TRIGONOMETRY

CONTENTS

- Right Angle Triangle
- Trigonometric Ratio (T.R.) of some Specific Angles
- Trigonometric Ratios of Complementary Angles
- Trigonometric Identities

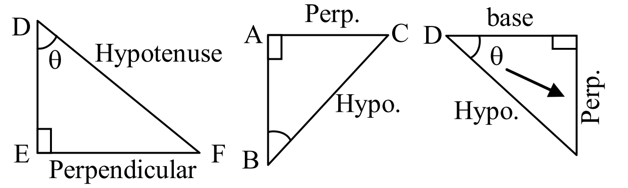
Trigonometry is the branch of mathematics in which we study of relationships between the sides & angles of a triangle.

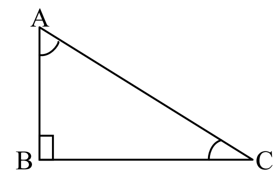
Fact :	In Greek words :
	Tri = three
	gon = sides
	metron = measure

The ratio of sides of a right angle triangle with respect to acute angles are called "Trigonometric ratios of the angle".

➢ RIGHT ANGLE TRIANGLE

1. A Δ having one angle equal to 90º is called right angle Δ.
2. The sum of other two acute (Less than 90º) angles is 90º. (or both acute angles are complementary)
3. The side opposite to 90º, is called hypotenuse, it is longest side in Δ.
4. The side opposite to given one acute angle is perpendicular.
5. The rest (IIIrd) side is base.



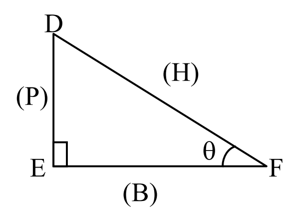


	Hypotenuse	Perpendicular	Base
for ∠A	AC	BC	AB
for ∠C	AC	AB	BC

The trigonometry ratio are

sine of ∠θ, cosine of ∠θ, tangent of ∠θ, cotangent of ∠θ, secant of ∠θ, cosecant of ∠θ.

These ratios are abbreviated as sin θ, cos θ, tan θ, cot θ, sec θ, cosec θ and the relation with sides are



sin θ	= P/H = DE/DF
cos θ	= B/H = EF/DF
tan θ	= P/B = DE/EF
cot θ	= B/P = EF/DE
sec θ	= H/B = DF/EF
cosec θ	= H/P = DF/DE

By above table $\sin\theta = \frac{1}{\text{cosec}\,\theta}$, $\cos\theta = \frac{1}{\sec\theta}$,

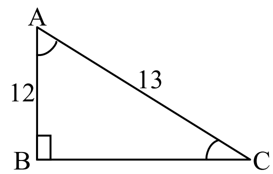
$\tan\theta = \frac{1}{\cot\theta}$ also $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{P/H}{B/H} = \frac{P}{B}$

∴ we can say ‘‘Trigonometric Ratio’’ represents ratio between acute angles & sides of triangle.

❖ EXAMPLES ❖

Ex.1 If ABC is right angle triangle, $\angle B = 90^\circ$, AB = 12 cm, AC = 13 cm then find sin A and cos C.

Sol. Using Pythagoras theorem



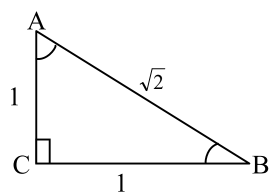
$BC = \sqrt{AC^2 - AB^2} = \sqrt{169 - 144} = 5$ cm

$\therefore \sin A = \dfrac{BC}{AC} = \dfrac{5}{13}$

$\cos C = \dfrac{AB}{AC} = \dfrac{12}{13}$ **Ans.**

Ex.2 If $\sin A = \dfrac{1}{\sqrt{2}}$ in right triangle ABC, then find value of tan A, cosec A, tan B, cosec B.

Sol.



Θ $\sin A = \dfrac{1}{\sqrt{2}} = \dfrac{BC}{AB}$

$\therefore AC = \sqrt{AB^2 - BC^2} = \sqrt{(\sqrt{2}k)^2 - (k)^2}$

$= \sqrt{2k^2 - k^2} = \sqrt{k^2} = k$

$\therefore \tan A = \dfrac{BC}{AC} = \dfrac{k}{k} = 1$

$\text{cosec } A = \dfrac{1}{\sin A} = \dfrac{\sqrt{2}k}{k} = \sqrt{2}$

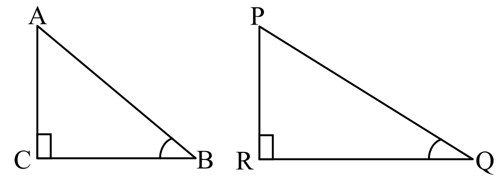
$\tan B = \dfrac{AC}{BC} = \dfrac{k}{k} = 1$

$\text{cosec } B = \dfrac{AB}{AC} = \dfrac{\sqrt{2}k}{k} = \sqrt{2}$

Ex.3 If $\angle B$ and $\angle Q$ are acute angles such that sin B = sin Q, then prove that $\angle B = \angle Q$.

[NCERT]

Sol. Let us consider two right triangles ABC and PQR where sin B = sin Q.



We have $\sin B = \dfrac{AC}{AB}$

and $\sin Q = \dfrac{PR}{PQ}$

Then $\dfrac{AC}{AB} = \dfrac{PR}{PQ}$

Therefore, $\dfrac{AC}{PR} = \dfrac{AB}{PQ} = k$, say ....(1)

Now, using Pythagoras theorem,

$BC = \sqrt{AB^2 - AC^2}$

and $QR = \sqrt{PQ^2 - PR^2}$

So, $\dfrac{BC}{QR} = \dfrac{\sqrt{AB^2 - AC^2}}{\sqrt{PQ^2 - PR^2}}$

$= \dfrac{\sqrt{k^2PQ^2 - k^2PR^2}}{\sqrt{PQ^2 - PR^2}}$

$= \dfrac{k\sqrt{PQ^2 - PR^2}}{\sqrt{PQ^2 - PR^2}} = k$ .....(2)

From (1) and (2), we have

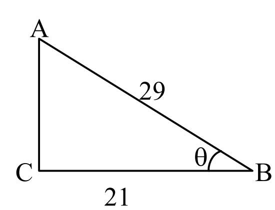
$\dfrac{AC}{PR} = \dfrac{AB}{PQ} = \dfrac{BC}{QR}$

Then, by using Theorem, $\Delta ACB \sim \Delta PRQ$ and therefore, $\angle B = \angle Q$.

Ex.4 Consider ΔACB, right-angled at C, in which AB = 29 units, BC = 21 units and $\angle ABC = \theta$ (see figure). Determine the value of

(i) $\cos^2\theta + \sin^2\theta$,

(ii) $\cos^2\theta - \sin^2\theta$ **[NCERT]**



Sol. In ΔACB, we have

$$AC = \sqrt{AB^2 - BC^2} = \sqrt{(29)^2 - (21)^2}$$

$$= \sqrt{(29-21)(29+21)} = \sqrt{(8)(50)}$$

$$= \sqrt{400} = 20 \text{ units}$$

So, $\sin\theta = \frac{AC}{AB} = \frac{20}{29}$, $\cos\theta = \frac{BC}{AB} = \frac{21}{29}$

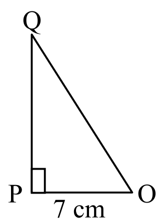
Now, (i) $\cos^2\theta + \sin^2\theta = \left(\frac{20}{29}\right)^2 + \left(\frac{21}{29}\right)^2$

$$= \frac{20^2 + 21^2}{29^2} = \frac{400+441}{841} = 1,$$

and (ii) $\cos^2\theta - \sin^2\theta = \left(\frac{21}{29}\right)^2 - \left(\frac{20}{29}\right)^2$

$$= \frac{(21+20)(21-20)}{29^2} = \frac{41}{841}.$$

Ex.5 In Δ OPQ, right-angled at P, OP = 7 cm and OQ – PQ = 1 cm (see figure). Determine the values of sin Q and cos Q. **[NCERT]**



Sol. In ΔOPQ, we have

$OQ^2 = OP^2 + PQ^2$

i.e $(1 + PQ)^2 = OP^2 + PQ^2$

i.e. $1 + PQ^2 + 2PQ = OP^2 + PQ^2$

i.e. $1 + 2PQ = 7^2$

i.e. PQ = 24 cm and OQ = 1 + PQ = 25 cm

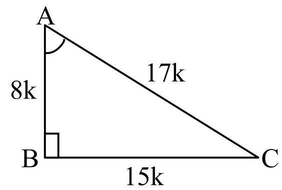
So, $\sin Q = \frac{7}{25}$ and $\cos Q = \frac{24}{25}$

Note :

1. The values of sin θ & cos θ are always less than or equal to 1 & greater than or equal to –1.
2. Value of tan θ & cot θ lie between $-\infty$ to $+\infty$.
3. sin A, cos A, etc. are not product of sin and A.
4. $(\sin A)^2 \neq \sin A^2$ etc.

Ex.6 Given 15 cot A = 8, find sin A and sec A. **[NCERT]**

Sol. $\cot A = \frac{8}{15} = \frac{\text{base}}{\text{perpendicular}}$



$$AC = \sqrt{AB^2 + BC^2} = \sqrt{64k^2 + 225k^2}$$

$$= \sqrt{289k^2} = 17k$$

$$\sin A = \frac{BC}{AC} = \frac{15k}{17k} = \frac{15}{17}$$

$$\sec A = \frac{AC}{AB} = \frac{17k}{8k} = \frac{17}{8}$$ **Ans.**

Ex.7 Given $\sec\theta = \frac{13}{12}$, calculate all other trigonometric ratios. **[NCERT]**

Sol. Θ $\sec\theta = \frac{13}{12} = \frac{\text{Hypotenuse}}{\text{Base}}$

$\therefore$ perpendicular $= \sqrt{(13k)^2 - (12k)^2}$

$$= \sqrt{(169-144)k^2} = 5\text{ k}$$

$\therefore$ $\sin\theta = \frac{P}{H} = \frac{5k}{13k} = \frac{5}{13}$

$\cos\theta = \frac{1}{\sec\theta} = \frac{12}{13}$

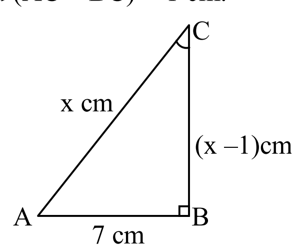
$\tan\theta = \frac{P}{B} = \frac{5k}{12k} = \frac{5}{12}$

$\cot\theta = \frac{B}{P} = \frac{12k}{5k} = \frac{12}{5}$

$\text{cosec}\,\theta = \frac{H}{P} = \frac{13k}{5k} = \frac{13}{5}$

Ex.8 In ΔABC, right –angled at B, AB = 7 cm and (AC – BC) = 1 cm. Find the values of sin C and cos C.

