3 Matrices and Determinants

KEY FACTS

1. A Matrix (plural-matrices) is a rectangular array of real numbers, arranged in rows and columns.
The general form of a matrix with m rows and n columns is

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & - & - & a_{1n} \\ a_{21} & a_{22} & a_{23} & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{m1} & a_{m2} & a_{m3} & - & - & a_{mn} \end{bmatrix}$$

which is written in a compact form as $A_{m\times n} = [a_{ij}]_{m\times n}$

Hence $\boldsymbol{a_{ij}}$ is the element of the ***i*th row** and ***j*th column**.

For example, a_{12} is element in 1st row and 2nd column.
a_{34} is the element in the 3rd row and 4th column.

Order of a Matrix is the ordered pair having first component as the number of rows and the second component as the number of columns in the matrix. Thus, **a matrix of order $m \times n$ has m rows and n columns** and is called an **$m \times n$ (read "m by n") matrix.**

Thus the generalised form of a 3 × 3 matrix is :

$$A_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

2. Types of Matrices:

(*i*) **Rectangular Matrix.** Any $m \times n$ matrix, where $m \neq n$ is called a rectangular matrix.

For example, $\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ -6 & 2 \end{bmatrix}$ of the order 3×2 is a rectangular matrix.

(*ii*) **Row Matrix.** A matrix having only one row is called a row matrix.

For example, $[3\ \ 7\ \ 1\ \ -2]_{1\times 4}$, $[2\ \ -3]_{1\times 2}$ are row matrices.

(*iii*) **Column Matrix.** A matrix which has a single column is called a column matrix.

For example, $\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}_{3\times 1}$, $\begin{bmatrix} 5 \\ 7 \\ 1 \\ -6 \end{bmatrix}_{4\times 1}$ are column matrices.

(*iv*) **Square Matrix.** A matrix in which the number of rows is equal to the number of columns is called a square matrix. An $m \times m$ matrix is termed as a square matrix of order m.

For example, $\begin{bmatrix} 1 & 3 \\ 6 & 7 \end{bmatrix}_{2\times 2}$ is a square matrix of order 2.

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times 3}$ is a square matrix of order 3.

(*v*) **Diagonal Matrix.** It is a square matrix all of whose elements except those in the leading diagonal are zero. The leading diagonal elements of square matrix $A = [a_{ij}]_{m\times n}$ are a_{11}, a_{22}, a_{33}, ______, a_{mn}

For example, $\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}_{2\times 2}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{bmatrix}_{3\times 3}$ are all diagonal matrices.

A diagonal matrix of order n, having d_1, d_2, ______, d_n as diagonal elements may be denoted by diagonal $[d_1, d_2$, _____, $d_n]$

Thus, the diagonal matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ may be denoted by diagonal $[1 \quad -2 \quad 7]$

(*vi*) **Scalar Matrix.** A square matrix in which the diagonal elements are all equal, all other elements being zeros is called a scalar matrix.

For example, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a scalar matrix of order 3.

(*vii*) **Unit Matrix or Identity Matrix.** A square matrix in which each diagonal element is unity, all other elements being zeros, is called a unit matrix or an identity matrix.

Unit matrix of order n is denoted by I_n.

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(*viii*) **Zero or Null Matrix.** A matrix each of whose elements is zero is called a zero matrix.

For example, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are all null matrices.

(*ix*) **Sub-Matrix.** A matrix obtained by deleting the rows or columns or both of a matrix is called a sub-matrix.

For example, $A = \begin{bmatrix} 5 & 7 \\ -1 & 2 \end{bmatrix}$ is a sub matrix of $B = \begin{bmatrix} 5 & 7 & 3 \\ -1 & 2 & 0 \\ 4 & 1 & 0 \end{bmatrix}$ obtained by deleting the third row and third column of matrix B.

(*x*) **Comparable Matrices.** Two matrices A and B are said to be comparable if they are of the same order, i.e., they have the same number of rows and same number of columns.

For example, $\begin{bmatrix} 1 & 0 & -3 \\ 2 & 7 & 4 \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$ are comparable matrices each of order 2×3.

(*xi*) **Triangular Matrix.** A square matrix of order n is called a triangular matrix if its diagonal elements are all equal to zero.

For example, $\begin{bmatrix} 0 & 1 & 2 \\ -2 & 0 & 3 \\ -4 & -3 & 0 \end{bmatrix}$ is a triangular matrix of order 3.

(*a*) **Upper Triangular Matrix.** A square matrix in which the elements below the principal diagonal are all zero is called upper triangular matrix, *i.e.,* $a_{ij} = 0$ for all $i > j$.

For example, $\begin{bmatrix} 1 & -4 & 7 \\ 0 & -6 & 8 \\ 0 & 0 & 9 \end{bmatrix}$ is an upper triangular matrix.

Principal diagonal

(*b*) **Lower Triangular Matrix.** A square matrix in which the elements above the principal diagonal are all zero is called lower triangular matrix, *i.e.,* $a_{ij} = 0$ for all $i < j$.

For example, $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 4 & 0 \\ 8 & 2 & 1 \end{bmatrix}$ is a lower triangular matrix.

3. Equality of Matrices: Two matrices A and B are equal if and only if both matrices are of same order and each element of one is equal to the corresponding element of the other, *i.e.*,

$A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are said to be equal if $a_{ij} = b_{ij}$ for all i,j

For example, $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 4/2 & 2-1 \\ \sqrt{9} & 0 \end{bmatrix}$, but $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$.

4. Addition and Subtraction of Matrices.

(*i*) **Addition of Matrices.** The sum of two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ of the same order $m \times n$ is $(A + B)_{m \times n}$ in which the element at ***i*th row and *j*th column** is $a_{ij} + b_{ij}$ for all $1 \le i \le m$ and $1 \le j \le n$.

Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then

$A + B = [a_{ij} + b_{ij}]_{m \times n}$

For example, $\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 2 \\ -3 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 3+1 & 1+3 & 2+2 \\ 2+(-3) & 1+1 & 4+(-4) \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 \\ -1 & 2 & 0 \end{bmatrix}$.

Caution: The sum of two matrices of different orders is not defined.

(*ii*) **Negative of a matrix.** The negative of a matrix $A_{m \times n}$ denoted by $-A_{m \times n}$ is the matrix formed by replacing each entry in the matrix $A_{m \times n}$ by its additive inverse.

Thus, if $A = [a_{ij}]_{m \times n}$ be any matrix, then its negative $-A = [-a_{ij}]_{m \times n}$.

For example if $A = \begin{bmatrix} 3 & -1 \\ 2 & -2 \\ -4 & 5 \end{bmatrix}$, then $-A = \begin{bmatrix} -3 & 1 \\ -2 & 2 \\ 4 & -5 \end{bmatrix}$.

(*iii*) **Additive Inverse.** For each matrix $A = [a_{ij}]_{m \times n}$, there exists a matrix $-A = [-a_{ij}]_{m \times n}$ (negative of matrix A), called the additive inverse of A, such that $A + (-A) = O$ (null matrix)

Thus, the additive inverse of $\begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 4 & -3 \\ -2 & -1 \end{bmatrix}$ and $\begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -3 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$ (null matrix).

(*iv*) **Subtraction of matrices.** If A and B are matrices of the same order, then the sum $B + (-A)$ is called the difference or subtraction of B and A is denoted by $B - A$.

