Chapter 4

Torsion of Shafts

CHAPTER HIGHLIGHTS

- *Introduction*
- *Torsional equation*
- *Power transmitted*
- *Shaft combinations*
- *Combined bending and torsion*

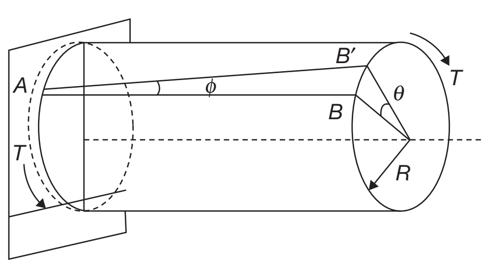
Introduction

Torsion of Circular Shafts

When a moment is applied on a shaft about its axis, it is twisted about its axis. The shaft is then said to be in torsion. The applied moment is called 'twisting moment' or 'torsional moment'. Shafts transmitting power, springs, etc., are some of the examples.

In actual practice, a member/shaft may be subjected to combined effect of torsion, axial forces and bending moments. It is said to be under pure torsion if only torsional moments are acting.

Torsional Equation



When a torsional moment is applied on the shaft, the effects are:

1. There is an angular displacement of a cross-section of one end with respect to the other end.
2. Shearing stresses are set up on any cross-section perpendicular to the axis.

Refer to the given figure. Line AB is twisted to a position AB'. The surface of the shaft is moved by angle ϕ. The cross-section at B is twisted by an angle θ.

Here, ϕ = The shear strain

θ = Angle of twist

It can be seen that:

$$R\theta = L\phi$$

But, $$\phi = \frac{q_s}{G}$$

Where

q_s = Shear stress at surface

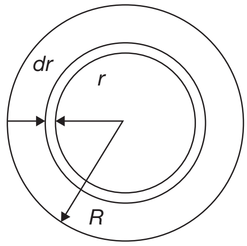
G = Modulus of rigidity

$$\therefore R\theta = L\frac{q_s}{G},$$

or $$\boxed{\frac{q_s}{R} = \frac{G\theta}{L}}$$

$\frac{G\theta}{L}$ being constant, it can be seen that shear stress is directly proportional to radius.

$$\therefore \frac{q_s}{R} = \frac{q}{r}$$



Considering an elemental ring of thickness 'dr' shearing resistance of the ring $= q2\pi r\,dr$.

$\therefore$ Total resisting moment, $T = \int_0^R \frac{q_s}{R} 2\pi r^3 dr$

Since,

$$q = \frac{q_s}{R} r$$

$$T = \frac{q_s \pi R^4}{R\ 2} = q_s \frac{\pi R^3}{2} = \frac{q_s}{R} J,$$

where $J = \frac{\pi R^4}{2}$, the polar moment of inertia.

Or $$\frac{q_s}{R} = \frac{T}{J}$$

$\therefore$ $$\boxed{\frac{q_s}{R} = \frac{T}{J} = \frac{G\theta}{L}}$$

This may be compared with the equation for bending moment.

$$\frac{f}{y} = \frac{M}{I} = \frac{E}{R}$$

Polar Modulus and Stiffness

$$T = q_s \frac{J}{R} = q_s Z_p$$

Z_p = The polar modulus.

Also,

$$\frac{T}{J} = \frac{G\theta}{L} \text{ or } GJ = \frac{TL}{\theta}$$

GJ is called the torsional rigidity or stiffness. It is the torque required for a twist of one radian per unit length of the shaft.

For hollow shafts:

$$J = \frac{\pi}{2}(R^4 - r^4)$$

$$= \frac{\pi}{32}(D^4 - d^4)$$

Power Transmitted

Power transmitted by a shaft $= \frac{2\pi NT}{60}$ NM/s or W.

Where, $N = rpm$

Consider a solid shaft and hollow shaft made of the same material, same length and weight—Torque carrying capacity is to be compared.

Since mass remains the same:

$$\rho v_h = \rho v_s$$

$$v_h = v_s$$

$$\frac{\pi}{4}(do^2 - di^2) \times l = \frac{\pi}{4} d^2 \times l, \ l \text{ being the same}$$

$$do^2 - di^2 = d^2 \quad (1)$$

But, $\frac{T}{J} = \frac{f_s}{R}$. Torque carrying capacity depends on $\frac{f_s J}{R}$.