Sol. Consider ΔABC in which $\angle$B = 90º, AB = 7 cm and (AC – BC) = 1 cm.



Let AC = x cm.

Then, BC = (x – 1) cm

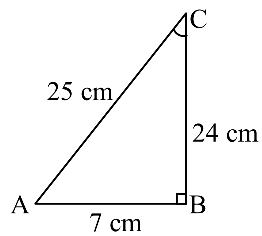
By Pythagoras theorem, we have :

$AB^2 + BC^2 = AC^2 \Rightarrow (7)^2 + (x - 1)^2 = x^2$

$\Rightarrow$ $49 + x^2 - 2x + 1 = x^2$

$\Rightarrow$ $2x = 50$

$\Rightarrow$ $x = 25$



$\therefore$ AC = 25 cm, BC = (25 – 1)cm = 24 cm and AB = 7 cm.

For T-ratios of $\angle C$, we have

base = BC = 24 cm,

perpendicular = AB = 7 cm and

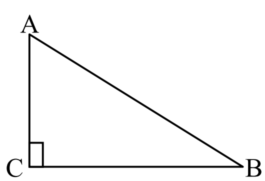
hypotenuse = AC = 25 cm.

$\therefore \sin C = \frac{AB}{AC} = \frac{7}{25}$ and $\cos C = \frac{BC}{AC} = \frac{24}{25}$.

Ex.9 If $\angle A$ and $\angle B$ are acute angles such that $\cos A = \cos B$, then show that $\angle A = \angle B$. **[NCERT]**

Sol. Θ $\cos A = \cos B$

$\frac{AC}{AB} = \frac{BC}{AB}$



$\therefore$ AC = BC

$\therefore$ Δ is an isosceles Δ

$\therefore$ $\angle A = \angle B$ Proved.

Ex.10 If $\cot\theta = \frac{7}{8}$, evaluate : **[NCERT]**

(i) $\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$, (ii) $\cot^2\theta$

Sol. Θ $\cot\theta = \frac{7}{8} = \frac{\text{Base}}{\text{Perpendicular}} = \frac{B}{P}$

$\therefore H = \sqrt{(8k)^2 + (7k)^2} = \sqrt{(64+49)}k$

$= \sqrt{113}\,k$

$\therefore \sin\theta = \frac{P}{H} = \frac{8k}{\sqrt{113}k} = \frac{8}{\sqrt{113}}$

$\cos\theta = \frac{B}{H} = \frac{7k}{\sqrt{113}k} = \frac{7}{\sqrt{113}}$

(i) $\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$

$$= \frac{\left(1+\frac{8}{\sqrt{113}}\right)\left(1-\frac{8}{\sqrt{113}}\right)}{\left(1+\frac{7}{\sqrt{113}}\right)\left(1-\frac{7}{\sqrt{113}}\right)}$$

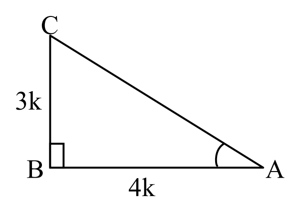
$$= \frac{(\sqrt{113}+8)(\sqrt{113}-8)}{(\sqrt{113}+7)(\sqrt{113}-7)}$$

$$= \frac{113-64}{113-49} = \frac{49}{64}$$ **Ans.**

(ii) $\cot^2\theta = \left(\frac{B}{P}\right)^2 = \left(\frac{7k}{8k}\right)^2 = \frac{49}{64}$ **Ans.**

Ex.11 If $3\cot A = 4$, check whether $\frac{1-\tan^2 A}{1+\tan^2 A} = \cos^2 A - \sin^2 A$ or not. **[NCERT]**

Sol. Θ $\cot A = \frac{4}{3}$ $\therefore \tan A = \frac{3}{4}$



$\therefore AC = \sqrt{AB^2 + BC^2} = \sqrt{16k^2 + 9k^2}$

$= \sqrt{25k^2} = 5k$

$\therefore \sin A = \frac{3k}{5k} = \frac{3}{5}$

$\cos A = \frac{4k}{5k} = \frac{4}{5}$

$LHS = \frac{1-\tan^2 A}{1+\tan^2 A}$

$$= \frac{1-\left(\frac{3}{4}\right)^2}{1+\left(\frac{3}{4}\right)^2} = \frac{1-\frac{9}{16}}{1+\frac{9}{16}}$$

$$= \frac{(16-9)/16}{(16+9)/16} = \frac{7}{25}$$

RHS = $\cos^2 A - \sin^2 A$

$$= \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$

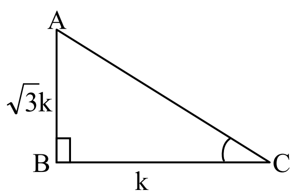
LHS = RHS

Ex.12 In triangle ABC, right-angled at B, if $\tan A = \frac{1}{\sqrt{3}}$, find the value of : **[NCERT]**

(i) sin A cos C + cos A sin C

(ii) cos A cos C – sin A sin C

Sol. $\tan A = \frac{1}{\sqrt{3}} = \frac{P}{B}$



$$\therefore AC = \sqrt{(\sqrt{3}k)^2 + (k)^2} = \sqrt{3k^2 + k^2} = 2k$$

$$\therefore \sin A = \frac{BC}{AC} = \frac{k}{2k} = \frac{1}{2};$$

$$\sin C = \frac{AB}{AC} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2};$$

$$\cos A = \frac{AB}{AC} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2};$$

$$\cos C = \frac{BC}{AC} = \frac{k}{2k} = \frac{1}{2}$$

(i) sin A cos C + cos A sin C

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{4} + \frac{3}{4} = 1$$

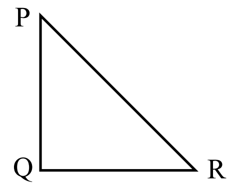
(ii) cos A cos C – sin A sin C

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$$

Ex.13 In ΔPQR, right-angled at Q, PR + QR = 25 cm and PQ = 5 cm. Determine the values of sin P, cos P and tan P. **[NCERT]**

Sol.



$\because$ PR + QR = 25 cm

PQ = 5 cm

Let PR = x cm

$\therefore$ QR = (25 – x) cm

Using Pythagoras theorem

$PR^2 = PQ^2 + QR^2$

$x^2 = 5^2 + (25 - x)^2$

$x^2 = 25 + 625 + x^2 - 50x$

$\Rightarrow$ 50x = 650

$\Rightarrow$ x = 13 cm = PR

$\therefore$ QR = 25 – 13 = 12 cm.

$$\sin P = \frac{QR}{PR} = \frac{12k}{13k} = \frac{12}{13}$$

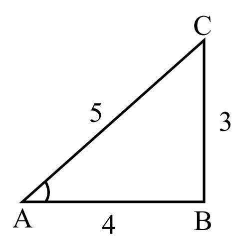
$$\cos P = \frac{PQ}{PR} = \frac{5k}{13k} = \frac{5}{13}$$

$$\tan P = \frac{QR}{PQ} = \frac{12k}{5k} = \frac{12}{5}$$ **Ans.**

Ex.14 If $\sin A = \frac{3}{5}$, find cos A and tan A.

Sol. Since $\sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{3}{5}$, so

We draw a triangle ABC, right angled at B such that



Perpendicular = BC = 3 units,

and, Hypotenuse = AC = 5 units.

By Pythagoras theorem, we have

$AC^2 = AB^2 + BC^2$

$\Rightarrow$ $5^2 = AB^2 + 3^2$

$\Rightarrow$ $AB^2 = 5^2 - 3^2$

$\Rightarrow$ $AB^2 = 16$ $\Rightarrow$ AB = 4

When we consider the t-ratio of ∠A, we have

Base = AB = 4, Perpendicular = BC = 3, Hypotenuse = AC = 5.

$\therefore \quad \cos A = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{4}{5}$

and, $\tan A = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{3}{4}$

Ex.15 If cosec A = $\sqrt{10}$, find other five trigonometric ratios.

Sol. Since cosec A = $\dfrac{\text{Hypotenuse}}{\text{Perpendicular}} = \dfrac{\sqrt{10}}{1}$,

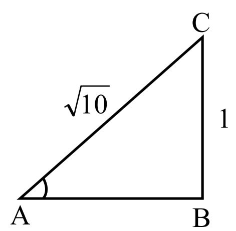
so we draw a right triangle ABC, right angled at B such that

Perpendicular = BC = 1 unit. and,

Hypotenuse = AC = $\sqrt{10}$ units.

By Pythagoras theorem, we have

$AC^2 = AB^2 + BC^2$



$\Rightarrow \quad (\sqrt{10})^2 = AB^2 + 1^2$

$\Rightarrow \quad AB^2 = 10 - 1 = 9$

$\Rightarrow \quad AB = \sqrt{9} = 3$

When we consider the trigonometric ratios of ∠A, we have

Base = AB = 3, Perpendicular = BC = 1, and Hypotenuse = AC = $\sqrt{10}$.

$\therefore \quad \sin A = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{1}{\sqrt{10}}$;

$\cos A = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{3}{\sqrt{10}}$;

$\tan A = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{1}{3}$;

$\sec A = \dfrac{\text{Hypotenuse}}{\text{Base}} = \dfrac{\sqrt{10}}{3}$;

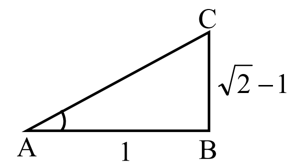
and $\cot A = \dfrac{\text{Base}}{\text{Perpendicular}} = \dfrac{3}{1} = 3$

Ex.16 If $\tan A = \sqrt{2} - 1$, show that $\sin A \cos A = \dfrac{\sqrt{2}}{4}$.

Sol. Since $\tan A = \dfrac{\text{Perpendicular}}{\text{Base}} = \dfrac{\sqrt{2}-1}{1}$, so we draw a right triangle ABC, right angled at B such that Base = AB = 1 and Perpendicular = BC = $\sqrt{2} - 1$.

By Pythagoras theorem, we have

$AC^2 = AB^2 + BC^2$



$\Rightarrow \quad AC^2 = 1^2 + (\sqrt{2}-1)^2$

$\Rightarrow \quad AC^2 = 1 + 2 + 2 - 2\sqrt{2}$

$\Rightarrow \quad AC^2 = 4 - 2\sqrt{2} \Rightarrow AC = \sqrt{4-2\sqrt{2}}$

Now, $\sin A = \dfrac{BC}{AC} = \dfrac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}}$, and

$\cos A = \dfrac{AB}{AC} = \dfrac{1}{\sqrt{4-2\sqrt{2}}}$

$\therefore \sin A \cos A = \dfrac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}} \times \dfrac{1}{\sqrt{4-2\sqrt{2}}}$

$= \dfrac{\sqrt{2}-1}{4-2\sqrt{2}} = \dfrac{\sqrt{2}-1}{2\sqrt{2}(\sqrt{2}-1)} = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$.

