Chapter 14
Waves

Waves

The patterns, which move without the actual physical transfer or flow of matter as a whole, are called waves.

The waves we come across are mainly of three types:

(a) Mechanical waves,

(b) Electromagnetic waves and

(c) Matter waves.

Mechanical waves

Mechanical waves are governed by Newton’s laws, and require a material medium for their propagation., such as water, air, rock, etc.
E.g, water waves, sound waves, seismic waves, etc.

Electromagnetic waves

The electromagnetic waves do not require any medium for their propagation.
All electromagnetic waves travel through vacuum at the same speed of light c, 3x 10^8 ms^{-1}
E.g, visible light, ultraviolet light, radio waves, microwaves, x-rays etc.

Matter waves

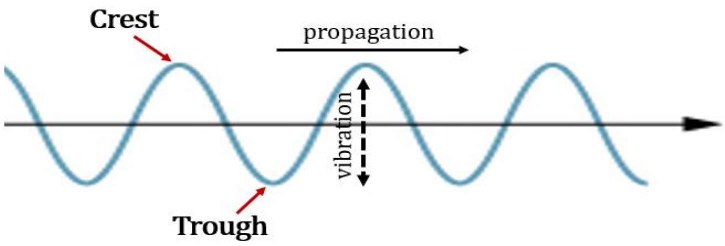
Matter waves are associated with moving electrons, protons, neutrons and other fundamental particles, and even atoms and molecules. These are the constituents of matter and hence such wave are called matter waves. Matter waves associated with electrons are employed in electron microscopes.

Transverse and Longitudinal Waves

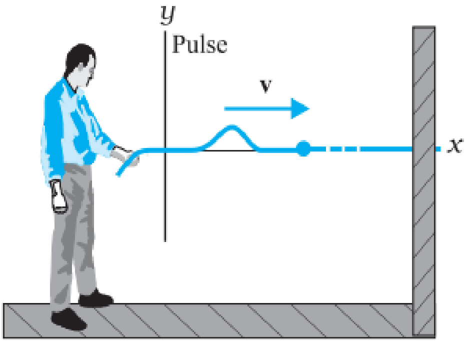
Mechanical waves can be transverse or longitudinal depending on the relationship between the directions of vibrations of particles in the medium and that of the propagation of wave.

Transverse waves

In transverse waves, the constituents of the medium oscillate perpendicular to the direction of wave propagation.



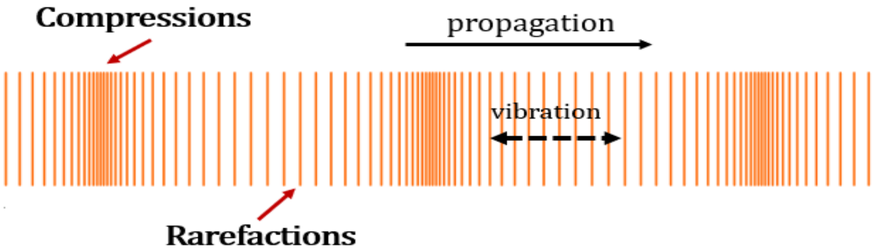
- They travel in the form of crests and troughs.
- Transverse waves can be propagated only in solids and strings, and not in fluids.
- E.g, Waves on a stretched string,



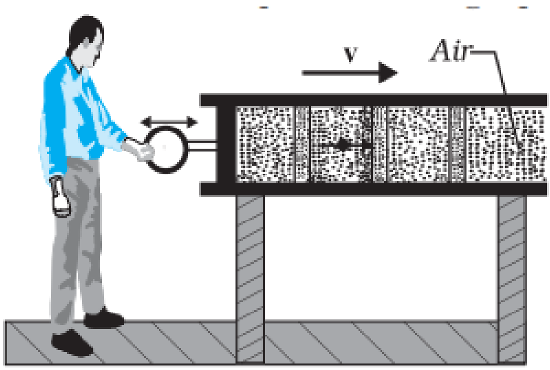
A single pulse is sent along a stretched string. As each element of the string move perpendicular to the direction in which the wave travels, the wave is a transverse wave.

Longitudinal waves

In longitudinal waves the constituents of the medium oscillate along the direction of wave propagation.



- They travel in the form of compressions and rarefactions.
- Longitudinal waves can propagate in all elastic media,i.e,solids,liquids nd gases.
- E.g, sound waves, vibrations in a spring.



A sound wave is set up in an air filled pipe by moving a piston back and forth. As the oscillations of an element of air are parallel to the direction in which the wave travels, the wave is a longitudinal wave.

Capillary waves and Gravity waves

The waves on the surface of water are of two kinds: capillary waves and gravity waves.