Since, f_s is same for both the shafts. It depends $\frac{J}{R}$ – the polar modulus.

$$T_h \,\alpha\, \frac{J_h}{R} = \frac{\pi}{32} \frac{(do^4 - di^4)}{do} \times 2 = \frac{\pi}{16} \frac{do^4 \left[1 - \left(\frac{di}{do}\right)^4\right]}{do}$$

$$T_h \,\alpha\, \frac{\pi}{16} do^3 [1 - k^4] - (1), \text{ where } k = \frac{di}{do}$$

$$T_s \propto \frac{\pi}{16} d^3$$

$$\therefore \quad \frac{T_h}{T_s} = \frac{do^3[1-k^4]}{d^3} = \frac{do^3(1-k^4)}{(do^2 - di^2)^{3/2}}$$

$$= \frac{do^3\left[1-k^4\right]}{do^3\left[1-k^2\right]^{3/2}}$$

$$\frac{T_h}{T_s} = \frac{1-k^4}{(1-k^2)^{3/2}} = \frac{1-k^4}{(1-k^2)\sqrt{1-k^2}}$$

$$= \frac{(1+k^2)(1-k^2)}{(1-k^2)\sqrt{1-k^2}} = \frac{1+k^2}{\sqrt{1-k^2}}$$

$$\frac{T_h}{T_s} > 1$$

$\therefore$ Torque carrying capacity of the hollow shaft is more than that of the solid shaft, provided:

1. They are made of the same material.
2. They are of the same mass.
3. They are of the same length.

$$\boxed{\frac{T_h}{T_s} = \frac{1+k^2}{\sqrt{1-k^2}}}$$

Comparison of Stiffness

A solid shaft and hollow shaft made of the same material, the same mass, and length may be considered. It is required to compare their stiffness.

We have the equation, $\dfrac{T}{J} = \dfrac{N\theta}{l}$

$\dfrac{T}{\theta}$ = the stiffness = $\dfrac{NJ}{l}$; 'N' and 'l' being constant,

Stiffness $\propto J$

$(\text{Stiffness})_{\text{solid}} \propto J_s$

$(\text{Stiffness})_{\text{hollow}} \propto J_H$

$$\therefore \quad \frac{(\text{Stiffness})_{\text{hollow}}}{(\text{Stiffness})_{\text{solid}}} = \frac{J_u}{J_s} = \frac{\frac{\pi}{64} do^4(1-k^4)}{\frac{\pi}{64} d^4}$$

$$= \frac{do^4(1-k^4)}{d^4}$$

$\therefore$ But, $d^2 = do^2 - di^2$

$$\therefore \quad \frac{(\text{Stiffness})_{\text{hollow}}}{(\text{Stiffness})_{\text{solid}}} = \frac{do^4(1-k^4)}{(do^2-di^2)^2}$$

$$= \frac{do^4(1-k^4)}{do^4[1-k^2]^2}$$

$$= \frac{(1-k^2)(1+k^2)}{(1-k^2)^2} = \frac{1+k^2}{1-k^2} > 1$$

$\therefore$ Hollow shaft is more stiff when compared to the solid shaft, provided:

1. They are of the same mass.
2. They are of the same length.
3. They are of the same material.

Shaft Combinations

A shaft may consists of various small shafts of different cross-sectional areas or different materials.

The shaft combination may be:

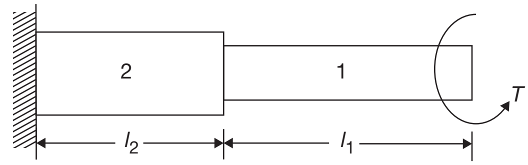
1. Shafts in series or stepped shafts.
2. Shafts in parallel or composite shafts.
3. Indeterminate shafts.

In analyzing these shafts, some points to be noted are:

1. At fixed end, a torque is developed to keep the shaft in equilibrium.
2. At the ends of any portion, the torque developed is equal and opposite.
3. At common point between two portions, angle of twist remains the same.

Shafts in Series

One end fixed and torque applied at the free end.