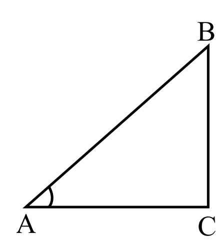
> ◈ $\sin^2\theta = (\sin\theta)^2$
> $\cos^2\theta = (\cos\theta)^2$
> $\tan^2\theta = (\tan\theta)^2$
> $\text{cosec}^2\theta = (\text{cosec}\,\theta)^2$
> $\sec^2\theta = (\sec\theta)^2$
> $\cot^2\theta = (\cot\theta)^2$

❖ EXAMPLES ❖

Ex.17 In a Δ ABC right angled at C, if $\tan A = \dfrac{1}{\sqrt{3}}$ and $\tan B = \sqrt{3}$. Show that

sin A cos B + cos A sin B = 1.

Sol. Let us draw a ΔABC, right angled at C in which $\tan B = \sqrt{3}$ and $\tan A = \dfrac{1}{\sqrt{3}}$.



Now, $\tan A = \frac{1}{\sqrt{3}}$

$\Rightarrow \frac{BC}{AC} = \frac{1}{\sqrt{3}} \quad \left[\Theta \tan A = \frac{BC}{AC}\right]$

$\Rightarrow$ BC = x and AC = $\sqrt{3}$x ....(i)

And, $\tan B = \sqrt{3}$

$\Rightarrow \frac{AC}{BC} = \frac{\sqrt{3}}{1} \quad \left[\Theta \tan B = \frac{AC}{BC}\right]$

$\Rightarrow$ AC = $\sqrt{3}$ x and BC = x ....(ii)

From (i) and (ii), we have

BC = x, AC = $\sqrt{3}$ x

By Pythagoras theorem, we have

$AB^2 = AC^2 + BC^2$

$\Rightarrow AB^2 = (\sqrt{3}x)^2 + x^2$

$\Rightarrow AB^2 = 3x^2 + x^2$

$\Rightarrow AB^2 = 4x^2$

$\Rightarrow AB = 2x$

When we find the t-rations of ∠A, we have Base = AC = $\sqrt{3}$ x, Perpendicular = BC = x, and Hypotenuse = AB = 2x.

$\therefore \sin A = \frac{BC}{AB} = \frac{x}{2x} = \frac{1}{2}$ and

$\cos A = \frac{AC}{AB} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}$

When we consider the t-ratios of ∠B, we have Base = BC = x, Perpendicular = AC = $\sqrt{3}$ x, and Hypotenuse = AB = 2x.

$\therefore \cos B = \frac{BC}{AB} = \frac{x}{2x} = \frac{1}{2}$ and

$\sin B = \frac{AC}{AB} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}$

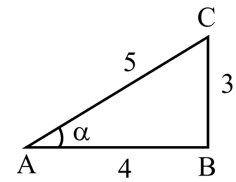
Now,

$\sin A \cos B + \cos A \sin B = \frac{1}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}$

$= \frac{1}{4} + \frac{3}{4} = 1.$

Ex.18 If $\sec \alpha = \frac{5}{4}$, evaluate $\frac{1 - \tan\alpha}{1 + \tan\alpha}$.

Sol. Since $\sec \alpha = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{5}{4}$, so we draw a right triangle ABC, right angled at B such that Hypotenuse = AC = 5 units, Base = AB = 4 units, and ∠BAC = α.



By Pythagoras theorem, we have

$AC^2 = AB^2 + BC^2$

$\Rightarrow 5^2 = 4^2 + BC^2$

$\Rightarrow BC^2 = 5^2 - 4^2 = 9$

$\Rightarrow BC = \sqrt{9} = 3$

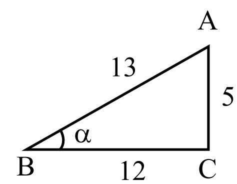
$\therefore \tan \alpha = \frac{BC}{AB} = \frac{3}{4}$

Now, $\frac{1 - \tan\alpha}{1 + \tan\alpha} = \frac{1 - \frac{3}{4}}{1 + \frac{3}{4}} = \frac{\frac{1}{4}}{\frac{7}{4}} = \frac{1}{7}.$

Ex.19 If $\cot B = \frac{12}{5}$, prove that

$\tan^2 B - \sin^2 B = \sin^4 B. \sec^2 B.$

Sol. Since $\cot B = \frac{\text{Base}}{\text{Perpendicular}} = \frac{12}{5}$, so we draw a right triangle ABC, right angled at C such that Base = BC = 12 units.
Perpendicular = AC = 5 units.



By Pythagoras theorem, we have

$AB^2 = BC^2 + AC^2$

$\Rightarrow AB^2 = 12^2 + 5^2 = 169$

$\Rightarrow AB = \sqrt{169} = 13$

$\therefore \sin B = \frac{AC}{AB} = \frac{5}{13}$, $\tan B = \frac{AC}{BC} = \frac{5}{12}$

and $\sec B = \frac{AB}{BC} = \frac{13}{12}$

Now, LHS = $\tan^2 B - \sin^2 B = (\tan B)^2 - (\sin B)^2$

$= \left(\frac{5}{12}\right)^2 - \left(\frac{5}{13}\right)^2 = \frac{25}{144} - \frac{25}{169}$

$= 25\left(\frac{1}{144} - \frac{1}{169}\right) = 25\left(\frac{169 - 144}{144 \times 169}\right)$

$= 25 \times \frac{25}{144 \times 169} = \frac{25 \times 25}{144 \times 169}$

$= \frac{5^2 \times 5^2}{12^2 \times 13^2}$ ....(i) and, RHS = $\sin^4 B \sec^2 B$

$= (\sin B)^4 (\sec B)^2$

$= \left(\frac{5}{13}\right)^4 \times \left(\frac{13}{12}\right)^2$

$= \frac{5^4}{13^2 \times 12^2}$

$= \frac{5^2 \times 5^2}{13^2 \times 12^2}$ ....(ii)

From (i) and (ii), we have

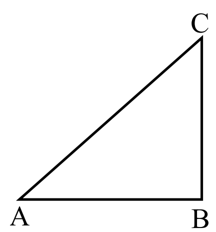
$\tan^2 B - \sin^2 B = \sin^4 B \sec^2 B$.

Ex.20 In a right triangle ABC, right angled at B, the ratio of AB to AC is 1 : $\sqrt{2}$. Find the values of

(i) $\frac{2\tan A}{1+\tan^2 A}$ and (ii) $\frac{2\tan A}{1-\tan^2 A}$

Sol. We have, AB : AC = 1 : $\sqrt{2}$ i.e. $\frac{AB}{AC} = \frac{1}{\sqrt{2}}$

$\therefore$ AB = x and AC = $\sqrt{2}$ x.



By Pythagoras theorem, we have

$AC^2 = AB^2 + BC^2$

$\Rightarrow (\sqrt{2}\,x)^2 = x^2 + BC^2$

$\Rightarrow BC^2 = 2x^2 - x^2 = x^2$

$\Rightarrow BC = x$

$\therefore \tan A = \frac{BC}{AB} = \frac{x}{x} = 1$

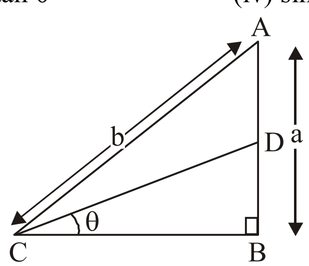
Now, $\frac{2\tan A}{1+\tan^2 A} = \frac{2\times 1}{1+1^2} = \frac{2}{2} = 1$

Now, $\frac{2\tan A}{1-\tan^2 A} = \frac{2\times 1}{1-1} = \frac{2}{0}$, which is undefined.

Ex.21 In fig. AD = DB and $\angle$B is a right angle. Determine

(i) $\sin\theta$ (ii) $\cos\theta$

(iii) $\tan\theta$ (iv) $\sin^2\theta + \cos^2\theta$



Sol. We have,

AB = a

$\Rightarrow$ AD + DB = a [Θ AD = DB]

$\Rightarrow$ AD + AD = a

$\Rightarrow$ 2AD = a $\Rightarrow$ AD = $\frac{a}{2}$

Thus, AD = DB = $\frac{a}{2}$

By Pythagoras theorem, we have

$AC^2 = AB^2 + BC^2$

$\Rightarrow b^2 = a^2 + BC^2$

$\Rightarrow BC^2 = b^2 - a^2 \Rightarrow BC = \sqrt{b^2 - a^2}$

Thus, in ΔBCD, we have

Base = BC = $\sqrt{b^2 - a^2}$

and Perpendicular = BD = $\frac{a}{2}$

Applying Pyhthagoras theorem in ΔBCD, we have

$BC^2 + BD^2 = CD^2$

$\Rightarrow (\sqrt{b^2 - a^2})^2 + \left(\frac{a}{2}\right)^2 = CD^2$

$\Rightarrow CD^2 = b^2 - a^2 + \frac{a^2}{4}$

$\Rightarrow CD^2 = \frac{4b^2 - 4a^2 + a^2}{4}$

$\Rightarrow CD = \frac{\sqrt{4b^2 - 3a^2}}{2}$

Now,

(i) $\sin\theta = \frac{BD}{CD}$

$\Rightarrow \sin\theta = \frac{a/2}{\frac{\sqrt{4b^2 - 3a^2}}{2}} = \frac{a}{\sqrt{4b^2 - 3a^2}}$

(ii) $\cos\theta = \frac{BC}{CD}$

$\Rightarrow \cos\theta = \frac{\sqrt{b^2 - a^2}}{\frac{\sqrt{4b^2 - 3a^2}}{2}} = \frac{2\sqrt{b^2 - a^2}}{\sqrt{4b^2 - 3a^2}}$

(iii) $\tan\theta = \frac{BD}{CD}$

$\Rightarrow \tan\theta = \frac{a/2}{\sqrt{b^2 - a^2}} = \frac{a}{2\sqrt{b^2 - a^2}}$, and

(iv) $\sin^2\theta + \cos^2\theta$

$$= \left(\frac{a}{\sqrt{4b^2 - 3a^2}}\right)^2 + \left(\frac{2\sqrt{b^2 - a^2}}{\sqrt{4b^2 - 3a^2}}\right)^2$$

$$= \frac{a^2}{4b^2 - 3a^2} + \frac{4(b^2 - a^2)}{4b^2 - 3a^2}$$

$$= \frac{4b^2 - 3a^2}{4b^2 - 3a^2} = 1$$

➢ TRIGONOMETRIC RATIO (T.R.) OF SOME SPECIFIC ANGLES

The angles 0°, 30°, 45°, 60°, 90° are angles for which we have values of T.R.

