CHAPTER 07

Energy Methods of Analysis

7.1 Application of Minimum Potential Energy

"Among the all Geometrically compatible state of structure which satisfy deflection boundary condition and force equilibrium requirement will have final stable condition when its total potential energy is minimum".

Consider a portal frame shown in figure 7.1.

If reactions at support A are H_A, R_A and M_A and reaction at support B is R_B. It is clear that above portal frame is indeterminate and degree of indeterminacy $D_S = 4 - 3 = 1$

Assuming vertical reaction R_B as redundant (say R). For any value of R, the reactions H_A, R_A, and M_A can be calculated by the conditions of equilibrium. But true value of redundant will be when for which the total potential energy is minimum.

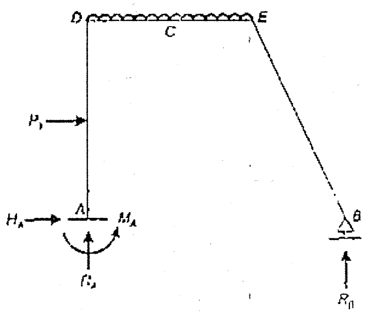


Fig.7.1

If U is total strain energy stored in frame. Then total strain energy will be minimum, when,

$$\frac{\partial U}{\partial R} = 0 \quad \text{(Compatibility condition)}$$

It is an application of castigliano's theorem and based on principal of least work.

NOTE: Strain energy method is a force method which is suitable when redundants are less i.e., D_S is less.

7.2 Algorithm for Analysis

(*i*) Find degree of static indeterminacy D_S and identify redundants i.e. unknown reactions or member forces.

(*ii*) Find other reactions by using equations of equilibrium in terms of redundants.

(*iii*) Consider the redundants as variable and use compatibility of minimum potential energy with respect to redundant

- If only one reaction is redundant (say R)

$$\frac{\partial U}{\partial R} = 0 \quad \ldots(i)$$

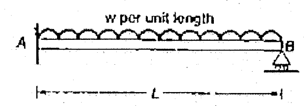
- If two reactions are redundants (say R and M)

$$\frac{\partial U}{\partial R} \text{ and } \frac{\partial U}{\partial M} = 0 \quad \ldots(ii)$$

(*iv*) Solving above compatibility conditions find true value of redundants.

(*v*) Using all reactions, find B.M. and S.F. in each member and draw corresponding BMD and SFD.

Example 7.1 A propped cantilever of span L carries a uniformly distributed load of 'w' per unit length over the entire span. Find the propped reaction at B. Use principle of minimum potential energy. Consider effect of bending only.



Solution:

Let R be the propped reaction at B. The BM at a section x from B is given by,

$$M_x = Rx - \frac{wx^2}{2}$$

Total strain energy stored in beam,

$$U = \int \frac{M_x^2\,dx}{2EI} = \int_0^L \frac{\left(Rx - \frac{wx^2}{2}\right)^2 dx}{2EI}$$

The true value of redundant will be when, the total strain energy stored in beam is minimum

$$\therefore \quad \frac{\partial U}{\partial R} = 0$$

$$\Rightarrow \quad \frac{1}{2EI}\int_0^L 2\left(Rx - \frac{wx^2}{2}\right)\cdot x\,dx = 0$$

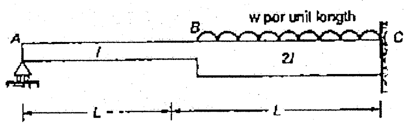
$$\therefore \quad \int_0^L Rx^2 dx - \int_0^L \frac{wx^3}{2}dx = 0$$

$$\Rightarrow \quad R\left[\frac{x^3}{3}\right]_0^L - w\left[\frac{x^4}{8}\right]_0^L = 0$$

$$\Rightarrow \quad \frac{RL^3}{3} - \frac{wL^4}{8} = 0$$

$$\Rightarrow \quad R = \frac{3}{8}wL\ (\uparrow)$$

Example 7.2 A stepped beam ABC, simply supported at A and fixed at C as shown in the figure carries a uniformly distributed load of intensity 'w' per unit length over BC. Determine the vertical reaction at A using energy method.



Solution:

We know that strain energy stored in the beam,

$$U = \int \frac{M^2 dx}{2EI}$$

Total strain energy stored in beam $U = U_{AB} + U_{BC}$

Portion AB:

$$M_x(x \text{ from } A) = R \cdot x \qquad [0 \le x \le L]$$

$$\therefore \quad U_{AB} = \int_0^L \frac{(R \cdot x)^2 dx}{2EI} = \frac{R^2}{2EI}\int_0^L x^2 dx$$

$$\Rightarrow \quad U_{AB} = \frac{R^2}{2EI}\left[\frac{x^3}{3}\right]_0^L = \frac{R^2L^3}{6EI}$$

$$\Rightarrow \quad \frac{\partial U_{AB}}{\partial R} = \frac{RL^3}{3EI} \qquad ...(i)$$

Portion BC:

$$M_x(x \text{ from } B) = R(L+x) - \frac{wx^2}{2} \qquad [0 \le x \le L]$$

$$\therefore \quad U_{BC} = \int_0^L \frac{\left[R(L+x) - \frac{wx^2}{2}\right]^2 dx}{2E \cdot 2I}$$

$$\Rightarrow \quad U_{BC} = \frac{1}{4EI}\int_0^L \left[R(L+x) - \frac{wx^2}{2}\right]^2 dx$$

$$\Rightarrow \quad \frac{\partial U_{BC}}{\partial R} = \frac{2}{4EI}\int_0^L \left[R(L+x) - \frac{wx^2}{2}\right](L+x)dx$$

$$\Rightarrow \quad \frac{\partial U_{BC}}{\partial R} = \frac{1}{4EI}\int_0^L [2R(L+x)^2 - wx^2(L+x)]dx$$

$$\Rightarrow \quad \frac{\partial U_{BC}}{\partial R} = \frac{1}{4EI}\int_0^L [2R(L^2 + x^2 + 2Lx) - wLx^2 - wx^3]dx$$

$$\Rightarrow \quad \frac{\partial U_{BC}}{\partial R} = \frac{1}{4EI}\left[2R\left[L^2x + \frac{x^3}{3} + 2\frac{Lx^2}{2}\right]_0^L - wL\left[\frac{x^3}{3}\right]_0^L - w\left[\frac{x^4}{4}\right]_0^L\right]$$

$$\Rightarrow \quad \frac{\partial U_{BC}}{\partial R} = \frac{1}{4EI}\left[2R\left[L^3 + \frac{L^3}{3} + L^3\right] - \frac{wL^4}{3} - \frac{wL^4}{4}\right]$$

$$\Rightarrow \quad \frac{\partial U_{BC}}{\partial R} = \frac{1}{4EI}\left[\frac{14RL^3}{3} - \frac{7wL^4}{12}\right] \qquad ...(ii)$$

According to the principle of minimum potential energy, the true value of redundant will be when, total potential energy stored in a beam is minimum.

$$\therefore \quad \frac{\partial U}{\partial R} = 0$$

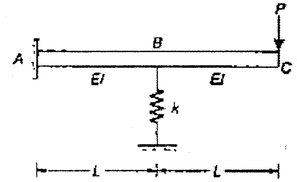
$$\Rightarrow \quad \frac{\partial U_{AB}}{\partial R} + \frac{\partial U_{BC}}{\partial R} = 0 \qquad [\because U = U_{AB} + U_{BC}]$$

From equation (*i*) and (*ii*),

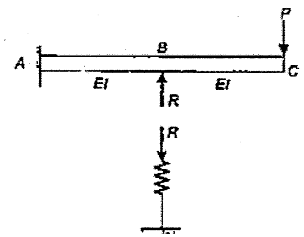
$$\Rightarrow \quad \frac{RL^3}{3EI} + \frac{1}{4EI}\left[\frac{14RL^3}{3} - \frac{7wL^4}{12}\right] = 0$$

$$\Rightarrow \quad R = \frac{7wL}{72}(\uparrow)$$

Example 7.3 Using the force (flexibility/compatibility) method, analyse the structure shown in figure below:



Solution:



Let the stiffness of spring be K.

