is orbiting the earth in a circular orbit radius r. It starts losing energy due to small air resistance at the rate of C J/s. Then the time taken for the satellite to reach the earth is_____.

Q.16 A satellite is orbiting the Earth of mass M in equatorial plane in a circular orbit having radius 2R and same sense of rotation as that of the Earth. Find duration of time for which a man standing on the equator will be able to see the satellite continuously. Assume that the man can see the satellite when it is above horizon. Take Earth's angular velocity $= \omega$

Q.17 A launching pad with a spaceship is moving along a circular orbit of the moon, whose radius R is triple that of moon Rm. The ship leaves the launching pad with a relative velocity equal to the launching pad's initial orbital velocity v_0 and the launching pad then falls to the moon. Determine the angle θ with the horizontal at which the launching pad crashes into the surface if its mass is twice that of the spaceship m.

Q.18 A body moving radially away from a planet of mass M, when at distance r from planet, explodes in such a way that two of its many fragments move in mutually perpendicular circular orbits around the planet. What will be

(a) Their velocity in circular orbits

(b) Maximum distance between the two fragments before collision and

(c) Magnitude of their relative velocity just before they collide.

Q.19 A cord of length 64 m is used to connect a 100 kg astronaut to spaceship whose mass is much larger than that of the astronaut. Estimate the value of the tension in the cord. Assume that the spaceship is orbiting near earth's surface. Assume that the spaceship and the astronaut fall on a straight line from the earth's center. The radius of the earth is 6400km.

Q.20 Imagine a planet of mass M with a small moon of mass m and radius a orbiting it and keeping the same face toward it. If the moon now approaches the planet, there will be a critical distance from the planet's center at which loose material lying on the moon's surface will be lifted off. Show that this distance is given by $r_e = a(3M/m)^{1/3}$. This critical distance is called Roche's limit.

Q.21 A hypothetical planet of mass M has three moons each of equal mass 'm' revolving in the same circular orbit of radius R. The masses are equally spaced and thus form an equilateral triangle. Find:

(i) The total P.E. of the system

(ii) The orbital speed of each moon such that they maintain this configuration.

Q.22 A remote sensing satellite is revolving in an orbit of radius x over the equator of earth. Find the area on earth's surface in which satellite cannot send message.

Q.23 A pair of stars rotate about a common center of mass. One of the stars has a mass M which is twice as large as the mass m of the other. Their centers are a distance d apart, d being large compared to the size of either star.

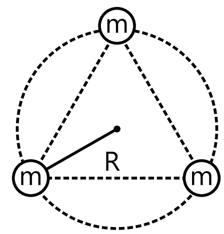
(a) Derive an expression for the period of rotation of the star about their common center of mass in terms of d, m, G.

(b) Compare the angular momentum of the two stars about their common center of mass by calculating the ratio L_m/L_M.

(c) Compare the kinetic energies of the two stars by calculating the ratio K_m/K_M.

Q.24 Assume that a geosynchronous communications satellite is in orbit at the longitude of Mumbai. You are in Mumbai and want to pick up its signals. In what direction should you point the axis of your parabolic antenna? The latitude of Mumbai $30°$ N.

Q.25 The fastest possible rate of rotation of a planet such that for which the gravitational force on material at the equator barely provides the centripetal force needed for the rotation. Show that the corresponding shortest period of rotation is given by $T = \sqrt{\frac{3\pi}{G\rho}}$, where ρ is the density of the planet, assumed to be homogeneous.



Exercise 2

Multiple Correct Choice Type

Q.1 Assuming the earth to be a sphere of uniform density the acceleration due to gravity

(A) At a point outside the earth is inversely proportional to the square of its distance from the center.

(B) At a point outside the earth in inversely proportional to its distance from the center.

(C) At a point inside is zero.

(D) At a point inside is proportional to its distance from the center.

Q.2 Two masses m_1 and $m_2 (m_1 < m_2)$ are released from rest from a finite distance. They start under their mutual gravitational attraction.

(A) Acceleration of m_1 is more than that of m_2.

(B) Acceleration of m_1 is more than that of m_1.

(C) Center of mass of system will remain at rest in all reference frames.

(D) Total energy of system remains constant.

Q.3 Inside a hollow isolated spherical shell

(A) Everywhere gravitational potential is zero.

(B) Everywhere gravitational field is zero.

(C) Everywhere gravitational potential is same.

(D) Everywhere gravitational field is same.

Q.4 When a satellite in a circular orbit around the earth centers the atmospheric region, it encounters small air resistance to its motion. Then

(A) Its kinetic energy increases.

(B) Its kinetic energy decreases.

(C) Its angular momentum about the earth decreases.

(D) Its period of revolution around the earth increases.

Q.5 A communications Earth satellite

(A) Goes round the earth from east to west.

(B) Can be in the equatorial plane only.

(C) Can be vertically above any place on the earth.

(D) Goes round the earth from west to east.

Q.6 An earth satellite is moved from one stable circular orbit to another larger and stable circular orbit. The following quantities increase for the satellite as a result of this change:-

(A) Gravitational potential energy

(B) Angular velocity

(C) Linear orbital velocity

(D) Centripetal acceleration

Q.7 A satellite S is moving in an elliptical orbit around the earth. The mass of the satellite is very small compared to the mass of the earth.

(A) The acceleration of S is always directed towards the center of the earth.

(B) The angular momentum of S about the center of the earth changes in direction, but its magnitude remains constant.

(C) The total mechanical energy of S varies periodically with time.

(D) The linear momentum of S remains constant in magnitude.

Q.8 If a satellite orbits as close to the earth's surface as possible,

(A) Its speed is maximum

(B) Time period of its rotation is minimum

(C) The total energy of the 'earth plus satellite' system is minimum

(D) The total energy of the 'earth plus satellite' system is maximum

Q.9 For a satellite to orbit around the earth, which of the following must be true?

(A) It must be above the equator at some time.

(B) Its cannot pass over the poles at any time

(C) Its height above the surface cannot exceed 36,000 km

(D) Its period of rotation must be $> 2\pi\sqrt{R/g}$ where R is radius of earth

Q.10 Two satellites s_1 & s_2 of equal masses revolve in the same sense around a heavy planet in coplanar circular orbit of radii R & 4R.

(A) The ratio of period of revolution s_1 & s_2 is 1:8

(B) Their velocities are in the ratio 2:1

(C) Their angular momentum about the planet are in the ratio 2:1

(D) The ratio of angular velocities of s_1 w.r.t. s_2 when all three are in same line is 9:5

Assertion Reasoning Type

(A) Statement-I is true, statement-II is true and statement-II is correct explanation for statement-I.

(B) Statement-I is true, statement-II is true and statement-II is NOT the correct the explanation for statement-I.

(C) Statement-I is true, statement-II is false.

(D) Statement-I false, statement-II is true.

Q.11 Statement-I: Moon revolving around earth does not come despite earth's gravitational attraction.

Statement-II: A radially outward force balances earth's force of attraction during revolution of moon.

Q.12 Statement-I: Time period of simple pendulum in an orbiting geostationary satellite in infinite.

Statement-II: Earth's gravitational field becomes negligible at large distance from it.

Q.13 Statement-I: Geostationary satellite may be setup in equatorial plane in orbits of any radius more than earth's radius.

Statement-II: Geostationary satellite have period of revolution of 24 hrs.

Q.14 Statement-I: For the calculation of gravitational force between any two uniform spherical shells, they can always be replaced by particles of same mass placed at respective centers.

Statement-II: Gravitational field of a uniform spherical shell out side it is the same as that of particle of same mass placed at its center of mass.

Q.15 Statement-I: It takes more fuel for a spacecraft to travel from the earth to moon than for the return trip.

Statement-II: Potential energy of spacecraft at moon's surface is greater than that at earth surface.

Comprehension Type

Paragraph 1:

Two uniform spherical stars made of same material have radii R and 2R. Mass of the smaller planet is m. They start moving from rest towards each other from a large distance under mutual force of gravity. The collision between the stars is inelastic with coefficient of restitution ½.

Q.16 Kinetic energy of the system just after the collision is:

(A) $\frac{8Gm^2}{3R}$ (B) $\frac{2Gm^2}{3R}$

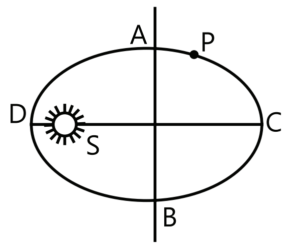
(C) $\frac{4Gm^2}{3R}$ (D) Cannot be determined

Q.17 The maximum separation between their centers after their first collision

(A) 4 R (B) 6 R (C) 8 R (D) 12 R

Paragraph 2:

The given figure shows the orbit of a planet P round the sun S, AB and CD are the minor and major axes of the ellipse.



Q.18 If t_1 is the time taken by the planet to travel along ACB and t_2 the time along BDA, then

(A) $t_1 = t_2$ (B) $t_1 > t_2$

(C) $t_1 < t_2$ (D) Nothing can be concluded

Q.19 If U is the potential energy and K kinetic energy then $|U| > |K|$ at

(A) Only D (B) Only C

(C) Both D & C (D) Neither D nor C

Paragraph 3:

During the formation of stars from clouds of hydrogen gas in space, due to gravitational force of attraction, volume of gas decreases, which in turn heats the gas. Specific heat capacity of gas is S, universal gravitational constant is G and mass in a hydrogen cloud is M.

Q.20 If radius of gas cloud decreases from R to $R/2$, the increment in temperature of gas is (assume No loss of energy outside due to radiations, and clouds are spherical in shape)

(A) $\frac{GM}{RS}$ (B) $\frac{3GM}{5RS}$ (C) $\frac{3GS}{5MR}$ (D) $-\frac{3GM}{RS}$

Q.21 Assume the initial temperature of gas is 0 K and thermonuclear reactions will start at T_0 K temperature, the minimum mass of gas required so that thermonuclear reactions start when radius of cloud becomes half of initial radius (R). Assume uniform temperature in entire volume of gas.

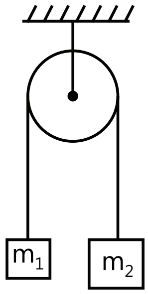
(A) $\frac{5}{3}\frac{SRT_0}{G}$ (B) $\frac{3}{5}\frac{SRT_0}{G}$

(C) $\frac{SRT_0}{G}$ (D) None

Paragraph 4:

In some parts of universe, it is found that acceleration produced in a body is inversely proportional to the square of its mass and directly proportional to the net force (F) according to equation $a = c\frac{F}{m^2}$ where c is constant, whose magnitude is 1, if m is measured in kg, a is measured in m/s² and F is in Bose. Also action and reaction force are equal and opposite and on different interacting bodies.

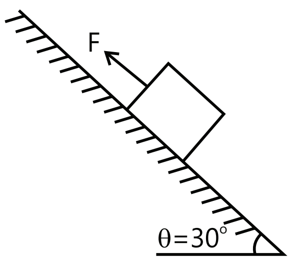
Q.22 In the given figure shown, two blocks of mass m_1 = 2 kg and m_2 = 4 kg are attached via an ideal massless string over frictionless mass less pulley. If acceleration due to gravity g = 5 m/s².The tension in the string is



(A) 6 Bose (B) 1.67 Bose

(C) 3 Bose (D) 32 Bose

Q.23 In the given figure a block of mass m=2 kg is placed on smooth inclined plane. The minimum value of force F needed to support the block is $\left(g = 5m/s^2\right)$



(A) Zero, Newton (B) 10 Bose

(C) 20 Bose (D) 10 Newton

Previous Years' Questions

Q.1. **Statement-I:** An astronaut in an orbiting space station above the earth experiences weightlessness.

Statement-II: An object moving around the earth under the influence of earth's gravitational force is in a state of 'free-fall' ***(2008)***

(A) If statement-I is true, statement-II is true; statement-II is the correct explanation for statement-I

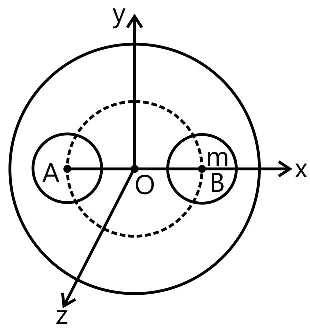
(B) If statement-I is true; statement-II is true; statement-II is not a correct is true; statement-I

(C) If statement-I is true; statement-II is false

(D) If statement-I is false; statement-II is true

Q.2 A solid sphere of uniform density and radius 4 units is located with its center at the origin O of coordinates. Two spheres of equal radii 1 unit, with their centers at A (-2, 0, 0) and B (2, 0, 0) respectively, are taken out of the solid leaving behind spherical cavities as shown in the given figure. ***(1993)***

Then



(A) the gravitational field due to this object at the origin is zero.