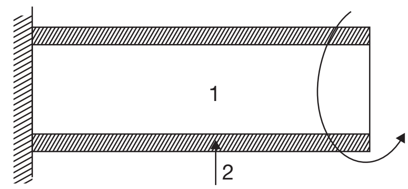
Here, torque transmitted by each shaft is same.

That is, $T_1 = T_2 = T$

Angle of twist:

$$\theta = \theta_1 + \theta_2 = \frac{T}{J}\left(\frac{l_1}{G_1} + \frac{l_2}{G_2}\right)$$

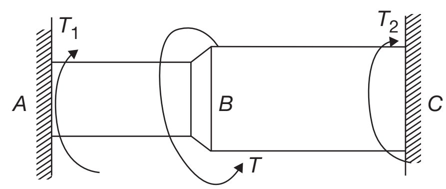
Shafts in Parallel



In this case, angle of twist is same for each shaft, that is, $\theta = \theta_1 = \theta_2$, and torque, $T = T_1 + T_2$.

Indeterminate Shafts

The shaft is fixed at both ends and torque is applied at a common point.



Torque T is applied at the point B torque T_1 and T_2 are developed at the ends.

Here, $T_1 + T_2 = T$ and $\theta_{1B} = \theta_{2B}$

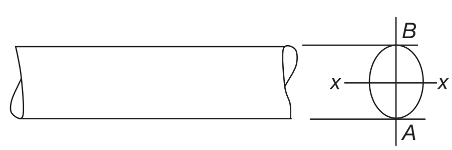
Torsion of a Tapering Shaft

It can be shown that the angle of twist in a tapering shaft of length L and end radii r_1 and r_2 when a constant torque T is acting, is

$$\theta = \frac{2TL}{3\pi G} \frac{r_1^2 + r_1 r_2 + r_2^2}{r_1^3 r_2^3}$$

Combined Bending and Torsion

A shaft is generally subjected to torsional shear stresses. But due to self weight, eccentric thrust, etc., there may be bending moments also.



Bending stresses and torsional shear stresses are maximum at the extreme fibres A and B.

Bending stress: $f = \frac{M}{I} y = \frac{32M}{\pi d^3}$

Shear stress: $q = \frac{T}{J} R = \frac{16T}{\pi d^3}$

Maximum principal stress:

$$p_1 = \frac{f}{2} + \sqrt{\left(\frac{f}{2}\right)^2 + q^2}$$

$$= \frac{1}{2}\left(\frac{32M}{\pi d^3}\right) + \sqrt{\frac{1}{4}\left(\frac{32M}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right]$$

Let, M_e be the equivalent bending moment.

Then, $p_{\max} = \frac{32M_e}{\pi d^3} = p_1$

$$\therefore M_e = \frac{1}{2}\left[M + \sqrt{M^2 + T^2}\right]$$

Maximum shear stress:

$$q_{\max} = \sqrt{\left(\frac{f}{2}\right)^2 + q^2} = \sqrt{\frac{1}{4}\left(\frac{32M}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

If T_e is the equivalent twisting moment, then:

$$q_{\max} = \frac{16T_e}{\pi d^3}$$

$$\therefore T_e = \sqrt{M^2 + T^2}.$$

Strain Energy in Torsion

Strain energy, $U = \frac{1}{2} T\theta$

It can be shown that $U = \frac{q_s{}^2}{4G} \times \text{Volume}$

SOLVED EXAMPLES

Example 1

A steel drive shaft 2 m long with 6 cm outer and 4 cm inner diameter transmits 150 kW at 1500 rpm.
Taking modulus of rigidity:

$$G = 8 \times 10^6 \text{ N/cm}^2$$

Determine the maximum shear stress.

Solution

Power transmitted $= \frac{2\pi NT}{60}$ W

i.e., $150 \times 10^3 = \frac{2\pi \times 1500 \times T}{60}$

$T = 954.93$ N/m

$= 95493$ N/cm

$$\frac{T}{J} = \frac{q_s}{R}$$

$$J = \frac{\pi}{32}(D^4 - d^4)$$

$$= \frac{\pi}{32}(6^4 - 4^4) = 102.1 \text{ cm}^4$$

$$= 102.1 \text{ cm}^4$$

$$\frac{T}{J} = \frac{95493}{102.1} = 935.29$$

Maximum shear stress:

$$q = \frac{TR}{J} = 935.29 \times 3 \text{ N/cm}^2$$

$$= 2805.87 \text{ N/cm}^2.$$

Example 2

In the above problem, find the angle of twist of the shaft.