Capillary waves

Capillary waves are ripples of fairly short wavelength, not more than a few centimetres. The restoring force that produces them is the surface tension of water.


Gravity waves

Gravity waves have wavelengths typically ranging from several metres to several hundred metres. The restoring force that produces these waves is the pull of gravity, which tends to keep the water surface at its lowest level.

Travelling or Progressive Wave

A wave, transverse or longitudinal, is said to be travelling or progressive if it travels from one point of the medium to another.

Displacement Relation in a Progressive Wave along a String (transverse wave)

A progressive wave travelling along the positive direction of the x-axis can be represented as

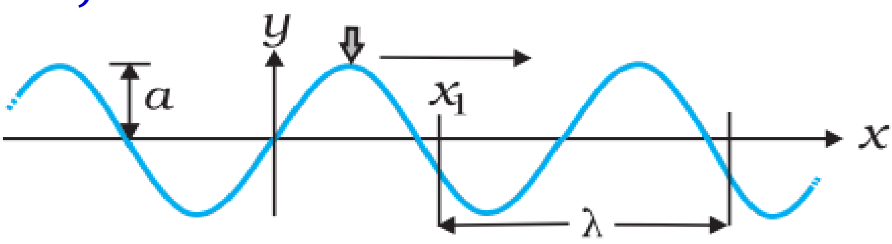
$$y(x,t) = a \sin(kx - \omega t + \phi)$$

A progressive wave travelling along the negative direction of the x-axis can be represented as

$$y(x,t) = a \sin(kx + \omega t + \phi)$$

$$\overbrace{y\,(x,\,t)}^{\textit{Displacement}} = \overbrace{a}^{\textit{Amplitude}} \sin\overbrace{(\underset{\substack{\uparrow \\ \textit{Angular} \\ \textit{Wave} \\ \textit{Number}}}{kx} - \underset{\substack{\uparrow \\ \textit{Angular} \\ \textit{Frequency}}}{\omega t} + \underset{\substack{\uparrow \\ \textit{Initial} \\ \textit{Phase} \\ \textit{Angle}}}{\phi})}^{\text{Phase}}$$

Graphical variation of displacement wih time for a progressive wave (Transverse wave)



Crest

A point of maximum positive displacement in a wave, is called crest.

Trough

A point of maximum negative displacement is called trough.

Amplitude

The amplitude a of a wave is the magnitude of the maximum displacement of the elements from their equilibrium positions as the wave passes through them.

Since ‘**a**’ is a magnitude, it is a positive quantity, even if the displacement is negative.

Phase

The phase of the wave is the argument $(kx - \omega t + \phi)$ of the oscillatory term sine . It describes the state of motion as the wave sweeps through a string element at a particular position x. It changes linearly with time t.

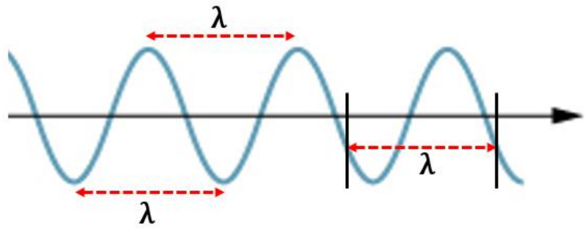
Phase Constant

The constant ϕ is called the initial phase angle. The value of ϕ is determined by the initial displacement(at, $t=0$)and velocity of the element (at, $x = 0$).

Wavelength and Angular Wave Number

Wavelength

The wavelength λ of a wave is the minimum distance between two consecutive troughs or crests or two consecutive points in the same phase of wave motion.



Propagation Constant or Angular Wave Number

Propagation constant or Angular Wave Number is defined as

$$k = \frac{2\pi}{\lambda}$$

Its SI unit is radian per metre or rad m^{-1}

Period, Angular Frequency and Frequency

Period

The period of oscillation T of a wave is defined as the time taken by any element to complete one oscillation.

Angular Frequency

Angular Frequency of a wave is given by

$$\omega = \frac{2\pi}{T}$$

Its SI unit is rad s^{-1}

From this equation, $T = \frac{2\pi}{\omega}$

Frequency

It is the number of oscillations per unit time made by an element as the wave passes through it.

Frequency is the reciprocal of period.

$$\nu = \frac{1}{T}$$

$$\nu = \frac{\omega}{2\pi}$$

It is usually measured in hertz.

Example

A wave travelling along a string is described by,
y(x, t) = 0.005 sin (80.0 x – 3.0 t), in which the numerical constants are in SI units (0.005 m, 80.0 rad m^{-1}, and 3.0 rad s^{-1}). Calculate

(a) the amplitude,
(b) the wavelength,
(c) the period and frequency of the wave.
(d) Calculate the displacement y of the wave at a distance x = 30.0 cm and time t = 20 s ?