For example, if $L = \begin{bmatrix} 2 & 0 \\ -3 & 6 \end{bmatrix}$ and $M = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$, then

$$L - M = L + (-M) = \begin{bmatrix} 2 & 0 \\ -3 & 6 \end{bmatrix} + \begin{bmatrix} -1 & +2 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 2-1 & 0+2 \\ -3+0 & 6-4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}.$$

(*v*) Properties of Matrix Addition.

If A, B and C belong to the set $S_{m \times n}$ of all $m \times n$ matrices with real numbers as elements, then

I. $A + B \in S_{m \times n}$ **Closure property of addition**

II. $A + B = B + A$. **Commutative law of addition**

$$\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$$

III. $(A + B) + C = A + (B + C)$ **Associative law of addition**

IV. The matrix $O_{m \times n}$ has the property that for every matrix $A_{m \times n}$. **Additive – identity law**

$$A + 0 = A \text{ and } 0 + A = A$$

$$\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \end{bmatrix}$$

V. For every matrix $A_{m \times n}$, there exists a matrix $-A_{m \times n}$, such that

$A + (-A) = 0$ and $(-A) + A = 0$

$$\begin{bmatrix} 4 & -5 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} -4 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

VI. If A, B and C are three matrices of the same order, then,

$A + B = A + C \Rightarrow B = C$ **Left Cancellation Law**

$B + A = C + A \Rightarrow B = C$ **Right Cancellation Law**

5. Scalar Multiplication. The product of a real number or scalar ***k*** and a matrix $A = [a_{ij}]_{m \times n}$ is a matrix whose elements are the products of k and corresponding elements of A, *i.e.*

$kA = [ka_{ij}]_{m \times n} \quad \forall \quad i, j$

For example, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$.

Properties of scalar multiplication of matrices

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be the two given matrices and k_1, k_2, k_3 the scalars, *i.e.*, $k_1, k_2, k_3 \in R$, then

I. $k_1 (A + B) = k_1A + k_1B$

II. $(k_1 + k_2)A = k_1A + k_2A$

III. $(k_1k_2)A = k_1(k_2A) = k_2(k_1A)$

IV. $1A = A$ and $(-1)A = -A$

V. $(-k)A = -kA$

VI. $OA = O$ and $k_1O = O$

6. Multiplication of Matrices

Matrices need to be **conformable** or **compatible** for multiplication, *i.e.*, **for the product** of two matrices. A and B, ***AB* to exist the number of columns of matrix *A* should be equal to the number of rows of *B*.**

Then, the product matrix AB has the same number of rows as A and same number of columns as B.

$\therefore \; A_{m \times p} \times B_{p \times n} = C_{m \times n}$

Note: If two matrices are conformable for the matrix multiplication $A \times B$ then it does not necessarily imply that they are conformable for the order $B \times A$ also.

The matrix multiplication follows the *"multiply row by column"* process which can be shown diagrammatically as:

Let $A = [a_{ij}]_{2\times3}$, $B = [b_{ij}]_{3\times2}$, Then

No. of columns of $A = 3$ = No. of rows of $B \Rightarrow AB$ exists and is of order 2×2

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} \\ \textbf{1}\boldsymbol{st}\textbf{ row of } \boldsymbol{A}\times\textbf{1}\boldsymbol{st}\textbf{ col. of } \boldsymbol{B} & \textbf{1}\boldsymbol{st}\textbf{ row of } \boldsymbol{A}\times\textbf{2}\boldsymbol{nd}\textbf{ col. of } \boldsymbol{B} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} \\ \textbf{2}\boldsymbol{nd}\textbf{ row of } \boldsymbol{A}\times\textbf{1}\boldsymbol{st}\textbf{ col. of } \boldsymbol{B} & \textbf{2}\boldsymbol{nd}\textbf{ row of } \boldsymbol{A}\times\textbf{2}\boldsymbol{nd}\textbf{ col. of } \boldsymbol{B} \end{bmatrix}$$

Similarly, No. of columns of $B = 2$ = No. of rows of $A \Rightarrow BA$ exists and is of order 3×3.

$$BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times2} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \end{bmatrix}_{2\times3} = \begin{bmatrix} b_{11}a_{11}+b_{12}a_{21} & b_{11}a_{12}+b_{12}a_{22} & b_{11}a_{13}+b_{12}a_{33} \\ b_{21}a_{11}+b_{22}a_{21} & b_{21}a_{12}+b_{22}a_{22} & b_{21}a_{13}+b_{22}a_{33} \\ b_{31}a_{11}+b_{32}a_{21} & b_{31}a_{12}+b_{32}a_{22} & b_{31}a_{13}+b_{32}a_{33} \end{bmatrix}_{3\times3}$$

For example, $\begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}_{2\times3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}_{3\times2} = \begin{bmatrix} 3\times1+1\times2+2\times3 & 3\times(-1)+1\times1+2\times1 \\ 1\times1+0\times2+1\times3 & 1\times(-1)+0\times1+1\times1 \end{bmatrix}_{2\times2}$

$$= \begin{bmatrix} 3+2+6 & -3+1+2 \\ 1+0+3 & -1+0+1 \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ 4 & 0 \end{bmatrix}_{2\times2}$$

and $\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}_{3\times2} \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}_{2\times3} = \begin{bmatrix} 1\times3+(-1)\times1 & 1\times1+(-1)\times0 & 1\times2+(-1)\times1 \\ 2\times3+1\times1 & 2\times1+1\times0 & 2\times2+1\times1 \\ 3\times3+1\times1 & 3\times1+1\times0 & 3\times2+1\times1 \end{bmatrix}_{3\times3}$

$$= \begin{bmatrix} 3-1 & 1-0 & 2-1 \\ 6+1 & 2+0 & 4+1 \\ 9+1 & 3+0 & 6+1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 7 & 2 & 5 \\ 10 & 3 & 7 \end{bmatrix}_{3\times3}$$

(*i*) The product of matrices is not commutative

- **(*a*)** Wherever AB exists, BA is not always defined. For example, if A be a 5×4 matrix and B be a 4×3 matrix, then AB is defined and is of order 5×3, while BA is not defined. (No. of columns of $B = 3 \neq$ No. of rows of $A = 5$).
- **(*b*)** If AB and BA are both defined, it is not necessary that they are of the same order. For example, if A be a 4×3 matrix and B be a 3×4 matrix, then AB is defined and is a 4×4 matrix. BA is also defined but is a 3×3 matrix. AB and BA being of different orders $AB \neq BA$.
- **(*c*)** Even if AB and BA are both defined and are of the same order, it is not necessary $AB = BA$.

(*ii*) Matrix multiplication is associative if conformability is assured, *i.e.*, $A(BC) = (AB)C$

(*iii*) Matrix multiplication is distributive with respect to matrix addition $A(B + C) = AB + AC$

Also, it can be proved that: $(B + C)A = BC + CA$, $A(B - C) = AB - AC$, $(B - C)A = BA - CA$

(*iv*) The product of two matrices can be zero without either factor being a zero matrix.

For example, Let $A = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ and $B = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$ where, $a \neq 0, b \neq 0, c \neq 0, d \neq 0$

Then, $AB = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0\times c+a\times0 & 0\times d+a\times0 \\ 0\times c+b\times0 & 0\times d+b\times0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$

(*v*) Existence of Identity Matrix: For each square matrix A of order n, we have an ***identity matrix I of order n*** such that $A_{n\times n} I_{n\times n} = A_{n\times n} = I_{n\times n} A_{n\times n}$.