Total strain energy stored in system,

$$U = U_{AB} + U_{BC} + U_{spring}$$

For true value of R,

$$\frac{\partial U}{\partial R} = \frac{\partial U_{AB}}{\partial R} + \frac{\partial U_{BC}}{\partial R} + \frac{\partial U_{spring}}{\partial R} \qquad ...(i)$$

Strain energy in portion BC:

$$M_x(x \text{ from } C) = -P \cdot x \qquad [0 \le x \le L]$$

$$\therefore \quad U_{BC} = \int_0^L \frac{(-Px)^2 dx}{2EI} = \frac{P^2}{2EI}\int_0^L x^2 dx$$

$$\Rightarrow \quad U_{BC} = \frac{P^2L^3}{6EI}$$

$$\Rightarrow \quad \frac{\partial U_{BC}}{\partial R} = 0 \qquad ...(iii)$$

Strain energy in portion AB:

$$M_x(x \text{ from } C) = -P \cdot x + R(x - L) \qquad [L \le x \le 2L]$$

$$\therefore \quad U_{AB} = \int_L^{2L} \frac{\{-Px+R(x-L)\}^2\,dx}{2EI}$$

$$\Rightarrow \quad \frac{\partial U_{AB}}{\partial R} = \int_L^{2L} \frac{2\{-Px+R(x-L)\}\cdot(x-L)}{2EI}\,dx$$

$$\Rightarrow \quad \frac{\partial U_{AB}}{\partial R} = \int_L^{2L} \frac{1}{EI}\left[-Px(x-L)+R(x-L)^2\right]dx$$

$$\Rightarrow \quad \frac{\partial U_{AB}}{\partial R} = \frac{1}{EI}\left[-\frac{Px^3}{3}+\frac{PLx^2}{2}+\frac{R(x-L)^3}{3}\right]_L^{2L}$$

$$\Rightarrow \quad \frac{\partial U_{AB}}{\partial R} = \frac{1}{EI}\left[-\frac{8PL^3}{3}+2PL^3+\frac{RL^3}{3}+\frac{PL^3}{3}-\frac{PL^3}{2}-0\right]$$

$$\Rightarrow \quad \frac{\partial U_{AB}}{\partial R} = \frac{1}{EI}\left[\frac{14PL^3-19PL^3}{6}+\frac{RL^3}{3}\right]$$

$$\Rightarrow \quad \frac{\partial U_{AB}}{\partial R} = \frac{1}{EI}\left[\frac{RL^3}{3}-\frac{5PL^3}{6}\right] \qquad ...(iii)$$

Strain energy stored in spring:

$$U_{spring} = \frac{R^2}{2k}$$

$$\therefore \quad \frac{\partial U_{spring}}{\partial R} = \frac{2R}{2k} = \frac{R}{k} \qquad ...(iv)$$

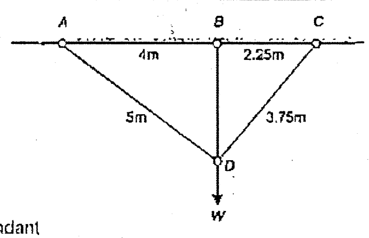
From equation (*i*), (*ii*), (*iii*) and (*iv*),

$$\Rightarrow \quad \frac{\partial U_{AB}}{\partial R}+\frac{\partial U_{BC}}{\partial R}+\frac{\partial U_{spring}}{\partial R} = 0$$

$$\Rightarrow \quad \frac{1}{EI}\left[\frac{RL^3}{3}-5\frac{PL^3}{6}\right]+0+\frac{R}{k} = 0$$

$$\Rightarrow \quad R = \frac{5PL^3}{2L^3-\frac{6EI}{k}}$$

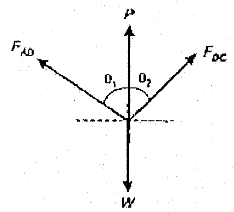
Example 7.4 Three wires *AD*, *BD* and *CD* having the same cross-sectional area and of the same material, support a load '*W*' as shown in figure. Determine the tension in the three wires shown. Also show that the horizontal movement of *D* equals one-seventh of the extension of BD.



Solution:

(i) **Force in wires:** Let axial force in wire *BD* is redundant say $F_{BD} = P$ (tension)

Consider equilibrium of joint *D*,



Here, $\sin\theta_1 = \frac{4}{5} = 0.8$

and $\sin\theta_2 = \frac{2.25}{3.75} = 0.6$

$\cos\theta_1 = \frac{3}{5} = 0.6$

and $\cos\theta_2 = \frac{3}{3.75} = 0.8$

$$\Sigma F_x = 0$$

$$\Rightarrow \quad F_{DC}\sin\theta_2 - F_{AD}\sin\theta_1 = 0$$

$$\Rightarrow \quad F_{DC}\sin\theta_2 = F_{AD}\sin\theta_1$$

$$\Rightarrow \quad F_{DC}\times 0.6 = F_{AD}\times 0.8$$

$$\Rightarrow \quad 0.75F_{DC} = F_{AD} \qquad ...(i)$$

$$\Sigma F_y = 0$$

$$\Rightarrow \quad P+F_{AD}\cos\theta_1+F_{DC}\cos\theta_2-W = 0$$

$$\Rightarrow \quad P+F_{AD}\times 0.6+F_{DC}\times 0.8 = W$$

$$\Rightarrow \quad 0.6F_{AD}+0.8F_{DC} = W-P \qquad ...(ii)$$

From eq. (i) and (ii),

$$\Rightarrow \quad 0.6\times 0.75F_{DC}+0.8F_{DC} = W-P$$

$$\Rightarrow \quad 0.45F_{DC}+0.8F_{DC} = W-P$$

$$\Rightarrow \quad 1.25F_{DC} = W-P$$

$$\Rightarrow \quad F_{DC} = \frac{(W-P)}{1.25} = 0.8(W-P)$$

From eq. (i), we get

$$F_{AD} = 0.75\times\frac{(W-P)}{1.25} = 0.6(W-P)$$

Total strain energy stored in system,

$$U = U_{AD}+U_{BD}+U_{CD}$$

$$= \frac{F_{AD}^2\times L_{AD}}{2AE}+\frac{F_{BD}^2\times L_{BD}}{2AE}+\frac{F_{DC}^2\times L_{DC}}{2AE}$$

$$= \frac{1}{2AE}\left[[0.6(W-P)]^2\times 5+P^2\times 3+[0.8(W-P)]^2\times 3.75\right]$$

$$= \frac{1}{2AE}\left[1.8(W-P)^2+3P^2+2.4(W-P)^2\right]$$

$$= \frac{1}{2AE}\left[4.2(W-P)^2+3P^2\right]$$

The true value of redundant force will be that for which total strain energy stored in system is minimum

$$\therefore \quad \frac{\partial U}{\partial P} = 0$$

$\Rightarrow \quad \frac{1}{2AE}\left[4.2\times 2(W-P)\times -1+3\times 2P\right]=0$

$\Rightarrow \quad -8.4\,(W-P)+6P=0$

$\Rightarrow \quad P=0.583\,W(T)$

Hence force in member AD,

$F_{AD}=0.6\,(W-P)$

$=0.6\,(W-0.583\,W)=0.250\,W(T)$

Force in member DC,

$F_{DC}=0.8\,(W-P)$

$=0.8\,(W-0.583\,W)=0.333\,W(T)$

Thus force in members are

$F_{BD}=0.583\,W(T)$

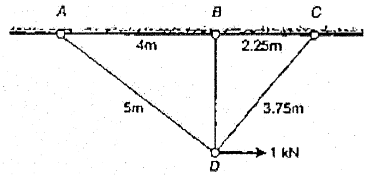
$F_{AD}=0.250\,W(T)$

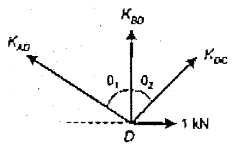
$F_{DC}=0.333\,W(T)$

(ii) Vertical movement of point D, Δ_{VD} = extension in wire BD

$$=\frac{P_{BD}\times L_{BD}}{AE}=\frac{0.583W\times 3}{AE}=\frac{1.75W}{AE} \qquad \text{...(i)}$$

For horizontal movement of D, remove external loading and apply unit load in horizontal direction as shown below:





Consider equilibrium of joint D,

Let force k_{BD} is redundant,

$k_{BD}=k$ (say)

$\Sigma F_x=0$

$\Rightarrow \quad 1+k_{DC}\sin\theta_2-k_{AD}\sin\theta_1=0$

$\Rightarrow \quad 0.6\,k_{DC}-0.8\,k_{AD}=-1 \qquad \text{...(A)}$

$\Sigma F_y=0$

$\Rightarrow k_{AD}\cos\theta_1+k+k_{DC}\times\cos\theta_2=0$

$\Rightarrow \quad 0.6k_{AD}+k+0.8\,k_{DC}=0$

$\Rightarrow \quad 0.6k_{AD}=-0.8\,k_{DC}-k$

$\Rightarrow \quad k_{AD}=\frac{-0.8k_{DC}-k}{0.6}=-1.33\,k_{DC}-1.67\,k$

From equation (A)

$$0.6k_{DC}-0.8\left[\frac{-0.8k_{DC}-k}{0.6}\right]=-1$$

$\Rightarrow \quad k_{DC}=\frac{-(1+1.33k)}{1.667}=-(0.6+0.8k)$

$\therefore \quad k_{AD}=-1.33\times\frac{-(1+1.33k)}{1.667}-1.67k=0.80-0.6\,k$

Strain energy stored in system,

$$U=\frac{(0.6+0.8k)^2\times 5}{2AE}+\frac{k^2\times 3}{2AE}+\frac{(0.80-0.6k)^2\times 3.75}{2AE}$$

$$=\frac{1}{2AE}\left[5(0.6+0.8k)^2+3k^2+3.75(0.8-0.6k)^2\right]$$

For true value of k, $\quad \frac{\partial U}{\partial k}=0$

$$\therefore \frac{\partial U}{\partial k}=\frac{1}{2AE}\left[5\times 2(0.6+0.8k)\times 0.8+3\times 2k+3.75\times 2(0.8-0.6k)\times(-0.6)\right]=0$$

$\Rightarrow 8(0.6+0.8\,k)+6\,k-4.5\,(0.8-0.6\,k)=0$

$\Rightarrow \quad 1.2+15.1k=0$

$\Rightarrow \quad k=-0.08$ kN (C)