Answer

$$y(x, t) = 0.005 \sin (80.0 x - 3.0 t)$$

The general expression for a travelling wave is

$$y (x,t) = a \sin (kx - \omega t + \phi)$$

Comparing these equations

(a) Amplitude , **a=0.005m**

(b) k=80 rad m^{-1}

but , $k=\frac{2\pi}{\lambda}$

$$\frac{2\pi}{\lambda} = 80$$

$$\lambda = \frac{2\pi}{80} = 0.0785 \text{ m}$$

$$\boldsymbol{\lambda = 7.85\ cm}$$

(c) ω=3

$$\frac{2\pi}{T} = 3$$

$$T = \frac{2\pi}{3} = \mathbf{2.09\ s}$$

Frequency, $\nu = 1/T$ =1/2.09

$$\boldsymbol{\nu} = \mathbf{0.48\ Hz}$$

(d) y(x, t) = 0.005 sin (80.0 x – 3.0 t)

x = 30.0 cm =0.3m

t = 20 s

y(x, t) = 0.005 sin (80.0 x 0.3 – 3.0 x20)

= (0.005 m) sin (–36)

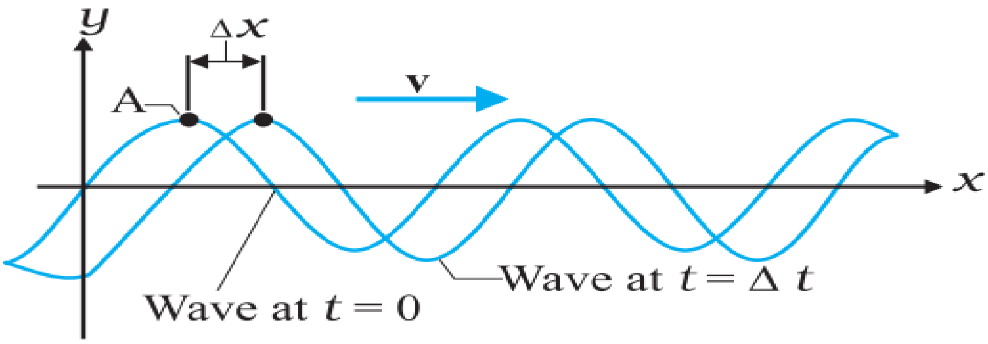
= (0.005 m) sin ($-36 + 12\pi$)

12 π is added ,so tht (–36 + 12 π) becomes positive

= (0.005 m) sin (1.699)

= (0.005 m) sin (97^0) = 5 mm

The Speed of a Travelling Wave



Consider a wave propagating in positive x direction with initial phase $\phi = 0$

$$y(x, t) = a \sin(kx - \omega t)$$

As the wave moves, each point of the waveform (say A) retains its displacement y. This is possible only when the argument $(kx - \omega t)$ is constant.

$$(kx - \omega t) = \text{constant}$$

$$\frac{d}{dt}(kx - \omega t) = 0$$

$$k\frac{dx}{dt} - \omega\frac{dt}{dt} = 0$$

$$\frac{dx}{dt} = \frac{\omega}{k}$$

$$\mathbf{v} = \frac{\boldsymbol{\omega}}{\boldsymbol{k}}$$

$$\omega = 2\pi\nu \ , \quad k = \frac{2\pi}{\lambda}$$

$$v = \frac{2\pi\nu}{\frac{2\pi}{\lambda}}$$

$$\mathbf{v = \nu\lambda}$$

This is a general relation valid for all progressive waves.

The speed of a wave is related to its wavelength and frequency , but it is determined by the properties of the medium.

Speed of a Transverse Wave on Stretched String

The speed of transverse waves on a string is determined by two factors,

(i) the linear mass density or mass per unit length, μ, and

(ii) the tension T

$$\mathbf{v} = \sqrt{\frac{T}{\mu}}$$

The speed of a wave along a stretched ideal string does not depend on the frequency of the wave.

Example

A steel wire 0.72 m long has a mass of 5.0×10^{-3} kg. If the wire is under a tension of 60 N, what is the speed of transverse waves on the wire ?

$$v=\sqrt{\frac{T}{\mu}}$$

$$\mu = \frac{M}{l}$$

$$=\frac{5.0 \times 10^{-3}}{0.72}$$

$$= 6.9 \times 10^{-3} \text{ kg } m^{-1}$$

$$T = 60 \text{ N}$$

$$v=\sqrt{\frac{60}{6.9 \times 10^{-3}}} = 93 \text{ m } s^{-1}$$

Speed of a Longitudinal Wave(Speed of Sound)

The longitudinal waves in a medium travel in the form of compressions and rarefactions or changes in density, ρ.