For example, $\underset{\mathbf{A_2}}{\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}}\underset{\mathbf{I_2}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 2\times1+3\times0 & 2\times0+3\times1 \\ -1\times1+1\times0 & -1\times0+1\times1 \end{bmatrix} = \underset{\mathbf{A_2}}{\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}}$

$\underset{\mathbf{I_2}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}\underset{\mathbf{A_2}}{\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}} = \begin{bmatrix} 1\times2+0\times-1 & 1\times3+0\times1 \\ 0\times2+1\times-1 & 0\times3+1\times1 \end{bmatrix} = \underset{\mathbf{A_2}}{\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}}$

Thus for any matrix A, there exists an identity matrix I such that $AI = A = IA$, whenever the products are defined.

(*vi*) Zero Matrix: For any matrix A, we have a zero matrix O such that, $AO = O = OA$, whenever these products are defined.

Thus, for any matrix $A_{p\times n}$, we have

$$O_{m\times p}\, A_{p\times n} = O_{m\times n} \text{ and } A_{p\times n}\, O_{n\times q} = O_{p\times q}.$$

(*vii*) Positive Integral Power of Matrices: The product $A.A$ is defined only when A is a square matrix.

$$\begin{aligned} A^2 &= A.\ A \\ A^3 &= A^2.\ A = A.\ A.\ A \\ A^4 &= A^3.\ A = A.\ A.\ A.\ A \\ \therefore\quad A^m &= (A.\ A.\ A.\ A \text{ ------- } m \text{ times}) \end{aligned}$$

Note: 1. Also for an identity matrix I of any order $I = I^2 = I^3 = I^4$ --------- $= I^n$.

2. For a square matrix A of order n,

$f(A) = a_0 I + a_1 A + a_2 A^2 +$ -------- $+ a_n A^n$ is a matrix polynomial.

If $f(A)$ is a zero matrix, then A is the **root** or **zero** of polynomial $f(x)$

7. Transpose Matrix: *The matrix obtained from any given matrix A by interchanging its rows and columns is called the transpose of the given matrix and is denoted by* $\boldsymbol{A^T}$ *or* $\boldsymbol{A'}$.

If $A = [a_{ij}]_{m\times n} \Rightarrow A^T = [a_{ji}]_{n\times m}$

For example, if $A = \begin{bmatrix} 3 & 6 & 2 \\ 1 & -1 & 5 \end{bmatrix}_{3\times2}$, then A^T or $A' = \begin{bmatrix} 3 & 1 \\ 6 & -1 \\ 2 & 5 \end{bmatrix}_{2\times3}$

$A = \begin{bmatrix} 4 & -1 & 2 \\ 7 & 6 & 5 \\ -3 & -2 & 0 \end{bmatrix}$, then $A' = \begin{bmatrix} 4 & 7 & -3 \\ -1 & 6 & -2 \\ 2 & 5 & 0 \end{bmatrix}$

Note:
- $(i\text{-}j)$th element of A = $(j\text{-}i)$th element of A^T
- If order of A is $m\times n$, then order of A^T is $n\times m$

Properties of Transpose Matrix

I. If A is any matrix, then $\boldsymbol{(A')' = A}$

Let $A = \begin{bmatrix} 4 & -2 & -3 \\ 1 & 6 & 5 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 4 & 1 \\ -2 & 6 \\ -3 & 5 \end{bmatrix} \Rightarrow (A')' = \begin{bmatrix} 4 & -2 & -3 \\ 1 & 6 & 5 \end{bmatrix}$

II. If **A and $\boldsymbol{B}$ are two matrices of the same order, then $\boldsymbol{(A + B)' = A' + B'}$**

For example, Let $A = \begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 6 & -5 \\ 0 & 8 \end{bmatrix}$

$$A + B = \begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 6 & -5 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 2+6 & 0+(-5) \\ 1+0 & -3+8 \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ 1 & 5 \end{bmatrix}$$

$$(A + B)' = \begin{bmatrix} 8 & 1 \\ -5 & 5 \end{bmatrix}$$

$$A' + B' = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2+6 & 1+0 \\ 0+(-5) & -3+8 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ -5 & 5 \end{bmatrix}.$$

III. If ***A* is *m*×*p*** matrix and ***B* is *p*×*n*** matrix then $\boldsymbol{(AB)' = B'A'}$.

For example, let $A = \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & -2 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}_{3\times 1} \begin{bmatrix} 3 & 1 & -2 \end{bmatrix}_{1\times 3} = \begin{bmatrix} 1\times 3 & 1\times 1 & 1\times -2 \\ -5\times 3 & -5\times 1 & -5\times -2 \\ 7\times 3 & 7\times 1 & 7\times -2 \end{bmatrix}_{3\times 3} = \begin{bmatrix} 3 & 1 & -2 \\ -15 & -5 & 10 \\ 21 & 7 & -14 \end{bmatrix}$$

$$\therefore\ (AB)' = \begin{bmatrix} 3 & -15 & 21 \\ 1 & -5 & 7 \\ -2 & 10 & -14 \end{bmatrix}$$

Now $$A' = \begin{bmatrix} 1 & -5 & 7 \end{bmatrix},\ B' = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

So, $$B'A' = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}_{3\times 1} \begin{bmatrix} 1 & -5 & 7 \end{bmatrix}_{1\times 3} = \begin{bmatrix} 3\times 1 & 3\times -5 & 3\times 7 \\ 1\times 1 & 1\times -5 & 1\times 7 \\ -2\times 1 & -2\times -5 & -2\times 7 \end{bmatrix}_{3\times 3} = \begin{bmatrix} 3 & -15 & 21 \\ 1 & -5 & 7 \\ -2 & 10 & -14 \end{bmatrix}.$$

$$\Rightarrow\ (AB)' = B'A'$$

IV. If ***A* is a matrix** and ***k*** is a **scalar**, then $\boldsymbol{(kA)' = kA'}$.

8. (*i*) Symmetric Matrices: A square matrix $A = [a_{ij}]$ is said to be **symmetric** if its $(i - j)$th element is equal to its $(j - i)$th element, *i. e.*, if $\boldsymbol{a_{ij} = a_{ji}\ \forall\ i, j}$.

$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 4 & 7 \\ 4 & 5 & 0 \\ 7 & 0 & 3 \end{bmatrix}$ are symmetric matrices

then $a_{12} = a_{21}$ $\qquad a_{12} = a_{21},\ a_{13} = a_{31},\ a_{23} = a_{32}$

> **Note:** • Symmetric matrices are always square matrices.
> • For a matrix A to be symmetric, it is necessary that the matrix is equal to its transpose, *i.e.,* $A^T = A$.
> • Diagonal matrices are always symmetric.

(*ii*) Skew-Symmetric Matrices: A square matrix $A = [a_{ij}]$ is said to be skew-symmetric if the $(i - j)th$ element of A is the negative of $(j - i)th$ element of A, *i.e.*, if $a_{ij} = -a_{ji}\ \forall\ i, j$.

Thus, the matrix $\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ -b & c & 0 \end{bmatrix}$ is a skew symmetric matrix.

Each element on the principal diagonal of a skew-symmetric matrix is zero as:

$$a_{ij} = -a_{ji}\ \forall\ i, j$$
$$\Rightarrow\ a_{ii} = -a_{ii}\ \forall\ i = j \quad \Rightarrow \quad 2a_{ii} = 0 \Rightarrow a_{ii} = 0\ \forall\ i$$
$$\Rightarrow\ a_{11} = a_{22} = a_{33} = \ldots\ldots\ldots\ldots = a_{nn} = 0.$$

> **Note: 1.** For a skew-symmetric matrix, it is necessary that $A' = -A$.
> **2.** A matrix which is both symmetric and skew symmetric is a square null matrix.
> A is symmetric as well as skew symmetric.
> $\Rightarrow A' = A$ and $A' = -A$.
> $\Rightarrow A = -A \Rightarrow 2A = 0 \Rightarrow A = 0$.