Solution

$$\frac{T}{J} = \frac{G\theta}{L}$$

Angle of twist $\theta = \frac{TL}{JG} = 935.29 \times \frac{200}{8 \times 10^6}$

$= 0.0234$ radian

$= 1.34° \left(1 \text{ rad} = \frac{180°}{\pi}\right)$

Example 3

A hollow shaft is to transmit a torque 3500 N-m. The diametral ratio of the hollow shaft is 0.5. The permissible shear stress of the material is 80 MPa. The inside diameter of the shaft is:

(A) 28 m
(B) 31 mm
(C) 25.5 mm
(D) 35 mm

Solution

$$T = \frac{f_s \pi\ do^3 (1 - k^4)}{16}$$

$$700 \times 1000 = \frac{80 \times \pi\ do^3 (1 - 0.5^4)}{16}$$

$700{,}000 = \pi\, do^3\ [1 - 0.0625]$

$do = 61.94 = 62$ mm

$di = 31$ mm.

Hence, the correct answer is option (B).

NOTES

1. Torque carrying capacity of a shaft is represented by the polar modulus $\left(\frac{J}{R}\right)$.

   Stiffness of the shaft is represented by the polar moment of inertia (J).

2. Torque carried by solid shaft of diameter 'd'.

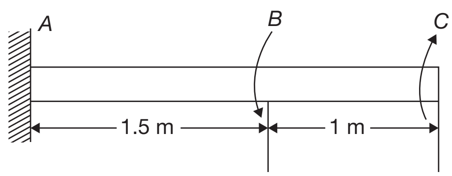
   $T = f_s \frac{\pi d^3}{16}$

   Torque carried by hollow shaft:

   $T = f_s \frac{\pi do^3}{16}(1-k^4)$, where k-diameter ratio $\frac{di}{do}$ for the hollow shaft.

Example 4

A 2.5 m long steel shaft of circular cross-section is subjected to torques as shown in the following figure.



Torque at B = 500 Nm (anti-clockwise).
Torque at C = 1000 Nm (clockwise).
Determine the diameter of the shaft if permissible shear stress is 6000 kN/m^2.

Modulus of rigidity = 80 GN/m^2

Solution

$$\frac{T}{J} = \frac{q_s}{R}, \text{ or } q_s = \frac{16T}{\pi D^3}$$

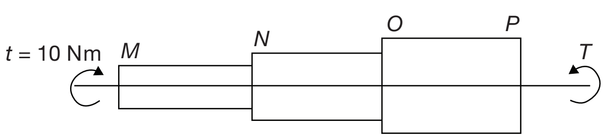
Maximum value of torque is to be considered for selecting the diameter.
For section BC, torque is 1000 Nm.
For section AB, torque is 1000 – 500 = 500 Nm

$$D^3 = \frac{16T}{\pi \times q_s} = \frac{16\times 1000}{\pi \times 6000\times 10^3} = 0.849\times 10^{-3}$$

D = 0.0947 m = 9.47 cm.

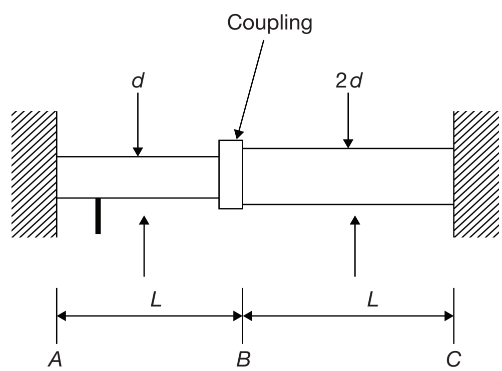
Exercises

1. A torque of 10 Nm is transmitted through a stepped shaft as shown in the figure. The torsional stiffnesses of individual sections of lengths MN, NO and OP are 20 N m/rad, 30 N m/rad and 60 Nm/rad respectively. The angular deflection between the ends M and P of the shaft is