Hence, $\quad k_{AD}=0.80+0.6\times 0.08=0.848$ kN

$k_{DC}=-(0.6+0.8\,k)=-(0.6-0.8\times 0.08)=-0.536$ kN

Horizontal deflection of joint D

$$\Delta_{HD}=\sum\frac{PkL}{AE}$$

$$\Rightarrow \quad AE\Delta_{HD}=(P_{AD}\times k_{AD}\times L_{AD})+(P_{BD}\times k_{BD}\times L_{BD})+(P_{DC}\times k_{DC}\times L_{DC})$$

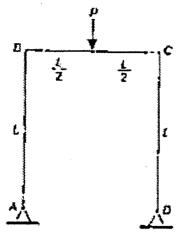
$$=(0.25W\times 0.848\times 5)-(0.583W\times 0.08\times 3)-(0.333W\times 0.536\times 3.75)$$

$\therefore \quad \Delta_{HD}=\frac{0.25W}{AE}$

Hence, $\quad \frac{\Delta_{VD}}{\Delta_{HD}}=\frac{\frac{1.75W}{AE}}{\frac{0.25W}{AE}}=7$

$\therefore \quad \Delta_{HD}=\frac{1}{7}\Delta_{VD}$ (Proved)

Example 7.5 Analyse the portal frame shown in figure and draw BMD. Use strain energy method.



Solution:

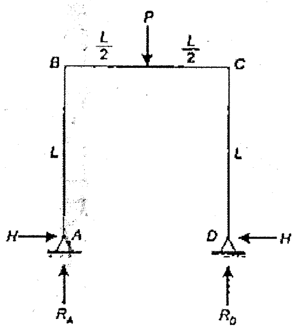
For portal frame,

$D_S=4-3=1$

Take horizontal reaction H as redundant.

For true value of redundant, the strain energy stored in frame should be minimum

$$\therefore \quad \frac{\partial U}{\partial H} = \int \frac{2M_x \frac{\partial M_x}{\partial H} ds}{2EI} = 0$$



$$\Rightarrow \quad \int \frac{M_x \frac{\partial M_x}{\partial H} ds}{2EI} = 0 \quad ...(i)$$

if there are more than one members, then

$$\sum \int \frac{M_x \frac{\partial M_x}{\partial H} ds}{2EI} = 0$$

$$\Sigma F_x = 0$$

$$\Rightarrow \quad H_A = H_D = H \text{ (say)}$$

$$\Sigma F_y = 0$$

$$\Rightarrow \quad R_A + R_B = P \quad ...(ii)$$

Also,

$$\Sigma M_D = 0$$

$$\Rightarrow \quad R_A \times L - P \times \frac{L}{2} = 0$$

$$\Rightarrow \quad R_A = \frac{P}{2} \text{ and } R_D = \frac{P}{2}$$

Take outer face as reference.

Member	M_x	$\frac{\partial M_x}{\partial H}$	Limit
AB	$-Hy$	$-y$	0 to L
BE	$\frac{P}{2}x - HL$	$-L$	0 to L/2

Now,

$$\sum \int \frac{M_x \frac{\partial M_x}{\partial H} ds}{2EI} = 0$$

$$\Rightarrow \quad \frac{1}{EI} \times 2\left[\int_0^L (-H \cdot y) \cdot (-y) dy + \int_0^{L/2} \left(\frac{P}{2}x - HL\right)(-L) dx\right] = 0$$

$$\Rightarrow \quad H \int_0^L y^2 dy - L \int_0^{L/2} \left(\frac{P}{2}x - HL\right) dx = 0$$

$$\Rightarrow \quad \frac{HL^3}{3} - L \times \left[\frac{P}{2}\frac{x^2}{2} - HLx\right]_0^{L/2} = 0$$

$$\Rightarrow \quad \frac{HL^3}{3} - \frac{PL^3}{16} + \frac{HL^3}{2} = 0$$

$$\Rightarrow \quad H = \frac{3}{40}P$$

Bending moment diagram:

Portion AB:

$$M_y \text{ (y from A)} = -Hy \text{ (linear)} \qquad [0 \le y \le L]$$

at $y = 0$; $\quad M_A = 0$

at $y = L$; $\quad M_B = -H \cdot L = -\frac{3}{40}PL$

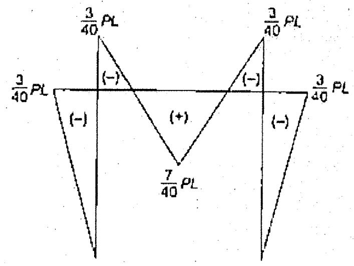
Portion BE:

$$M_x \text{ (x from B)} = \frac{P}{2}x - H \cdot L \qquad \left[0 < x \le \frac{L}{2}\right]$$

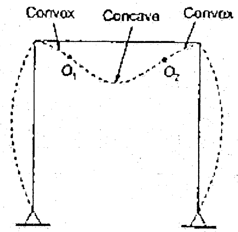
$$\Rightarrow \quad M_x = \frac{P}{2}x - \frac{3}{40}PL \text{ (linear)}$$

at $x = 0$; $\quad M_B = -\frac{3}{40}PL$

at $x = L/2$; $\quad M_E = \frac{P}{2} \times \frac{L}{2} - \frac{3P}{40} \times L = \frac{7}{40}PL$



BMD



Elastic Curve

7.3 Unit Load Method

Unit load method also referred to as method of virtual work was developed by John Bernoulli in 1717. In this method a virtual unit load is applied at point of deflection (Δ) and in the direction of deflection.

By applying principal of work and energy.

External virtual work = Internal virtual work

$$\Delta = \int_0^L \frac{Mm_1}{EI} dx \quad ...(i)$$

where structure consist more than one members, then

$$\Delta = \sum \frac{Mm_1}{EI} dx$$

where, M = Bending moment at any section due to given loading.

m_1 = Bending moment at any section is applied, when external loading is removed and a unit load is applied at a point where deflection is required

The unit load method can also be used to find rotation at any point in a structure. Rotation at any point can be found by

$$\theta = \sum \frac{Mm_2}{EI} dx$$

where, M = Bending moment at any section due to given loading.

m_2 = Bending moment at any section, when external loading is removed and a unit concentrated moment is applied at a point where rotation is required.

7.3.1 Algorithm to Find Slope/Deflection

(i) Find external support reactions.

(ii) Find expression of M for all members due to given loading.

(iii) Remove external loading and apply unit load or unit moment at that point where slope or deflection is required.

(iv) Find expression of m for all members due to unit load or unit moment.

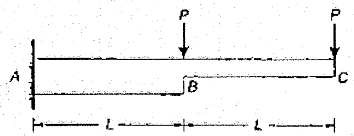
(v) Apply formulae for slope and deflection.

$$\Delta = \sum \frac{Mm_1 dx}{EI}$$

$$\theta = \sum \frac{Mm_2}{EI} dx$$

NOTE: Unit load method can be applied for both prismatic and non-prismatic members.

Example 7.6 Determine the deflection and rotation at the free end of the cantilever shown in figure. Use unit load method.



Solution:

For given beam,

Deflection at free end is given by

Portion	CB	BA
Origin	C	B
Limit	0 – L	0 – L
M	–Px	–P(L + x) + Px
m_1	–x	–(x + L)
m_2	–1	–1
MOI	I	2 I

$$\Delta = \sum \frac{Mm_1}{EI} dx$$

$$\Delta = \int_0^L \frac{(-Px)(-x)}{EI} dx + \int_0^L \frac{[P(L+x)+Px](x+L)}{E(2I)} dx$$

$$= \int_0^L \frac{Px^2 dx}{EI} + \int_0^L \frac{[P(x+L)^2 + Px(x+L)] dx}{2EI}$$

$$= \int_0^L \frac{Px^2 dx}{EI} + \int_0^L \frac{P}{2EI}[x^2 + L^2 + 2Lx + x^2 + Lx]dx$$

$$= \int_0^L \frac{Px^2 dx}{EI} + \int_0^L \frac{P}{2EI}[2x^2 + 3Lx + L^2]dx$$

$$= \frac{P}{2EI}\left[\int_0^L 2x^2 dx + \int_0^L (2x^2 + 3Lx + L^2)dx\right]$$

$$= \frac{P}{2EI}\left[2\left[\frac{x^3}{3}\right]_0^L + \left[2\cdot\frac{x^3}{3} + 3\cdot\frac{Lx^2}{2} + L^2\cdot x\right]_0^L\right]$$

$$= \frac{P}{2EI}\left[2\cdot\frac{L^3}{3} + \frac{2}{3}L^3 + \frac{3}{2}L^3 + L^3\right]$$

$$\Delta = \frac{23PL^3}{12EI}$$

Rotation at free end is given by,

$$\theta = \sum \frac{Mm_2}{EI} dx$$

$$= \int_0^L \frac{(-Px)(-1)dx}{EI} + \int_0^L \frac{[-P(L+x)+Px](-1)dx}{2EI}$$

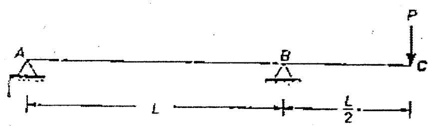
$$= \int_0^L \frac{Px dx}{EI} + \int_0^L \frac{[P(L+x)+Px]}{2EI} dx$$

$$= \frac{P}{2EI}\left[\int_0^L 2x dx + \int_0^L (2x + L)dx\right]$$

$$= \frac{P}{2EI}\left[2\left[\frac{x^2}{2}\right]_0^L + 2\left[\frac{x^2}{2}\right]_0^L + L[x]_0^L\right]$$

$$\theta = \frac{P}{2EI}\left[2\cdot\frac{L^2}{2} + 2\cdot\frac{L^2}{2} + L\cdot L\right] = \frac{3PL^2}{2EI}$$

Example 7.7 Find the vertical deflection of the free end for the Beam shown in figure. Use unit load method.