Properties of Symmetric and Skew-Symmetric Matrices:

I. The sum of two symmetric matrices is a symmetric matrix.

Let A, B be two symmetric matrices. Then,

$A' = A$, $B' = B$

$\therefore$ $(A + B)' = A' + B' = (A + B) \Rightarrow A + B$ is symmetric.

II. The sum of two skew symmetric matrices is a skew symmetric matrix.

Let, A, B be two skew symmetric matrices. Then,

$A' = -A$, $B' = -B$.

Then, $(A + B)' = A' + B' = -A + (-B) = -(A + B)$

$\Rightarrow (A + B)$ is skew symmetric.

III. For a scalar *k* and

(*a*) a symmetric matrix A, kA is a symmetric matrix.

(*b*) a skew symmetric matrix A, kA is a skew symmetric matrix.

IV. If A be any square matrix, then $A + A'$ is symmetric and $A - A'$ is skew symmetric.

- $(A + A')' = A' + (A')'$
  $= A' + A$

$\Rightarrow A + A'$ is symmetric.

- $(A - A')' = A' - (A')'$
  $= A' - A = -(A - A')$

$\Rightarrow A - A'$ is skew symmetric.

V. Every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew–symmetric matrix.

Given, a square matrix A, it can be expressed as the sum of a symmetric matrix and a skew symmetric matrix as under:

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

where $\frac{1}{2}(A + A')$ is a symmetric matrix and

$\frac{1}{2}(A - A')$ is a skew symmetric matrix.

VI. For the symmetric matrices A, B, if AB is a symmetric matrix, then $AB = BA$ and vice versa.

Given, $A' = A$, $B' = B$ and $(AB)' = AB$ ...(*i*)

Now $(AB)' = B'A' = BA$ ...(*ii*)

From (i) and (ii) $AB = BA$

If $AB = BA \Rightarrow AB = B'A' \Rightarrow AB = (AB)'$ $(\because A^1 = A, B^1 = B)$

$\Rightarrow AB$ is a symmetric matrix.

VII. All positive integral powers of a symmetric matrix are symmetric.

If $A' = A$, then $(A^n)' = A^n$.

$(A^n)' = (A.A.A.A. \ldots n \text{ times})'$

$= A'.A'.A'.A' \ldots n$ times $(\because (A)' = A)$

$= A.A.A.A \ldots n$ times $= A^n$.

$\Rightarrow A^n$ is symmetric.

DETERMINANTS

9. Determinant of a square matrix

Associated with each square matrix A having real number entries is a real number called the determinant of A and is denoted by δA or ΔA or $|A|$ or $\det(A)$.

- For a square matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ of order 2

  det $(A) = \Delta A = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$

Note: The determinant of a matrix of order 1, *i.e.*, $[a]$ is a itself.

If $A = [-3]$, then det. $A = |A| = |-3| = -3$.

Caution: The determinant $|-3| = -3$ should not be confused with the absolute value $|-3| = 3$.

- For a square matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ of order 3, the value of determinant A is calculated as explained below.

  Some new concepts that will be used are:

Minors and Cofactors

Minors: The minor of an element in a determinant is a determinant that is left after removing the row and column which intersect at the element, and is of order one less than that of the given determinant.

$\therefore$ In the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Minor of $a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, Minor of $b_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, Minor of $c_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$\therefore$ ***In general, the minor M_{ij} of element a_{ij} is the value of the determinant obtained by deleting the ith row and jth column of the given determinant.***

Cofactor. The cofactor of an element or element a_{ij} is the minor of a_{ij} with appropriate sign. Thus,

$$\textbf{Cofactor of } a_{ij} = A_{ij} = (-1)^{i+j} M_{ij} \textbf{ (minor)}$$

where i and j are respectively the row number and column number of the element.

$\therefore$ In the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Cofactor of $a_1 = (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = b_2 c_3 - b_3 c_2$

Cofactor of $a_2 = (-1)^{2+1} \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = -(b_1 c_3 - b_3 c_1)$

Cofactor of $b_1 = (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = -(a_2 c_3 - a_3 c_2)$

Thus the value of a determinant can be determined by expanding it along any row or column.

Now, the value of a determinant of order 3 can be written as: Expanding along row 1 (R_1)

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= a_1$(its cofactor) $+ b_1$(its cofactor) $+ c_1$(its cofactor) $= a_1$(its cofactor) $+ a_2$(its cofactor) $+ a_3$(its cofactor)

$= a_2$(its cofactor) $+ b_2$(its cofactor) $+ c_2$(its cofactor) $= b_1$(its cofactor) $+ b_2$(its cofactor) $+ b_3$(its cofactor)

$= a_3$(its cofactor) $+ b_3$(its cofactor) $+ c_3$(its cofactor) $= c_1$(its cofactor) $+ c_2$(its cofactor) $+ c_3$(its cofactor)

Note:
- The ordered pairs used are (an element, minor) for the same row or column.
- While expanding a determinant by any row or column using minors, we may keep in mind the following scheme of Signs for a determinant of order 3.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

- If a row or column of a determinant consists of all zeros, the value of the determinant is zero.

10. Singular matrix. A square matrix A is said to be singular if det $[A] = 0$, otherwise it is a non-singular matrix. For example,

(*a*) Let $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

Then, $|A| = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = (1\times(-1)) - (2 \times 3) = -1 - 6 = -7 \neq 0$

$\therefore$ A is a non-singular matrix.

(*b*) Let $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ Then,

$$|A| = 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \quad \text{(Expanding along } R_1\text{)}$$

$$= 0 + (-1)(-1) + (-1) \times 1 = 1 - 1 = 0$$

$\therefore |A| = 0 \Rightarrow A$ is a singular matrix.

11. Properties of Determinants.

Determinants have some properties which help in simplifying the process of finding the value of the determinant. In fact in some cases using these properties, we can find the value of the determinant without ever expanding along a given row or column. We shall list the properties here and show them with the help of examples.

Property I: ***If each entry in any row, or each entry in any column of a determinant is 0, then the determinant is equal to 0.***

For example, $\begin{vmatrix} 3 & 1 & 5 \\ 0 & 0 & 0 \\ 2 & 4 & -2 \end{vmatrix} = 0\begin{vmatrix} 1 & 5 \\ 4 & -2 \end{vmatrix} - 0\begin{vmatrix} 3 & 5 \\ 2 & -2 \end{vmatrix} + 0\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix}$ (Expanding by R_2)

$= 0$

Property II: ***If the rows be changed in columns and columns into rows, the determinant remains unaltered.***

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\text{LHS} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \quad \text{(Expanding along } C_1\text{(col. 1))}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$\text{RHS} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \quad \text{(Expanding along } R_1\text{(row 1)).}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Hence LHS = RHS.

Property III: ***If any two rows (or columns) of a determinant are interchanged the resulting determinant is the negative of the original determinant.***

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}$$

For example, $\begin{vmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ 3 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$ (Expanding along R_1)

$$= 1(2-4) - 2(4-12) - 3(2-3) = -2 + 16 + 3 = 17$$

$$\begin{vmatrix} 2 & 1 & -3 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 2(4-12) - 1(2-4) - 3(3-2) = -16 + 2 - 3 = \mathbf{-17.}$$

Note: If any line (row or column) of a determinant Δ be passed over in parallel lines, the resulting determinant $\Delta' = (-1)^m \Delta$.