(B) the gravitational field at the point B (2, 0, 0) is zero

(C) the gravitational potential is the same at all points of circle $y^2 + z^2 = 36$

(D) the gravitational potential is the same at all points on the circle $y^2 + z^2 = 4$

Q.3 The magnitude of the gravitational field at distance r_1 and r_2 from the center of a uniform sphere of radius R and mass M are F_1 and F_2 respectively. Then ***(1994)***

(A) $\frac{F_1}{F_2} = \frac{r_1}{r_2}$ if $r_1 < R$ and $r_2 < R$

(B) $\frac{F_1}{F_2} = \frac{r_2^2}{r_1^2}$ if $r_1 > R$ and $r_2 > R$

(C) $\frac{F_1}{F_2} = \frac{r_1^3}{r_2^3}$ if $r_1 < R$ and $r_2 < R$

(D) $\frac{F_1}{F_2} = \frac{r_1^2}{r_2^2}$ if $r_1 < R$ and $r_2 < R$

Q.4 Two satellites S_1 and S_2 revolve round a planet in coplanar circular orbits in the same sense. Their periods of revolution are 1 h and 8 h respectively. The radius of the orbit of S_1 is 10^4 km. when S_2 is closed to S_1. Find

(a) the speed of S_2 relative to S_1,

(b) the angular speed of S_2 as actually observed by an astronaut in S_1 ***(1986)***

Q.5 Three particles, each of mass m, are situated at the vertices of an equilateral triangle of side length a. The only forces acting on the particles are their mutual gravitational forces. It is desired that each particle moves in a circle while maintaining the original mutual separation a. Find the initial velocity that should be given to each particle and also the time period of circular motion ***(1988)***

Q.6 An artificial satellite is moving in circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the earth. ***(1990)***

(a) Determine the height of the satellite above the earth's surface.

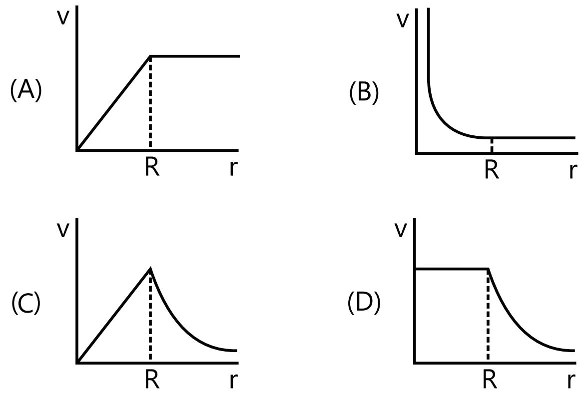
(b) If the satellite is stopped suddenly in its orbit and allowed to fall freely onto the earth, find the speed with which it hits surface of the earth.

Q.7 Distance between the centers of two stars is $10a$. The masses of these stars are M and 16 M and their radii a and 2a respectively. A body of mass m is fired straight from the surface of the larger star towards the surface of the smaller star. What should be its minimum initial speed to reach the surface of the smaller star? Obtain the expression in terms of G, M and a. ***(1996)***

Q.8 There is a crater of depth $\frac{R}{100}$ on the surface of the moon (radius R). A projectile is fired vertically upward from the crater with velocity, which is equal to the escape velocity v from the surface of the moon. Find the maximum height attained by the projectile. ***(2003)***

Q.9 Gravitational acceleration on the surface of a planet is $\frac{\sqrt{6}}{11}g$, where g is the gravitational acceleration on the surface of the earth. The average mass density of the planet is $\frac{2}{3}$ times that of the earth. If the escape speed on the surface of the earth is taken to be 11 km/s the escape speed on the surface of the planet in km/s will be? ***(2010)***

Q.10 A spherically symmetric gravitational system of particles has a mass density $\rho = \begin{cases} \rho_0 \text{ for } r \le R \\ 0 \text{ for } r > R \end{cases}$ where ρ_0 is a constant. A test mass can undergo circular motion under the influence of the gravitational field of particles. Its speed v as a function of distance r from the center of the system is represented by **(2008)**



Q.11 Two spherical planets P and Q have the same uniform density ρ, masses M_P and M_Q, an surface areas A and 4A, respectively. A spherical planet R also has uniform density ρ and its mass is $(M_P + M_Q)$. The escape velocities from the planets P, Q and R, are V_P, V_Q and V respectively. Then ***(2012)***

(A) $V_Q > V_R > V_P$ (B) $V_R > V_Q > V_P$

(C) $\frac{V_R}{V_P} = 3$ (D) $\frac{V_P}{V_Q} = \frac{1}{2}$

Q.12 A planet of radius R = $\frac{1}{10} \times$ (radius of Earth) has the same mass density as Earth. Scientists dig a well of depth $\frac{R}{5}$ on it and lower a wire of the same length and of linear mass density 10^{-3} kgm^{-1} into it. If the wire is not touching anywhere, the force applied at the top of the wire by a person holding it in place is (take the radius of Earth = 6×10^6 m and the acceleration due to gravity of Earth is 10 ms^{-2}) **(2014)**

(A) 96 N (B) 108 N (C) 120 N (D) 150 N

Q.13 In an experiment to determine the acceleration due to gravity g, the formula used for the time period of a periodic motion is $T = 2\pi = \sqrt{\frac{7(R-r)}{5g}}$. The values of R and r are measured to be (60 ± 1) mm and (10 ± 1) mm, respectively. In five successive measurements, the time period is found to be 0.52 s, 0.56 s, 0.57 s, 0.54 s and 0.59 s. The least count of the watch used for the measurement of time period is 0.01 s. Which of the following statement(s) is (are) true? ***(2016)***

(A) The error in the measurement of r is 10%

(B) The error in the measurement of T is 3.57%

(C) The error in the measurement of T is 2%

(D) The error in the determined value of g is 11%

Questions

JEE Main/Boards

Exercise 1

Q.23 Q.31 Q.33
Q.35

Exercise 2

Q. 2 Q.13 Q.17
Q.20

JEE Advanced/Boards

Exercise 1

Q.1 Q.5 Q.6
Q.11 Q.16 Q.21

Exercise 2

Q.1 Q.2 Q.5
Q.6 Q.7 Q.8
Q.10 Q.11 Q.12
Q.16

Answer Key

JEE Main /Boards

Exercise 1

Q.3 No

Q.4 Zero

Q.9 $10\sqrt{10}$

Q.10 11 km/s

Q.13 One-Fourth

Q.14 x=2h

Q.15 0.732

Q.23 $v_A/v_P = r_P/r_A$; No, time for path BAC is greater than time for path CPB

Q.24 4 R

Q.25 129 days

Q.26 1.972×10^{30} kg

Q.27 $6.69 \times 10^{-11}\, m^2 kg^{-2}$

Q.28 6.025×10^{24} kg; 5.56×10^3 kg/m^3

Q.29 Increase by 5%

Q.30 2649.6 km

Q.31 66.12 kg wt; 8.89 kg wt; 5 kg wt.

Q.32 80 km

Q.33 $\frac{5GmM}{6R}$

Q.34 2.56×10^4 km

Q.35 (a) $7756.6\, m/s$; (b) 5373 s

Q.36 5064 s

Q.37 $2.354 \times 10^3\, ms^{-1}$

Q.38 $5.28\, kms^{-1}$

Exercise 2

Single Correct Choice Type

Q.1 B	**Q.2** A	**Q.3** B	**Q.4** B	**Q.5** C	**Q.6** A
Q.7 A	**Q.8** B	**Q.9** B	**Q.10** B	**Q.11** A	**Q.12** B
Q.13 C	**Q.14** A	**Q.15** C	**Q.16** D	**Q.17** D	**Q.18** C
Q.19 B	**Q.20** C	**Q.21** A	**Q.22** A	**Q.23** A	**Q.24** C

Previous Years' Questions

Q.1 C	**Q.2** A	**Q.3** B	**Q.4** B	**Q.5** D	**Q.6** C
Q.7 D	**Q.8** B	**Q.9** C	**Q.10** B	**Q.11** A	**Q.12** C
Q.13 D	**Q.14** A	**Q.15** B	**Q.16** C		

JEE Advanced/Boards

Exercise 1

Q.1 $\frac{GM^2}{3L^2}$

Q.2 $\frac{R_e k^2}{1-k^2}$

Q.3 $\sqrt{\frac{2GM}{a}\left(1-\frac{1}{\sqrt{2}}\right)}$

Q.4 $h=\frac{\sqrt{3}-1}{2}$

Q.5 (i) $\frac{4}{3}\sqrt{\frac{GM}{R}}$ (ii) $\frac{2}{3}\sqrt{\frac{2GM}{5R}}$

Q.6 (a) $E=\frac{\pi G\rho_0 R^3}{6}\left[\frac{1}{\left(x-(R/2)\right)^2}-\frac{8}{x^2}\right]$, (b) $E=\frac{GM}{2R^2}$

Q.7 2R, $3R\left[3-\sqrt{3}\right]$

Q.8 $\sqrt{6.4}$ km

Q.9 $\frac{2\pi R^{3/2}\left(6\sqrt{6}\right)}{\sqrt{GM}\left(2\sqrt{2}+3\sqrt{3}\right)}$

Q.10 R

Q.11 $\frac{6GM^2}{2a}$

Q.12 $2\times\sqrt{R^3/GM}$

Q.13 1.6 hours if it is rotating from west to east, $24/17$ hours if it is rotating from east to west

Q.14 1×10^5 J

Q.15 $t=\frac{GMm}{2C}\left(\frac{1}{R_e}-\frac{1}{r}\right)$

Q.16 $\frac{2\pi}{3\left(\sqrt{\frac{Gm}{8R^3}}-\omega_e\right)}$

Q.17 $\cos\theta=\frac{3}{\sqrt{10}}$

Q.18 (a) $\sqrt{\frac{GM}{r}}$; (b) $r\sqrt{2}$; (c) $\sqrt{\frac{2GM}{R}}$

Q.19 $T=3\times10^{-2}$N

Q.20 $r_e=a(3M/m)^{1/3}$

Q.21 (i) $-\frac{3GM}{R}\left(\frac{m}{\sqrt{3}}+m\right)$, (ii) $\left(\sqrt{\frac{GM}{R}\frac{(2\sqrt{3}+R)}{2\sqrt{3}}}\right)$

Q.22 $2\pi R^2\left(1+\frac{R}{x}\right)$

Q.23 (a) $T=\frac{2\pi d^{3/2}}{\sqrt{3GM}}$ (b) 2 (c) 2

Q.24 $\cot^{-1}\left(\sqrt{3}-\frac{32}{105}\right)$ to vertical

Q.25 $T=\sqrt{\frac{3\pi}{G\rho}}$

Exercise 2

Multiple Correct Choice Type

Q.1 A, D **Q.2** A, D **Q.3** B, C, D **Q.4** A,C **Q.5** B, D **Q.6** A, C, D

Q.7 A, D **Q.8** A, B, C **Q.9** A, D **Q.10** A, B, D

Assertion Reasoning Type

Q.11 C **Q.12** B **Q.13** D **Q.14** D **Q.15** A

Comprehension Type

Paragraph 1: **Q.16** B **Q.17** A

Paragraph 2: **Q.18** B **Q.19** C

Paragraph 3: **Q.20** B **Q.21** A

Paragraph 4: **Q.22** D **Q.23** B

Previous Years' Questions

Q.1 A **Q.2** A **Q.3** B **Q.4** (a) $-\pi\times10^4$ km/h (b) 3×10^{-4} rad/s

Q.5 $v=\sqrt{\frac{Gm}{a}}, T=2\pi\sqrt{\frac{a^3}{3Gm}}$ **Q.6** (a) 6400 km (b) 7.9 km/s

Q.7 $\frac{3\sqrt{5}}{2}\sqrt{\frac{GM}{a}}$ **Q.8** 99.5 R **Q.9** 3 **Q.10** C

Q.11 B, D **Q.12** B **Q.13** A

Solutions

JEE Main/Boards

Exercise 1

Sol 1: Newton's law of gravitation is called a universal law because it is applicable anywhere in the universe.

Sol 2: The value is same on moon. G is called universal gravitational constant, which is constant anywhere in the universe

Sol 3: The packet doesn't reach the earth (theoretically). Because in a satellite, centrifugal force balances the gravitational force on it. The same will happen with the packet, which has same initial velocity as that of satellite.

Sol 4: Its reading will be zero. A spring balance shows the net force the hanging body exerts on it net force by body = Mass × Acceleration

∴ The net acceleration is zero as the centrifugal and gravitational forces balance each other, it reads zero.

Sol 5: Escape velocity is always constant for a given celestial body. Escape velocity of earth is 11.2 km s^{-1}. It is same irrespective of mass.

Sol 6: Weight = Mass × Net acceleration It will be zero, if net acceleration is zero like a satellite.

Sol 7: Gravitational force (f) = $\frac{GMm}{r^2}$

$\therefore f \propto m$

∴ Acceleration due to gravity $g = \frac{f}{m}$

$$g = \frac{GM}{r^2}$$

It is independent of m.