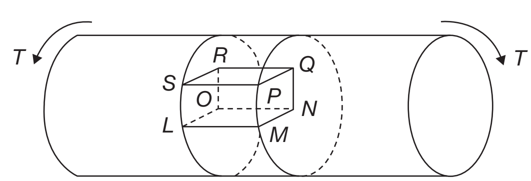
   (A) 0.5 rad
   (B) 1.0 rad
   (C) 5.0 rad
   (D) 10.0 rad

2. The two shafts AB and BC, of equal length and diameters d and $2d$, are made of the same material. They are joined at B through a shaft coupling, while the ends A and C are built-in (cantilevered). A twisting moment T is applied to the coupling. If T_A and T_C represent the twisting moments at the ends A and C, respectively, then



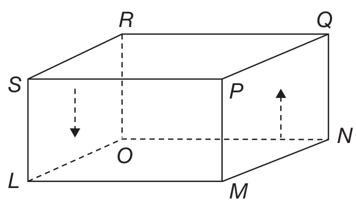
   (A) $T_C = T_A$ (B) $T_C = 8T_A$
   (C) $T_C = 16T_A$ (D) $T_A = 16T_C$

3. A shaft is subjected to torsion as shown in the following figure:

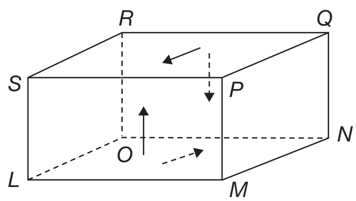


Which of the following figures represents the shear stress on the element *LMNOPQRS*?

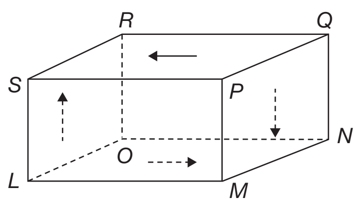
(A)



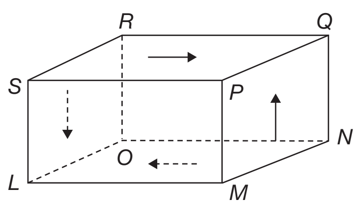
(B)



(C)



(D)



4. A long shaft of diameter d is subjected to twisting moment T at its ends. The maximum normal stress acting at its cross-section is equal to

(A) zero (B) $\dfrac{16T}{\pi d^3}$

(C) $\dfrac{32T}{\pi d^3}$ (D) $\dfrac{64T}{\pi d^3}$

5. A circular solid shaft of span L = 5 m is fixed at one end and free at the other end. A twisting moment T = 100 kN-m is applied at the free end. The torsional rigidity GJ per unit angular twist is 50000 kN-m^2/rad. Following statements are made for this shaft:

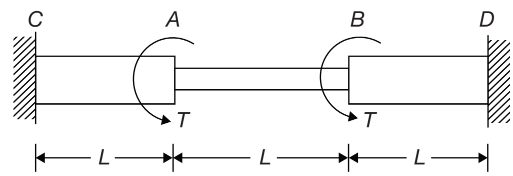
I. The maximum rotation is 0.01 rad

II. The torsional strain energy is 1 kN-m.

With reference to the above statements, which of the following applies?

(A) Both statements are true
(B) Statement I is true but II is false
(C) Statement II is true but I is false
(D) Both the statements are false

6. A circular shaft shown in the figure is subjected to torsion T at two points A and B. The torsional rigidity of portions CA and BD is GJ_1 and that portion AB is GJ_2. The rotations of shaft at points A and B are θ_1 and θ_2. The rotation θ_1 is



(A) $\dfrac{TL}{GJ_1 + GJ_2}$ (B) $\dfrac{TL}{GJ_1}$

(C) $\dfrac{TL}{GJ_2}$ (D) $\dfrac{TL}{GJ_1 - GJ_2}$

7. For a power transmission shaft transmitting power P at N rpm, its diameter is proportional to

(A) $\left(\dfrac{P}{N}\right)^{1/3}$ (B) $\left(\dfrac{P}{N}\right)^{1/2}$

(C) $\left(\dfrac{P}{N}\right)^{2/3}$ (D) $\left(\dfrac{P}{N}\right)$

8. A circular section rod ABC is fixed at ends A and C. It is subjected to torque T at B. $AB = BC = L$ and the polar moment of inertia of portions AB and BC are $2J$ and J respectively. If G is the modulus of rigidity, what is the angle of twist at point B?