EI is constant

Solution:

(i) Support Reactions due to given loading:

Let R_A and R_B be the vertical reactions at A and B

$$\Sigma F_y = 0$$

$$\Rightarrow \quad R_A + R_B = P \quad \ldots(i)$$

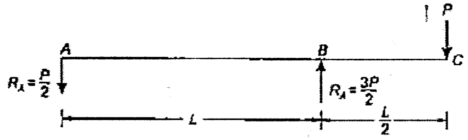
$$\Sigma M_B = 0$$

$$\Rightarrow \quad R_A \times L + P \times \frac{L}{2} = 0$$

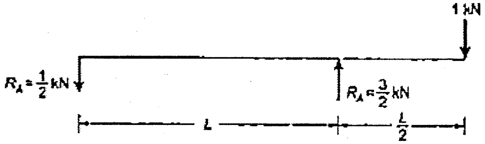
$$\Rightarrow \quad R_A = -\frac{P}{2} \text{ or } \frac{P}{2}(\downarrow)$$

From (i), we get

$$R_B = P - R_A = \frac{3P}{2}(\uparrow)$$



(ii) Support reactions due to unit load at C. Similarly



Portion	AB	CB
Origin	A	C
Limit	$0 - L$	$0 - \frac{L}{2}$
M	$-\frac{P}{2}x$	$-Px$
m_1	$-\frac{x}{2}$	$-x$
EI	EI	EI

where M = B.M at any section due to given loading

m_1 = B.M at any section due to unit load at C

vertical deflection at C,

$$\Delta_C = \int \frac{Mm_1}{EI} ds$$

$$EI\Delta_C = \int_0^L \left(-\frac{Px}{2}\right)\left(-\frac{x}{2}\right)dx + \int_0^{L/2} (-Px)(-x)\,dx$$

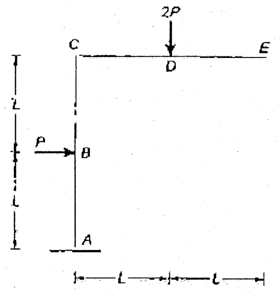
$$= \int_0^L \frac{Px^2 dx}{4} + \int_0^{L/2} Px^2 dx$$

$$= \frac{P}{4}\left[\frac{x^3}{3}\right]_0^L + P\left[\frac{x^3}{3}\right]_0^{\frac{L}{2}}$$

$$= \frac{P}{4}\left[\frac{L^3}{3}\right] + P\left[\frac{L^3}{24}\right] = \frac{PL^3}{8}$$

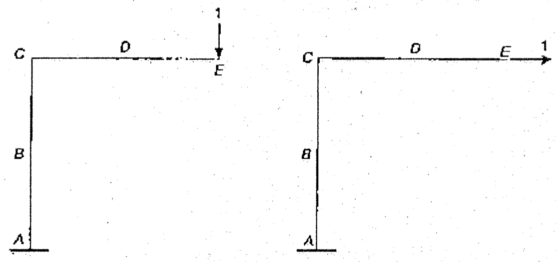
$$\therefore \quad \Delta_C = \frac{PL^3}{8EI}(\downarrow)$$

Example 7.8 Determine the vertical and horizontal deflection at the free end of the bent shown in figure. Assume flexural rigidity EI throughout is bent is uniform.



Solution:

For vertical and horizontal deflection at E apply unit load at E in vertical and horizontal direction respectively.



For given bent,

Taking outer face as reference face

Portion	ED	DC	CB	BA
Origin	E	D	C	B
Limit	$O - L$	$O - L$	$O - L$	$O - L$
M	0	$-2Px$	$-2PL$	$-2PL - Px$
m_1	$-x$	$-(L + x)$	$-2L$	$-2L$
m_2	0	0	$-x$	$-(x + L)$
MOI	I	I	I	I

where M = Expression of B.M due to given loading

m_1 = Moments due to vertical unit load at the free end

m_2 = Moments due to horizontal unit load at the free end

The vertical deflection of free end is given by,

$$\Delta_{EV} = \sum \frac{Mm_1}{EI} dx$$

$$= 0 + \int_0^L \frac{(-2Px)(-L + x)dx}{EI} + \int_0^L \frac{(-2PL)(-2PL)dx}{EI} + \int_0^L \frac{(-2PL + Px)\cdot(-2L)dx}{EI}$$

$$= \frac{1}{EI}\left[\int_0^L \left[2Px - 2Px^2 + 4PL^2 + 4PL^2 + 2PLx\right]dx\right]$$

$$= \frac{1}{EI}\int_0^L\left[-2Px^2 + 4PLx + 8PL^2\right]dx = \frac{2P}{EI}\int_0^L\left[-x^2 + 2Lx + 4L^2\right]dx$$

$$= \frac{2P}{EI}\left\{\left[-\frac{x^3}{3}\right]_0^L + 2L\left[\frac{x^2}{2}\right]_0^L + 4L^2[x]_0^L\right\}$$

$$= \frac{2P}{EI}\left[-\frac{L^3}{3} + L^3 + 4L^3\right] = \frac{28PL^3}{3EI}(\downarrow)$$

The horizontal deflection of free end is given by

$$\Delta_{EH} = \sum \frac{Mm_2 dx}{EI}$$

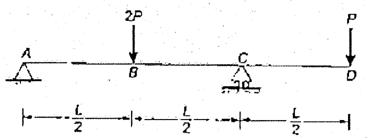
$$= 0 + 0 + \int_0^L \frac{(-2PL)(-x)dx}{EI} + \int_0^L -\frac{(2PL + Px)\cdot[-(x+L)]dx}{EI}$$

$$= \frac{2P}{EI}\int_0^L Lx\,dx + \int_0^L \frac{(2PLx + Px^2 + 2PL^2 + PLx)dx}{EI}$$

$$= \frac{2P}{EI}\int_0^L Lx\,dx + \frac{P}{EI}\int_0^L \left(x^2 + 3Lx + 2L^2\right)dx$$

$$= \frac{P}{EI}\left[\int_0^L (5Lx + x^2 + 2L^2)dx\right] = \frac{P}{EI}\cdot\left[\frac{5L^3}{2} + \frac{L^3}{3} + 2L^3\right] = \frac{29PL^3}{6EI}(\rightarrow)$$

Example 7.9 Determine the rotation at *A* and vertical deflection at end *D* in the overhanging beam shown in figure.

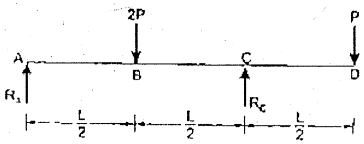


EI is constant

Solution:

(i) Rotation at End A

Reactions at supports due to given loading:



$$\Sigma M_B = 0$$

$$\Rightarrow \quad R_A \times L = -2P \times \frac{L}{2} + P \times \frac{L}{2} = 0$$

$$\Rightarrow \quad R_A = P - \frac{P}{2} = \frac{P}{2}(\uparrow)$$

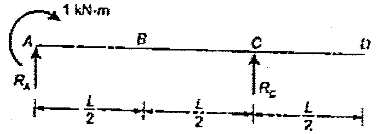
$$\Sigma F_y = 0$$

$$\Rightarrow \quad R_A + R_C = 3P \qquad \text{...(i)}$$

$$\Rightarrow \quad R_C = 3P - R_A = 3P - \frac{P}{2}$$

$$\Rightarrow \quad R_C = \frac{5P}{2}(\uparrow)$$

Reactions at supports due to unit moment at A:



$$\Sigma F_y = 0$$

$$\Rightarrow \quad R_A + R_B = 0 \qquad \text{...(ii)}$$

Also,

$$\Sigma M_C = 0$$

$$\Rightarrow \quad R_A \times L + 1 = 0$$

$$\Rightarrow \quad R_A = -\frac{1}{L} = \frac{1}{L}(\downarrow)$$

From (i), we get

$$R_B = \frac{1}{L}(\uparrow)$$

Portion	AB	BC	DC
Origin	*A*	*B*	*D*
Limit	$0-\frac{L}{2}$	$0-\frac{L}{2}$	$0-\frac{L}{2}$
M	$\frac{Px}{2}$	$\frac{P}{2}\left(\frac{L}{2}+x\right)-2Px$	$-Px$
m_2	$1-\frac{x}{L}$	$1-\left(\frac{L}{2}+x\right)\frac{1}{L}$	0
EI	*EI*	*EI*	2*EI*

where, M = BM at any section due to given loading

m_2 = BM at any section due to unit moment at A

Rotation at A,

$$\theta_A = \int \frac{Mm_2 ds}{EI}$$

$$EI\theta_A = \int_0^{\frac{L}{2}} \frac{Px}{2}\left(1 - \frac{x}{L}\right)dx + \int_0^{\frac{L}{2}} \frac{P(L-6x)}{4}\cdot\frac{P(L-2x)}{2L} + 0$$

$$= \int_0^{\frac{L}{2}}\left[\frac{Px}{2} - \frac{Px^2}{2L} + \frac{PL^2}{8L} - Px + \frac{3Px^2}{2L}\right]$$