∠A	0°	30°	45°	60°	90°
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
cot A	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
cosec A	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

- $\sin\theta \uparrow$ when $\theta \uparrow$, $0 \le \theta \le 90°$
- $\cos\theta \downarrow$ when $\theta \uparrow$, $0 \le \theta \le 90°$
- tan θ, cot θ are not defined for θ = 90° & 0 respectively.
- cosec θ, sec θ are not defined when θ = 0 & 90° respectively.
- sin θ = cos θ for only θ = 45°
- $\because$ $180° = \pi^c$
- $\therefore$ $30° = \left(\frac{\pi}{6}\right)^c$; $45° = \left(\frac{\pi}{4}\right)^c$

  $60° = \left(\frac{\pi}{3}\right)^c$; $90° = \left(\frac{\pi}{2}\right)^c$

❖ EXAMPLES ❖

Ex.22 Evaluate each of the following in the simplest form :

(i) sin 60º cos 30º + cos 60º sin 30º

(ii) sin 60º cos 45º + cos 60º sin 45º

Sol. (i) sin 60º cos 30º + cos 60º sin 30º

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1$$

(ii) sin 60º cos 45º + cos 60º sin 45º

$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Ex.23 Evaluate the following expression :

(i) $\tan 60^\circ \operatorname{cosec}^2 45^\circ + \sec^2 60^\circ \tan 45^\circ$

(ii) $4\cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + \cos^2 90^\circ$.

Sol. (i) $\tan 60^\circ \operatorname{cosec}^2 45^\circ + \sec^2 60^\circ \tan 45^\circ$

$\tan 60^\circ (\operatorname{cosec} 45^\circ)^2 + (\sec 60^\circ)^2 \tan 45^\circ$

$$= \sqrt{3} \times (\sqrt{2})^2 + (2)^2 \times 1$$

$$= \sqrt{3} \times 2 + 4 = 4 + 2\sqrt{3}$$

(ii) $4\cot^2 45^\circ - \sec^2 60^\circ + \sin^2 60^\circ + \cos^2 90^\circ$

$$= 4(\cot 45^\circ)^2 - (\sec 60^\circ)^2 + (\sin 60^\circ)^2 + (\cos 90^\circ)^2$$

$$= 4 \times (1)^2 - (2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + 0$$

$$= 4 - 4 + \frac{3}{4} + 0 = \frac{3}{4}$$

Ex.24 Show that :

(i) $2(\cos^2 45^\circ + \tan^2 60^\circ) - 6(\sin^2 45^\circ - \tan^2 30^\circ) = 6$

(ii) $2(\cos^4 60^\circ + \sin^4 30^\circ) - (\tan^2 60^\circ + \cot^2 45^\circ) + 3\sec^2 30^\circ = \frac{1}{4}$

Sol. (i) $2(\cos^2 45^\circ + \tan^2 60^\circ) - 6(\sin^2 45^\circ - \tan^2 30^\circ)$

$$= 2\left(\left(\frac{1}{\sqrt{2}}\right)^2 + (\sqrt{3})^2\right) - 6\left(\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right)$$

$$= 2\left(\frac{1}{2} + 3\right) - 6\left(\frac{1}{2} - \frac{1}{3}\right) = 2\left(\frac{1+6}{2}\right) - 6\left(\frac{3-2}{6}\right)$$

$$= 2 \times \frac{7}{2} - 6 \times \frac{1}{6} = 7 - 1 = 6$$

(ii) $2(\cos^4 60^\circ + \sin^4 30^\circ) - (\tan^2 60^\circ + \cot^2 45^\circ) + 3\sec^2 30^\circ$

$$= 2\left(\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4\right) - \left((\sqrt{3})^2 + (1)^2\right) + 3\left(\frac{2}{\sqrt{3}}\right)^2$$

$$= 2\left(\frac{1}{16} + \frac{1}{16}\right) - (3 + 1) + 3 \times \frac{4}{3}$$

$$= 2 \times \frac{1}{8} - 4 + 4 = \frac{1}{4}$$

Ex.25 Find the value of x in each of the following :

(i) $\tan 3x = \sin 45^\circ \cos 45^\circ + \sin 30^\circ$

(ii) $\cos x = \cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ$

Sol.(i) $\tan 3x = \sin 45^\circ \cos 45^\circ + \sin 30^\circ$

$\Rightarrow \tan 3x = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{2}$

$\Rightarrow \tan 3x = \frac{1}{2} + \frac{1}{2}$

$\Rightarrow \tan 3x = 1$

$\Rightarrow \tan 3x = \tan 45^\circ$

$\Rightarrow 3x = 45^\circ \Rightarrow x = 15^\circ$

(ii) $\cos x = \cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ$

$\Rightarrow \cos x = \frac{1}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \times \frac{1}{2}$

$\Rightarrow \cos x = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}$

$\Rightarrow \cos x = \frac{\sqrt{3}}{2}$

$\Rightarrow \cos x = \cos 30^\circ$

$\Rightarrow x = 30^\circ$

Ex.26 If x = 30°, verify that

(i) $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

(ii) $\sin x = \sqrt{\frac{1-\cos 2x}{2}}$

Sol.(i) When x = 30°, we have 2x = 60° .

$\therefore \tan 2x = \tan 60^\circ = \sqrt{3}$

And, $\frac{2\tan x}{1-\tan^2 x} = \frac{2\tan 30^\circ}{1-\tan^2 30^\circ}$

$= \frac{2\times\frac{1}{\sqrt{3}}}{1-\left(\frac{1}{\sqrt{3}}\right)^2}$

$= \frac{2/\sqrt{3}}{1-\frac{1}{3}} = \frac{2/\sqrt{3}}{2/3} = \frac{2}{\sqrt{3}}\times\frac{3}{2} = \sqrt{3}$

$\therefore \tan 2x = \frac{2\tan x}{1-\tan^2 x}$

(ii) When x = 30°, we have 2x = 60°.

$\therefore \sqrt{\frac{1-\cos 2x}{2}} = \sqrt{\frac{1-\cos 60^\circ}{2}}$

$= \sqrt{\frac{1-\frac{1}{2}}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

And, $\sin x = \sin 30^\circ = \frac{1}{2}$

$\therefore \sin x = \frac{\sqrt{1-\cos 2x}}{2}$.

Ex.27 Find the value of θ in each of the following :

(i) $2\sin 2\theta = \sqrt{3}$ (ii) $2\cos 3\theta = 1$

Sol.(i) $2\sin 2\theta = \sqrt{3}$

$\Rightarrow \sin 2\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \sin 2\theta = \sin 60^\circ$

$\Rightarrow 2\theta = 60^\circ \Rightarrow \theta = 30^\circ$

(ii) $2\cos 3\theta = 1$

$\Rightarrow \cos 3\theta = \frac{1}{2}$

$\Rightarrow \cos 3\theta = \cos 60^\circ$

$\Rightarrow 3\theta = 60^\circ \Rightarrow \theta = 20^\circ$.

Ex.28 If θ is an acute angle and sin θ = cos θ, find the value of $2\tan^2\theta + \sin^2\theta - 1$.

Sol. $\sin\theta = \cos\theta$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\cos\theta}{\cos\theta}$

[Dividing both sides by cos θ]

$\Rightarrow \tan\theta = 1$

$\Rightarrow \tan\theta = \tan 45^\circ \Rightarrow \theta = 45^\circ$

$\therefore 2\tan^2\theta + \sin^2\theta - 1$

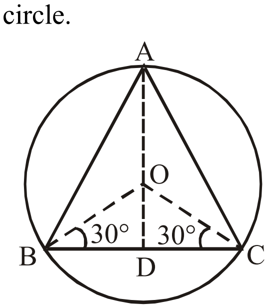
$= 2\tan^2 45^\circ + \sin^2 45^\circ - 1$

$= 2(1)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 - 1$

$= 2 + \frac{1}{2} - 1 = \frac{5}{2} - 1 = \frac{3}{2}$.

Ex.29 An equilateral triangle is inscribed in a circle of radius 6 cm. Find its side.

Sol. Let ABC be an equilateral triangle inscribed in a circle of radius 6 cm. Let O be the centre of the circle.



Then, OA = OB = OC = 6 cm.

Let OD be perpendicular from O on side BC. Then, D is mid-point of BC and OB and OC are bisectors of $\angle B$ and $\angle C$ respectively.

$\therefore \quad \angle OBD = 30°$

In ΔOBD, right angled at D, we have

$\angle OBD = 30°$ and OB = 6 cm.

$\therefore \quad \cos \angle OBD = \frac{BD}{OB} \Rightarrow \cos 60° = \frac{BD}{6}$

$\Rightarrow \quad BD = 6 \cos 60° = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$ cm.

$\Rightarrow \quad BC = 2\ BD = 2(3\sqrt{3})\text{cm} = 6\sqrt{3}$ cm.

Hence, the side of the equilateral triangle is $6\sqrt{3}$ cm.

Ex.30 Using the formula,
$\sin(A - B) = \sin A \cos B - \cos A \sin B$, find the value of sin 15º.

Sol. Let A = 45º and B = 30º. Then A – B = 15º. Putting A = 45º and B = 30º in the given formula, we get

$\sin(45º - 30º) = \sin 45º \cos 30º - \cos 45º \sin 30º$

or, $\sin(45º - 30º) = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \Rightarrow \sin 15º = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Ex.31 If $\tan(A + B) = \sqrt{3}$ and $\tan(A - B) = \frac{1}{\sqrt{3}}$;
$0° < A + B \le 90°$; $A > B$, find A and B.

Sol. $\tan(A + B) = \sqrt{3} = \tan 60°$

& $\tan(A - B) = 1/\sqrt{3} = \tan 30°$

A + B = 60° …….(1)
A – B = 30° …….(2)

2A = 90° ⇒ A = 45° **Ans.**
adding (1) & (2)

A + B = 60

A – B = 30

Sub fact equation (2) from (1)

A + B = 60
A – B = 30
– + –

2B = 30°
⇒ B = 15°. **Ans.**

Note : $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin(A + B) \ne \sin A + \sin B$.

➤ TRIGONOMETRIC RATIOS OF COMPLEMENTARY ANGLES

Θ We know complementary angles are pair of angles whose sum is 90°

Like 40°, 50° ; 60°, 30° ; 20°, 70° ; 15°, 75° ; etc,

Formulae :

$\sin(90° - \theta) = \cos\theta$, $\cot(90° - \theta) = \tan\theta$

$\cos(90° - \theta) = \sin\theta$, $\sec(90° - \theta) = \text{cosec}\,\theta$

$\tan(90° - \theta) = \cot\theta$, $\text{cosec}(90° - \theta) = \sec\theta$

Ex.32 Evaluate $\frac{\tan 65°}{\cot 25°}$. **[NCERT]**

Sol. Θ 65° + 25° = 90°

$\therefore \frac{\tan 65°}{\cot 25°} = \frac{\tan(90° - 25°)}{\cot 25°} = \frac{\cot 25°}{\cot 25°} = 1$

Ans.