- **The speed of propagation of a longitudinal wave in a fluid**

$$\mathbf{v}=\sqrt{\frac{B}{\rho}}$$

B= the bulk modulus of medium

ρ = the density of the medium

- **The speed of a longitudinal wave in a solid bar**

$$\mathbf{v}=\sqrt{\frac{Y}{\rho}}$$

Y =Young's modulus

ρ=density of the medium,

- **The speed of a longitudinal wave in an ideal gas**

Case1 -Newtons Formula

Newton assumed that, the pressure variations in a medium during propagation of sound are isothermal.

$$\mathbf{v}=\sqrt{\frac{B}{\rho}}$$

For isothermal process

$$PV = \text{constant}$$

$$V\Delta P + P\Delta V = 0$$

$$V\Delta P = -P\Delta V$$

$$-\frac{V\Delta P}{\Delta V} = P$$

$$B = P$$

$$v=\sqrt{\frac{P}{\rho}}$$

This relation was first given by Newton and is known as **Newton's formula.**

Case 2- Laplace correction to Newton's formula.

Laplace that the pressure variations in the propagation of sound waves are so fast that there is little time for the heat flow to maintain constant temperature. These variations, therefore, are adiabatic and not isothermal.

$$\mathbf{v}=\sqrt{\frac{\mathbf{B}}{\boldsymbol{\rho}}}$$

For adiabatic processes

$$PV^{\gamma} = \text{constant}$$

$$\Delta\, PV^{\gamma} = 0$$

$$P\gamma V^{\gamma-1}\Delta V + V^{\gamma}\,\Delta\, P = 0$$

$$\gamma P\, V^{\gamma-1}\Delta V = -V^{\gamma}\,\Delta\, P$$

$$\gamma\, P = -\frac{V^{\gamma}\Delta P}{V^{\gamma-1}\Delta V}$$

$$\gamma\, P = -\,\frac{\Delta P}{V^{-1}\Delta V}$$

$$\gamma\, P = -\,\frac{V\Delta P}{\Delta V} = B$$

$$B = \gamma\, P$$

$$\mathbf{v}=\sqrt{\frac{\boldsymbol{\gamma}\,\mathbf{P}}{\boldsymbol{\rho}}}$$

This modification of Newton's formula is referred to as the **Laplace correction.**

$$\gamma = \frac{C_P}{C_V} \qquad , \quad \text{For air } \gamma = \frac{7}{5}.$$

The speed of sound in air at STP = 331.3 m s^{-1}

The Principle of Superposition of Waves

When two or more waves overlap, the resultant displacement is the algebraic sum of the displacements due to each wave.

let y_1(x, t) and y_2 (x, t) be the displacements individual waves, then resultant displacement y (x, t) is,

$$\mathbf{y\,(x,t) = y_1(x,t) + y_2\,(x,t)}$$

Consider two wave travelling in positive x direction having same amplitude, same angular frequency and wavenumber and therefor same wavelength and speed. The waves differ only in their initial phase ϕ

$$y_1(x, t) = a \sin (kx - \omega t)$$
$$y_2(x, t) = a \sin (kx - \omega t + \phi)$$

Now, applying the superposition principle, the resultant displacement is

$$y(x, t) = a \sin (kx - \omega t) + a \sin (kx - \omega t + \phi)$$

$$\sin A + \sin B = 2 \sin \left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

Applying this relation the resultant displacement is,

$$\mathbf{y(x, t) = [2a \cos \tfrac{\phi}{2}] \sin (kx - \omega t + \tfrac{\phi}{2}) \text{ ----------(1)}}$$

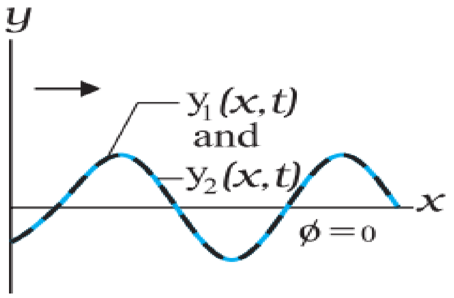
The resultant wave is also a sinusoidal wave, travelling in the positive direction of x-axis.