Here both fall at same time

Sol 8: $U = \frac{GMm}{r}$

weight W = mg

$$= m.\left(\frac{GM}{r^2}\right) = \frac{1}{r}\left(\frac{GMm}{r}\right)$$

$$W = \frac{U}{r}$$

Hence weight = $\frac{U}{r}$

Sol 9: $T \propto r^{3/2}$

$$\frac{T_1}{T_2} = \left(\frac{r_1}{r_2}\right)^{\frac{3}{2}} = \left(\frac{10^{11}}{10^{10}}\right)^{\frac{3}{2}} = 10\sqrt{10}$$

Sol 10: Escape speed is still 11 km s^{-1} because escape speed is irrespective of angle of launch (of course not towards ground). We calculate escape velocity by

$$\frac{1}{2}m\,V^2 = \frac{GMm}{r}$$

(Kinetic energy + Potential energy)

= it Irrespective of angle of each.

Sol 11: Acceleration due to gravity

$$g = \frac{GM}{r^2}$$

$$g \propto \frac{1}{r^2}$$

Hence it is inversely proportional to r^2

Sol 12: $f \propto \frac{1}{r^2}$

$$f_2 = f_1.\left(\frac{r_1}{r_2}\right)^2$$

$$\frac{r_1}{r_2} = \frac{1}{2}$$

$$\therefore f_2 = \frac{f_1}{4} = \frac{1}{4}N$$

∴ Gravitational force 0.25 N

Sol 13: If he goes to a height r, his distance from center is 2r

i.e., $\frac{r_1}{r_2} = \frac{1}{2}$

$w \propto \frac{1}{r^2}$ (w = Weight)

$\frac{w_2}{w_1} = \left(\frac{r_1}{r_2}\right)^2$

$\Rightarrow \frac{w_2}{w_1} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$,

i.e., his weight quadrates

Sol 14: $g = \frac{GM}{r^2}$ for $r \geq r_0$ (r_0 is radius of earth)

$\frac{dg}{dr} = -\frac{GM}{2r^3}$

$\Rightarrow \Delta g = -\frac{GM}{2r^3}.\Delta r$

$\Rightarrow \Delta g = -\frac{GM}{2r_0^3}.h$

$g = \frac{GM}{r_0^3} r_1 \qquad r_1 \leq r_0$

$\Delta g = \frac{GM}{r_0^3}(r_1 - r_0)$

$r_1 - r_0 = -x$

$\Rightarrow \Delta g = \frac{GM}{r_0^3}.(-x)$

$\Rightarrow \frac{GM}{r_0^3}(-x) = \frac{GM}{r_0^3}(-2h)$

$\Rightarrow x = 2h$

Sol 15: $\frac{f_1}{f_2} = \left(\frac{r_2}{r_1}\right)^2$

$r_1 = r_0$, r_2 = Distance of rocket from center of earth

$\frac{f_2}{f_1} = \frac{1}{3}$

$\Rightarrow 3 = \left(\frac{r_2}{r_1}\right)^2 \Rightarrow \frac{r_2}{r_1} = \sqrt{3}$

Height of rocket n = $r_2 - r_1$

$\frac{r_2}{r_1} = \sqrt{3}$

$\frac{r_2}{r_1} - 1 = \sqrt{3} - 1$

$\Rightarrow \frac{r_2 - r_1}{r_1} = \sqrt{3} - 1$

$\Rightarrow \frac{h}{r_1} = 0.732$

Sol 16: No. gravitational force $f = \frac{GMm}{r_2}$

$\Rightarrow f \propto \frac{1}{r^2}$

$\frac{1}{r^2}$ will never become zero.

So force doesn't become zero.

Sol 17: Gravity is the force with which a body pulls another body towards its center.

Acceleration due to gravity is the acceleration which it produces in the body due to force of gravity

$g = \frac{GM}{r^2}$

Sol 18: $g = \frac{GM}{r^2}$

m = Mass of earth

r = Radius of earth

$m = \frac{r^2 g}{G}$

= We can calculate g by physical means, r is known, hence M can be calculated

Mean density of earth = $\frac{M}{\frac{4}{3}\pi r^3}$

∵ M can be calculated, mean density can also be calculated.

Sol 19: Escape velocity is the velocity with which when a body is projected from the surface of a celestial body, it crosses its potential barrier and escapes into out space for bodies to escape total energy ≥ 0

i.e. K. E + P.E ≥ 0

At escape velocity

K.E + P.E = 0

$\Rightarrow \frac{1}{2}mv^2 - \frac{GMm}{r} = 0$

$\Rightarrow v = \sqrt{\frac{2GM}{r}}$

Sol 20: Gravity field is a field in which a body produces a force on another body.

Intensity of gravitational force is the force which a body attracts a body of unit mass

Intensity = $\frac{GM}{r^2}$ ... (i)

$= \frac{GM}{r^2} = g$

Sol 21: Kepler's law $T^2 \propto r^3$

$T = \frac{2\pi r}{v}$; $\Rightarrow \frac{r^2}{v^2} \propto r^3$

$\Rightarrow v^2 \propto \frac{1}{r}$ ...... (i)

For a planetary motion

PE + KE = 0

$\Rightarrow \frac{1}{2} mv^2 + PE = 0$

$\Rightarrow PE = -\frac{1}{2}mv^2$; $\Rightarrow PE \propto -\frac{m}{r}$

Gravitational force = $\frac{d}{dr}$ P.E

$\Rightarrow F \propto \frac{d}{dr}\left(-\frac{m}{r}\right)$

$\Rightarrow f \propto \frac{m}{r^2} \Rightarrow f = \frac{km}{r^2}$ (k-some constant)

Let two bodies m_1, m_2 exert gravity on each other.

$f_{12} = \frac{k_2 m_2}{r^2} \Rightarrow f_{12} \propto m_1$ ...... (i)

$f_{21} = \frac{k_1 m_1}{r^2} \Rightarrow f_{21} \propto m_2$ ...... (ii)

Where f_{12} is force on body 1 by body 2 similarly f_{21} defined

but $|f_{12}| = |f_{21}|$ newton's third law

$\Rightarrow t_{12} \propto m_2$

$\Rightarrow f_{12} = k_3 \frac{m_1 m_2}{r^2}$ (k_3-some constant)

$\Rightarrow f \propto \frac{m_1 m_2}{r^2}$

Hence newton's law of gravity is deduced.

Sol 22: Newton's law of universal gravitation states that every point mass in the universe attracts every other point mass with a force that is directly proportional to the product of their masses and inversely proportional to the square of distance between them.

$f \propto \frac{m_1 m_2}{r^2}$

Gravitational constant (G) is a empirical physical constant involved in the calculation of gravitational force between two bodies. In simple terms it can be sides the proportionality constant for Newton's law of gravitation

$G = 6.67 \times 10^{-4}\ N\left(\frac{m}{kg}\right)^2$

Dimensional formula $[G] = \frac{fL^2}{m^2}$

$f = \frac{GMm}{r^2}$

$[f] = \frac{[G][M][m]}{[r]^2} = \frac{FL^2}{m^2}.m.m.L^{-2} = f$

It supports Newton's law empirically

Sol 23: The distance travelled $\overrightarrow{dx}$ in time dt is $\overrightarrow{dv}$

$= \vec{v}.dt$

Area swept by radius vector

$dA = \frac{1}{2}\vec{r} \times \vec{v}\ dt$

$dA_A = \frac{1}{2} r_A v_A dt$ and

$dA_P = \frac{1}{2} r_p v_p dt$

$\because \frac{dA}{dt}$ = Constant, (Kepler's 2nd law)

$\Rightarrow r_A v_A = r_P v_P$

The time taken is different as

area of SBAC ± SCPB

Sol 24: $T^2 \propto r^3$

$\Rightarrow \frac{r_1}{r_2} = \left(\frac{T_1}{T_2}\right)^{\frac{2}{3}}$; $\Rightarrow r_1 = r_2\left(\frac{T_1}{T_2}\right)^{\frac{2}{3}}$

gen rL = R, $T_1 = 32$, $T_2 = 256$

$\Rightarrow r_1 = 4R$

Sol 25: $\because T_1 = T_2 = \left(\frac{r_1}{r_2}\right)^{\frac{3}{2}}$ (kepler's 3rd law)

$\frac{r_1}{r_2} = \frac{1}{2}$

$\therefore T_1 = 365\left(\frac{1}{2}\right)^{\frac{3}{2}} = 129$ days

1 year would have 129 days.

Sol 26: For planetary motion,

$\frac{1}{2}mv^2 = \frac{GMm}{r}$; $\Rightarrow V = \sqrt{\frac{2GM}{r}}$

$V = \frac{2\pi r}{T}$; $\Rightarrow T = 2\pi\sqrt{\frac{\pi^3}{2GM}}$

$\Rightarrow M = \frac{(2\pi)^2.r^3}{2GT^2}$

$= \frac{4.\pi^2.(1.49\times10^{11})^3}{2\times6.66\times10^{-11}\times(365\times24\times3600)^2}$

$= 1.972 \times 10^{30}$ kg

Sol 27: $M = \frac{(2\pi)^2 r^3}{2GT^2}$; $\Rightarrow G = \frac{(2\pi)^2 r^3}{2MT^2}$

$r = 1.5 \times 10^{11}$ m

T = 365.3 days = 365.3 × 24 × 3600 seconds

$M = 2 \times 10^{30}$ Kg

$\Rightarrow G = \frac{4(\pi)^2.(1.5\times10^{11})^3}{2\times2\times10^{30}\times(365.3\times24\times3600)^2}$

$= 6.69\times 10^{-11}\ m^2/Kg^{-2}$

Note: For calculation purpose, you may take $\pi^2 = 10$

Sol 28: $g = \frac{GM}{r^2}$

$M = \frac{gr^2}{G}$

$= \frac{9.8\times(6.37\times10^6)^2}{(6.66\times10^{-11})} = 6.025 \times 10^{24}$ kg

Mean density $= \frac{M}{\frac{4}{3}\pi r^3} = \frac{gr^2}{G}.\frac{1}{\frac{4}{3}\pi r^3}$

$= \frac{3}{4}\frac{g}{Gr} = \frac{3}{4}\times\frac{9.8}{6.66\times10^{-11}\times6.37\times10^6}$

Mean density (e) = $5.56 \times 10^3\ kg/m^3$

Sol 29: $g = \frac{GM}{r^2}$; $\frac{dg}{dr} = -\frac{2GM}{r^3}$

$\Rightarrow \Delta g = -\frac{2GM}{r^3}\Delta r$; $\frac{\Delta g}{g} = \frac{\frac{-2GM}{r^3}}{\frac{GM}{r^2}}\Delta r$

$\frac{\Delta g}{g} = -2\left(\frac{\Delta r}{r}\right)$; $\frac{\Delta g}{g}\times 100 = -2\frac{\Delta r}{r}\times 100$

$\frac{\Delta r}{r}\times 100 = 2.5\%$; $\Rightarrow \frac{\Delta g}{g}\times 100 = -2(-2.5)$

= 5%

∴ Acceleration due to gravity increases by 5%

Note :- Try focusing on the sign convention. It you get confused, use common sense which implies when body gets denser, its g increases like a black hole, etc.