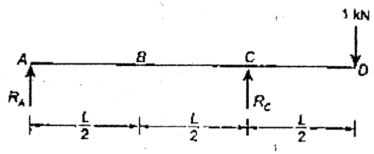
$$= \left[\frac{P}{2}\left(\frac{x^2}{2}\right) - \frac{P}{2L}\left[\frac{x^3}{3}\right] + \frac{PL}{8}[x] - P\left(\frac{x^2}{2}\right) + \frac{3P}{2L}\left(\frac{x^3}{3}\right)\right]_0^{\frac{L}{2}}$$

$$= \frac{PL^2}{16} - \frac{PL^3}{48} + \frac{PL^2}{16} - \frac{PL^2}{8} + \frac{3PL^3}{48} = \frac{PL^2}{24}$$

$$\theta_A = \frac{PL^2}{24EI}(\circlearrowright)$$

(ii) Vertical deflection at D:

Reactions at supports due to unit load at A:



$\Sigma F_y = 0$; $\quad R_A + R_C = 1 \quad$ ...(i)

$\Sigma_C = 0$; $\quad R_A \times L + 1 \times \frac{L}{2} = 0$

$$R_A = -\frac{1}{2}\text{kN or } \Sigma M_C = \frac{1}{2}\text{kN}(\downarrow)$$

From (i), we get

$$R_C = 1 - R_A = 1 + \frac{1}{2}$$

$$R_C = \frac{3}{2}\text{kN}(\uparrow)$$

where, M = B.M at any section due to given loading

m_1 = B.M at any section due to unit load at D.

Portion	AB	BC	DC
Origin	A	B	D
Limit	$0-\frac{L}{2}$	$0-\frac{L}{2}$	$0-\frac{L}{2}$
M	$\frac{Px}{2}$	$\frac{P}{2}\left(\frac{L}{2}+x\right)-2Px$	$-Px$
m_1	$-\frac{x}{2}$	$-\frac{1}{2}\left(\frac{L}{2}+x\right)$	$-1x$
EI	EI	EI	2EI

Vertical deflection at D,

$$\Delta_{VD} = \int \frac{Mm_1 ds}{EI}$$

$$EI\Delta_{VD} = \int_0^{\frac{L}{2}}\left(\frac{Px}{2}\right)\left(\frac{-x}{2}\right)dx + \int_0^{\frac{L}{2}}\frac{P(L-6x)}{4} \times \frac{-(L+2x)}{4}dx + \int_0^{\frac{L}{2}}(-Px)(-x)dx$$

$$= \frac{-P}{4}\int_0^{\frac{L}{2}}x^2dx - \frac{P}{16}\int_0^{\frac{L}{2}}(L-6x)(L+2x)dx + P\int_0^{\frac{L}{2}}x^2dx$$

$$= \frac{-P}{4}\int_0^{\frac{L}{2}}x^2dx - \frac{P}{16}\int_0^{\frac{L}{2}}\left(L^2 - 4Lx - 12x^2\right)dx + P\int_0^{\frac{L}{2}}x^2dx$$

$$= \frac{-P}{4}\left[\frac{x^3}{3}\right]_0^{\frac{L}{2}} - \frac{P}{16}\left[L^2 \cdot x - 4L\frac{x^2}{2} - \frac{12x^3}{3}\right]_0^{\frac{L}{2}} + P\left[\frac{x^3}{3}\right]_0^{\frac{L}{2}}$$

$$= \frac{-PL^3}{96} - \frac{P}{16}\left[\frac{L^3}{2} - \frac{L^3}{2} - \frac{L^3}{2}\right] + \frac{PL^3}{24}$$

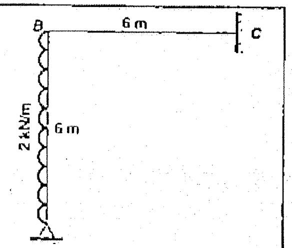
$$= \frac{-PL^3}{96} + \frac{PL^3}{32} + \frac{PL^3}{24}$$

$$= \frac{1}{6}PL^3$$

$$\therefore \quad \Delta_{VD} = \frac{PL^3}{16EI}(\downarrow)$$

Illustrative Examples

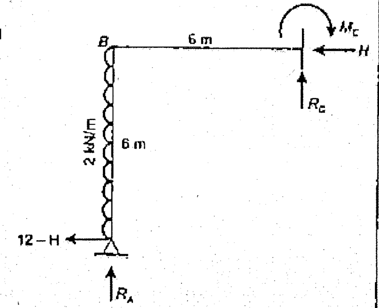
Example 7.10 Analyse the frame shown below and draw BMD. Use strain energy method. EI is constant.



Solution:

Let the vertical reactions at A and C be R_A and R_C respectively.

Reactions:



Let, $H_C = H$

$$\Sigma F_x = 0$$

$$\Rightarrow \quad H_A - 2 \times 6 + H = 0$$

Also $\quad \Sigma F_y = 0$

$$\Rightarrow \quad R_A + R_C = 0 \quad \text{...(a)}$$

Also, $\quad \Sigma M_A = 0$

$$\Rightarrow \quad M_C - 6 \times R_C - H \times 6 + 2 \times 6 \times 3 = 0$$

$$\Rightarrow \quad M_C - 6R_C - 6H + 36 = 0$$

$$\Rightarrow \quad M_C + 36 - 6H = 6R_C$$

$$\therefore \quad R_C = 6 - H + \frac{M_C}{6}$$

Hence, $\quad R_A = -R_C = H - \frac{M_C}{6} - 6$

Let redundants are H and M_C.

Now as we know, $\quad U = \int \frac{M^2 ds}{2EI}$

The value of H and M_C will be such that the total strain energy is minimum i.e.

$$\frac{\partial U}{\partial H} = 0$$

$$\Rightarrow \quad \frac{\partial U}{\partial H} = \int \frac{2M\frac{\partial M}{\partial H}ds}{2EI} = 0$$

$$\Rightarrow \quad \frac{\partial U}{\partial H} = \sum \int \frac{M\frac{\partial M}{\partial H}ds}{EI} = 0 \quad \text{...(i)}$$

Similarly, $\quad \frac{\partial U}{\partial M_C} = 0$

$$\therefore \quad \sum \int \frac{M.\frac{\partial M}{\partial M_C}ds}{EI} = 0$$

The B.M. (M) at any section and the corresponding values of $\frac{\partial M}{\partial H}$ and $\frac{\partial M}{\partial M_C}$ for the frame are tabulated below.

Member	Origin	M	$\frac{\partial M}{\partial H}$	$\frac{\partial M}{\partial M_C}$	Limit
AB	A	$(12-H)y - 2y\cdot\frac{y}{2}$	$-y$	0	0 to 6
BC	C	$\left(6-H+\frac{M_C}{6}\right)x - M_C$	$-x$	$\frac{x}{6}-1$	0 to 6

From equation (*i*), we get

$$\Rightarrow \int_0^6 \frac{\left[(12-H)y - 2y\cdot\frac{y}{2}\right](-y)\,dy}{EI} + \int_0^6 \frac{\left\{\left(6-H+\frac{M_C}{6}\right)x - M_C\right\}(-x)\,dx}{EI} = 0$$

$$\Rightarrow \int_0^6 (12y - Hy - y^2) + \int_0^6 \left(6x - Hx + \frac{M_C x}{6} - M_C\right)(-x)\,dx = 0$$

$$\Rightarrow \int_0^6 (-12y^2 + Hy^2 + y^3)\,dy + \int_0^6 \left(-6x^2 + Hx^2 - \frac{M_C x^2}{6} - M_C x\right)dx = 0$$

$$\Rightarrow \left[-12\frac{y^3}{3} + \frac{Hy^3}{3} + \frac{y^4}{4}\right]_0^6 + \left[-6\frac{x^3}{3} + \frac{Hx^3}{3} - \frac{M_C x^3}{18} + \frac{M_C x^2}{2}\right]_0^6 = 0$$

$$\Rightarrow -12\times\frac{6^3}{3} + \frac{H\times 6^3}{3} + \frac{6^4}{4} - \frac{6\times 6^3}{3} + \frac{H\times 6^3}{3} - \frac{M_C\times 6^3}{18} + \frac{M_C\times 6^2}{2} = 0$$

$$\Rightarrow -864 + 72H + 324 - 432 + 72H - 12M_C + 18M_C = 0$$

$$\Rightarrow 24H + M_C = 162 \qquad \text{...(ii)}$$

From equation (*ii*), we get

$$\Rightarrow \int_0^6 \frac{\left[(12-H)y - 2y\cdot\frac{y}{2}\right](0)\,dy}{EI} + \int_0^6 \frac{\left\{\left(6-H+\frac{M_C}{6}\right)x - M_C\right\}\left(\frac{x}{6}-1\right)dx}{EI} = 0$$