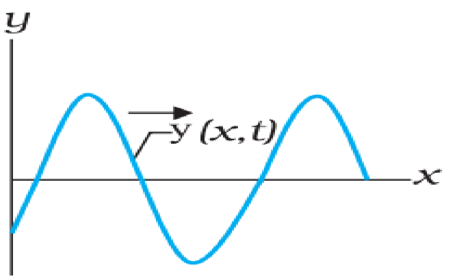
Initial phase of resultant wave = $\frac{\phi}{2}$

Amplitude of resultant wave is

$$\mathbf{A = 2a \cos \tfrac{\phi}{2} \text{ ------------(2)}}$$

If $\phi = 0$, i.e. ,the two waves are in phase,



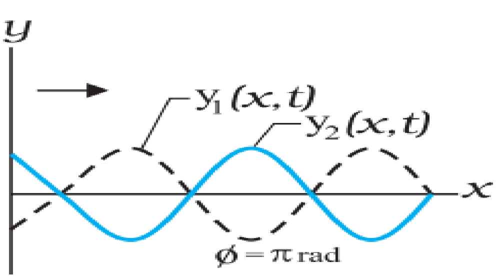


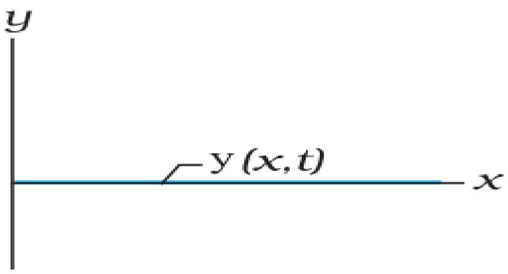
Resultant displacement, $\mathbf{y(x,t) = 2a \sin (kx - \omega t)}$

Amplitude , $\mathbf{A = 2a}$

which is the largest possible value of Amplitude A.

If $\phi = \pi$, i.e. the two waves are in phase,



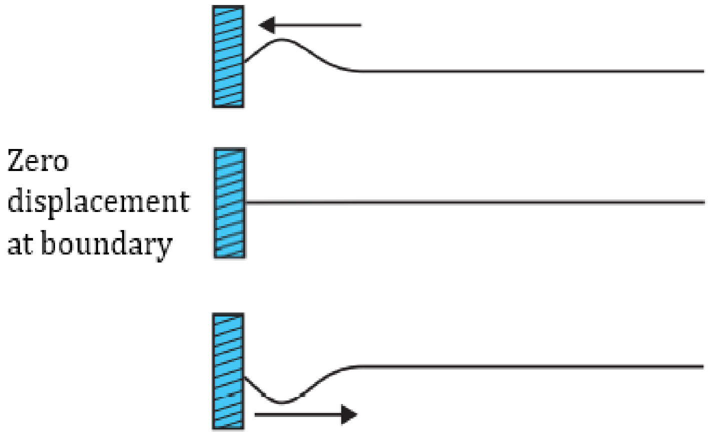


$$\mathbf{y(x, t) = 0}$$

$$\mathbf{A=0}$$

Reflection of Wave

The reflection at a rigid boundary



- The reflected wave will have a phase reversal i.e, a phase difference of π radian or 180^0.
- There will be no displacement at the boundary as the string is fixed there.

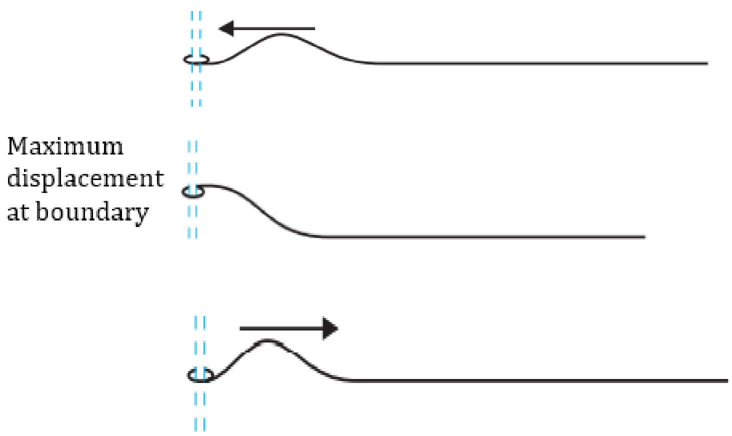
Incident wave, $\boldsymbol{y_i(x, t) = a \sin (kx - \omega t)}$

Reflected wave, $y_r(x, t) = a \sin (kx + \omega t + \pi)$

sin(180+θ)=-sin θ

$$\boldsymbol{y_r(x, t) = - a \sin (kx + \omega t)}$$

The Reflection at an Open Boundary



- The reflected wave will have same sign (no phase reversal) and amplitude as the incident wave.
- There will be maximum displacement at the boundary(twice the amplitude of either of the pulses)

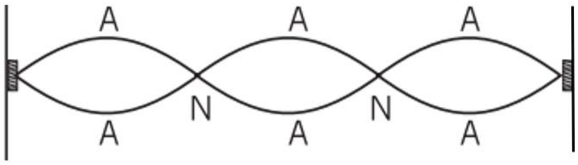
Incident wave, $\boldsymbol{y_i(x, t) = a \sin (kx - \omega t)}$

Reflected wave, $\boldsymbol{y_r(x, t) = a \sin (kx + \omega t)}$.