Sol 30: $g \propto \frac{1}{r^2}$

$\frac{g_1}{g_2} = \left(\frac{r_2}{r_1}\right)^2$; $\Rightarrow r_2 = r_1\sqrt{\frac{g_1}{g_2}}$

$\frac{g_1}{g_2} = 2$; $\Rightarrow r_2 = r_1\sqrt{2}$

Height $= r_2 - r_1 = r_1.(\sqrt{2}-1)$

= 2649.6 km

Sol 31: $wt \propto \frac{1}{r^2}$

$w_2 = w_1.\left(\frac{r_1}{r_2}\right)^2$

$r_1 = 6000$ km

$r_2 = 6600, 18000, 2400$ ($r_2 = r_1 + h$)

$w_1 = 80$ kg wt.

for $r_2 = 6600$

$w_2 = 80\left(\frac{6000}{6600}\right)^2 = 66.12$ kg wt

for $r_2 = 18000$

$w_2 = 80\left(\frac{6000}{18000}\right)^2 = 8.89$ kg wt

for $r_2 = 24000$

$w_2 = 80\left(\frac{6000}{24000}\right)^2 = 5$ kg.wt

Sol 32: g at a depth x, $g_x = \frac{GM}{r_0^3}(r_0 - x)$

g at a height h, $g_h = \frac{GM}{(r_0+h)^2}$

$g_x = gh$

By substituting we get solution

But for some intelligent manipulation

$gx = \frac{gm}{r_0^3}(r_0 - x) \Rightarrow \Delta g = \frac{-GMx}{r_0^3}$

$g = \frac{GM}{r_0^3}$

$\Delta g = -\frac{2GM}{x^2}\Delta r$ (Differentiation)

$\therefore$ g is equal $\Rightarrow \Delta g$ is equal

$\Rightarrow -\frac{2GM}{r_0^3}\Delta r = -\frac{GM}{r_0^3}x ; \Rightarrow \Delta r = \frac{x}{2}$

$\Delta r = n$

$x = 160\text{km} ; \Rightarrow h = \frac{160}{2} = 80$ km

$\Rightarrow$ It is same at a height 80 km

Note: $h << r_0$ is assumed hence we could apply this method of differentiation

Sol 33: Energy required = Total change in energy Initial energy = $-\frac{GMm}{R}$

Find energy = P. E + K. E

$= -\frac{GMm}{3R} + \frac{1}{2}mv^2$

(r = 3R because altitude = 2R)

For orbital motion, centrifugal force = Gravitational for

$\frac{mv^2}{r} = \frac{GMm}{r^2}$

$\frac{1}{2}mv^2 = \frac{GMm}{2r}$

$= \frac{GMm}{6R}$ (substitute r = 3R)

$\Rightarrow$ Final energy = $\frac{GMm}{6R} - \frac{GMm}{3R} = -\frac{GMm}{6R}$

$\Rightarrow$ Energy required = $-\frac{GMm}{6R} - \left(\frac{GMm}{R}\right)$

$= \frac{5GMm}{6R}$

Sol 34: Kinetic energy = Change in potential energy

$\Rightarrow \frac{1}{2}mv^2 = -\frac{GMm}{r} - \left(\frac{GMm}{R}\right)$

$\Rightarrow \frac{1}{r} = \frac{1}{R} - \frac{1}{2}\frac{v^2}{GM}$

$\frac{1}{r} = \frac{1}{R} - \frac{1}{2}\frac{v^2}{gR^2}$

$\frac{1}{r} = \frac{1}{6400} - \frac{1}{2} \times \frac{10^2}{9.8 \times 10^{-3} \times (6400)^2}$

$r = 2.56 \times 10^4$ km

Sol 35: Orbital velocity $v = \sqrt{\frac{GM}{r}}$

$= \sqrt{\frac{GM}{R+H}} = \sqrt{\frac{gR^2}{R+H}}$

$g = 9.8\ \text{ms}^{-1}$

$R = 6.38 \times 10^{-6}$

$H = 250\ \text{km} = 2.5 \times 10^5$

$v = 7756.6\ \text{ms}^{-1}$

$T = \frac{2\pi r}{v} = \frac{2\pi(R+H)}{v}$

$T = 5373$ s

Sol 36: Let orbital velocity = v

$\Rightarrow v = \sqrt{\frac{GM}{R}}$ where m = Mass, R = Radius of point

$T = \frac{2\pi R}{v} = 2\pi\sqrt{\frac{R^3}{GM}}$ .... (i)

$M = \frac{4}{3}\pi R^3\rho$

$\Rightarrow T = 2\pi\sqrt{\frac{R^3}{G.\frac{4}{3}\pi R^3\rho}}$

$= 2\pi\sqrt{\frac{3}{4\pi G\rho}} = 5064$ s

Sol 37: For escape velocity

$\frac{1}{2}mv_e^2 = \frac{GMm}{R}$

$v_e = \sqrt{\frac{2GM}{R}}$

$v_e = \sqrt{2gR}$

$= \sqrt{2 \times 1.63 \times 17 \times 10^6}$

$= 2.354 \times 10^3\ \text{ms}^{-1}$

Sol 38: $v_e \propto \sqrt{\frac{M}{R}}$

$\frac{v_m}{v_e} = \sqrt{\frac{M_m}{R_m} \times \frac{R_e}{M_e}}$

$v_m = ve\sqrt{\left(\frac{M_m}{M_e}\right) \times \left(\frac{D_e}{R_m}\right)}$

given $= \frac{M_m}{M_e} = \frac{1}{9}$

$\frac{R_e}{R_m} = 2$

$\Rightarrow v_m = v_e\sqrt{\frac{2}{9}} = 2\sqrt{\frac{2}{9}} = 5.28\ kms^{-1}$

M_m = Mass of planet mass

R_m = Radius of planet mass

Exercise 2

Single Correct Choice Type

Sol 1: (B) $g \propto \frac{1}{r^2}$

$\frac{g_1}{g_2} = \left(\frac{r_2}{r_1}\right)^2 ; \Rightarrow \frac{r_2}{r_1} = \sqrt{\frac{g_1}{g_2}} = \sqrt{4}$

$\Rightarrow r^2 = 2R ; \Rightarrow$ altitude $h = r_2 - R$

$h = R$

Sol 2: (A) Net acceleration at equator

$g' = g - R\omega^2$

($R\omega^2$ is radial acceleration)

$\therefore$ Weight at equator $mg' = mg - mR\omega^2$

acceleration at a depth d

$\Rightarrow g_d = \frac{g(R-d)}{R} = g - \frac{g}{R}d$

given $mg_d = mg'$

$\Rightarrow mg - mR\omega^2 = mg - \frac{mgd}{R} \Rightarrow d = \frac{R^2\omega^2}{g}$

Sol 3: (B) $g = \frac{GM}{r^2}$

$m = \frac{4}{3}\pi r^3\rho ; \Rightarrow g = \frac{4}{3}\pi r r$

$r\rho$ = constant

$\therefore \frac{r_2}{r_1} = \frac{\rho_1}{\rho_2} = \frac{1}{5}\left(\frac{\rho_2}{\rho_1} = 5 \text{ given}\right)$

$\therefore$ Radius to be changed by a factor of $\frac{1}{5}$

Sol 4: (B) $T = 2\pi\sqrt{\frac{\ell}{g}}$

$T \propto \frac{1}{\sqrt{g}} ; g = \frac{GM}{r^2}$

$\frac{1}{\sqrt{g}} = \frac{r}{\sqrt{M}} ; \frac{T_1}{T_2} = \frac{r_1}{r_2}\sqrt{\frac{M_2}{M_1}}$

$T_2 = T_1\left(\frac{r_2}{r_1}\sqrt{\frac{M_1}{M_2}}\right)$

$\frac{r_2}{r_1} = 2; \frac{M_1}{M_2} = \frac{1}{2}; T_1 = 2$seconds

$\therefore T_2 = (2\times 2)\left(\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}$ second

Note: Time period of a seconds pendulum is 2 seconds.

Sol 5: (C) Inside the shell, the gravitational field due to sphere is zero, but there is gravity due to particle.

Sol 6: (A) $g - \frac{v^2}{R} = \frac{g}{2} ; \Rightarrow \frac{v^2}{R} = \frac{g}{2}$

$v_e = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{1}{2}\frac{GM}{R^2}(4R)}$

$= \sqrt{\left(\frac{g}{2}\right)(4R)} = \sqrt{\frac{v^2}{R}(4R)}$

$v_e = 2v$

Sol 7: (A) $v \propto \sqrt{\frac{M}{R}}$

$\frac{M}{R} = \frac{\frac{4}{3}\pi.R^3\rho}{R} \propto R^2 ; \Rightarrow v \propto R$

$\Rightarrow \frac{v_A}{v_B} = \frac{R_A}{R_B} = 2$

Sol 8: (B) $v_e = \sqrt{\frac{2GM}{R}}$

P.E at surface = $\frac{GMm}{R}$

P.E at centre of earth = $\frac{3GMm}{2R}$

KE = ΔPE = $\frac{1}{2}\frac{GMm}{R}$

$\frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{R}$; ⇒ $v = \sqrt{\frac{GM}{R}}$

$= \frac{v_e}{\sqrt{2}}$

Sol 9: (B) Potential at surface = Potential at center for hollow sphere

Potential P = $-\frac{GM}{r}$

r = radius

Let $P_0 = -\frac{GM}{r_0}$

new potential P = $-\frac{GM}{\frac{r_0}{2}} = -\frac{2GM}{r_0}$

$P < P_0$

∴ Decreases

Sol 10: (B) $g \propto \frac{1}{r^2}$

Sol 11: (A) In A, both 5M forces, cancel each other hence net force is proportional to (3M – M) and M (at center)

The same is for B.

∴ $F_A = F_B$

Similarly $f_C \propto (3m-m)2m$ ∴ $f_C > f_B$

same is for F_D

∴ $F_C = f_D$

∵ $F_A = F_B < F_C = F_D$

Sol 12: (B) It moves tangentially as there is no centripetal force.

Sol 13: (C) Time period of a geo stationary satellite is 24 hrs but due to the given situation, it moves twice above same point in one day

∴ Time for successful interval interval = $\frac{24}{2}$ = 12 Hrs

Sol 14: (A) Let centre of mass be at a distance ℓ from 4m

⇒ $4m\ell = m(d - \ell)$

⇒ $\ell = \frac{d}{5}$

⇒ Orbital radius of 4 m = $\frac{d}{5}$, m = $\frac{4d}{5}$

Both bodies have same angular velocities

⇒ $\frac{v}{r}$ = constant

⇒ v = Kr (K = constant)

$v_{am} = = \frac{kd}{5}$, $v_m = \frac{4kd}{5}$

$$\frac{KE_{am}}{KE_m} = \frac{\frac{1}{2}(4m)(v_{4m})^2}{\frac{1}{2}m(v_m)^2} = 4\left(\frac{1}{4}\right)^2 = \frac{1}{4}$$

Sol 15: (C) C-options defines the information about the escape velocities

B-option it is elliptical path

Sol 16: (D) Change in potential energy

$\Delta E = \frac{GMm}{R} - \frac{GMm}{2R} = \frac{GMm}{2R}$

Final velocity or orbital velocity

$v_0 = \sqrt{\frac{GM}{2R}}$

Change in P.E = Change in K.E.

$\frac{GMm}{2R} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$

$\frac{1}{2}mv^2 = \frac{GMm}{2R} + \frac{1}{4}m\,.\,\frac{GM}{R}$

$\frac{1}{2}mv^2 = \frac{3}{4}\frac{GMm}{R}$

$\frac{1}{2}mv^2 = \frac{3}{4}mgR$

∴ Energy required is $\frac{3}{4}$ mgR

Sol 17: (D) K.E $\propto \frac{1}{r}$

$\therefore$ A is K.E.

Total energy > Potential energy,

total energy, potential energy $\propto -\frac{1}{r}$

$\therefore$ C is total energy

B is potential energy

Sol 18: (C) Kepler's 2nd law, areal velocity is constant.

Sol 19: (B) Let final velocity of 4 M be V_1

$5MV = 4MV_1 - MV$

(Conservation of linear momentum)

$\Rightarrow V_1 = \frac{3}{2}V$

now $V = \sqrt{\frac{GM}{r}}$

$V_e = \sqrt{\frac{2GM}{r}} = \sqrt{2}V = V_1 > \sqrt{2}V$

i.e., $V_1 > V_e$

$\Rightarrow$ Body gets unbound

Sol 20: (C) $T \propto r^{3/2}$

$\frac{T_2}{T_1} = \frac{1}{2}; \left(\frac{r_2}{r_1}\right)^{\frac{3}{2}} = \frac{1}{2}$

$\Rightarrow r_2 = r_1\left(\frac{1}{2}\right)^{\frac{2}{3}} = \frac{r}{(4)^{\frac{1}{3}}}$

Sol 21: (A) Consider the planet to be at one of the vertex.

Let its distance from sun be r, velocity be v.

Area covered in time dT

$dA = \frac{1}{2}rv.\,dT; \frac{2dA}{dt} = vr$

$mvr = \frac{2mdA}{dt}; \frac{dA}{dt} = \frac{A}{T}$

$\therefore$ Angular momentum $= \frac{2mA}{T}$

Sol 22: (A) $\frac{dA}{dt} \propto rv$

$V \propto \frac{1}{\sqrt{r}}; \therefore \frac{dA}{dt} \propto \sqrt{r}$

$\therefore$ ratio of their area velocity

$= \sqrt{\frac{r_1}{r_2}} = \sqrt{\frac{1}{4}} = 1:2$

Sol 23: (A) Field due to gravity $E_1 = \frac{GM}{R^2}$

$m = \frac{4}{3}\pi R^3; \therefore E_1 = \frac{4}{3}\pi G\rho R$

Field due to infinite plate $E_2 = 2\pi\rho tG$

$(t = H) = 2\pi\rho HG$

$E_1 = E_2$

$\therefore \frac{4}{3}\pi G\rho R = 2\pi\rho HG; \Rightarrow H = \frac{2R}{3}$

Sol 24: (C) Maximum speed occurs at least distance

$\frac{dA}{dt} = \frac{1}{2}r_{min}v_{max.}$

$4 \times 10^{16} = \frac{1}{2} \times 2 \times 10^{12} \times v$

$v = 4 \times 10^4\ ms^{-1}$

$\therefore v = 40\ kms^{-1}$

Previous Years' Questions

Sol 1: (C) $g = \frac{GM}{R^2}$

or $g \propto \frac{1}{R^2}$

g will increase if R decreases

Sol 2: (A) $\Delta U = \frac{mgh}{1+\frac{h}{R}}$

Given, h = R

$\Delta U = \frac{mgR}{1+\frac{R}{R}} = \frac{1}{2}mgR$

Sol 3: (B) $\frac{mv^2}{R} \propto R^{-5/2} \therefore v \propto R^{-3/4}$

Now, $T = \frac{2\pi R}{v}$

or $T^2 \propto \left(\frac{R}{v}\right)^2$

or $T^2 \propto \left(\frac{R}{R^{-3/4}}\right)^2$

or $T^2 \propto R^{7/2}$

Sol 4: (B) From Kepler's third law

$T^2 \propto r^3$ or $T \propto (r)^{3/2}$

$$\therefore \frac{T_2}{T_1} = \left(\frac{r_2}{r_1}\right)^{3/2}$$

or $T_2 = T_1\left(\frac{r_2}{r_1}\right)^{3/2} = (365)\left(\frac{1}{2}\right)^{3/2}$

$T_2 \approx 129$ days

Sol 5: (D) $T \propto \frac{1}{\sqrt{g}}$ i.e., $\frac{T_2}{T_1} = \sqrt{\frac{g_1}{g_2}}$

where g_1 = Acceleration due to gravity on

Earth's surface

= g

g_2 = Acceleration due to gravity at a height

h = R from earth's surface = g/4

$$\left[\text{Using } g(h)\frac{g}{\left(1+\frac{h}{R}\right)^2}\right]\frac{T_2}{T_1} = \sqrt{\frac{g}{g/4}} = 2$$

Sol 6: (C) Time period of a satellite very close to earth's surface is 84.6 min. Time period increases as the distance of the satellite from the surface of earth increase. So, time period of spy satellite orbiting a few 100 km above the earth's surface should be slightly greater than 84.6 min. Therefore. The most appropriate option is (C) or 2 h.