For example, if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta' = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}$, then

$\Delta' = (-1)^2 \Delta = \Delta.$

Property IV: ***If any two rows (or columns) in a determinant are identical, the determinant is equal to zero.***

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

$$\Delta = a_1\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_1\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_1\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= a_1(b_1c_2 - b_2c_1) - b_1(a_1c_2 - a_2c_1) + c_1(a_1b_2 - b_1a_2)$$

$$= a_1b_1c_2 - a_1b_2c_1 - b_1a_1c_2 + b_1a_2c_1 + c_1a_1b_2 - b_1a_2c_1 = 0$$

Property V: ***If all the elements of any row (or column) be multiplied by a non-zero real number k, then the value of the new determinant is k times the value of the original determinant.***

Thus if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_2 \end{vmatrix}$ and $\Delta' = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then $\Delta = k\Delta'$

For example, $\begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 6 & -3 & 15 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ 3(2) & 3(-1) & 3(5) \end{vmatrix} = 3\begin{vmatrix} 1 & 3 & -1 \\ 2 & 4 & 1 \\ 2 & -1 & 5 \end{vmatrix}$

Note: If two parallel lines (rows or columns) be such that the elements of one are equi-multiples of the elements of the other, the determinant is equal to zero.

Let $\Delta = \begin{vmatrix} 1 & 3 & 2 \\ 1 & 3 & 4 \\ 2 & 6 & 8 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & 2 \\ 1 & 3 & 4 \\ 1 & 3 & 4 \end{vmatrix} = 2 \times 0 = 0$ (Two identical rows)

Here elements of row 2 (R_3) are equimultiples of elements of R_3.

Property VI: ***If each entry in a row (or column) of a determinant is written as the sum of two or more terms, then the determinant can be written as the sum of two or more determinants.***

i.e,
$$\Delta = \begin{vmatrix} a_1+x_1 & b_1 & c_1 \\ a_2+x_2 & b_2 & c_2 \\ a_3+x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}$$

Property VII: ***If each entry of one row (or column) of a determinant is multiplied by a real number k and the resulting product is added to the corresponding entry in another row (or column respectively) in the determinant, then the resulting determinant is equal to the original determinant.***

If
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3+ka_1 & b_3+kb_1 & c_3+kc_1 \end{vmatrix} \text{ Then } \Delta = \Delta'.$$

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3+ka_1 & b_3+kb_1 & c_3+kc_1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ ka_1 & kb_1 & kc_1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \times 0 \qquad (\because \text{ Two rows are identical})$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta.$$

This property can be generalised as:

If to each element of a line (row or column) of a determinant be added the equi-multiples of the corresponding elements of one or more parallel lines, the determinant remains unaltered.

$$\begin{vmatrix} a_1+la_2+ma_3 & a_2 & a_3 \\ b_1+lb_2+mb_3 & b_2 & b_3 \\ c_1+lc_2+lc_3 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

This property provides very powerful methods for simplifying the process for calculating the values of determinants.

$R_1 \longrightarrow R_1 + m\,R_2 + n\,R_3$ means to the first row, we add m times the second row and n times the third row. Similarly, $C_3 \longrightarrow C_3 - C_1$ means subtracting corresponding elements of column 1 from elements of column 3 and placing them in place of elements of column 3.

For example, $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 43-7\times6 & 1 & 6 \\ 35-7\times4 & 7 & 4 \\ 17-7\times2 & 3 & 2 \end{vmatrix}$ and so $C_1 \longrightarrow C_1 - 7C_3 = \begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 3 & 3 & 2 \end{vmatrix} = 0$, C_1 and C_2 being identical.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix} \text{ Operating } C_1 \longrightarrow C_1 + C_2 + C_3$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} \text{ (Taking out } (a+b+c) \text{ common)}$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \text{ Operating } R_2 \longrightarrow R_2 - R_1,\ R_3 \longrightarrow R_3 - R_1$$

$= (a + b + c).\ \{(c - b)(b - c) - (a - b)(a - c)\}$ expanding along column 1

$= (a + b + c)\ \{bc - b^2 - c^2 + cb - (a^2 - ab - ac + bc)\}$

$= (a + b + c)\ \{ab + bc + ca - a^2 - b^2 - c^2\}.$

Property VIII : ***Product of two determinants***

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

12. Adjoint of a square matrix

Let $A = [a_{ij}]_{3 \times 3}$ be the given square matrix of order 3.

Then,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

If A_{ij} be the cofactor a_{ij}, the adjoint of matrix A denoted by adj. A is defined as:

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

So, *the adjoint of a square matrix A is the transpose of the matrix obtained by replacing each element of A by its cofactor in |A| (det A).*

For example,

(*a*) If $A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$, then to find adj A, we find the cofactors.

$A_{11} = (-1)^{+1}\ |-2| = -2$ (Missing out entries in first row and first column)

$A_{12} = (-1)^{1+2}\ |4| = -4$

$A_{21} = (-1)^{2+1}\ |-1| = +1$

$A_{22} = (-1)^{2+2}\ |3| = 3$

$$\therefore \text{ adj. } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$$

(*b*) Find the adjoint of matrix $A = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 1 \\ 3 & 2 & 2 \end{bmatrix}$

Let A_{ij} be the cofactor of a_{ij}. Then,

$$A_{11} = +\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = (4 - 2) = 2$$

Remember the signs of cofactor by

$$(-1)^{i+j} \text{ or } \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$A_{12} = -\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = -(8 - 3) = -5$$

$$A_{13} = \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} = (8 - 6) = 2$$

$$A_{21} = -\begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} = -(8 - 6) = -2 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = (2 - 9) = -7$$

$$A_{23} = -\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = -(2 - 12) = 10 \qquad A_{31} = \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = (4 - 6) = -2$$

$$A_{32} = -\begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} = -(1-12) = 11 \qquad A_{33} = \begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} = (2-16) = -14$$

$$\therefore \text{ adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 2 & -2 & -2 \\ -5 & -7 & 11 \\ 2 & 10 & -14 \end{bmatrix}$$

Properties of adjoint of a square matrix

1. If A be a square matrix of order n, then (adj A) $A = A$ (adj A) $= |A|\, I_n$, where $|A|$ = determinant value of matrix A and I_n is the identity matrix of order n.

For a 3 × 3 square matrix, let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \text{Then adj } A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

where A_1, B_1, C_1 ..... are the respective cofactors of a_1, b_1, c_1 ....

$$\therefore \quad A \text{ (adj } A) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$(\because\ a_1A_2 + b_1B_2 + c_1C_2 = a_1(b_1c_3 - b_3c_1) + b_1(a_1c_3 - a_3c_1) + c_1(a_1b_3 - a_3b_1) = 0)$

Similarly for all the entries besides the entries of the principal diagonal.

$$= |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\, I_3$$

Also it can be shown similarly that (*adj* A) $A = |A|\ \mathrm{I}_3$

> **Note:** Adj $I = I$ and Adj $0 = 0$.