$$\Rightarrow \int_0^6 \frac{\left(6x - Hx + \frac{M_C x}{6} - M_C\right)\left(\frac{x}{6}-1\right)}{EI}\,dx = 0$$

$$\Rightarrow \int_0^6 \left(x^2 - \frac{Hx^2}{6} + \frac{M_C x^2}{36} - \frac{M_C x}{6} - 6x + Hx - \frac{M_C x}{6} + M_C\right)dx = 0$$

$$\Rightarrow \left[\frac{x^3}{3} - \frac{Hx^3}{18} + \frac{M_C x^3}{108} - \frac{M_C x^2}{12} - \frac{6x^2}{2} + \frac{Hx^2}{2} - \frac{M_C x^2}{12} + M_C x\right]_0^6$$

$$\Rightarrow \left[\frac{6^3}{3} - \frac{H\times 6^3}{18} + \frac{M_C\times 6^3}{108} - \frac{M_C\times 6^2}{12} - \frac{6\times 6^2}{2} + \frac{H\times 6^2}{2} - \frac{M_C\times 6^2}{12} + 6M_C\right] = 0$$

$$\Rightarrow 72 - 12H + 2M_C - 3M_C - 108 + 18H - 3M_C + 6M_C = 0$$

$$\Rightarrow M_C + 3H = 18 \qquad \text{...(iii)}$$

Solving (*ii*) and (*iii*), we get

$$H = 6.8571\text{ kN}$$

$$M_C = -2.571\text{ kN-m}$$

$$H_C = H = 6.8571\text{ kN}$$

$$H_A = 12 - H = 5.1429\text{ kN}$$

$$R_A = H - \frac{M_C}{6} - 6 = 6.8571 + \frac{2.571}{6} - 6$$

$$R_A = 1.2856\text{ kN}$$

$$R_C = 6 - H + \frac{M_C}{6} = 6 - 6.8571 + \left(\frac{-2.571}{6}\right)$$

Bending moment diagram:

Taking outer face as reference face.

Portion AB:

$$M_y(y\text{ from }A) = H_A y - 2y\cdot\frac{y}{2} = 5.1429y - y^2 \qquad [0 \le y \le 6]$$

at $y = 0$; $M_A = 0$

at $y = 6$; $M_B = 5.1429\times 6 - 6^2 = -5.143$ kN-m

For M_{max}, $\frac{dM_y}{dy} = 0$

$$5.1429 - 2y = 0$$

$$y = 2.571\text{ m}$$

$$\therefore \quad M_{max} = 5.1429\times 2.571 - (2.571)^2 = 6.61\text{ kN-m}$$

For M_y to be zero,

$$5.1429y - y^2 = 0$$

$$y(5.1429 - y) = 0$$

$y = 0$ and $y = 5.1429$ m

Portion CB:

$$M_x(x\text{ from }C) = R_C x - M_C \quad [0 \le x \le 6]$$

$$M_x = -1.2856x + 2.571$$

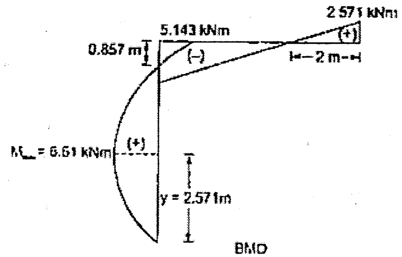
at $x = 0$; $M_C = 2.571$ kN-m

at $x = 6$; $M_B = -1.2856\times 6 + 2.571 = -5.143$ kN-m

For M_x to be zero,

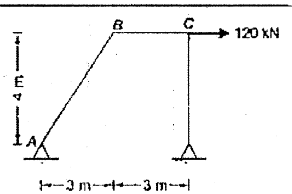
$$-1.2856x + 2.571 = 0$$

$$x = 1.999\text{ m} \simeq 2\text{ m}$$



BMD

Example 7.11 Analyse the portal frame shown in figure and draw BMD using strain energy method. EI is constant.



Solution:

Reaction:

Consider $H_A = H$ as redundant

$$\Sigma F_x = 0$$

$\Rightarrow$ $H_A + H_D = 120$

$\Rightarrow$ $H_D = 120 - H (\leftarrow)$ ...(i)

$$\Sigma F_y = 0$$

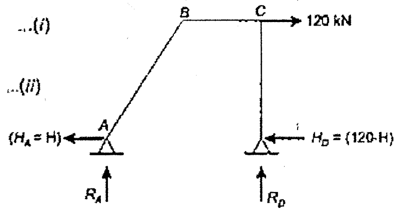
$\Rightarrow$ $R_A + R_D = 0$ ...(ii)

$$\Sigma M_D = 0$$

$\Rightarrow$ $R_A \times 6 + 120 \times 4 = 0$

$\Rightarrow$ $R_A = -80 \text{ kN} (\downarrow)$

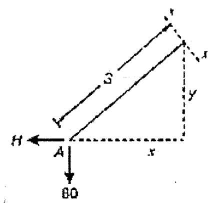
and $R_D = 80 \text{ kN} (\uparrow)$



The value of H will be such that the total strain energy is minimum

$$\therefore \quad \frac{\partial U}{\partial H} = \sum \int \frac{M \frac{\partial M}{\partial H} ds}{EI} = 0$$

The B.M (M) at any section and corresponding value of $\frac{\partial M}{\partial H}$ for the frame are tabulated below.



For member AB:

Consider a section at a distance S for A along the member

$$M = -80x + Hy$$

where, $x = S\cos\theta = S \times \frac{3}{5}$

$y = S\sin\theta = S \times \frac{4}{5}$

$\therefore$ $M = -80 \times \frac{3}{5} S + H \times \frac{4}{5} S$

$M = -48\,S + 0.8\,HS$

Member	Origin	M	$\frac{\partial M}{\partial H}$	Limit
AB	A	$-48\,S + 0.8\,HS$	0.8 S	0 to 5
BC	B	$-80(3 + x) + 4H$	4	0 to 3
DC	D	$-(120 - H)\,y$	y	0 to 4

$$\therefore \quad \Sigma \int \frac{M \frac{\partial M}{\partial H} ds}{EI} = 0$$

$$\Rightarrow \int_0^5 (-48S + 0.8HS)\cdot(0.8S)\,ds + \int_0^3 [-80(3+x) + 4H]\cdot 4\,dx + \int_0^4 -[120 - H]\,y \cdot y\,dy = 0$$

$$\Rightarrow \int_0^5 (-38.4S^2 + 0.64HS^2)\,dS + \int_0^3 [-960 - 320x + 16H]\,dx + \int_0^4 [-120y^2 + Hy^2]\,dy = 0$$

$$\Rightarrow 38.4\left[\frac{S^3}{3}\right]_0^5 + 0.64H\left[\frac{S^3}{3}\right]_0^5 - 960[x]_0^3 - 320\left[\frac{x^2}{2}\right]_0^3 + 16H[x]_0^3 - 120\left[\frac{y^3}{3}\right]_0^4 + H\left[\frac{y^3}{3}\right]_0^4 = 0$$

$\Rightarrow$ $-1600 + 26.67\,H - 2880 - 1440 + 48\,H - 2560 + 21.33\,H = 0$

$\Rightarrow$ $96\,H - 8480 = 0$

$\Rightarrow$ $H = 88.33 \text{ kN}$

Bending Moment Diagram:

Portion AB:

M_s (S from A) $= -48S + 0.8\,HS$ $[0 \le S \le 5]$

$M_s = -48S + 0.8 \times 83.33S$

at $S = 0$, $M_A = 0$

at $S = 5$ m; $M_B = -48 \times 5 + 0.8 \times 83.33 \times 5 = +113.32$ kN-m

Portion BC:

M_x (x from B) $= -80(3 + x) + 83.33 \times 4$ $[0 \le x \le 3]$

$M_x = 93.32 - 80x$

at $x = 0$, $M_B = 93.32$ kN-m

at $x = 3$, $M_C = 93.32 - 80 \times 3$

$= -146.68$ kN-m

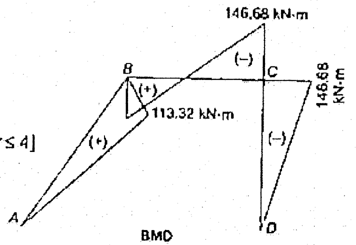
Portion DC:

M_y (y from D) $= -[120 - H]\,y$ $[0 \le y \le 4]$

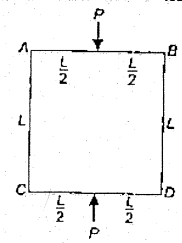
$M_y = -[120 - 83.33]\,y$

at $y = 0$, $M_D = 0$

at $y = 4$, $M_C = -146.68$ kN-m



Example 7.12 Analyse the closed frame as shown in figure. Also draw bending moment diagram. EI is constant.



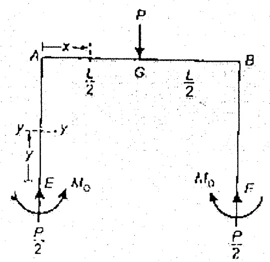
Solution:

Consider equilibrium of one half position as shown below.

Consider M_0 be the redundant.