Sol 7: (D) In case of binary star system angular velocity and hence the time period of both the stars are equal.

Sol 8: (B) In circular orbit of a satellite, potential energy

= – 2 × (kinetic energy)

$= -2 \times \frac{1}{2}mv^2 = -mv^2$

Just to escape from the gravitational pull, its total mechanical energy should be zero therefore, its kinetic energy should be + mv^2

Sol 9: (C)

$$V_{esc} = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2G\times 10M}{R/10}} = 10\times 11 = 110 km/s$$

Sol 10: (B) $g = GM/r^2$

Sol 11: (A) $g' = \frac{GM}{(R+h)^2}$, acceleration due to gravity at height h

$$\Rightarrow \frac{g}{9} = \frac{GM}{R^2}\frac{R^2}{(R+h)^2} = g\left(\frac{R}{R+h}\right)^2$$

$$\Rightarrow \frac{1}{9} = \left(\frac{R}{R+h}\right)^2 \Rightarrow \frac{R}{R+h} = \frac{1}{3}$$

$$\Rightarrow 3R = R+h \Rightarrow 2R = h$$

Sol 12: (C) Position of the null point from mass m,

$$x = \frac{r}{1+\sqrt{\frac{4m}{m}}} = \frac{r}{3}$$

$$V = -Gm\left(\frac{3}{r}+\frac{12}{2r}\right) = -9\frac{Gm}{r}$$

Sol 13: (D) To launch the spaceship out into free space, from energy conservation,

$$\frac{-GMm}{R} + E = 0$$

$$E = \frac{GMm}{R} = \left(\frac{GM}{R^2}\right)mR = mgR$$

$= 6.4 \times 10^{10}$ J

Sol 14: (A) $E_f = \frac{1}{2}mv_0^2 - \frac{GmM}{3R} = \frac{1}{2}m\frac{GM}{3R} - \frac{GmM}{3R}$

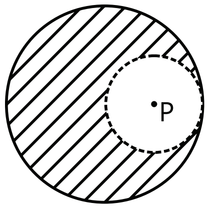
$$= \frac{GmM}{3R}\left(\frac{1}{2}-1\right) = \frac{-GMm}{6R}$$

$$E_i = \frac{-GMm}{R} + K$$

$$E_i = E_f$$

$$K = \frac{5GmM}{6R}$$

Sol 15: (B) Potential at point P due to complete solid sphere



$$= -\frac{GM}{2R^3}\left(3R^2 - \left(\frac{R}{2}\right)^2\right)$$

$$= -\frac{GM}{2R^3}\left(3R^2 - \frac{R^2}{4}\right)$$

$$= -\frac{GM}{2R^3}\left(\frac{11R^2}{4}\right) = -\frac{11GM}{8R}$$

Potential at point P due to cavity part

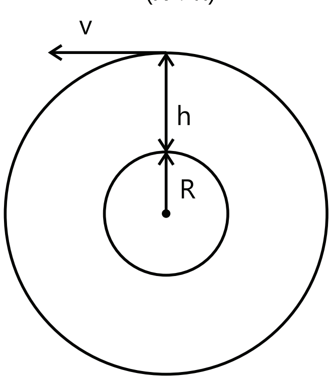
$$= -\frac{3}{2}\frac{G\frac{M}{8}}{\frac{R}{2}} = -\frac{-3GM}{8R}$$

So potential due to remaining part at point P

$$= \frac{-11GM}{8R} - \left(\frac{-3GM}{8R}\right)$$

$$= \frac{-11GM + 3GM}{8R} = \frac{-GM}{R}$$

Sol 16: (C) $\frac{GmM}{(R+h)^2} = \frac{GMm}{R}$



$$v = \sqrt{\frac{GM}{R}}$$

$$\frac{1}{2}mv_1^2 - \frac{GMm}{R} = 0$$

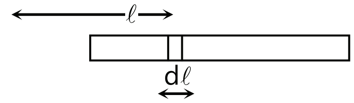
$$v_1 = \sqrt{\frac{2GM}{R}}$$

$$\Delta V = \sqrt{\frac{GM}{R}}(\sqrt{2}-1) = \sqrt{gR}(\sqrt{2}-1)$$

JEE Advanced/Boards

Exercise 1

Sol 1:



consider a small strip of rod of length $d\ell$ at a distance ℓ from the small mass. Let the mass of strip be dm

df = $\frac{GM}{\ell^2}$dm (df= force of attraction between the strip and small mass)

dm = σ dl

where σ is linear density of rod

$\sigma = \frac{M}{2L}$

$\Rightarrow df = \frac{GM}{\ell^2}.\ \sigma dl$

integrating from L to 3L

$$\int_0^f df = \int_L^{3L}\frac{GM}{\ell^2}\sigma\, d\ell$$

$$f = GM\sigma.\ -\frac{1}{L}\Big|_2^{32}$$

$$f = GM\sigma.\ \frac{2}{3L}$$

$$= GM\ .\ \frac{M}{2L}.\ \frac{2}{3L}$$

$$F = \frac{GM^2}{3L^2}$$

Note: Try understanding the boundary conditions. It is most important aspect of physics. Here it is integrated from L to 3L because the rod starts from distance L till distance 3L from the small mass.

Sol 2: $v_c = \sqrt{\frac{2GM}{R}}$

Kinetic energy = Change in potential energy

$$\frac{1}{2}mv^2 = -\frac{GMm}{r} - \left(-\frac{GMm}{R}\right)$$

$$\frac{1}{2}m.(k\ .\ v_e)^2 = \frac{GMm}{R} - \frac{GMm}{r}$$

$\Rightarrow \frac{1}{R} - \frac{1}{r} = \frac{1}{2} \cdot m \cdot k^2 \cdot 2\frac{GM}{R} \cdot \frac{1}{GMm}$

$\Rightarrow \frac{1}{R} - \frac{1}{r} = \frac{k^2}{R}$

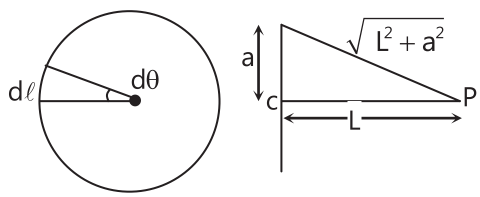
$\Rightarrow r = \frac{R}{1-k^2}$

height = r – R

$= \frac{R}{1-R^2} - R = \frac{k^2R}{1-k^2}$

Hence it will rise to a height of $\frac{k^2R}{1-k^2}$

Sol 3: Consider a small path on the ring of length $d\ell$, which subtends an angle $d\theta$ at the center. Let its mass be dM



Ring Side view of ring

$d\ell = a\, d\theta$

$dM = \sigma\, d\ell$

r = Linear density of ring = $\frac{M}{2\pi a}$

$\Rightarrow dM = \frac{M}{2\pi a} \cdot a\, d\theta = \frac{M}{2\pi} d\theta$

Let the particle be at a distance < along the axis from center.

Potential energy due to → Mass patch

$d_E = \frac{GM}{r}.dM$ (m = mass of particles)

$= \frac{GM}{r} \cdot \frac{M}{2\pi} d\theta$

$r = \sqrt{L^2+a^2}$

$\int_0^E dE = \int_0^{2\pi} \frac{GMm}{\sqrt{L^2+a^2}} \cdot \frac{1}{2\pi} d\theta$

$E = \frac{GMm}{\sqrt{K^2+a^2}} \times \frac{2\pi}{2\pi} = \frac{GMm}{L^2+a^2}$

Kinetic energy = Change in potential energy

$\frac{1}{2}mv^2 = \frac{GMm}{a} - \frac{GMm}{\sqrt{a^2+L^2}}$

$\frac{1}{2}mv^2 = \frac{GMm}{a\sqrt{(a^2+\ell^2)}} (\sqrt{a^2+L^2} - a)$

$v = \sqrt{\frac{2GM}{a\sqrt{a^2+\ell^2}}(\sqrt{a^2+\ell^2} - a)}$

Here given $\ell = a$

$\Rightarrow v = \sqrt{\frac{2GM}{a}\frac{(\sqrt{2}-1)}{\sqrt{2}}}$

$v = \sqrt{\frac{2GM}{a}\left(1-\frac{1}{\sqrt{2}}\right)}$

Sol 4: Let the height be h

$g = \frac{GM}{(R+h)^2}$ (above the surface)

g below the surface

$g = \frac{GM(R-h)}{R^3}$

$\Rightarrow \frac{GM}{(R+h)^2} = \frac{GM(R-h)}{R^3}$

$\Rightarrow h^3 + h^2R - hR^2 = 0$

h = 0 (which is an obvious solution)

$h^2 + hR - R^2 = 0$

$h = -\frac{R \pm \sqrt{3R^2}}{2}$

$h = \frac{\sqrt{3}-1}{2}$ ($\because$ h > 0)

Sol 5: (i) Maximum height = 8R

⇒ Distance from center of earth (r)

= 8R + R

= 9 R

Kinetic energy = Change in potential energy

$\frac{1}{2}mv^2 = -\frac{GMm}{9R} - \left(-\frac{GMm}{R}\right)$

$\frac{1}{2}mv^2 = GMm\left(\frac{8}{9R}\right)$

$v = \sqrt{\frac{16GM}{9R}}$; $v = \frac{4}{3}\sqrt{\frac{GM}{R}}$

(ii) Half minimum height = 4R

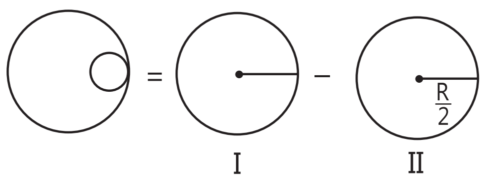
$\Rightarrow r = 4R + R = 5R$

$\Rightarrow \frac{1}{2}mv^2 = -\frac{GMm}{9R} - \left(-\frac{GMm}{5R}\right)$

$\Rightarrow v = \sqrt{\frac{8GM}{45R}}$; $v = \frac{2}{3}\sqrt{\frac{2GM}{5R}}$

Sol 6: We use principal of superposition gravitation field due to sphere I

$E_1 = -\frac{GM}{x^2}$; x > R,



$= -\frac{GM}{R^3}.x$; x < R

Let M_2 mass of sphere II

Here the center is at $\frac{R}{2}$ hence distance function is $R - \frac{R}{2}$.