Ex.33 Without using trigonometric tables, evaluate the following :

(i) $\frac{\cos 37º}{\sin 53º}$ (ii) $\frac{\sin 41º}{\cos 49º}$ (iii) $\frac{\sin 30º17'}{\cos 59º43'}$

Sol.(i) We have

$\frac{\cos 37º}{\sin 53º} = \frac{\cos(90º - 53º)}{\sin 53º} = \frac{\sin 53º}{\sin 53º} = 1$

[Θ cos(90º – θ) = sin θ]

(ii) We have,

$\frac{\sin 41º}{\cos 49º} = \frac{\sin(90º - 49º)}{\cos 49º} = \frac{\cos 49º}{\cos 49º} = 1$

[Θ sin (90º – θ) = cos θ]

(iii) We have,

$\frac{\sin 30º17'}{\cos 59º43'} = \frac{\sin(90º - 59º43')}{\cos 59º43'} = \frac{\cos 59º43'}{\cos 59º43'} = 1.$

Ex.34 Without using trigonometric tables evaluate the following :

(i) $\sin^2 25º + \sin^2 65º$ (ii) $\cos^2 13º - \sin^2 77º$

Sol.(i) We have,

$\sin^2 25º + \sin^2 65º = \sin^2(90º - 65º) + \sin^2 65º$

$= \cos^2 65º + \sin^2 65º = 1$

[Θ sin (90º – θ) = cos θ]

(ii) We have,

$\cos^2 13º - \sin^2 77º = \cos^2(90º - 77º) - \sin^2 77º$

$= \sin^2 77º - \sin^2 77º = 0$

[Θ cos (90º – θ) = sin θ]

Ex.35 Without using trigonometric tables, evaluate the following :

(i) $\frac{\cot 54^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot 70^\circ} - 2$

(ii) sec 50° sin 40° + cos 40° cosec 50°

Sol.(i) We have,

$\frac{\cot 54^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot 70^\circ} - 2$

$= \frac{\cot(90^\circ - 36^\circ)}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot(90^\circ - 20^\circ)} - 2$

$= \frac{\tan 36^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\tan 20^\circ} - 2 = 1 + 1 - 2 = 0$

(ii) We have,

sec50° sin40° + cos40° cosec50°

= sec(90° – 40°) sin40° + cos40° cosec(90° – 40°)

= cosec40° sin40° + cos40°sec40°

$= \frac{\sin 40^\circ}{\sin 40^\circ} + \frac{\cos 40^\circ}{\cos 40^\circ} = 1 + 1 = 2$

Ex.36 Express each of the following in terms of trigonometric ratios of angles between 0° and 45°;

(i) cosec 69° + cot 69°

(ii) sin 81° + tan 81°

(iii) sin 72° + cot 72°

Sol.(i) We have,

cosec 69° + cot 69°

= cosec (90° – 21°) + cot (90° – 21°)

= sec 21° + tan 21°

[Θ cosec (90° – θ) = sec θ and cot (90° – θ) = tan θ]

(ii) We have,

sin 81° + tan 81°

= sin (90° – 9°) + tan (90° – 9°)

= cos 9° + cot 9°

[Θ sin (90° – θ) = cos θ and tan (90° – θ) = cot θ]

(iii) We have,

sin 72° + cot 72°

= sin (90° – 18°) + cot (90° – 18°)

= cos 18° + tan 18°

[Θ sin (90° – 18°) = cos 18° and tan (90° – 18°) = cot 18°]

Ex.37 Without using trigonometric tables, evaluate the following :

$\frac{\sin^2 20^\circ + \sin^2 70^\circ}{\cos^2 20^\circ + \cos^2 70^\circ} + \frac{\sin(90^\circ - \theta)\sin\theta}{\tan\theta} + \frac{\cos(90^\circ - \theta)\cos\theta}{\cot\theta}$

Sol. $\frac{\sin^2 20^\circ + \sin^2 70^\circ}{\cos^2 20^\circ + \cos^2 70^\circ} + \frac{\sin(90^\circ - \theta)\sin\theta}{\tan\theta} + \frac{\cos(90^\circ - \theta)\cos\theta}{\cot\theta}$

$= \frac{\sin^2 20^\circ + \sin^2(90^\circ - 20^\circ)}{\cos^2 20^\circ + \cos^2(90^\circ - 20^\circ)} + \frac{\sin(90^\circ - \theta)\sin\theta}{\tan\theta} + \frac{\cos(90^\circ - \theta)\cos\theta}{\cot\theta}$

$= \frac{\sin^2 20^\circ + \cos^2 20^\circ}{\cos^2 20^\circ + \sin^2 20^\circ} + \frac{\cos\theta\sin\theta}{\frac{\sin\theta}{\cos\theta}} + \frac{\sin\theta\cos\theta}{\frac{\cos\theta}{\sin\theta}}$

$\left[\begin{array}{r}\sin(90^\circ - \theta) = \cos\theta \text{ and} \\ \cos(90^\circ - \theta) = \sin\theta\end{array}\right]$

$= \frac{1}{1} + \cos^2\theta + \sin^2\theta = 1 + 1 = 2$

Ex.38 If tan 2θ = cot (θ + 6°), where 2θ and θ + 6° are acute angles, find the value of θ.

Sol. We have,

tan 2θ = cot (θ + 6°)

⇒ cot(90° – 2θ) = cot (θ + 6°)

⇒ 90° – 2θ = θ + 6° ⇒ 3θ = 84°

⇒ θ = 28°

Ex.39 If A, B, C are the interior angles of a triangle ABC, prove that $\tan \frac{B+C}{2} = \cot \frac{A}{2}$

Sol. In ΔABC, we have

A + B + C = 180°

⇒ B + C = 180° – A

⇒ $\frac{B+C}{2} = 90^\circ - \frac{A}{2}$

⇒ $\tan\left(\frac{B+C}{2}\right) = \tan\left(90^\circ - \frac{A}{2}\right)$

⇒ $\tan\left(\frac{B+C}{2}\right) = \cot\frac{A}{2}$

Ex.40 If tan 2A = cot (A – 18°), where 2A is an acute angle, find the value of A. **[NCERT]**

Sol. tan 2A = cot (A – 18°)

cot (90° – 2A) = cot (A – 18°)

(Θ cot (90° – θ) = tan θ)

90° – 2A = A – 18°

3A = 108°

A = 36° **Ans.**

Ex.41 If tan A = cot B, prove that A + B = 90°.

Sol. $\because$ tan A = cot B

tan A = tan (90° – B)

A = 90° – B

A + B = 90°. Proved

Ex.42 If A, B and C are interior angles of a triangle ABC, then show that

$\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$ **[NCERT]**

Sol. $\because$ A + B + C = 180° (a.s.p. of Δ)

B + C = 180° – A

$\left(\frac{B+C}{2}\right) = 90° - \frac{A}{2}$

$\sin\left(\frac{B+C}{2}\right) = \sin\left(90° - \frac{A}{2}\right)$

$\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}$ Proved.

Ex.43 Express sin 67° + cos 75° in terms of trigonometric ratios of angles between 0° and 45°.

Sol. $\because$ 23 = 90 – 67 & 15 = 90 – 75

∴ sin 67° + cos 75°

= sin (90 – 23)° + cos (90 – 15)°

= cos 23° + sin 15°. **Ans.**

➤ TRIGONOMETRIC IDENTITIES

(1) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ (linear)

(2) $\sin^2\theta + \cos^2\theta = 1$

(3) $1 + \tan^2\theta = \sec^2\theta$

(4) $1 + \cot^2\theta = \text{cosec}^2\theta$

(2)–(4): square identites

❖ EXAMPLES ❖

Ex.44 Prove the following trigonometric identities :

(i) $(1 - \sin^2\theta)\sec^2\theta = 1$

(ii) $\cos^2\theta(1 + \tan^2\theta) = 1$

Sol.(i) We have,

LHS = $(1 - \sin^2\theta)\sec^2\theta = \cos^2\theta\sec^2\theta$

$[\because 1 - \sin^2\theta = \cos^2\theta]$

$= \cos^2\theta.\left(\frac{1}{\cos^2\theta}\right)$ $\left[\because \sec\theta = \frac{1}{\cos\theta}\right]$

= 1 = RHS

(ii) We have,

LHS = $\cos^2\theta(1 + \tan^2\theta)$

$= \cos^2\theta . \sec^2\theta$

$[\because 1 + \tan^2\theta = \sec^2\theta]$

$= \cos^2\theta.\left(\frac{1}{\cos^2\theta}\right)$ $\left[\because \sec\theta = \frac{1}{\cos\theta}\right]$

Ex.45 Prove the following trigonometric identities :

(i) $\frac{\sin\theta}{1-\cos\theta} = \text{cosec}\theta + \cot\theta$

(ii) $\frac{\tan\theta+\sin\theta}{\tan\theta-\sin\theta} = \frac{\sec\theta+1}{\sec\theta-1}$

Sol.(i) We have,

LHS = $\frac{\sin\theta}{(1-\cos\theta)} \times \frac{(1+\cos\theta)}{(1+\cos\theta)}$

[Multiplying numerator and denominator by (1 + cosθ)]

$= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} = \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta}$

$[\because 1 - \cos^2\theta = \sin^2\theta]$

$= \frac{1+\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$

= cosecθ + cotθ = RHS

$\left[\because \frac{1}{\sin\theta} = \text{cosec}\,\theta \text{ and } \frac{\cos\theta}{\sin\theta} = \cot\theta\right]$

(ii) We have,

LHS = $\frac{\tan\theta+\sin\theta}{\tan\theta-\sin\theta}$

$= \frac{\frac{\sin\theta}{\cos\theta}+\sin\theta}{\frac{\sin\theta}{\cos\theta}-\sin\theta} = \frac{\sin\theta\left(\frac{1}{\cos\theta}+1\right)}{\sin\theta\left(\frac{1}{\cos\theta}-1\right)}$

$\frac{\frac{1}{\cos\theta}+1}{\frac{1}{\cos\theta}-1} = \frac{\sec\theta+1}{\sec\theta-1}$ = RHS

Ex.46 Prove the following identities :

(i) $(\sin\theta + \text{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 = 7 + \tan^2\theta + \cot^2\theta$

(ii) $(\sin\theta + \sec\theta)^2 + (\cos\theta + \text{cosec}\theta)^2 = (1 + \sec\theta\,\text{cosec}\theta)^2$

(iii) $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$

Sol.(i) We have,

LHS = $(\sin\theta + \text{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2$

$= (\sin^2\theta + \text{cosec}^2\theta + 2\sin\theta\,\text{cosec}\theta)$

$(\cos^2\theta + \sec^2\theta + 2\cos\theta\sec\theta)$

$= \left(\sin^2\theta + \operatorname{cosec}^2\theta + 2\sin\theta.\frac{1}{\sin\theta}\right)$

$+ \left(\cos^2\theta + \sec^2\theta + 2\cos\theta.\frac{1}{\cos\theta}\right)$

$= (\sin^2\theta + \operatorname{cosec}^2\theta + 2) + (\cos^2\theta + \sec^2\theta + 2)$

$= \sin^2\theta + \cos^2\theta + \operatorname{cosec}^2\theta + \sec^2\theta + 4$

$= 1 + (1 + \cot^2\theta) + (1 + \tan^2\theta) + 4$

$[\because \operatorname{cosec}^2\theta = 1 + \cot^2\theta, \sec^2\theta = 1 + \tan^2\theta]$

$= 7 + \tan^2\theta + \cot^2\theta$ = RHS.