The true value of redundant M will be that for which total strain energy is minimum i.e.

$$\frac{\partial U}{\partial M_0} = 0$$



$$\Rightarrow \quad \frac{\partial}{\partial M_0} \int \frac{M_x^2\,ds}{2EI} = 0$$

$$\Rightarrow \quad \int \frac{2 \cdot M_x \cdot \frac{\partial M_x}{\partial M_0} ds}{2EI} = 0$$

Also, $$\sum \int \frac{M_x \cdot \frac{\partial M_x}{\partial M_0} \cdot ds}{EI} = 0 \quad ...(i)$$

The BM (M_x) at any section and the corresponding values of $\frac{\partial M_x}{\partial M_0}$ for the frame are tabulated below.

From equation (*i*), we get

Member	Origin	M_x	$\frac{\partial M_x}{\partial M_0}$	Limit
EA	E	$-M_0$	-1	0 to $\frac{L}{2}$
AG	A	$\frac{P}{2}x - M_0$	-1	0 to $\frac{L}{2}$

$$\Rightarrow \quad 2\int_0^{L/2} \frac{(-M_0)(-1)\,dy}{EI} + 2\int_0^{L/2} \frac{\left(\frac{P}{2}x - M_0\right)(-1)\,dx}{EI} = 0$$

$$\Rightarrow \quad \int_0^{L/2} M_0\,dy - \int_0^{L/2} \left(\frac{P}{2}x - M_0\right)dx = 0$$

$$\Rightarrow \quad M_0[y]_0^{L/2} - \frac{P}{2}\left[\frac{x^2}{2}\right]_0^{L/2} + M_0[x]_0^{L/2} = 0$$

$$\Rightarrow \quad M_0\left[\frac{L}{2}\right] - \frac{P}{4}\left(\frac{L}{2}\right)^2 + M_0\left[\frac{L}{2}\right] = 0$$

$$\Rightarrow \quad M_0 L - \frac{PL^2}{16} = 0$$

$$\Rightarrow \quad M_0 = \frac{PL}{16}$$

Bending moment diagram:

Taking outer face as reference face.

Portion *EA*:

$$M_y\ (y \text{ from } E) = -M_0 \quad \left[0 \le y \le \frac{L}{2}\right]$$

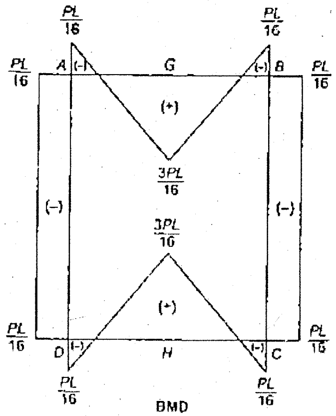
$$M_y = -\frac{PL}{16} \quad \text{(Constant)}$$

Portion *AG*:

$$M_x\ (x \text{ from } A) = \frac{P}{2}x - \frac{PL}{16} \quad \left[0 \le x \le \frac{L}{2}\right]$$

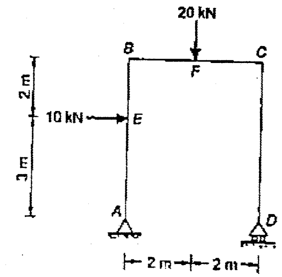
at $x = 0$, $\quad M_A = -\frac{PL}{16}$

at $x = \frac{L}{2}$, $\quad M_G = \frac{PL}{4} - \frac{PL}{16} = \frac{3PL}{16}$



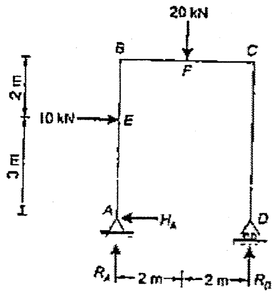
Since loading is symmetrical about *G*. Hence BMD will be symmetrical about *G*.

Example 7.13 Determine the horizontal displacement and rotation at roller support in the frame shown in figure. *EI* is constant.



Solution:

Support reactions:



$$\Sigma F_x = 0$$

$\Rightarrow$ $\quad 10 - H_A = 0$

$\therefore$ $\quad H_A = 10$ kN (←)

$$\Sigma F_y = 0$$

$\Rightarrow$ $\quad R_A + R_D = 20$

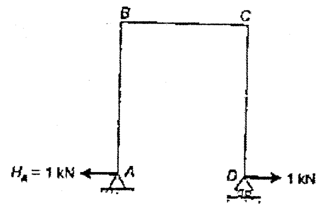
Also, $\quad \Sigma M_D = 0$

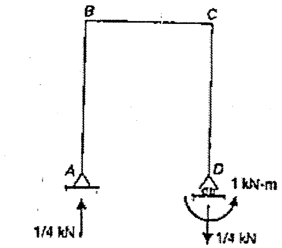
$\Rightarrow$ $\quad R_A \times 4 + 10 \times 3 - 20 \times 2 = 0$

$\Rightarrow$ $\quad R_A = 2.5$ kN (↑)

and $\quad R_D = 17.5$ kN (↑)

For displacement and rotation of roller support, remove external loading and apply horizontal unit load and unit moment separately as shown below.





Taking outer face as reference face.

Portion	AE	EB	BF	FC	CD
Origin	A	E	B	C	D
Limit	0 – 3	0 – 2	0 – 2	0 – 2	0 – 5
M	$10x$	$10(3+x) - 10x = 30$	$10 \times 5 - 10 \times 2 + 2.5x) = 30 + 2.5x$	$17.5x$	0
m_1	x	$1(3+x)$	5	5	x
m_2	0	0	$1/4x$	$1 - x/4$	1
EI	EI	EI	EI	EI	EI

where, M = B.M at any section due to given loading

m_1 = B.M at any section when only unit is applied at D

m_2 = B.M at any section when only unit moment is applied at D

For horizontal displacement of roller

$$\Delta_{DH} = \sum \int \frac{Mm_1 dx}{EI}$$

$$= \int_0^3 \frac{(10x)x\,dx}{EI} + \int_0^2 \frac{30(x+3)dx}{EI} + \int_0^2 \frac{(30+2.5x)5\,dx}{EI} + \int_0^2 \frac{(17.5x)5\,dx}{EI} + 0$$

$$= \frac{1}{EI}\left[\int_0^3 10x^2\,dx + \int_0^2 (90+30x) + \int_0^2 (150+12.5x)dx + \int_0^2 87.5x\,dx\right]$$

$$= \frac{1}{EI}\left[10\cdot\left[\frac{x^3}{3}\right]_0^3 + 90[x]_0^2 + 30\left[\frac{x^2}{2}\right]_0^2 + 150[x]_0^2 + 12.5\left[\frac{x^2}{2}\right]_0^2 + 87.5\left[\frac{x^2}{2}\right]_0^2\right]$$

$$= \frac{1}{EI}\left[\frac{10}{3}\times(3)^3 + 90\times(2) + \frac{30}{2}\times(2)^2 + 150\times(2) + \frac{12.5}{2}(2)^2 + \frac{87.5}{2}(2)^2\right]$$

$$= \frac{1}{EI}[90+180+60+300+25+175]$$

$\therefore$ $$\Delta_{DH} = \frac{830}{EI} \quad (\rightarrow)$$

For rotation at D,

$$\theta_D = \sum \int \frac{Mm_2\,dx}{EI}$$

$$= 0 + 0 + \int_0^2 \frac{(30+2.5x)\cdot\frac{x}{4}dx}{EI} + \int_0^2 \frac{(17.5x)\left(1-\frac{x}{4}\right)dx + 0}{EI}$$

$$= \frac{1}{EI}\left[\int_0^2 (7.5x+0.625x^2)dx + \int_0^2 (17.5x - 4.375x^2)\,dx\right]$$

$$= \frac{1}{EI}\left[\int_0^2 (25x - 3.75x^2)\,dx\right]$$

$$= \frac{1}{EI}\left[25\times\frac{x^2}{2} - 3.75\frac{x^3}{3}\right]_0^2 = \frac{1}{EI}\left[\frac{25\times 2^2}{2} - \frac{3.75\times 2^3}{3}\right]$$

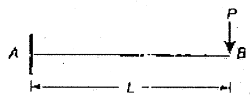
$\therefore$ $$\theta_D = \frac{40}{EI} (\circlearrowright)$$

Summary

- The strain energy stored in bending is
$$U = \int \frac{M_x^2\,ds}{2EI}$$
- The strain energy stored due to shear force is
$$U = \int \frac{S^2\,ds}{2\,ArG}$$
where, Ar = Reduced area due to shear
G = Shear modulus
- The strain energy due to axial duo axial force is
$$U = \int \frac{P^2 dx}{2AE}$$
- The strain energy due to torque is
$$U = \int \frac{T_x^2\,dx}{2GI_P}$$
- If a structure have redundant reaction, then the true value of redundant will be that for which the total strain energy stored in minimum.
i.e. $$\frac{\partial U}{\partial R} = 0$$
- According to unit load method, the deflection of any point is given by
$$\Delta = \int \frac{Mm_1}{EI}\,ds$$
- According to unit load method, the rotation of any point is given by
$$\theta = \int \frac{Mm_2\,ds}{EI}$$
Where, m_1 = BM at any section due to unit load at that point
m_2 = BM at any section due to unit moment at that point