Assume sphere is uniform

$\Rightarrow M = \frac{4}{3}\pi R^3$

$m_2 = \frac{4}{3}\pi\left(\frac{R}{2}\right)^3 = \frac{1}{8}\left(\frac{4}{3}\pi R^3\right) = \frac{M}{8}$

$E_2 = -\frac{GM_2}{\left(x - \frac{R}{2}\right)^2}$ x < 0, x > R

$-\frac{GM_2}{\left(\frac{R}{2}\right)^3}\left(x - \frac{R}{2}\right)$ 0 < x < R

$= -\frac{GM}{8\left(R - \frac{R}{2}\right)^2}$ x <0, x > R

$-\frac{GM_2}{R^3}\left(x - \frac{R}{2}\right)$

for x > R

$E = E_1 - E_2$

$= -\frac{GM}{x^2} + \frac{GM}{8\left(x - \frac{R}{2}\right)^2}$

$= \frac{GM}{8}\left(\frac{1}{\left(k - \frac{R}{2}\right)^2} - \frac{8}{x^2}\right)$

$M = \frac{4}{3}\pi R^3 \rho_0$

$\Rightarrow E = \frac{\pi G \rho_0 R^3}{6}\left(\frac{1}{\left(x - \frac{R}{2}\right)^2} - \frac{8}{x^2}\right)$

for x < R

$E = E_1 - E_2$

$= -\frac{GM}{R^3}x + \frac{GM}{R^3}\left(x - \frac{R}{2}\right)$

$E = \frac{GM}{2R^2}$

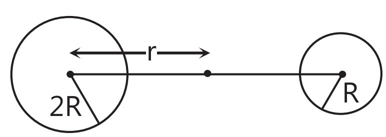
If is independent of x, hence uniform

Sol 7: Potential energy due to planet, at a distance r from its cents

$P.E_1 = -\frac{G(3M)m}{r}$; 2R < r < 5R

Potential due to plant 2

$P.E_2 = -\frac{GMm}{(6R - r)}$; R < r < 4R



$P.E = P.E_1 + P.E_2$

$= -\frac{3GMm}{r} - \frac{GMm}{6R - r}$

$E(r) = -GMm\left(\frac{3}{r} + \frac{1}{6R - r}\right)$

Differentiating

$$\frac{dE}{dr} = -GMm\left[-\frac{3}{r^2} - \frac{1}{(6R-r)}(-1)\right]$$

For maximum E_1 $\frac{dE}{dr} = 0$

$$\Rightarrow -\frac{3}{r^2} + \frac{1}{(6R-r)^2} = 0$$

$$\Rightarrow r = \frac{6\sqrt{3}R}{\sqrt{3}+1}$$

For particle to reach other side it is sufficient if its velocity is zero at

$$r_0 = \frac{6\sqrt{3}R}{\sqrt{3}+1}$$

i.e., K.E = 0 at this point

$\therefore$ Speed is minimum at $r_0 = \frac{6\sqrt{3}R}{\sqrt{3}+1}$

$$= 3R(3-\sqrt{3})$$

Potential energy at x = 2R

$\therefore$ $PE_1 + PE_2$

$$= \frac{G.3M}{2R} - \frac{GMm}{6R-R} = -\frac{17GMm}{10R}$$

Potential energy at x = 5 R

$\Rightarrow PE_1 + PE_2$

$$= -\frac{G(3M)m}{5R} - \frac{GMm}{6R-R}$$

$$= -\frac{8}{5}\frac{GMm}{R} \quad PE(2r) < PE(5R)$$

$PE(r_0) - PE(2r) > PE(r_0) - PE(5R)$

Hence it has maximum speed at x = 2R

Sol 8: Maximum range = $\frac{v^2}{g}$

$$b = \frac{v^2}{g}$$

$$v = \sqrt{bg} = \sqrt{\frac{bGM}{r_e^2}}$$

$$M = \frac{4}{3}\pi r^3\rho$$

$$\Rightarrow v = \sqrt{\frac{bG}{r_e^2}.\frac{4}{3}\pi r_e^3\rho}$$

$$v = \sqrt{\frac{4\pi bG\, r_e\rho}{3}}$$

Escape velocity $V_e = \sqrt{\frac{2GM}{r}}$

$$= \sqrt{\frac{2G}{r_P}.\frac{4}{3}\pi r_p^3.2\rho} = \sqrt{\frac{16\pi\rho r_P^2}{3}}$$

v= ve

$$\Rightarrow \sqrt{\frac{4\pi\, bG\, r_e\rho}{3}} = \sqrt{\frac{16\pi\rho r_P^2}{3}}$$

$$\Rightarrow r_P = \sqrt{\frac{br_e}{2}}$$

$$= \frac{\sqrt{4\times10^{-3}\times6400}}{2}$$

$r_P = \sqrt{6.4}$ km

Maximum radius of planet is $\sqrt{6.4}$ km

Sol 9: $V = \sqrt{\frac{GM}{r}}$ where V = Orbital velocity

$V_1 = \sqrt{\frac{GM}{2R}}$, $V_2 = \sqrt{\frac{GM}{3R}}$ (r = R + height)

Angular velocity $\omega = \frac{V}{R}$

$$\omega_1 = \frac{V}{2R}\sqrt{\frac{GM}{2R}}; \omega_2 = \frac{1}{3R}\sqrt{\frac{GM}{3R}}$$

Relative angular velocity $\omega_R = \omega_1 + \omega_2$

$$= \sqrt{\frac{GM}{R^3}}\left(\frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}}\right)$$

$$t = \frac{2\pi}{wR} = \frac{2\pi}{\sqrt{\frac{GM}{R^3}}\frac{2\sqrt{2}+3\sqrt{3}}{6\sqrt{6}}}$$

$$\therefore t = \frac{2\pi R^{\frac{3}{2}}(6\sqrt{6})}{\sqrt{GM}(2\sqrt{2}+3\sqrt{3})}$$

Sol 10: Let d be distance between them. Distance of centre of mass from m

$$r_1 = \frac{Md}{m+M}; r_1 = \frac{2d}{3}$$

$F = \frac{GMm}{d^2} = \frac{2GM^2}{d^2}$

Gravitational force = Centrifugal force

$\frac{2GM^2}{d^2} = \frac{mv_1^2}{\frac{2d}{3}}$; $\Rightarrow v_1 = \sqrt{\frac{4GM}{3d}}$

$T = \frac{2\pi}{v_1} r_1 = \frac{2\pi\frac{2d}{3}}{\sqrt{\frac{4GM}{3d}}} = \frac{2\pi d^{\frac{3}{2}}}{\sqrt{3GM}}$

$m = \frac{M_s}{3}$ (M_s = Mass of surfs)

$\therefore T = \frac{2\pi d^{\frac{3}{2}}}{\sqrt{GM_s}}$

Time period of earth $T_e = \frac{2\pi d^{\frac{3}{2}}}{\sqrt{GM_s}}$

given $T = T_e$; $\Rightarrow d = R$

Sol 11: Total energy = $\sum_{i<j} \frac{-G.M_iM_j}{r_{ij}}$

$= -G\left(\frac{M^2}{a}\right) \times (\Sigma 3) = -\frac{6GM^2}{a}$

Final energy = $-\frac{6GM^2}{2a}$

Change in energy = $-\frac{6GM^2}{2a} + \frac{6GM^2}{a}$

$= \frac{6GM^2}{2a}$

$\therefore$ Work done is $\frac{6GM^2}{2a}$

Sol 12: Potential = $\frac{GM}{r}$

$\therefore$ Change in potential = $GMm\left(\frac{1}{R} - \frac{1}{2R}\right)$

$= \frac{GMm}{2R}$

K.E. = Charge in P.E

$\frac{1}{2}mv^2 = \frac{GMm}{2R}$; $v = \sqrt{\frac{GM}{R}}$

Inside sphere v is constant

$\therefore$ Time = $\frac{2R}{v} = 2\sqrt{\frac{R^3}{GM}}$

Sol 13: $\omega_1 = \frac{2\pi}{T}$

Angular velocity of earth

$\omega_e = \frac{2\pi}{T_0}$ (T_0 = 24 Hz)

$\omega_r = \omega_1 + \omega_e$ or $\omega_1 - \omega_e$

$T = \frac{2\pi}{\omega_r} = \frac{2\pi}{\frac{2\pi}{T} \pm \frac{2\pi}{T_0}} = \left(\frac{T_0 \pm T}{T_0 T}\right)^{-1}$

$= \left(\frac{24 \pm 1.5}{24 \times 1.5}\right)^{-1} = \left(\frac{17}{24}\right)^{-1}$ hrs, $\left(\frac{5}{8}\right)^{-1}$ hrs

Sol 14: For a satellite

$|K.E| = \frac{1}{2}|P.E.| = |\text{total energy}|$

$E_1 = -2 \times 10^5$ J

$\Rightarrow U_1 = -4 \times 10^{-5}$, $K_1 = 2 \times 10^5$

$U_2 = -2 \times 10^{-5}$

$\Rightarrow E_1 = -1 \times 10^{-5}$ J

$\Delta E = 1 \times 10^5$ J

$\therefore$ Energy required is 10^5 J

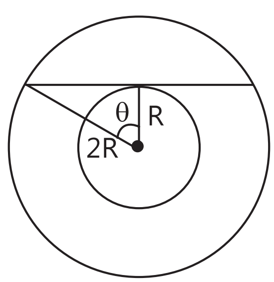
Sol 15: Total energy = $-\frac{GMm}{2r}$, r = Radius

$\therefore$ Change is energy = $\frac{GMm}{2}\left(\frac{1}{R_e} - \frac{1}{r}\right)$

t = Change in energy

$\Rightarrow t = \frac{GMm}{2C}\left(\frac{1}{R_e} - \frac{1}{r}\right)$

Sol 16:



Angle of view = 2θ

$= 2\cos^{-1}\frac{R}{2R} = 2(60°) = 120° = \frac{2\pi}{3}$

Angular velocity of earth $\omega_e = \frac{2\pi}{T_0}$

(T_0 = 24 Hrs)

Angular velocity of satellite $\omega = \sqrt{\frac{GM}{8R^3}}$

$\left(\omega = \sqrt{\frac{GM}{r}}\right)$

Relative angular velocity $\omega_r = \omega - \omega_e$

$= \sqrt{\frac{GM}{8R^3}} - \omega_e$; $T = \frac{\theta}{\omega} = \frac{2\pi}{3\left(\sqrt{\frac{Gm}{8R^3}} - \omega_e\right)}$

Sol 17: Let final velocity of launch pad be x

$\Rightarrow 3mv_0 = m(x + v_0) + 2m(x)$

$\Rightarrow x = \frac{2v_0}{3}$ ..... (i)

Angular momentum = mvr

$= m \cdot \frac{2v_0}{3} \cdot (3R_m)$

$L = 2mv_0 R_m$ ..... (i)

Angular momentum is constant

$\therefore mvx \cdot R_m = L$

$\Rightarrow mv_x \cdot R_m = 2mv_0R_m$

$v_x = 2v_0$ ...... (ii)

$\Delta KE = \Delta Pt$

$\Rightarrow \frac{1}{2}mv^2 - \frac{1}{2}mx^2 = GMm\left(\frac{1}{R_m} - \frac{1}{3R_m}\right)$

$\Rightarrow v^2 = x^2 + 2GM\left(\frac{2}{3R_m}\right)$

Now satellite equals

$v = \sqrt{\frac{GM}{r}}$; $\Rightarrow v_0 = \sqrt{\frac{GM}{3R_m}}$

$\Rightarrow \frac{GM}{3R_m} = v_0^2$; $\Rightarrow v^2 = x^2 + 4v_0^2$

$x = \frac{2v_0}{3}$; $\Rightarrow v = \frac{2}{3}\sqrt{10}\, v_x$

$\cos\theta = \frac{v_x}{v}$; $\Rightarrow \theta = \cos^{-1}\frac{2v_0}{\frac{2\sqrt{10}}{3}v_0}$

$\Rightarrow \cos\theta = \frac{3}{\sqrt{10}}$ $\Rightarrow \theta = \cos^{-1}\frac{3}{\sqrt{10}}$

Sol 18: (a) Orbital velocity $V = \sqrt{\frac{GM}{R}}$

(b) At maximum distance, they are mutually perpendicular radially about the winter of planet

$\therefore$ Maximum distance = $\sqrt{2}\, r$

(c) Their relative velocity = $\sqrt{2}\, V = \sqrt{\frac{2GM}{R}}$

Sol 19: Gravitational force

$F = \frac{GMm}{r^2}$

$\Rightarrow \Delta F = -\frac{GMm}{r^3}\Delta r$

$\therefore$ Net force at a height $r + \Delta r$

$= \frac{GMm}{r^2}\left(1 - \frac{2\Delta r}{r}\right)$

Centrifugal force $f = mrw^2$

$\Delta f = mw^2\Delta r$

$\therefore$ Net centrifugal force $- f + \Delta f$

$= mrw^2\left(1 + \frac{\Delta r}{r}\right)$

$= mrw^2\left(1 + \frac{\Delta r}{r}\right) - \frac{GMm}{r^2}\left(1 - \frac{2\Delta r}{r}\right)$

$= mrw^2\frac{\Delta r}{r} + \frac{GMm}{r^2}\frac{2\Delta r}{r}$

$mrw^2 = \frac{GMm}{r^2} = mg$ ($\therefore$ satellite moon)

$\therefore T = 3Mg\frac{\Delta r}{r} = 3 \times 100 \times 10 \times \frac{64 \times 10^{-3}}{6400}$

$T = 3 \times 10^{-2}$ N

Sol 21: (i) $P.E = \left(-\frac{GMm}{R}\right)3 + 3\left(-\frac{GMm}{\sqrt{3}R}\right)$

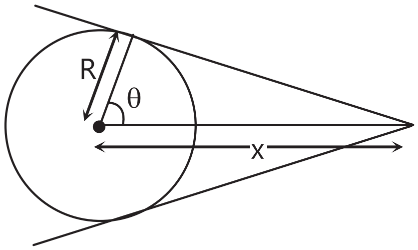
$= -\frac{3GM}{R}\left(\frac{m}{\sqrt{3}} + m\right)$