(ii) We have,

LHS $= (\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2$

$= \left(\sin\theta + \frac{1}{\cos\theta}\right)^2 + \left(\cos\theta + \frac{1}{\sin\theta}\right)^2$

$= \sin^2\theta + \frac{1}{\cos^2\theta} + \frac{2\sin\theta}{\cos\theta} + \cos^2\theta + \frac{1}{\sin^2\theta} + \frac{2\cos\theta}{\sin\theta}$

$= (\sin^2\theta + \cos^2\theta) + \left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right) + 2\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$

$= (\sin^2\theta + \cos^2\theta) + \left(\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta}\right) + \frac{2(\sin^2\theta + \cos^2\theta)}{\sin\theta\cos\theta}$

$= 1 + \frac{1}{\sin^2\theta\cos^2\theta} + \frac{2}{\sin\theta\cos\theta}$

$= \left(1 + \frac{1}{\sin\theta\cos\theta}\right)^2 = (1 + \sec\theta\operatorname{cosec}\theta)^2$ = RHS

(iii) We have, LHS $= \sec^4\theta - \sec^2\theta$

$= \sec^2\theta(\sec^2\theta - 1) = (1 + \tan^2\theta)(1 + \tan^2\theta - 1)$

$[\because \sec^2\theta = 1 + \tan^2\theta]$

$= (1 + \tan^2\theta)\tan^2\theta = \tan^2\theta + \tan^4\theta$ = RHS.

Ex.47 Prove the following identities :

(i) $\cos^4 A - \cos^2 A = \sin^4 A - \sin^2 A$

(ii) $\cot^4 A - 1 = \operatorname{cosec}^4 A - 2\operatorname{cosec}^2 A$

(iii) $\sin^6 A + \cos^6 A = 1 - 3\sin^2 A\cos^2 A$.

Sol.(i) We have,

LHS $= \cos^4 A - \cos^2 A = \cos^2 A(\cos^2 A - 1)$

$= -\cos^2 A(1 - \cos^2 A) = -\cos^2 A\sin^2 A$

$= -(1 - \sin^2 A)\sin^2 A = -\sin^2 A + \sin^4 A$

$= \sin^4 A - \sin^2 A$ = RHS

(ii) We have,

LHS $= \cot^4 A - 1 = (\operatorname{cosec}^2 A - 1)^2 - 1$

$[\because \cot^2 A = \operatorname{cosec}^2 A - 1$

$\therefore \cot^4 A = (\operatorname{cosec}^2 A - 1)^2]$

$= \operatorname{cosec}^4 A - 2\operatorname{cosec}^2 A + 1 - 1$

$= \operatorname{cosec}^4 A - 2\operatorname{cosec}^2 A$ = RHS

(iii) We have,

LHS $= \sin^6 A + \cos^6 A = (\sin^2 A)^3 + (\cos^2 A)^3$

$= (\sin^2 A + \cos^2 A)\{(\sin^2 A)^2 + (\cos^2 A)^2 - \sin^2 A\cos^2 A)\}$

$[\because a^3 + b^3 = (a + b)(a^2 - ab + b^2)]$

$= \{(\sin^2 A)^2 + (\cos^2 A)^2 + 2\sin^2 A\cos^2 A - \sin^2 A\cos^2 A\}$

$= [(\sin^2 A + \cos^2 A)^2 - 3\sin^2 A\cos^2 A]$

$= 1 - 3\sin^2 A\cos^2 A$ = RHS

Ex.48 Prove the following identities :

(i) $\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\sin^2 A} = \frac{1}{\sin^2 A\cos^2 A} - 2$

(ii) $\frac{\cos A}{1 - \tan A} + \frac{\sin^2 A}{\sin A - \cos A} = \sin A + \cos A$

(iii) $\frac{(1 + \sin\theta)^2 + (1 - \sin\theta)^2}{\cos^2\theta} = 2\left(\frac{1 + \sin^2\theta}{1 - \sin^2\theta}\right)$

Sol.(i) We have,

LHS $= \frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\sin^2 A} = \frac{\sin^4 A + \cos^2 A}{\sin^2 A\cos^2 A}$

[on taking LCM]

$= \frac{(\sin^2 A)^2 + (\cos^2 A)^2 + 2\sin^2 A\cos^2 A - 2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}$

$= \frac{(\sin^2 A + \cos^2 A)^2 - 2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}$

$= \frac{1 - 2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}$

$= \frac{1}{\sin^2 A\cos^2 A} - 2$ = RHS

(ii) We have,

LHS $= \frac{\cos A}{1 - \tan A} + \frac{\sin^2 A}{\sin A - \cos A}$

$= \frac{\cos A}{1 - \frac{\sin A}{\cos A}} + \frac{\sin^2 A}{\sin A - \cos A}$

$= \dfrac{\dfrac{\cos A}{\cos A - \sin A}}{\cos A} + \dfrac{\sin^2 A}{\sin A - \cos A}$

$= \dfrac{\cos^2 A}{\cos A - \sin A} + \dfrac{\sin^2 A}{\sin A - \cos A}$

$= \dfrac{\cos^2 A}{\cos A - \sin A} - \dfrac{\sin^2 A}{\cos A - \sin A}$

$= \dfrac{\cos^2 A - \sin^2 A}{\cos A - \sin A}$

$= \dfrac{(\cos A + \sin A)(\cos A - \sin A)}{\cos A - \sin A}$

$= \cos A + \sin A$ = RHS

(iii) We have,

LHS $= \dfrac{(1+\sin\theta)^2 + (1-\sin\theta)^2}{\cos^2\theta}$

$= \dfrac{(1+2\sin\theta+\sin^2\theta) + (1-2\sin\theta+\sin^2\theta)}{\cos^2\theta}$

$= \dfrac{2+2\sin^2\theta}{\cos^2\theta} = \dfrac{2(1+\sin^2\theta)}{1-\sin^2\theta} = 2\left(\dfrac{1+\sin^2\theta}{1-\sin^2\theta}\right)$

= RHS.

Ex.49 Prove the following identities :

(i) $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1 = 0$

(ii) $(\sin^8\theta - \cos^8\theta) = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

Sol.(i) We have,

LHS $= 2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$

$= 2[(\sin^2\theta)^3 + (\cos^2\theta)^3] - [3(\sin^2\theta)^2 + (\cos^2\theta)^2] + 1$

$= 2[(\sin^2\theta + \cos^2\theta)\{(\sin^2\theta)^2 + (\cos^2\theta)^2 - \sin^2\theta\cos^2\theta)]\}$

$- 3[(\sin^2\theta)^2 + (\cos^2\theta)^2 + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta] + 1$

$= 2[(\sin^2\theta)^2 + (\cos^2\theta)^2 + 2\sin^2\theta\cos^2\theta - 3\sin^2\theta\cos^2\theta]$

$-3[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta] + 1$

$= 2[(\sin^2\theta + \cos^2\theta)^2 - 3\sin^2\theta\cos^2\theta]$

$-3[1 - 2\sin^2\theta\cos^2\theta] + 1$

$= 2(1 - 3\sin^2\theta\cos^2\theta) - 3(1 - 2\sin^2\theta\cos^2\theta) + 1$

$= 2 - 6\sin^2\theta\cos^2\theta - 3 + 6\sin^2\theta\cos^2\theta + 1$

= 0 = RHS

(ii) We have,

LHS $= \sin^8\theta - \cos^8\theta = (\sin^4\theta)^2 - (\cos^4\theta)^2$

$= (\sin^4\theta - \cos^4\theta)(\sin^4\theta + \cos^4\theta)$

$= (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)(\sin^4\theta + \cos^4\theta)$

$= (\sin^2\theta - \cos^2\theta)\{(\sin^2\theta)^2 + (\cos^2\theta)^2 + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta\}$

$= (\sin^2\theta - \cos^2\theta)\{(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta\}$

$= (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$ = RHS

Ex.50 If (secA + tanA)(secB + tanB)(secC + tanC) = (secA – tanA)(secB – tanB)(secC – tanC) prove that each of the side is equal to ±1.

Sol. We have,

(secA + tanA)(secB + tanB)(secC + tanC)

= (secA – tanA)(secB – tanB)(secC – tanC)

Multiplying both sides by

(secA – tanA)(secB – tanB)(secC – tanC) we get

(secA + tanA) (secB + tanB) (secC + tanC) (secA – tanA) (secB – tanB) (secC – tanC)

$= (\sec A - \tan A)^2 (\sec B - \tan B)^2 (\sec C - \tan C)^2$

$\Rightarrow (\sec^2 A - \tan^2 A)(\sec^2 B - \tan^2 B)(\sec^2 C - \tan^2 C)$

$= (\sec A - \tan A)^2(\sec B - \tan B)^2(\sec C - \tan C)^2$

$\Rightarrow 1 = [(\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)]^2$

$\Rightarrow (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C) = \pm 1$

Similarly, multiplying both sides by

(secA + tanA)(secB + tanB)(secC + tanC),

we get

(secA + tanA)(secB + tanB)(secC + tanC) = ±1

Ex.51 If tanθ + sinθ = m and tanθ – sinθ = n, show that $m^2 - n^2 = 4\sqrt{mn}$.