Standing Waves and Normal Modes

Standing Waves

The interference of two identical waves moving in opposite directions produces standing waves.



Wave travelling in the positive direction of x-axis

$$y_1(x, t) = a \sin (kx - \omega t)$$

Wave travelling in the negative direction of x-axis

$$y_2(x, t) = a \sin (kx + \omega t)$$

By the principle of superposition

$$y(x, t) = y_1(x, t) + y_2(x, t) = a \sin (kx - \omega t) + a \sin (kx + \omega t)$$

$$y(x, t) = (2a \sin kx) \cos \omega t$$

This equation represents a standing wave, a wave in which the waveform does not move.

Amplitude of wave , $A = 2a \sin kx.$

Nodes and Antinodes

The positions of zero amplitude in a staning wave are called nodes and the positions of maximum amplitude are called antinodes.

Condition for Nodes

At nodes,the amplitude of standing wave is zero

$$2a \sin kx = 0$$

$$\sin kx = 0$$

$$kx = n\pi, \text{ for } n = 0, 1, 2, 3, ..$$

$$\text{But } k = \frac{2\pi}{\lambda}$$

$$\frac{2\pi}{\lambda} x = n\pi$$

$$x = n\frac{\lambda}{2}, \quad \text{for } n = 0, 1, 2, 3, ..$$

i.e., nodes are formed at locations $x=0, \frac{1\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, ...$

The nodes are separated by $\lambda/2$ and are located half way between pairs of antinodes.

Condition for Antinodes

At antinodes, the amplitude of standing wave is maximum.

$$2a \sin kx = \text{maximum}$$

$$\sin kx = \pm 1$$

$$kx = (n + \frac{1}{2})\ \pi, \text{ for } n = 0, 1, 2, 3, ..$$

$$\text{but } , k = \frac{2\pi}{\lambda}$$

$$\frac{2\pi}{\lambda} x = (n + \frac{1}{2})\ \pi$$

$$\mathbf{x = (n + \frac{1}{2})\ \frac{\lambda}{2}\ , \quad for\ \ n = 0, 1, 2, 3, ..}$$

i.e., antinodes are formed at locations x= $\frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$ **,.........**

The antinodes are separated by $\lambda/2$ and are located half way between pairs of nodes.

(1)Standing waves in a Stretched String fixed at both the ends

For a stretched string of length L, fixed at both ends, the two ends x= 0 and x= L of the string have to be nodes.

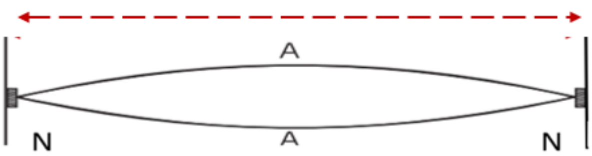
The condition for node at L

$$\mathbf{L = n\frac{\lambda}{2}, \quad for\ n = 1, 2, 3, ...}$$

Fundamental mode or the first harmonic

The oscillation mode with n=1 , the lowest frequency is called the fundamental mode or the first harmonic.

$$\mathbf{L = \frac{\lambda_1}{2}}$$



$$L = \frac{\lambda_1}{2}$$

$$\lambda_1 = 2L$$

$$\text{But } v = \nu\lambda ,$$

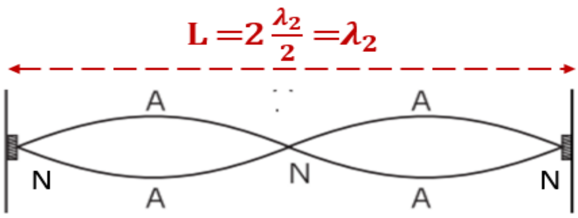
$$\nu = \frac{v}{\lambda}$$

$$\text{Frequency, } \nu_1 = \frac{v}{\lambda_1}$$

$$\boldsymbol{\nu_1 = \frac{v}{2L}} \quad \text{----------(1)}$$

The second harmonic

The second harmonic is the oscillation mode with n = 2.