(ii) Centrifugal force = Force towards center

$\frac{mv^2}{R} = \frac{GMm}{R^2} + \left(\frac{GMm}{(\sqrt{3}R)^2} \cdot \frac{\sqrt{3}}{2}\right)R$

$\Rightarrow v = \left(\sqrt{\frac{GM}{R}\frac{(2\sqrt{3} + R)}{2\sqrt{3}}}\right)$

Sol 22: Surface area of earth A = 4πR²



$\cos\theta = \frac{R}{x}$

Area covered by the satellite on surface of earth

$A_1 = 2\pi R^2(1 - \cos\theta)$

where θ is semi-vertical angle

∴ Area out of reach = $A - A_1$

$= 4\pi R^2 - 2\pi r^2 (1 - \cos\theta)$

$= 2\pi R^2(1 + \cos\theta)$

$= 2\pi R^2\left(1+\frac{R}{x}\right)$

Sol 23: (a) M = 2m

Center of mass from m $(r_1)\ \frac{md}{m+m}$

$= \frac{2d}{3}\ ; \Rightarrow r^2 = \frac{d}{3}$

Force between them (t)$=\frac{GMm}{d^2}$

$=\frac{2GM^2}{d^2}$

Let velocity of mass m be v_1

$\frac{mv_1^2}{r_1} = (F)\ ; \Rightarrow \frac{mv_1^2}{\frac{2d}{3}} = \frac{2GM^2}{d^2}$

$\Rightarrow v_1 = \sqrt{\frac{4GM}{3d}}\ ; T = \frac{2\pi}{w} = \frac{2\pi r}{v}$

$= \frac{2\pi\times\frac{2d}{3}}{\sqrt{\frac{4GM}{3d}}} = \frac{4\pi}{3}\sqrt{\frac{3d^3}{4GM}} = \frac{2\pi d^{3/2}}{\sqrt{3GM}}$

(b) $\frac{v_m}{r_m} = \frac{v_M}{r_M}$

$\frac{L_m}{L_M} = \frac{mv_m r_m}{2mv_m r_m} - \frac{1}{2}\left(\frac{r_m}{r_M}\right)^2 = \frac{1}{2}(2)^2$

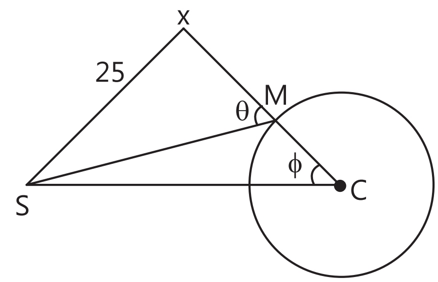
$\frac{L_m}{L_M} = 2$

(c) $k = \frac{1}{2}mv^2$

$\frac{k_m}{k_M} = \frac{\frac{1}{2}mv_m^2}{\frac{1}{2}mv_m^2} = \frac{m}{2m}\cdot\left(\frac{r_m}{r_M}\right)^2 = \frac{1}{2}(2)^2$

$\frac{k_m}{k_M} = 2$

Sol 24:



Let S be satellite, M Mumbai, C center of earth.

ϕ = 30°

let SC = r

MC = R

$\Rightarrow r\sin\phi = (r\cos\phi - R)\tan q$

$\Rightarrow \theta = \tan^{-1}\left(\frac{\sin\phi}{\cos\phi-\frac{R}{r}}\right)\ ; r = \left(\frac{T\sqrt{GM}}{2\pi}\right)^{\frac{2}{3}}$

T = 24 Hr

$\sqrt{GM} = R\sqrt{g}$

R = radius of earth

Upon substitution we get

$\theta = \cot^{-1}\left(\sqrt{3}-\frac{32}{105}\right)$

Sol 25: Angular velocity be ω

$mr\omega^2 = mg$

$\omega = \frac{2\pi}{T}\ ; mR.\frac{(2\pi)^2}{T^2} = mg$

$T = 2\pi\sqrt{\frac{R}{g}}$

$g = \frac{GM}{R^2}$

$= \frac{G}{R^2} \cdot \frac{4}{3}\pi R^3 r$

$\therefore T = 2\pi \sqrt{\frac{R}{GR.\frac{4}{3}\pi\rho}}$

$T = \sqrt{\frac{3\pi}{G\rho}}$

Exercise 2

Multiple Correct Choice Type

Sol 1: (A, D)

$g = \frac{GM}{r^2}; \quad r > R$

$\frac{GM}{R^2} r; \qquad r < R$

Sol 2: (A, D) $f_{12} = f_{21}$

$\Rightarrow m_1 a_1 = m_2 a_2$

$\Rightarrow$ if $m_1 < m_2$

$\therefore a_1 > a_2$

Total energy is constant, by law of conservation of energy centre of mass is in motion in the reference from of the masses.

Sol 3: (B, C, D) Everywhere gravitational field is zero which is same everywhere inside the spherical shell. Everywhere potential is same as $-\frac{dE}{dx} = 0$

i.e., gravitational field is same. Potential inside sphere is equal to that on surface.

Sol 4: (A, C) The satellite will always be in orbital motion at every instant

$|u| = 2|k|$

$|\Delta u| = 2|\Delta k|$

$u = -2k$

$\Rightarrow \Delta u = -2\Delta k$

$\because \Delta u$ is –ve

hence kinetic energy increases

$L = mvr$

$\Delta L = mvr\left(\frac{\Delta v}{v} + \frac{\Delta r}{r}\right)$

$U = -\frac{GMm}{r}$; $\Delta u = \frac{GMm}{r^2}\Delta r$

$k = \frac{1}{2}mv^2$

$\Delta k = mv\,\Delta u$

$\Delta u = -2\Delta k$

$\frac{GMm}{r^2}\Delta r = -2mv\Delta u$

$\frac{GMm}{r^2}\Delta r = 2\left(\frac{1}{2}mv^2\right)$

$\frac{\Delta r}{r} = -\frac{2\Delta v}{v}$; $\Rightarrow \Delta L = mvr\left(\frac{\Delta r}{2r}\right)$

$\Delta r < 0$

$\Rightarrow \Delta L < 0$

Sol 5: (B, D) Communication satellites are geo stationary.

Sol 6: (A, C, D) Only potential energy increase

$P.E = -\frac{GMm}{r}$

v decrease; $v \propto \frac{1}{\sqrt{r}}$

and hence angular velocity and centripetal acceleration decreases as r increases.

Sol 7: (A, D) Acceleration is always directed towards centre of earth. Centripetal force.

Sol 8: (A, B, C) $v \propto \frac{1}{r}$, $T \propto r^{3/2}$

$\therefore$ Speed is maximum and time period is minimum

Potential energy is minimum

$P.E \propto \left(-\frac{1}{r}\right)$

Sol 9: (A, D) Satellite has to be above equator at some time

$T = 2\pi\sqrt{\left(\frac{r}{R}\right)^2 . \frac{r}{g}}$; r = radius of orbit

$> 2\pi\sqrt{\frac{R}{g}}$

$\because r > R, \left(\frac{r}{R}\right)^2 > 1$

Sol 10: (A, B, D) $T \propto R^{3/2}$

$\therefore S_1 : S_2 = 1^{3/2} : 4^{3/2} = 1 : 8$

$v \propto \frac{1}{\sqrt{r}} \Rightarrow v_1 : v_2 = 4^{1/2} : 1^{1/2} = 2 : 1$

Angular momentum $L \propto r^{1/2}$

$L_1 : L_2 = 1 : 4^{1/2} = 1 : 2$

Let velocities be 2k, k

Relative velocities are 3k, k

i.e. $v_1 : v_2 = 3 : 1$

Relative radii = 4R + R, 4R – R

= 5R, 3R

i.e. $R_1 : R_2 = 4 : 3$.

$\omega = \frac{v}{R}$

$\therefore \omega_1 : \omega_2 = \frac{3}{5} : \frac{1}{3}$

$\omega_1 : \omega_2 = 9 : 5$

Assertion Reasoning Type

Sol 11: (C) There is no such real radial force. It only appears in moon's frame of reference as centrifugal force.

Sol 12: (B) Statement-I is true because there is no net acceleration downward in it.

Sol 13: (D) Geostationary satellites have fixed orbital radius and do have 24 hours of time period of revolution.

Sol 14: (D) Statement-I is only true long distances between them.

Sol 15: (A) For travel, energy required

= maximum P.E – P.E at surface

Comprehension type

Paragraph 1:

Sol 16: (B) Let P.E at ∞ = 0

Final P.E = $\frac{GM_1M_2}{d}$

final distance between center of masses

d = R + 2R = 3R

mass of small sphere is m

mass of smaller sphere

$= m\left(\frac{R_2}{R}\right)^3 = m\left(\frac{2R}{R}\right)^3$

= 8 m

Final energy = $\frac{GM.8m}{3R} = \frac{8GM^2}{3R}$

Let initial velocities be v_1, v_2

Let final velocities be v_3, v_4

since centre of mass is at rest

$\vec{v}_3 = -e\,\vec{v}_1$

(e = Coefficient of restitution)

$|V_3| = \frac{1}{2}|V_1|$

$\Rightarrow \frac{1}{2}mv_3^2 = \frac{1}{2}m\left(\frac{1}{2}v\right)^2$

$= \frac{1}{4}\left(\frac{1}{2}mv_1^2\right)$

Similarly $\frac{1}{2}mv_4^2 = \frac{1}{4}\left(\frac{1}{2}mv_2^2\right)$

$\therefore$ Final energy = $\frac{1}{2}mv_3^2 + \frac{1}{2}mv_4^2$

$= \frac{1}{4}\left(\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2\right)$

$= \frac{1}{4}\left(\frac{8Gm^2}{3R}\right)$

$= \frac{2Gm^2}{3R}$

Sol 17: (A) Change in P.E = kinetic energy

$$\frac{2GM^2}{3R} = P.E - \left(-\frac{8GM^2}{3R}\right)$$

$$\Rightarrow PE = \frac{6GM^2}{3R} = \frac{GM(8M)}{4R}$$

$\therefore$ Maximum distance between them is 4R

Note:- Try deriving the result

$\vec{V}_3 = -e\vec{V}_1$ used in the problem. Here centre of mass is at rest.

Paragraph 2:

Sol 18: (B) Area of ASBC > ASBD

$\therefore t_1 > t_2$

Kepler's 2^{nd} law

Sol 19: (C) |u| > |k| always

Because if |k| ≥ |u| body escapes from the suns gravitational force

Paragraph 3:

Sol 20: (B) Self energy of a uniform sphere of radius R and mass M is given by

$$E = -\frac{3GM^2}{5R}$$

$\therefore$ Change in energy

$$= -\frac{3GM^2}{5}\left(\frac{1}{R/2} - \frac{1}{R}\right)$$

$$= \frac{3GM^2}{5R}$$

Increase in temperature $= \frac{\text{energy}}{M.S}$

$$= \frac{3GM^2}{5R} \cdot \frac{1}{M.S}$$

$$= \frac{3GM}{5RS}$$

Sol 21: (A) $T_0 = \frac{3GM}{5RS}$

$$M = \frac{5SRT_0}{3G}$$

Note: Study the self-energy of objects here is a derivation.

Consider a sphere of density ρ and initial radius r initial mass $m = \frac{4}{3}\pi r^3\rho$

Let additional mass added

$dm = 4\pi r^2.dr\rho$

Increase is energy $dE = \frac{GMdm}{r}$

$$dE = \frac{G.\frac{4}{3}\pi r^3\rho.4\pi r^2 dr\rho}{r}$$

$$dE = \frac{(4\pi)^2}{3}\rho^2 . G. r^4 . dr$$

$$E = \int_0^R dE = \frac{(4\pi)^2}{3}\rho^2 . G . \frac{r^5}{5}$$

$\rho = \frac{M}{\frac{4}{3}\pi R^3}$ (M is final mass)

$$\Rightarrow E = \frac{m^2}{\left(\frac{4\pi}{3}\right)^2 R^6} \cdot \frac{R^5}{5} \times \frac{(4\pi)^2}{3}$$

$$E = \frac{3GM^2}{5R}$$

Paragraph 4:

Sol 22: (D) 1 Bose $= \frac{1}{c}$ Newton

Let T be tension in the string.