Sol. We have,

LHS $= m^2 - n^2 = (\tan\theta + \sin\theta)^2 - (\tan\theta - \sin\theta)^2$

$= 4\tan\theta\sin\theta$ $[\because (a+b)^2 - (a-b)^2 = 4ab]$

And, RHS $= 4\sqrt{mn}$

$= 4\sqrt{(\tan\theta + \sin\theta)(\tan\theta - \sin\theta)}$

$= 4\sqrt{\tan^2\theta - \sin^2\theta}$

$= 4\sqrt{\dfrac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta}$

$= 4\sqrt{\dfrac{\sin^2\theta - \sin^2\theta\cos^2\theta}{\cos^2\theta}}$

$= 4\sqrt{\dfrac{\sin^2\theta(1-\cos^2\theta)}{\cos^2\theta}} = 4\sqrt{\dfrac{\sin^4\theta}{\cos^2\theta}}$

$= 4\dfrac{\sin^2\theta}{\cos\theta} = 4\sin\theta\dfrac{\sin\theta}{\cos\theta} = 4\sin\theta\tan\theta$

Thus we have

LHS = RHS, i.e. $m^2 - n^2 = 4\sqrt{mn}$

Ex.52 If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, show that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

Sol. We have,

$\cos\theta + \sin\theta = \sqrt{2}\cos\theta$

$\Rightarrow (\cos\theta + \sin\theta)^2 = 2\cos^2\theta$

$\Rightarrow \cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta = 2\cos^2\theta$

$\Rightarrow \cos^2\theta - 2\cos\theta\sin\theta = \sin^2\theta$

$\Rightarrow \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta = 2\sin^2\theta$

$\Rightarrow (\cos\theta - \sin\theta)^2 = 2\sin^2\theta$

$\Rightarrow \cos\theta - \sin\theta = \sqrt{2}\sin\theta$

Ex.53 If $\sin\theta + \cos\theta = p$ and $\sec\theta + \text{cosec}\theta = q$, show that $q(p^2 - 1) = 2p$

Sol. We have,

LHS $= q(p^2 - 1)$

$= (\sec\theta + \text{cosec}\theta)\,[(\sin\theta + \cos\theta)^2 - 1]$

$= \left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)\{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1\}$

$= \left(\frac{\sin\theta + \cos\theta}{\cos\theta\sin\theta}\right)[1 + 2\sin\theta\cos\theta - 1)$

$= \left(\frac{\sin\theta + \cos\theta}{\cos\theta\sin\theta}\right)(2\sin\theta\cos\theta)$

$= 2(\sin\theta + \cos\theta) = 2p =$ RHS

Ex.54 If $\sec\theta + \tan\theta = p$, show that $\frac{p^2 - 1}{p^2 + 1} = \sin\theta$.

Sol. We have,

LHS $= \frac{p^2 - 1}{p^2 + 1} = \frac{(\sec\theta + \tan\theta)^2 - 1}{(\sec\theta + \tan\theta)^2 + 1}$

$= \frac{\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta - 1}{\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta + 1}$

$= \frac{(\sec^2\theta - 1) + \tan^2\theta + 2\sec\theta\tan\theta}{\sec^2\theta + 2\sec\theta\tan\theta + (1 + \tan^2\theta)}$

$= \frac{\tan^2\theta + \tan^2\theta + 2\sec\theta\tan\theta}{\sec^2\theta + 2\sec\theta\tan\theta + \sec^2\theta}$

$= \frac{2\tan^2\theta + 2\tan\theta\sec\theta}{2\sec^2\theta + 2\sec\theta\tan\theta}$

$= \frac{2\tan\theta(\tan\theta + \sec\theta)}{2\sec\theta(\sec\theta + \tan\theta)}$

$= \frac{\tan\theta}{\sec\theta} = \frac{\sin\theta}{\cos\theta\sec\theta} = \sin\theta =$ RHS

Ex.55 If $\frac{\cos\alpha}{\cos\beta} = m$ and $\frac{\cos\alpha}{\sin\beta} = n$ show that $(m^2 + n^2)\cos^2\beta = n^2$.

Sol. LHS $= (m^2 + n^2)\cos^2\beta$

$= \left(\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\alpha}{\sin^2\beta}\right)\cos^2\beta$

$\left[\because\ m = \frac{\cos\alpha}{\cos\beta} \text{ and } n = \frac{\cos\alpha}{\sin\beta}\right]$

$= \left(\frac{\cos^2\alpha\sin^2\beta + \cos^2\alpha\cos^2\beta}{\cos^2\beta\sin^2\beta}\right)\cos^2\beta$

$= \cos^2\alpha\left(\frac{1}{\cos^2\beta\sin^2\beta}\right)\cos^2\beta$

$= \frac{\cos^2\alpha}{\sin^2\beta} = \left(\frac{\cos\alpha}{\sin\beta}\right)^2 = n^2 =$ RHS

Ex.56 If $a\cos\theta + b\sin\theta = m$ and $a\sin\theta - b\cos\theta = n$, prove that $a^2 + b^2 = m^2 + n^2$.

Sol. We have,

RHS $= m^2 + n^2$

$= (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2$

$= (a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta)$

$+ (a^2\sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cos\theta)$

$= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$

$= a^2 + b^2 =$ LHS.

Ex.57 If $a\cos\theta - b\sin\theta = c$, prove that $a\sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

Sol. We have,

$(a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2$

$= (a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta)$

$+ (a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta)$

$= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$

$= a^2 + b^2$

$\Rightarrow c^2 + (a\sin\theta + b\cos\theta)^2 = a^2 + b^2$

$[\because a\cos\theta - b\sin\theta = c]$

$\Rightarrow (a\sin\theta + b\cos\theta)^2 = a^2 + b^2 - c^2$

$\Rightarrow a\sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$.

Ex.58 Prove that :

$(1 - \sin\theta + \cos\theta)^2 = 2(1 + \cos\theta)(1 - \sin\theta)$

Sol. $(1 - \sin\theta + \cos\theta)^2$

$= 1 + \sin^2\theta + \cos^2\theta - 2\sin\theta + 2\cos\theta - 2\sin\theta\cos\theta$

$= 2 - 2\sin\theta + 2\cos\theta - 2\sin\theta\cos\theta$

$= 2(1 - \sin\theta) + 2\cos\theta(1 - \sin\theta)$

$= 2(1 - \sin\theta)(1 + \cos\theta) =$ RHS

Ex.59 If $\sin\theta + \sin^2\theta = 1$, prove that $\cos^2\theta + \cos^4\theta = 1$.

Sol. We have,

$\sin\theta + \sin^2\theta = 1$

$\Rightarrow \sin\theta = 1 - \sin^2\theta$

$\Rightarrow \sin\theta = \cos^2\theta$

Now, $\cos^2\theta + \cos^4\theta = \cos^2\theta + (\cos^2\theta)^2$

$= \cos^2\theta + \sin^2\theta = 1$

Ex.60 Prove that :

$$\frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}+\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}=\frac{2}{2\sin^2\theta-1}$$

Sol. We have,

$$\text{LHS} = \frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}+\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}$$

$$= \frac{(\sin\theta-\cos\theta)^2+(\sin\theta+\cos\theta)^2}{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}$$

$$= \frac{2(\sin^2\theta+\cos^2\theta)}{\sin^2\theta-\cos^2\theta}$$

$[\because (a+b)^2+(a-b)^2 = 2(a^2+b^2)]$

$$= \frac{2}{\sin^2\theta-(1-\sin^2\theta)}$$

$$= \frac{2}{(2\sin^2\theta-1)} = \text{RHS.}$$

Ex.61 Express the ratios cos A, tan A and sec A in terms of sin A.

Sol. Since $\cos^2A + \sin^2A = 1$, therefore,

$\cos^2A = 1 - \sin^2A$, i.e., $\cos A = \pm\sqrt{1-\sin^2 A}$

This gives

$\cos A = \sqrt{1-\sin^2 A}$ (Why ?)

Hence,

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1-\sin^2 A}} \text{ and}$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1-\sin^2 A}}.$$

Ex.62 Prove that $\frac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1}=\frac{1}{\sec\theta-\tan\theta}$, using the identity $\sec^2\theta = 1 + \tan^2\theta$.

Sol.

$$\text{LHS} = \frac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1} = \frac{\tan\theta-1+\sec\theta}{\tan\theta+1-\sec\theta}$$

$$= \frac{(\tan\theta+\sec\theta)-1}{(\tan\theta-\sec\theta)+1}$$

$$= \frac{\{(\tan\theta+\sec\theta)-1\}(\tan\theta-\sec\theta)}{\{(\tan\theta-\sec\theta)+1\}(\tan\theta-\sec\theta)}$$

$$= \frac{(\tan^2\theta-\sec^2\theta)-(\tan\theta-\sec\theta)}{\{\tan\theta-\sec\theta+1\}(\tan\theta-\sec\theta)}$$

$$= \frac{-1-\tan\theta+\sec\theta}{(\tan\theta-\sec\theta+1)(\tan\theta-\sec\theta)}$$

$$= \frac{-1}{\tan\theta-\sec\theta} = \frac{1}{\sec\theta-\tan\theta},$$

which is the RHS of the identity, we are required to prove.

EXERCISE # 1

Q.1 If $\sin\alpha = \frac{1}{\sqrt{2}}$ and $\tan\beta = 1$, find the value of $\sin(\alpha + \beta)$, where α and β both are acute.

Q.2 If $\cos\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{\sqrt{3}}$, find the value of $\sin(\alpha + \beta)$, where α and β both are acute.

Q.3 Without using trigonometric tables evaluate the following :

(i) $\frac{\sin 20^\circ}{\cos 70^\circ}$ (ii) $\frac{\cos 19^\circ}{\sin 71^\circ}$ (iii) $\frac{\sin 21^\circ}{\cos 69^\circ}$

(iv) $\frac{\tan 10^\circ}{\cot 80^\circ}$ (v) $\frac{\sec 11^\circ}{\operatorname{cosec} 79^\circ}$ (vi) $\frac{\sin 20^\circ 30'}{\cos 69^\circ 30'}$

Q.4 Without using trigonometric tables evaluate the following :

(i) $\left(\frac{\sin 49^\circ}{\cos 41^\circ}\right)^2 + \left(\frac{\cos 41^\circ}{\sin 49^\circ}\right)^2$

(ii) $\frac{\cot 40^\circ}{\tan 50^\circ} - \frac{1}{2}\left(\frac{\cot 35^\circ}{\sin 55^\circ}\right)$

Q.5 Without using trigonometric tables evaluate the following :

(i) $\frac{\tan 35^\circ}{\cot 55^\circ} + \frac{\cot 78^\circ}{\tan 12^\circ} - 1$

(ii) $\operatorname{cosec}^2 67^\circ - \tan^2 23^\circ$

Q.6 Without using trigonometric tables evaluate the following :

(i) $\sin^2 20^\circ + \sin^2 70^\circ - \tan^2 45^\circ$

(ii) $\sec 50^\circ \sin 40^\circ + \cos 40^\circ \operatorname{cosec} 50^\circ$

Q.7 Without using trigonometric tables prove the following :