$$L = 2\frac{\lambda_2}{2} = \lambda_2$$

$$\lambda_2 = L$$

$$\text{Frequency, } \nu_2 = \frac{v}{\lambda_2}$$

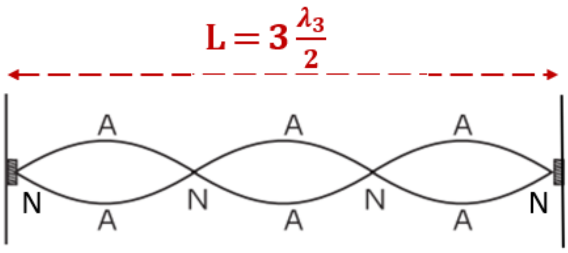
$$\nu_2 = \frac{v}{L}$$

$$\boldsymbol{\nu_2 = 2\frac{v}{2L}} \text{ -------------(2)}$$

$$\boldsymbol{\nu_2 = 2\nu_1}$$

The Third Harmonic

The third harmonic is the oscillation mode with n = 3.



$$L = 3\frac{\lambda_3}{2}$$

$$\lambda_3 = \frac{2L}{3}$$

$$\text{Frequency, } \nu_3 = \frac{v}{\lambda_3}$$

$$\nu_3 = \frac{v}{\frac{2L}{3}}$$

$$\boldsymbol{\nu_3 = 3\frac{v}{2L}} \text{ --------------(3)}$$

$$\boldsymbol{\nu_3 = 3\nu_1}$$

and so on.

$$\boldsymbol{\nu_1 : \nu_2 : \nu_3 = 1 : 2 : 3}$$

Thus all harmonics are possible in a stretched string fixed at both the ends.

(2) The modes of vibration in a closed pipe (system closed at one end and the other end open).

Eg: Resonance Column(Air columns such as glass tubes partially filled with water) .

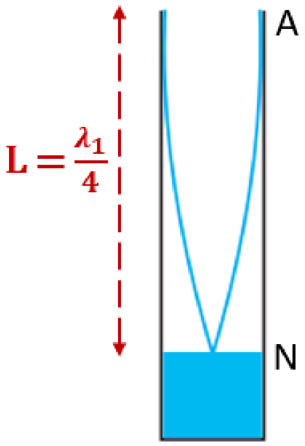
If the length of the air column is L, then closed end x=0 is a node and the open end, x = L, is an antinode.

The condition for antinode at L

$$L=\left(n+\frac{1}{2}\right)\frac{\lambda}{2} \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

Fundamental mode or the first harmonic

The oscillation mode with n=0 , fundamental mode or the first harmonic.



$$L = \frac{\lambda_1}{4}$$

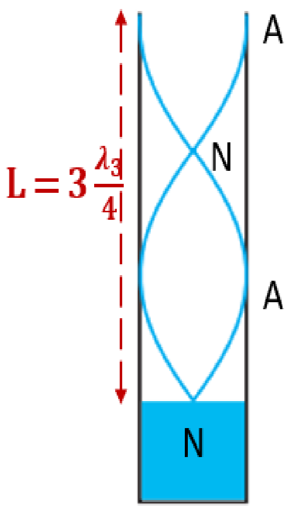
$$\lambda_1 = 4L$$

$$\text{Frequency, } \nu_1 = \frac{v}{\lambda_1}$$

$$\nu_1 = \frac{v}{4L} \text{ ------------(1)}$$

The Third Harmonic

The Third harmonic is the oscillation mode with n = 1.



$$L = 3\frac{\lambda_3}{4}$$

$$\lambda_3 = \frac{4L}{3}$$

$$\text{Frequency, } \nu_3 = \frac{v}{\lambda_3}$$

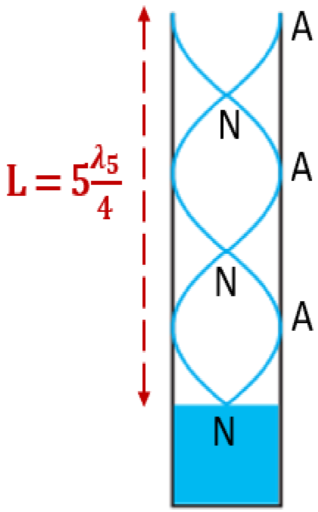
$$\nu_3 = \frac{v}{\frac{4L}{3}}$$

$$\nu_3 = 3\frac{v}{4L} \text{ --------------(2)}$$

$$\nu_3 = 3\nu_1$$

The Fifth Harmonic

The Fifth harmonic is the oscillation mode with n = 2.



$$L = 5\frac{\lambda_4}{4}$$

$$\lambda_4 = \frac{4L}{5}$$

$$\text{Frequency, } \nu_5 = \frac{v}{\lambda_5}$$

$$\nu_5 = \frac{v}{\frac{4L}{5}}$$

$$\boldsymbol{\nu_5 = 5\frac{v}{4L}}$$

$$\boldsymbol{\nu_5 = 5\nu_1}$$

And so on.

$$\boldsymbol{\nu_1 : \nu_3 : \nu_5 = 1 : 3 : 5}$$

Thus only odd harmonics are possible in a closed pipe.