Objective Brain Teasers

Q.1 The strain energy due to bending in the cantilever beam shown in the figure is



(a) $\frac{PL}{3EI}$ (b) $\frac{P^2L^3}{6EI}$

(c) $\frac{P^2L^3}{EI}$ (d) $\frac{P^2L^3}{2EI}$

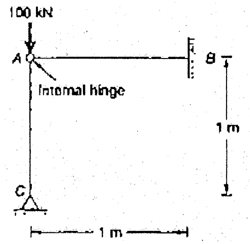
Q.2 The unit load method used in structural analysis is
(a) applicable only to statically indeterminate structures
(b) another name for stiffness method
(c) an extension of Maxwell's reciprocal theorem
(d) derived from Castigliano's theorem

Q.3 For a linear elastic structural system, minimization of potential energy yields
(a) compatibility conditions
(b) constitutive relations

(c) equilibrium equations

(d) strain-displacement relations

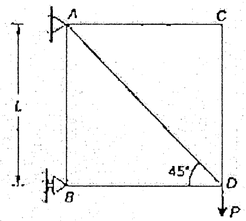
Q.4 Vertical reaction developed at B in the frame below due to the applied load of 100 kN (with 150,000 mm² cross-sectional area and 3.125×10^9 mm⁴ moment of inertia for both members) is



(a) 5.9 kN (b) 30.2 kN

(c) 66.3 kN (d) 94.1 kN

Q.5 The strain energy stored in member AB of the pin-jointed truss as shown in figure where E and A are same for all members, is



(a) $\frac{2P^2L}{AE}$ (b) $\frac{P^2L}{AE}$

(c) $\frac{P^2L}{2AE}$ (d) Zero

Q.6 Maxwell's reciprocal theorem is valid

(a) for all statically determinate structure

(b) for all structures

(c) for all elastic structures

(d) for all structures with linear force-displacement relation

Q.7 A simply supported beam of span L and flexural rigidity EI, carries a unit point load at its centre. The strain energy in the beam due to bending is

(a) $\frac{L^3}{48EI}$ (b) $\frac{L^3}{192EI}$

(c) $\frac{L^3}{96EI}$ (d) $\frac{L^3}{16EI}$

Q.8 Assertion (A): In the analysis of rigid frames by the energy method, it is usually considered sufficient to calculate the total strain energy due to flexure only.

Reason (R): The strain energies due to axial force and shear are normally insignificant when compared to that for flexure.

(a) both A and R are true and R is the correct explanation of A

(b) both A and R are true but R is not a correct explanation of A

(c) A is true but R is false

(d) A is false but R is true

Answers

1. (b) 2. (d) 3. (a) 4. (a) 5. (d)
6. (d) 7. (c) 8. (a)

Hints and Explanations:

1. (b)

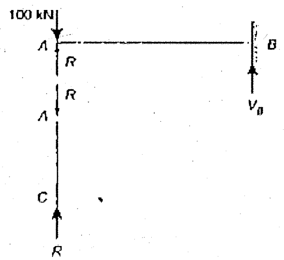
$$\text{Strain energy} = \int_0^L \frac{M_x^2\,dx}{2EI} = \int_0^L \frac{(Px)^2\,dx}{2EI} = \frac{P^2L^3}{6EI}$$

2. (d)

Unit load method is used to find deflection at any point of structure. It is derived from Castigliano's theorem.

$$\Delta = \frac{\partial U}{\partial Q} = \int M\frac{\partial M}{\partial Q}\frac{dx}{EI} = \int \frac{Mm\,dx}{EI}$$

4. (a)



Deflection at A in beam AB

= Compression in column AC

$$\frac{(100-R)L^3}{3EI} = \frac{RL}{AE}$$

$$\Rightarrow \frac{(100-R)\times(1\times1000)^3}{3\times3.125\times10^9} = \frac{R\times1000}{150000}$$

$\Rightarrow$ $100 - R = 0.0625$

$\Rightarrow$ $100 = 1.0625\,R$

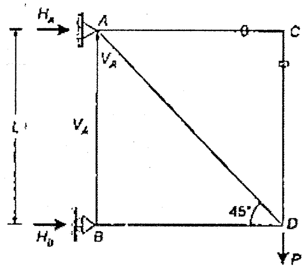
$\Rightarrow$ $R = 94.1$ kN

$\therefore$ $V_B = 100 - R$
$= 100 - 94.1$
$= 5.9$ kN

5. (d)

Degree of static indeterminacy

$D_s = m + r_e - 2j$
$= 5 + 3 - 2 \times 4$
$= 8 - 8 = 0$



$\therefore$ Truss is determinate.

Now, from equation of equilibrium

$\Sigma F_y = 0$

$\Rightarrow$ $V_A = P$

also, $\Sigma M_B = 0$

$\Rightarrow$ $P \times L + H_A \times L = 0$

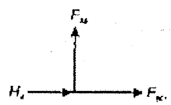
$\Rightarrow$ $H_A = -P$

$\Sigma F_x = 0$

$\Rightarrow$ $H_A + H_B = 0$

$\Rightarrow$ $H_B = P$

Consider joint B.



$\Sigma F_x = 0$

$\Rightarrow$ $F_{BD} = -P$

$\Sigma F_y = 0$

$\Rightarrow$ $F_{AB} = 0$

$\therefore$ strain energy y-stored in members AB,

$$U_{AB} = \int (F_{AB})^2 \frac{dx}{2AE} = 0$$

Alternate Method: Since at joint 'C' there is no any external force therefore force in member CA and CD will be zero.

Now at joint 'B' only horizontal support reaction is at B and that will be equal to force in the member BD. So force in the member BA must be zero.

Therefore strain energy in member AB

$$U_{AB} = \int (F_{AB})^2 \frac{dx}{2AE}$$

$\therefore$ $U_{AB} = 0$

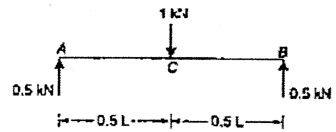
6. (d)

Maxwell's reciprocal theorem is valid for any structure which is linear elastic in which Hook's law is valid, temperature is constant and supports are unyielding.

Hence option (d) is correct.

7. (c)

$$\text{Strain energy due to bending} = \int \frac{M_x^2\,dx}{2EI}$$



$M_x = 0.5x$

$$U = 2U_{AC} = 2\int_0^{\frac{L}{2}} \frac{M_x^2\,dx}{2EI} = 2\int_0^{\frac{L}{2}} \frac{\left(\frac{1}{2}x\right)^2 dx}{2EI}$$

$$U = \frac{2}{2EI} \times \frac{1}{4}\int_0^{\frac{L}{2}} x^2\,dx = \frac{1}{4EI}\left[\frac{x^3}{3}\right]_0^{\frac{L}{2}}$$

$$U = \frac{L^3}{96EI}$$

Alternate Approach:

We know, strain energy,

U = work done

$$U = \frac{1}{2} \times P \times \Delta = \frac{1}{2} \times P \times \frac{PL^3}{48EI} = \frac{P^2L^3}{96EI}$$

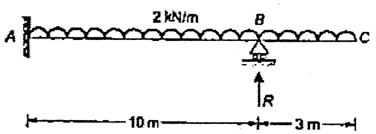
Here $P = 1$ kN,

$$\therefore \quad U = \frac{L^3}{96EI}$$

Hence option (c) is correct.

Conventional Practice Questions

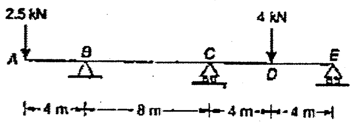
Q.1 Analyse the propped cantilever beam shown. Also draw bending moment diagram.



EI is constant.

Ans. R = 14.85 kN

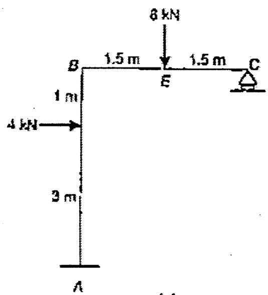
Q.2 Calculate the support reaction for following continuous beam and draw bending moment diagram. Use strain energy method.



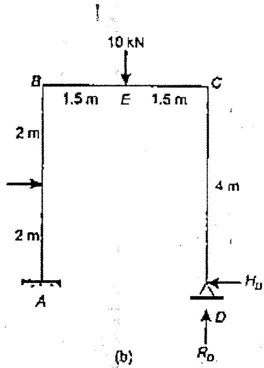
EI is constant.

Ans. R_B = 3.688 kN, R_C = 0.8736 kN, R_E = 1.938 kN

Q.3 Analyse the following portal frames and draw BMD.



(a)

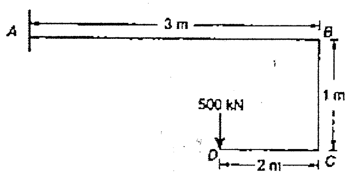


(b)

Ans. (*i*) M_A = −9.3 kN-m, H_A = −4 kN, R_A = 3.1 kN, R_C = 4.9 kN

(*ii*) R_D = 7.2608 kN, H_D = 1.3964 kN

Q.4 Determine the vertical deflection of end D. For the bend shown below. Take diameter of rod 100 mm and E = 200 kN/m².



Ans. Δ_D = 69.90 mm

■■■■