II. If A and B are two non-singular matrices of the same order, then **adj (AB) = (adj B) (adj A).**

13. Inverse of a square matrix

- If A and B are square matrices such that $AB = BA = I$, then B is called the inverse of A and is written as $A^{-1} = B$ and A is the inverse of B, written as $B^{-1} = A$.
  Thus, $A^{-1}A = A\,A^{-1} = I$.
- If A is a non-singular square matrix of order n, then

$$\boldsymbol{A^{-1} = \frac{1}{|A|} \text{ (adj } A) \qquad (|A| \neq 0)}$$

> **Note:** If $AB = BA = I$, then B is inverse of A, *i.e*, $B = A^{-1}$
> Also, we know by the property of adjoint of a square matrix that, A(adj A) = (adj A) $A = |A|\, I$
> $$\Rightarrow A\ \frac{(adj\ A)}{|A|} = \left(\frac{adj\ A}{|A|}\right) A = I \text{ (as } |A| \neq 0, A \text{ being singular)}$$
> $$\Rightarrow A^{-1} = \frac{1}{|A|}(adj\ (A))$$

- **Properties of matrices and inverses**

  I. If A and B are two non-singular matrices of order n, then AB is also a non - singular matrix of order n such that

  $\boldsymbol{(AB)^{-1} = B^{-1} A^{-1}}$

  (Reversal law for the inverse of a product)

  II. $(A^T)^{-1} = (A^{-1})^T$

For example, If $A = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 5 & 9 \\ -6 & 4 & 7 \end{bmatrix}$ and $AB = BA = I$, find B.

If $AB = BA = I$, then $B = A^{-1}$.

For A^{-1} to exists, A should be a non-singular matrix.

We have, $|A| = \begin{vmatrix} 3 & 0 & 2 \\ 1 & 5 & 9 \\ -6 & 4 & 7 \end{vmatrix} = 3\begin{vmatrix} 5 & 9 \\ 4 & 7 \end{vmatrix} - 0\begin{vmatrix} 1 & 9 \\ -6 & 7 \end{vmatrix} + 2\begin{vmatrix} 1 & 5 \\ -6 & 4 \end{vmatrix} = 3(35 - 36) + 2(4 + 30) = -3 + 68 = 65 \neq 0$

$\Rightarrow$ A is a non-singular matrix

Now we need to find adj A,

$\therefore \quad A_{11} = \begin{vmatrix} 5 & 9 \\ 4 & 7 \end{vmatrix} = (35 - 36) = -1$ $\qquad A_{12} = -\begin{vmatrix} 1 & 9 \\ -6 & 7 \end{vmatrix} = -(7 + 54) = -61$

$A_{13} = \begin{vmatrix} 1 & 5 \\ -6 & 4 \end{vmatrix} = 4 + 30 = 34$ $\qquad A_{21} = -\begin{vmatrix} 0 & 2 \\ 4 & 7 \end{vmatrix} = -(0 - 8) = 8$

$A_{22} = \begin{vmatrix} 3 & 2 \\ -6 & 7 \end{vmatrix} = 21 + 12 = 33$ $\qquad A_{23} = -\begin{vmatrix} 3 & 0 \\ -6 & 4 \end{vmatrix} = -(12 - 0) = -12$

$A_{31} = \begin{vmatrix} 0 & 2 \\ 5 & 9 \end{vmatrix} = 0 - 10 = -10$ $\qquad A_{32} = -\begin{vmatrix} 3 & 2 \\ 1 & 9 \end{vmatrix} = -(27 - 2) = -25$

$A_{33} = \begin{vmatrix} 3 & 0 \\ 1 & 5 \end{vmatrix} = 15$

$$\therefore \text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} -1 & -61 & 34 \\ 8 & 33 & -12 \\ -10 & -25 & 15 \end{bmatrix}^T = \begin{bmatrix} -1 & 8 & -10 \\ -61 & 33 & -25 \\ 34 & -12 & 15 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{65}\begin{bmatrix} -1 & 8 & -10 \\ -61 & 33 & -25 \\ 34 & -12 & 15 \end{bmatrix}.$$

14. Some more special matrices :

1. **Nilpotent Matrix:** *A square matrix A such that $A^n = 0$ is called a nilpotent matrix of order n.*

   If there exists a matrix such that $A^2 = 0$, then A is nilpotent of order 2.
2. **Idempotent Matrix:** A square matrix A, such that $A^2 = A$ is called an idempotent matrix.

   If $AB = A$ and $BA = B$, then A and B are idempotent matrices.
3. **Orthogonal Matrix:** A square matrix A, such that $AA^T = I$ is called an orthogonal matrix.

   If A is an orthogonal matrix, then $\boldsymbol{A^T = A^{-1}}$
4. **Involuntary Matrix:** A square matrix A, such that $A^2 = I$ is called an involuntary matrix.

Note: (*i*) A matrix A is involuntary $\Leftrightarrow (I - A)(I + A) = 0$

(*ii*) A matrix A is involuntary matrix then $A^{-1} = A$.

15. Application of matrices to the solution of linear equations

Consider the two simultaneous equations in two variables x and y.

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

These can be written in the matrix form as:

$$\underset{A}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}\underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

where $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is a 2×2 matrix and

$X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ are both 2×1 matrices and

Similarly the three simultaneous equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$ can be written in the matrix form as:

$$\underset{A}{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}}\underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}$$

where A is a square matrix of order 3 (3×3) and X and B are (3×1) column matrices.

Now, $\quad AX = B$

$\Rightarrow \quad A^{-1} AX = A^{-1} B$

$\Rightarrow \quad IX = A^{-1} B \Rightarrow X = A^{-1} B$

$\Rightarrow \quad X = \dfrac{1}{|A|}(adj\ A) . B$

The conditions for unique solution, no solution and infinite number of solution can be summarized as under:

$AX = B$

↓

Find $|A|$

- ↓ $|A| \neq 0$
  Unique solution consistent
  $X = A^{-1} B$
- ↓ $|A| = 0$
  No solution or infinite number of solutions
  ↓
  Find $(adj\ A)B$
  - ↓ $(adj\ A)B \neq 0$
    ↓
    No solution
    Inconsistent
  - ↓ $(adj\ A)\ B = 0$
    ↓
    Infinitely many solutions
    Consistent (dependent)

Special case: When $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

In this case, $|A| \neq 0 \Rightarrow x = 0, y = 0, z = 0$ we say that the system has **trivial** solution.

If $|A| = 0$, then the system has infinitely many solutions.

Note: We shall deal with equations in two variables only in this book.

Ex. 1. Use matrix method to solve the system of equations :
$4x - 3y = 11,\ 3x + 7y = -1$.

Sol. The given system of equations can be written in the matrix form as:

$$\underset{A}{\begin{bmatrix} 4 & -3 \\ 3 & 7 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} \underset{=}{=} \underset{B}{\begin{bmatrix} 11 \\ -1 \end{bmatrix}}$$

$$|A| = \begin{vmatrix} 4 & -3 \\ 3 & 7 \end{vmatrix} = 28 + 9 = 37 \neq 0$$

$\Rightarrow$ A is a non-singular matrix and the system has an unique solution.

$X = A^{-1} B$.

$\therefore$ We need to find adj A and hence the cofactors of A.

$A_{11} = 7, A_{12} = -3, A_{21} = -(-3) = 3, A_{22} = 4$

$$\therefore\ A^{-1} = \frac{1}{|A|}\, adj\ A = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T$$

$$= \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \frac{1}{37} \begin{bmatrix} 7 & 3 \\ -3 & 4 \end{bmatrix}$$

$$\therefore\ \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \begin{bmatrix} 7/37 & 3/37 \\ -3/37 & 4/37 \end{bmatrix} \begin{bmatrix} 11 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{77}{37} & -\frac{3}{37} \\ \frac{-33}{37} & -\frac{4}{37} \end{bmatrix} = \begin{bmatrix} \frac{74}{37} \\ \frac{-37}{37} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$\Rightarrow$ $\boldsymbol{x = 2, y = -1.}$

Ex. 2. Use matrix method to examine the consistency or inconsistency of the system of equations
$6x + 4y = 2,\ 9x + 6y = 3$.