(i) $\tan 20^\circ \tan 35^\circ \tan 45^\circ \tan 55^\circ \tan 70^\circ = 1$

(ii) $\sin 48^\circ \sec 42^\circ + \cos 48^\circ \operatorname{cosec} 42^\circ = 2$

(iii) $\sin 63^\circ \cos 27^\circ + \cos 63^\circ \sin 27^\circ = 1$

(iv) $\frac{\sin 70^\circ}{\cos 20^\circ} + \frac{\operatorname{cosec} 20^\circ}{\sec 70^\circ} - \cos 70^\circ \operatorname{cosec} 20^\circ = 1$

(v) $\frac{\cos 80^\circ}{\sin 10^\circ} + \cos 59^\circ \operatorname{cosec} 31^\circ = 2$

Q.8 Prove the following :

(i) $\sin\theta \sin(90^\circ - \theta) - \cos\theta\cos(90^\circ - \theta) = 0$

(ii) $\frac{\sin\theta\cos(90^\circ-\theta)\cos\theta}{\sin(90^\circ-\theta)} + \frac{\cos\theta\sin(90^\circ-\theta)\sin\theta}{\sin(90^\circ-\theta)} = 1$

(iii) $\frac{\sin\theta}{\sin(90^\circ-\theta)} + \frac{\cos\theta}{\cos(90^\circ-\theta)} = \sec\theta \operatorname{cosec}\theta$

(iv) $\sin(90^\circ - \theta)\cos(90^\circ - \theta) = \frac{\tan\theta}{1+\cot^2(90^\circ-\theta)}$

(v) $\frac{\cos(90^\circ-\theta)}{1+\sin(90^\circ-\theta)} + \frac{1+\sin(90^\circ-\theta)}{\cos(90^\circ-\theta)} = 2\operatorname{cosec}\theta$

(vi) $\frac{1}{1+\cos(90^\circ-\theta)} + \frac{1}{1-\cos(90^\circ-\theta)} = 2\operatorname{cosec}^2(90^\circ - \theta)$

(vii) $\sin^2(90^\circ - \theta)(1 + \cot^2(90^\circ - \theta)) = 1$

(viii) $\frac{\cos(90^\circ-\theta)\sec(90^\circ-\theta)\tan\theta}{\operatorname{cosec}(90^\circ-\theta)\sin(90^\circ-\theta)\cot(90^\circ-\theta)} + \frac{\tan(90^\circ-\theta)}{\cot\theta} = 2$

(ix) $\frac{\tan(90^\circ-A)\cot A}{\operatorname{cosec}^2 A} - \cos^2 A = 0$

(x) $\frac{\cos(90^\circ-A)\sin(90^\circ-A)}{\tan(90^\circ-A)} = \sin^2 A$

Q.9 Without using trigonometric tables, evaluate each of the following :

(i) $\sec^2 10^\circ - \cot^2 80^\circ + \frac{\sin 15^\circ\cos 75^\circ + \cos 15^\circ \sin 75^\circ}{\cos\theta\sin(90^\circ-\theta) + \sin\theta\cos(90^\circ-\theta)}$

(ii) $\sin(50^\circ + \theta) - \cos(40^\circ - \theta) + \tan 1^\circ \tan 10^\circ \tan 20^\circ \tan 70^\circ \tan 80^\circ \tan 89^\circ$

(iii) $\cot\theta\tan(90^\circ - \theta) - \sec(90^\circ - \theta)\operatorname{cosec}\theta + \sin^2 25^\circ + \sin^2 65^\circ + \sqrt{3}\,(\tan 5^\circ \tan 45^\circ \tan 85^\circ)$

(iv) $\cot\theta\tan(90^\circ - \theta) - \sec(90^\circ - \theta)\operatorname{cosec}\theta + \sqrt{3}\,(\tan 5^\circ \tan 30^\circ \tan 85^\circ) + \sin^2 25^\circ + \sin^2 65^\circ$

(v) $\frac{-\tan\theta\cot(90^\circ-\theta) + \sec\theta\operatorname{cosec}(90^\circ-\theta)}{\tan 10^\circ \tan 20^\circ \tan 45^\circ \tan 70^\circ \tan 80^\circ} + \frac{\sin^2 35^\circ + \sin^2 55^\circ}{\tan 10^\circ \tan 20^\circ \tan 45^\circ \tan 70^\circ \tan 80^\circ}$

Q.10 The round balloon of radius r subtends an angle α at the eye of the observer while the angle of elevation of its centre is β. Prove that the height of the centre of the balloon is r $\sin\beta$ cosec $\alpha/2$.

Q.11 If $\tan\theta = 8/15$ and $0^\circ < \theta < 90^\circ$, find $\sin\theta$.

Q.12 If $\sin\theta = 8/17$ and $0^\circ < \theta < 90^\circ$, find $\tan\theta$.

Q.13 If $\sin A = \frac{24}{25}$, find the value of tanA + secA, where $0^\circ < A < 90^\circ$.

Q.14 If $5 \tan\theta = 12$, find the value of $\frac{2\cos\theta + \sin\theta}{\sin\theta - \cos\theta}$.

Q.15 If $\tan\theta = \frac{3}{4}$, find the value of $\frac{1-\cos\theta}{1+\cos\theta}$.

Q.16 If $\tan\theta = \frac{12}{5}$, find the value of $\frac{1+\sin\theta}{1-\sin\theta}$.

Q.17 If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, by using $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, prove that $A + B = 45^\circ$

Q.18 If $4 \tan\theta = 3$, find the value of $\frac{4\sin\theta - 2\cos\theta}{4\sin\theta + 3\cos\theta}$.

Q.19 If $\operatorname{cosec}\theta = \frac{13}{12}$, find the value of $\frac{2\sin\theta - 3\cos\theta}{4\sin\theta - 9\cos\theta}$.

ANSWER KEY

1. 1	**2.** 1	**3.** (i) 1 , (ii) 1, (iii) 1, (iv) 1, (v) 1, (vi) 1
4. (i) 2, (ii) $\frac{1}{2}$	**5.** (i) 1, (ii) 1	**6.** (i) 0, (ii) 2
9. (i) 2, (ii) 1, (iii) $\sqrt{3}$, (iv) 1, (v) 2	**11.** $\frac{8}{17}$	**12.** $\frac{8}{15}$
13. 7	**14.** $\frac{22}{7}$	**15.** $\frac{1}{9}$
16. 25	**18.** $\frac{1}{6}$	**19.** 3

EXERCISE # 2

Q.1 Find the value of $\left(\frac{3\pi}{5}\right)$ radians in degrees.

Q.2 Find the value of 150º in radians.

Q.3 If $\sin\theta = \frac{5}{13}$, then find the values of $\tan\theta$ and $\sec\theta$.

Q.4 If $\tan\theta = \frac{x}{y}$, then find the value of $\left(\frac{x\sin\theta + y\cos\theta}{x\sin\theta - y\cos\theta}\right)$.

Q.5 If $5\tan\theta = 4$, find the value of $\left(\frac{5\sin\theta - 3\cos\theta}{5\sin\theta + 2\cos\theta}\right)$.

Q.6 If $16\cot x = 12$, then find the value of $\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)$.

Q.7 If $\tan\theta = (3/4)$ and $0 < \theta < 90^\circ$, then find the value of $(\sin\theta\cos\theta)$.

Q.8 If $8\tan x = 15$, then find the value of $(\sin x - \cos x)$.

Q.9 If $\tan\theta = \frac{1}{\sqrt{7}}$, then find the value of $\left(\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}\right)$.

Q.10 If $\cot\theta = \frac{1}{\sqrt{3}}$, then find the value of $\left(\frac{1-\cos^2\theta}{2-\sin^2\theta}\right)$.

Q.11 If $\tan\theta = \frac{4}{3}$, then find the value of $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$.

Q.12 If $3\cos\theta = 5\sin\theta$, then find the value of $\left(\frac{5\sin\theta - 2\sec^3\theta + 2\cos\theta}{5\sin\theta + 2\sec^3\theta - 2\cos\theta}\right)$.

Q.13 If $\tan\theta = (3/4)$, then find the value of $(\cos^2\theta - \sin^2\theta)$.

Q.14 Find the value of tan75º.

Q.15 If $\tan\theta = \frac{a}{x}$, then find the value of $\frac{x}{\sqrt{a^2 + x^2}}$.

Q.16 If $3\sin x + 5\cos x = 5$, then the value of $(3\cos x - 5\sin x)^2$.

Q.17 Find the value of $(\sin A + \cos A)^2 + (\sin A - \cos A)^2$.

Q.18 Find the value of $\sqrt{\frac{1+\sin A}{1-\sin A}}$.

Q.19 Find the value of $\sqrt{\frac{1-\sin A}{1+\sin A}}$.

Q.20 Find the value of $\sqrt{\frac{1-\cos x}{1+\cos x}}$.

Q.21 Find the value of $\sqrt{\frac{1+\cos x}{1-\cos x}}$.

Q.22 Find the value of $\sqrt{\frac{\sec x - \tan x}{\sec x + \tan x}}$.

Q.23 Find the value of $\left(\frac{\cot\theta}{\cot\theta - \cot 3\theta} + \frac{\tan\theta}{\tan\theta - \tan 3\theta}\right)$.

Q.24 Find the value of $\left(\frac{\sin A + \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A - \sin B}\right)$.

Q.25 Find the value of sin15º.

Q.26 Find the value of (sin40º – cos50º).

Q.27 If x = r sinA cosB, y = r sinA sinB and z = r cosA, then which is correct ?

(A) $x^2 + y^2 + z^2 = r^2$ (B) $x^2 - y^2 + z^2 = r^2$

(C) $x^2 + y^2 - z^2 = r^2$ (D) $-x^2 + y^2 + z^2 = r^2$

Q.28 Find the value of

(cot15º cot16º cot17º.... cot73º cot74º cot75º).

ANSWER KEY

1. 108°	**2.** $\left(\frac{5\pi}{6}\right)^c$	**3.** $\frac{5}{12}$ and $\frac{13}{12}$
4. $\frac{x^2+y^2}{x^2-y^2}$	**5.** $\frac{1}{6}$	**6.** $\frac{1}{7}$
7. $\frac{12}{25}$	**8.** $\frac{7}{17}$	**9.** $\frac{3}{4}$
10. $\frac{3}{5}$	**11.** $\frac{1}{3}$	**12.** $\frac{271}{979}$
13. $\frac{7}{25}$	**14.** $2-\sqrt{3}$	**15.** $\cos\theta$
16. 9	**17.** 2	
18. sec A + tan A	**19.** sec A – tan A	**20.** cosec x – cot x
21. cosec x + cot x	**22.** sec x – tan x	**23.** 1
24. 0	**25.** $\frac{\sqrt{3}-1}{2\sqrt{2}}$	**26.** 0
27. (A)	**28.** 1	