Let a_1, a_2 be acceleration of m_1, m_2 downward

$$a = \left(\frac{\frac{gm_1^2}{c} - T}{m_1^2}\right)c$$

where $\frac{gm_1^2}{c}$ is downward r gravitational force on m_1

Similarly $a_2 = \left(\frac{\frac{gm_2^2}{c} - T}{m_2}\right)c$

$a_1 + a_2 = 0$ by constrain equation

$$\left(\frac{gm_1^2 - T}{m_1^2}\right)c + \left(\frac{gm_2^2 - T}{m_2^2}\right) = 0$$

$$\Rightarrow T = 2g\left(\frac{m_1^2 m_2^2}{m_1^2 + m_2^2}\right) \cdot \frac{1}{c}$$

$$= \frac{2 \times 5 \times 2^2 \times 4^2}{2^2 + 4^2} \cdot \frac{1}{c}$$

$$T = \frac{32}{c}$$

T = 32 Bose

Note: If you do not know bose, try guessing what it could be. 1 newton is the force which is produced when an object of mass 1 kg moves with an acceleration of 1 ms^{-2}. Similarly define bose. This is the best assumption you can do with the given amount of information.

Sol 23: (B) Force due to gravity $F = \frac{gm^2}{c} = gm^2$ bose

force along slope $f_1 = f \sin\theta$

$f = f_1$

$= f \sin\theta$

$= gm^2 \sin\theta$ bose

$= 5 \times (2)^2 . \frac{1}{2}$

= 10 bose

Previous Years' Questions

Sol 1: (A) Force acting on astronaut is utilized in providing necessary centripetal force, thus he fells weightlessness, as he is in a state of free fall.

Sol 2: (A) The gravitational field is zero at the centre of a solid sphere. The small spheres can be considered as negative mass m located at A and B. The gravitational field due to these masses at O is equal and opposite. Hence, the resultant field at O is zero.

(c and d) → are correct because plane of these circles is y-z, i.e., perpendicular to x-axis i.e., potential at any point on these two circles will be equal due to the positive mass M and negative masses – m and – m.

Sol 3: (B) For $r \le R$. $F = \frac{GM}{R^3}$, r or $F \propto r$

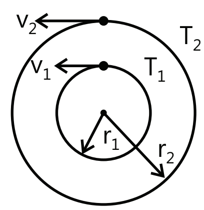
$\frac{F_1}{F_2} = \frac{r_1}{r_2}$ for $r_1 < R$ and $r_2 < R$

And for $r \le R$, $F = \frac{GM}{r^2}$ or $F \propto \frac{1}{r^2}$

i.e., $\frac{F_1}{F_2} = \frac{r_2^2}{r_1^2}$ for $r_1 > R$ and $r_2 > R$

Sol 4: $T \propto r^{3/2}$

or $r \propto T^{2/3}$



$$\frac{r_2}{r_1} = \left(\frac{T_2}{T_1}\right)^{2/3}$$

$$r_2 = \left(\frac{T_2}{T_1}\right)^{2/3} r_1 = \left(\frac{8}{1}\right)^{2/3}(10^4) = 4 \times 10^4 \text{ km}$$

$$\text{Now, } v_1 = \frac{2\pi r_1}{T_1} = \frac{(2\pi)(10^4)}{1} = 2\pi \times 10^4 \text{ km/h}$$

$$v_2 = \frac{2\pi r_2}{T_2} = \frac{(2\pi)(4 \times 10^4)}{8} = (\pi \times 10^4) \text{ km/h}$$

(a) Speed of S_2 relative to S_1

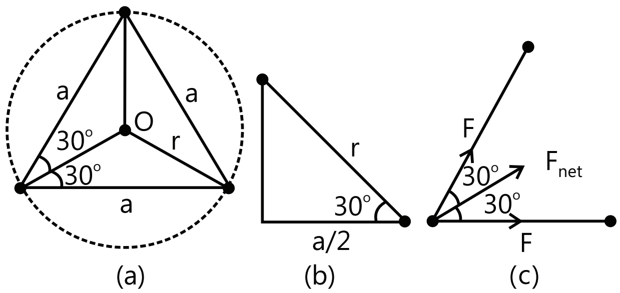
$= v_2 - v_1 = -\pi \times 10^4$ km/h

(b) Angular speed of S_2 as observed by S_1

$$\omega_r = \frac{|v_2 - v_1|}{|r_2 - r_1|} = \frac{\left(\pi \times 10^4 \times \frac{5}{18} \text{ m/s}\right)}{(3 \times 10^7 \text{ m})}$$

$$= 0.3 \times 10^{-3} \text{ rad/s} = 3 \times 10^{-4} \text{ rad/s}$$

Sol 5: Centre should be at O and radius r. We can calculate r from figure (b).



$$\frac{a/2}{r} = \cos 30° = \frac{\sqrt{3}}{2} \therefore r = \frac{a}{\sqrt{3}}$$

Further net force on any particle towards centre

$$F_{net} = 2F\cos 30°$$

$$= 2\left(\frac{Gm^2}{a^2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}\,Gm^2}{a^2}$$

This net force should be equal to $\frac{mv^2}{r}$

$$\therefore \frac{\sqrt{3}\,Gm^2}{a^2} = \frac{mv^2}{a/\sqrt{3}} \therefore v = \sqrt{\frac{Gm}{a}}$$

Time period of circular motion

$$T = \frac{2\pi r}{v} = \frac{2\pi\,(a/\sqrt{3})}{\sqrt{Gm/a}} = 2\pi\sqrt{\frac{a^3}{3\,Gm}}$$

Sol 6: (a) Orbital speed of a satellite at distance r from centre of earth,

$$v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{R+h}} \qquad \text{....(i)}$$

$$\text{Given, } v_0 = \frac{v_e}{2} = \frac{\sqrt{2GM/R}}{2} = \sqrt{\frac{GM}{2R}} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$h = R = 6400$ km

(b) Decrease in potential energy=increase in kinetic energy

$$\text{or } \frac{1}{2}mv^2 = \Delta U \therefore v = \sqrt{\frac{2\,(\Delta U)}{m}}$$

$$= \sqrt{\frac{2\left(\frac{mgh}{1+h/R}\right)}{m}} = \sqrt{gR}$$

$$(h = R) = \sqrt{9.8 \times 6400 \times 10^3} = 7919\text{ m/s} = 7.9\text{ km/s}$$

Sol 7: Let there are two stars 1 and 2 as shown below

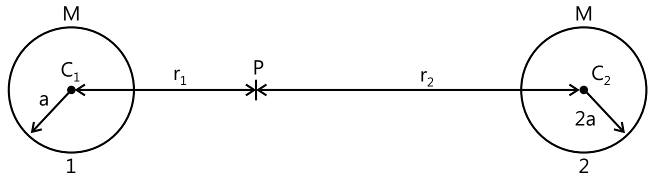
Let P is a point between C_1 and C_2, where gravitational field strength is zero or at P field strength due to star 1 is equal and opposite to the field strength due to star 2 Hence,

$$\frac{GM}{r_1^2} = \frac{G\,(16M)}{r_2^2} \text{ or } \frac{r_2}{r_1} = 4$$

$$r_1 + r_2 = 10\,a \therefore r_2 = \left(\frac{4}{4+1}\right)(10\,a) = 8a$$

and $r_1 = 2\,a$

Now, the body of mass m is projected from the surface of larger star towards the smaller one. Between C_2 and P it is attracted towards 2 and between C_1 and P it will be attracted towards 1

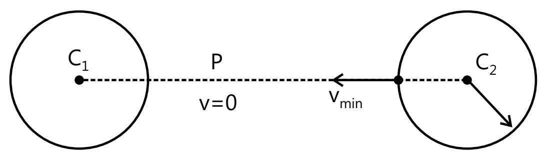


Therefore, the body should be projected to just cross point P because beyond that the particle is attracted towards the smaller star itself.

From conservation of mechanical energy $\frac{1}{2}mv^2_{min}$

= Potential energy of the body at P – Potential energy at the surface of larger star.

$$\therefore \frac{1}{2}mv^2_{min} = \left[-\frac{GMm}{r_1} - \frac{16\,GMm}{r_2}\right] - \left[-\frac{GMm}{10a-2a} - \frac{16\,GMm}{2a}\right]$$

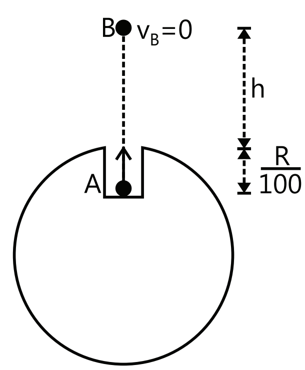


$$\text{Or } \frac{1}{2}mv^2_{min} = \left(\frac{48}{8}\right)\frac{GMm}{a}$$

$$\therefore v_{min} = \frac{3\sqrt{5}}{2}\left(\sqrt{\frac{GM}{a}}\right)$$

Sol 8: Speed of particle at A, v_A = escape velocity on the surface of earth

$$= \sqrt{\frac{2GM}{R}}$$



At highest point B, $v_B = 0$

Applying conservation of mechanical energy, decrease in kinetic energy = increase in gravitational potential energy or $\frac{1}{2}mv_A^2 = U_B - U_A = m(v_B - v_A)$

or $\frac{v_A^2}{2} = v_B - v_A$

$\therefore \frac{GM}{R} = -\frac{GM}{R+h} - \left[-\frac{GM}{R^3}\left(1.5R^2 - 0.5\left(R - \frac{R}{100}\right)^2\right)\right]$

or $\frac{1}{R} = -\frac{1}{R+h} + \frac{3}{2R} - \left(\frac{1}{2}\right)\left(\frac{99}{100}\right)^2 . \frac{1}{R}$

Solving this equation , we get

$h = 99.5\ R$

Sol 9: $g = \frac{GM}{R^2} = \frac{G\left(\frac{4}{3}\pi R^3\right)\rho}{R^2}$ or $g \propto \rho R$

or $R \propto \frac{g}{\rho}$

Now escape velocity, $v_e = \sqrt{2gR}$ or $v_e \propto \sqrt{gR}$

or $v_e \propto \sqrt{g \times \frac{g}{\rho}} \propto \sqrt{\frac{g^2}{\rho}}$

$\therefore (v_e)_{planet} = (11\,km\,s^{-1})\sqrt{\frac{6}{121} \times \frac{3}{2}} = 3\,km\,s^{-1}$

Sol 10: (C) For $r \le R$

$\frac{mv^2}{r} = \frac{GmM}{r^2}$ ...... (i)

Here, $M = \left(\frac{4}{3}\pi r^3\right)\rho_0$

Substituting in Eq. (i), we get

$v \propto r$

i.e., v-r graph is a straight line passing though origin For r > R

$\frac{mv^2}{r} = \frac{Gm\left(\frac{4}{3}\pi R^3\right)\rho_0}{r^2}$ or $v \propto \frac{1}{\sqrt{r}}$

The corresponding v-r graph will be as shown in option (c)

Sol 11: (B, D) $V_{es} = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2.G\rho\frac{4}{3}\pi R^3}{R}} = \sqrt{\frac{4G\rho}{3}}R$

$V_{es} \propto R$

Surface area of $P = A = 4\pi R_P^2$

Surface area of $Q = 4A = 4\pi R_Q^2$

$\Rightarrow R_Q = 2R_P$

Mass R is $M_R = M_P + M_Q$

$\rho\frac{4}{3}\pi R_R^3 = \rho\frac{4}{3}\pi R_P^3 + \rho\frac{4}{3}\pi R_Q^3 \Rightarrow R_R^3 = R_P^3 + R_Q^3 = 9R_P^3$

$R_R = 9^{1/3}R_P \Rightarrow R_R > R_Q > R_P$

Therefore $V_R > V_Q > V_P$

$\frac{V_R}{V_P} = 9^{1/3}$ and $\frac{V_P}{V_Q} = \frac{1}{2}$

Sol 12: (B) Inside planet

$g_i = g_s\frac{r}{R} = \frac{4}{3}G\pi r\rho$

Force to keep the wire at rest (F)

= Weight of wire

$= \int_{4R/5}^{R}(\lambda dr)\left(\frac{4}{3}G\pi r\rho\right) = \left(\frac{4}{3}G\pi\rho\right)\left(\frac{9\lambda}{50}\right)R^2$

Here, ρ = density of earth $= \frac{M_e}{\frac{4}{3}\pi R_e^2}$

Also, $R = \frac{R_e}{10}$; putting all values, F = 108 N

Sol 13: (A) Measured value of r = (10 ± 1) mm

$\Delta r = 1$ mm

Relative error $= \frac{\Delta r}{r} = \frac{1}{10} = 100\%$

Average value of

$\bar{T} = \frac{\sum_{i=l}^{n=5} T_i}{n} = \frac{(0.52 + 0.56 + 0.57 + 0.54 + 0.59)}{5}s$

$\Rightarrow \bar{T} = 0.556s \simeq 0.56s$

Relative error in time period $\simeq \frac{0.01}{0.56} = 1.79\%$

Reported value of (R - r) = (50 ± 2) mm

Relative error in (R - r) $= \frac{2}{50} = 4\%$

$T = 2\pi\sqrt{\frac{7(R-r)}{5g}} \Rightarrow \frac{\Delta g}{g} = 2\left(\frac{\Delta T}{T}\right) + \frac{\Delta(R-r)}{(R-r)}$

$\Rightarrow \frac{\Delta g}{g} = 7.57\%$