Sol. Writing the given system of equations in the matrix form, we have

$$\underset{A}{\begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} \underset{=}{=} \underset{B}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

Now, $$|A| = \begin{vmatrix} 6 & 4 \\ 9 & 6 \end{vmatrix} = 36 - 36 = 0$$

$\Rightarrow$ A is a singular matrix $\Rightarrow$ Either the system has no solution or infinite number of solutions.

To check that we find (adj A) B.

$A_{11} = 6, A_{12} = -9, A_{21} = -4, A_{22} = 6$

$$\therefore\quad \text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -9 & 6 \end{bmatrix}$$

$$(\text{adj } A)\ B = \begin{bmatrix} 6 & -4 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 - 12 \\ -18 + 18 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0.$$

Since (adj A) $B = 0$, the given system is consistent and has infinite number of solutions.

Let $y = k$ in the first equation. Then,

$$6x + 4k = 2 \Rightarrow 6x = 2 - 4k \Rightarrow x = \frac{2-4k}{6} = \frac{1}{3}(1-2k)$$

Putting this value of x in the second equation, we have

$9.\frac{1}{3}(1 - 2k) + 6k = 3 \quad \Rightarrow 3 - 6k + 6k = 3 \quad \Rightarrow 3 = 3$, which is true

Hence the given system has infinitely many solutions given by

$$\boldsymbol{x = \frac{1}{3}(1 - 2k), y = k}$$

Ex. 3. Use matrix method to examine the given system of equations for consistency or inconsistency.

$$\mathbf{3x - 2y = 5}$$
$$\mathbf{6x - 4y = 9}$$

Sol. Writing the given system of equations in the matrix form we have

$$\underset{A}{\begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} \underset{=}{=} \underset{B}{\begin{bmatrix} 5 \\ 9 \end{bmatrix}}$$

Now, $|A| = \begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix} = -12 - (-12) = -12 + 12 = 0$

$\Rightarrow$ A is a singular matrix

$\Rightarrow$ Either the system has no solution or infinitely many solutions.

To check that we find (adj A) B.

$$A_{11} = -4, A_{12} = -6, A_{21} = -(-2) = 2, A_{22} = 3$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix}$$

$$(\text{adj } A)\, B = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} -20+18 \\ -30+27 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \neq 0$$

$\Rightarrow$ The system of equations is inconsistent and has no solution.

16. Application of determinants to the solution of linear equations. (*Cramer's Rule*)

Consider the simultaneous equations,

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

$$\Rightarrow \quad a_1 x + b_1 y - c_1 = 0$$
$$a_2 x + b_2 y - c_2 = 0$$

Solving these equations by cross-multiplication method, we have

$$\frac{x}{-b_1c_2 + b_2c_1} = \frac{y}{-a_2c_1 + a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \quad x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

The solutions can be expressed in the determinant form as :

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{Dx}{D}, y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{Dy}{D}$$

The determinant D is the determinant of the coefficients of variables x and y, while in Dx, the coefficients of x, *i.e.*, a_1 and a_2 are replaced by constant terms c_1 and c_2 and in determinant Dy, the coefficients of y, *i.e.*, b_1 and b_2 are replaced by the constant terms.

The solutions to the above given equations will exist only when $D \neq 0$. Likewise, for the system of linear equations in three variables.

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

We have the solutions for x, y and z in the determinant form as:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{Dx}{D},\ y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{Dy}{D},\ z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{Dz}{D}$$

Here the important conditions are:

(***i***) For a solution to exist, $|D| \neq 0$

(***ii***) The constant terms are on the Right Hand Side of the given equations.

Conditions of consistency and inconsistency of linear equations.

A system of equations is said to be consistent if its solution exists whether unique or not, otherwise it is inconsistent.

The conditions for unique solution, infinitely many solutions and no-solution can be summarized as under:

(***i***) If $D \neq 0$, then the given system of equations is consistent and has a unique solution, namely,

$$x = \frac{Dx}{D},\ y = \frac{Dy}{D}$$

(***ii***) If $D = 0$ and $Dx = Dy = 0$, then the system may be consistent with infinitely many solutions or inconsistent.

(***iii***) If $D = 0$ and at least one of Dx and Dy is non-zero, then the system has no solution, *i.e.,* the system is inconsistent.

Note: We shall limit the conditions of consistency and inconsistency in this book only to equations with two variables.

Ex. 1. Solve $7x + 2y – 25 = 0$ and $2x – y – 4 = 0$ by Cramer's Rule.

Sol. The given equations are:

$$7x + 2y = 25$$
$$2x - y = 4$$

$$\therefore \quad D = \begin{vmatrix} 7 & 2 \\ 2 & -1 \end{vmatrix} = -7 - 4 = -11 \neq 0$$

$\because$ $D \neq 0$, therefore the solution exists.

$$Dx = \begin{vmatrix} 25 & 2 \\ 4 & -1 \end{vmatrix} = -25 - 8 = -33$$

$$Dy = \begin{vmatrix} 7 & 25 \\ 2 & 4 \end{vmatrix} = 28 - 50 = -22$$

$$x = \frac{Dx}{D} = \frac{-33}{-11} = 3,\ y = \frac{Dy}{D} = \frac{-22}{-11} = 2.$$

Hence, $x = 3$, $y = 2$

Ex. 2. Check whether the given system of equations is consistent or inconsistent.

(*i*) $x + 3y = 2$, $2x + 6y = 7$ **(*ii*) $2x + 7y = 9$, $4x + 14y = 18$**

Sol. (*i*) $x + 3y = 2$ ...(1)

$2x + 6y = 7$ ...(2)

$$D = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0$$

$$Dx = \begin{vmatrix} 2 & 3 \\ 7 & 6 \end{vmatrix} = 12 - 21 = -9 \neq 0$$

$$Dy = \begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} = 7 - 4 = 3 \neq 0$$

Since, $D = 0$ and $Dx \neq 0$, $Dy \neq 0$, (at least one of the determinant $Dx \neq 0$), the given system of equations is inconsistent, *i.e.*, it has no solution,

(*ii*) $2x + 7y = 9$ ...(1)

$4x + 14y = 18$ ...(2)

$$D = \begin{vmatrix} 2 & 7 \\ 4 & 14 \end{vmatrix} = 28 - 28 = 0$$

$$Dx = \begin{vmatrix} 9 & 7 \\ 18 & 14 \end{vmatrix} = 126 - 126 = 0$$

$$Dy = \begin{vmatrix} 2 & 9 \\ 4 & 18 \end{vmatrix} = 36 - 36 = 0$$

Since, $D = 0$ and $Dx = Dy = 0$, therefore the system has infinitely many solutions or is inconsistent.

Let $x = k$, Then from (1), $2k + 7y = 9 \Rightarrow y = \frac{9-2k}{7}$

Substituting this value in (2), $4(k) + 4\left(\frac{9-2k}{7}\right) = 18$

$\Rightarrow$ $4k + 18 - 4k = 18 \quad \Rightarrow 18 = 18.$

$\therefore$ The system has infinitely many solutions given by $x = k$, $y = \frac{9-2k}{7}$

SOLVED EXAMPLES

Ex. 1. If a matrix has 12 elements, what are the possible orders it can have?