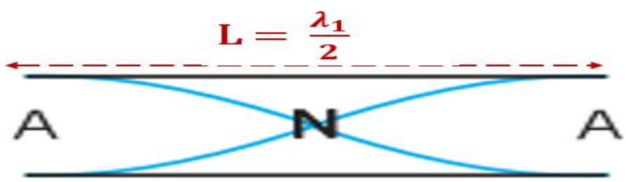
(3) The modes of vibration in a an open pipe (system open at both ends). Eg: Flute

For an open pipe of length L, antinodes are formed at both ends.

$$L = n\frac{\lambda}{2}, \quad \text{for } n = 1, 2, 3, \ldots$$

Fundamental Mode or The First Harmonic

The oscillation mode with n=1 , the lowest frequency is called the fundamental mode or the first harmonic.



$$L = \frac{\lambda_1}{2}$$

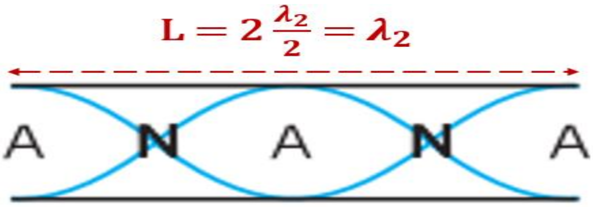
$$\lambda_1 = 2L$$

$$\text{Frequency, } \nu_1 = \frac{v}{\lambda_1}$$

$$\boldsymbol{\nu_1 = \frac{v}{2L}} \text{ ------------(1)}$$

The Second Harmonic

The second harmonic is the oscillation mode with n = 2.



$$L = 2\frac{\lambda_2}{2}$$

$$\lambda_2 = L$$

$$\text{Frequency,}\ \nu_2 = \frac{v}{\lambda_2}$$

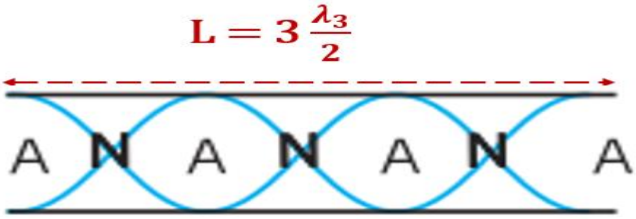
$$\nu_2 = \frac{v}{L}$$

$$\nu_2 = 2\frac{v}{2L} \text{------------(2)}$$

$$\nu_2 = 2\nu_1$$

The Third Harmonic

The third harmonic is the oscillation mode with n = 3.



$$L = 3\frac{\lambda_3}{2}$$

$$\lambda_3 = \frac{2L}{3}$$

$$\text{Frequency,}\ \nu_3 = \frac{v}{\lambda_3}$$

$$\nu_3 = \frac{v}{\frac{2L}{3}}$$

$$\nu_3 = 3\frac{v}{2L} \text{------------(3)}$$

$$\nu_3 = 3\nu_1$$

and so on.

$$\nu_1 : \nu_2 : \nu_3 = 1 : 2 : 3$$

Thus all harmonics are possible in an open pipe.

So open pipes are preferred over closed pipes in musical instruments.

Beats

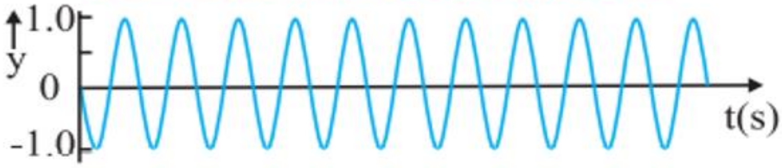
The periodic variations(wavering) of sound intensity when two waves of nearly same frequencies and amplitudes travelling in the same direction, are superimposed on each other is called beats.

These wavering of sound is also called waxing and waning.

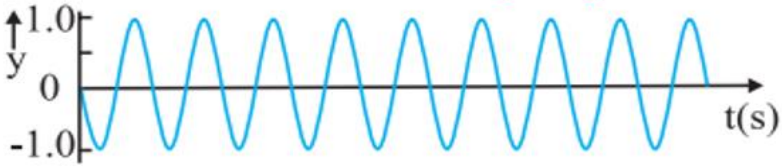
If v_1 *and* v_2 are the frequencies of superposing waves, the beat frequency

$$\boldsymbol{\nu_{beat} = \nu_1 - \nu_2}$$

A harmonic wave of frequency 11 Hz.



A harmonic wave of frequency 9 Hz



Superposition of two wave, producing beats of frequency 2Hz