(A) $\dfrac{TL}{3GJ}$ (B) $\dfrac{TL}{2GJ}$

(C) $\dfrac{TL}{GJ}$ (D) $\dfrac{2TL}{GJ}$

9. The diameter of a solid shaft made of mild steel, rotating at 250 rpm, is 45 mm. The shaft is designed to transmit 50 kW. What will be the factor of safety if the ultimate shear stress at yield is 427 N/mm^2?

(A) 6 (B) 5
(C) 4 (D) 3

10. The bending moment (M) and twisting moment (T) at four particular sections P, Q, R and S along the length of a shaft are as follows:

Section	P	Q	R	S
M(Nm)	10	40	20	15
T(Nm)	45	30	50	40

Which section is to be considered for designing the shaft?

(A) P (B) Q
(C) R (D) S

11. A shaft is subjected to simultaneous action of a torque T, bending moment M and an axial thrust F. Which one of the following statements is correct for this situation?

(A) One extreme end of the vertical diameter fibre is subjected to maximum compressive stress only.
(B) The opposite extreme end of the vertical diameter fibre is subjected to tensile/compressive stress only.
(C) Every point of the surface of that shaft is subjected to maximum shear stress only.
(D) Axial longitudinal fibre of the shaft is subjected to compressive stress only.

12. A circular shaft subjected to torsion undergoes a twist of 1° in a length of 1.6 m. If the maximum shear stress induced is 10,000 N/cm^2 and if modulus of rigidity is 8 $\times 10^6$ N/cm^2 then radius of the shaft is

(A) $\frac{27}{\pi}$ cm (B) $\frac{36}{\pi}$ cm
(C) $\frac{\pi}{36}$ cm (D) $\frac{\pi}{27}$ cm

Direction for questions 13 and 14:

A circular bar made of CI is to resist on occasional torque of 2.2 kN-m acting in transverse plane. The allowable stresses in compression, tension and shear are 100, 50, 35 MN/m^2 respectively.
(take $G = 40$ GN/m^2)

13. The diameter of the bar will be
(A) 64.8 mm (B) 68.4 mm
(C) 66.8 mm (D) 67.4 mm

14. The angle of twist under the applied torque per metre length of bar will be
(A) 1.86° (B) 1.26°
(C) 1.46° (D) 1.16°

15. A shaft subjected to a maximum bending stress of 80 N/mm^2 and maximum shearing stress equal to 30 N/mm^2 at a particular section. If the yield point in the tension of the material is 280 N/mm^2, and maximum shear stress theory of failure is used, then the factor of safety obtained will be
(A) 2.5 (B) 2.8
(C) 3.0 (D) 3.5

16. A circular shaft was subjected to torsion initially and then subjected to a bending moment. If the maximum bending stress and maximum torsional shear stress had same value, ratio of torque applied to bending moment is

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$
(C) $\frac{3}{2}$ (D) 2

17. A circular shaft of 100 mm diameter and 2 m length with modulus of rigidity 80 kN/mm^2 develops a maximum shear stress of 70 N/mm^2 while transmitting power. Strain energy stored by the shaft in joules is
(A) 186.6 (B) 211.4
(C) 240.5 (D) 252.3

18. A shaft with length to diameter ratio is 15 to transmit a torque of 35 kN-m. If the angle of twist for the shaft is not to exceed 1°, diameter of the shaft (in mm) required is ______.

(Assume shear modulus = 80 GN/m^2)

19. A circular shaft is subjected to a bending moment and a twisting moment. If the maximum principal stress is numerically twice the minimum principal stress, ratio of bending moment to twisting moment is

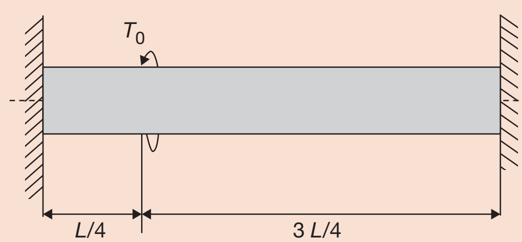
(A) $\frac{1}{\sqrt{8}}$ (B) $\frac{2}{\sqrt{8}}$
(C) $\frac{1}{\sqrt{6}}$ (D) $\frac{2}{\sqrt{6}}$

Previous Years' Questions

1. The maximum and minimum shear stresses in a hollow circular shaft of outer diameter 20 mm and thickness 2 mm, subjected to a torque of 92.7 Nm will be **[GATE, 2007]**
(A) 59 MPa and 47.2 MPa
(B) 100 MPa and 80 MPa
(C) 118 MPa and 160 MPa
(D) 200 MPa and 160 MPa

2. The maximum shear stress in a solid shaft of circular cross-section having diameter d is subjected to a torque T is τ. If the torque is increased by four times and the diameter of the shaft is increased by two times, the maximum shear stress in the shaft will be **[GATE, 2008]**
(A) 2τ (B) τ
(C) $\tau/2$ (D) $\tau/4$

3. A solid shaft of diameter, d and length L is fixed at both the ends. A torque, T_0 is applied at a distance, $L/4$ from the left end as shown in the figure given below:



The maximum shear stress in the shaft is **[GATE, 2009]**

(A) $\frac{16T_0}{\pi d^3}$ (B) $\frac{12T_0}{\pi d^3}$

(C) $\frac{8T_0}{\pi d^3}$ (D) $\frac{4T_0}{\pi d^3}$

4. A hollow circular shaft has an outer diameter of 100 mm and a wall thickness of 25 mm. The allowable shear stress in the shaft is 125 MPa. The maximum torque the shaft can transmit is **[GATE, 2009]**

(A) 46 kN-m (B) 24.5 kN-m
(C) 23 kN-m (D) 11.5 kN-m

5. A solid circular shaft of diameter d and length L is fixed at one end and free at the other end. A torque T is applied at the free end. The shear modulus of the material is G the angle of twist at the free end is **[GATE, 2010]**

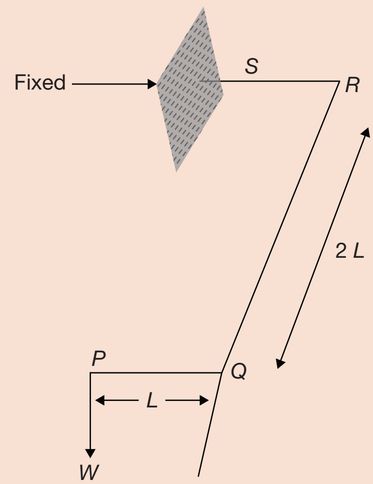
(A) $\frac{16TL}{\pi d^4 G}$

(B) $\frac{32TL}{\pi d^4 G}$

(C) $\frac{64TL}{\pi d^4 G}$

(D) $\frac{128TL}{\pi d^4 G}$

6. For the cantilever bracket, $PQRS$, loaded as shown in the a following figure ($PQ = RS = L$, and, $QR = 2L$), which of the following statements is FALSE? **[GATE, 2011]**



(A) The portion RS has a constant twisting moment with a value of $2WL$.
(B) The portion QR has a varying twisting moment with a maximum value of WL.
(C) The portion PQ has a varying bending moment with a maximum value of WL.
(D) The portion PQ has no twisting moment.

7. A hollow shaft of 1 m length is designed to transmit a power of 30 kW at 700 rpm. The maximum permissible angle of twist in the shaft is 1°. The inner diameter of the shaft is 0.7 times the outer diameter. The modulus of rigidity is 80 GPa. The outside diameter (in mm) of the shaft is ______. **[GATE, 2015]**

8. A hollow shaft ($d_0 = 2d_i$ where d_0 and d_i are the outer and inner diameters respectively) needs to transmit 20 kW power at 3000 rpm. If the maximum permissible shear stress is 30 MPa, d_0 is **[GATE, 2015]**

(A) 11.29 mm (B) 22.58 mm
(C) 33.87 mm (D) 45.16 mm

Answer Keys

Exercises

1. B	**2.** C	**3.** D	**4.** A	**5.** B	**6.** B	**7.** A	**8.** A	**9.** C	**10.** B
11. D	**12.** B	**13.** B	**14.** C	**15.** B	**16.** D	**17.** C	**18.** 156 to 157		**19.** A

Previous Years' Questions

1. B	**2.** C	**3.** B	**4.** C	**5.** B	**6.** B	**7.** 43 to 45	**8.** B