Sol. We know that a matrix of order $m \times n$ has mn elements. Hence to find all possible orders of a matrix having 12 elements, we will have to find all ordered pairs the product of whose components is 12. The possible ordered pairs satisfying the above given condition are (1, 12), (12, 1), (2, 6), (6, 2), (3, 4), (4, 3).

Hence, the possible orders are 1 × 12, 12 × 1, 2 × 6, 6 × 2, 3 × 4, and 4 × 3.

Ex. 2. Construct a 2 × 2 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = \frac{1}{2}|2i - 3j|$.

Sol. Let $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\therefore \quad a_{11} = \frac{1}{2}|2\times 1 - 3\times 1| = \frac{1}{2}|2-3| = \frac{1}{2}|-1| = \frac{1}{2}\times 1 = \frac{1}{2}$$

$$a_{12} = \frac{1}{2}|2\times 1 - 3\times 2| = \frac{1}{2}|2-6| = \frac{1}{2}|-4| = \frac{1}{2}\times 4 = 2$$

$$a_{21} = \frac{1}{2}|2\times2-3\times1|=\frac{1}{2}|4-3|=\frac{1}{2}\times1=\frac{1}{2}$$

$$a_{22} = \frac{1}{2}|2\times2-3\times2|=\frac{1}{2}|4-6|=\frac{1}{2}\times2=1$$

$$\therefore\ \mathbf{A} = \begin{bmatrix} \frac{1}{2} & 2 \\ \frac{1}{2} & 1 \end{bmatrix}.$$

Ex. 3. Find the values of x and y so that the matrices $A = \begin{bmatrix} 2x+1 & 3y \\ 0 & y^2-5y \end{bmatrix}$, $B = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$ may be equal ?

Sol. $A = [a_{ij}] = B = [b_{ij}] \Rightarrow a_{ij} = b_{ij}$

$\therefore\ 2x + 1 = x + 3 \Rightarrow x = 2$

$3y = y^2 + 2 \Rightarrow y^2 - 3y + 2 = 0 \Rightarrow (y-1)(y-2) = 0 \Rightarrow y = 1$ or 2

$y^2 - 5y = -6 \Rightarrow y^2 - 5y + 6 = 0 \Rightarrow (y-3)(y-2) = 0 \Rightarrow y = 3$ or 2

Since, $3y = y^2 + 2$ and $y^2 - 5y = -6$ must hold simultaneously, we take the common solution of the two equations, *i.e.*, $y = 2$.

$\therefore\ A = B \Rightarrow \mathbf{x = 2, y = 2}$

Ex. 4. Solve the equation $-2\left[x+\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\right] = 3x+\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, over $S_{3\times3}$.

Sol. Given equation becomes, $-2x + -2\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = 3x+\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -2 & -4 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 5x \qquad \Rightarrow 5x = \begin{bmatrix} -2-1 & -4-0 & -6-0 \\ 0-0 & -2-0 & -4-0 \\ 0-0 & 0-0 & -2-1 \end{bmatrix}$$

$$\Rightarrow 5x = \begin{bmatrix} -3 & -4 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & -3 \end{bmatrix} \qquad \Rightarrow \mathbf{x} = \begin{bmatrix} \mathbf{-3/5} & \mathbf{-4/5} & \mathbf{-6/5} \\ \mathbf{0} & \mathbf{-2/5} & \mathbf{-4/5} \\ \mathbf{0} & \mathbf{0} & \mathbf{-3/5} \end{bmatrix}$$

Ex. 5. If A = diag [3 –2 1] and B = diag [1 3 – 2], find 2A – 3B.

Sol. $A = \text{diag}\,[3\ \ -2\ \ 1] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\qquad B = \text{diag}\,[1\ \ 3\ \ -2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$$\Rightarrow 2A - 3B = 2\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 6-3 & 0 & 0 \\ 0 & -4-9 & 0 \\ 0 & 0 & 2+6 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -13 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \textbf{diag [3 \ \ –13 \ \ 8]}.$$

Ex. 6. Find a and b, if $\left\{3\begin{bmatrix}2 & 1 & -3\\ 1 & 4 & 2\end{bmatrix} - 2\begin{bmatrix}1 & -2 & 0\\ 2 & -1 & 3\end{bmatrix}\right\}\begin{bmatrix}2\\ 0\\ -1\end{bmatrix} = \begin{bmatrix}a\\ b\end{bmatrix}$.

Sol. LHS $= \left\{3\begin{bmatrix}2 & 1 & -3\\ 1 & 4 & 2\end{bmatrix} - 2\begin{bmatrix}1 & -2 & 0\\ 2 & -1 & 3\end{bmatrix}\right\}\begin{bmatrix}2\\ 0\\ -1\end{bmatrix} = \left\{\begin{bmatrix}6 & 3 & -9\\ 3 & 12 & 6\end{bmatrix} - \begin{bmatrix}2 & -4 & 0\\ 4 & -2 & 6\end{bmatrix}\right\}\begin{bmatrix}2\\ 0\\ -1\end{bmatrix}$

$= \begin{bmatrix}6-2 & 3+4 & -9-0\\ 3-4 & 12+2 & 6-6\end{bmatrix}\begin{bmatrix}2\\ 0\\ -1\end{bmatrix} = \begin{bmatrix}4 & 7 & -9\\ -1 & 14 & 0\end{bmatrix}\begin{bmatrix}2\\ 0\\ -1\end{bmatrix} = \begin{bmatrix}4\times2+7\times0-9\times-1\\ -1\times2+14\times0+0\times-1\end{bmatrix} = \begin{bmatrix}17\\ -2\end{bmatrix}$

Given, $\begin{bmatrix}17\\ -2\end{bmatrix} = \begin{bmatrix}a\\ b\end{bmatrix} \Rightarrow \boldsymbol{a = 17, b = -2}$.

Ex. 7. If A, B, C are three matrices such that $A = \begin{bmatrix}x & y & z\end{bmatrix}$, $B = \begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}$, $C = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$ **evaluate ABC**

Sol. $AB = \begin{bmatrix}x & y & z\end{bmatrix}_{1\times3}\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}_{3\times3} = \begin{bmatrix}xa+yh+zg & xh+yb+zf & xg+yf+zc\end{bmatrix}_{1\times3}$

$\Rightarrow ABC = \begin{bmatrix}xa+yh+zg & xh+yb+zf & xg+yf+zc\end{bmatrix}_{1\times3}\begin{bmatrix}x\\ y\\ z\end{bmatrix}_{3\times1}$

$= \begin{bmatrix}x(xa+yh+zg)+y(xh+yb+zf)+z(xg+yf+zc)\end{bmatrix}_{1\times1}$

$= [ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz]$.

Ex. 8. If $A = \begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}$ and $f(x) = x^2 - 4x + 7$, show that $f(A) = 0$. Use this result to find A^5.

Sol. $f(A) = A^2 - 4A + 7I_2$

$A^2 = A.A = \begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix} = \begin{bmatrix}2\times2+3\times-1 & 2\times3+3\times2\\ -1\times2+2\times-1 & -1\times3+2\times2\end{bmatrix} = \begin{bmatrix}4-3 & 6+6\\ -2-2 & -3+4\end{bmatrix} = \begin{bmatrix}1 & 12\\ -4 & 1\end{bmatrix}$

$\therefore\ f(A) = A^2 - 4A + 7I_2 = \begin{bmatrix}1 & 12\\ -4 & 1\end{bmatrix} - 4\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix} + 7\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$= \begin{bmatrix}1 & 12\\ -4 & 1\end{bmatrix} - \begin{bmatrix}8 & 12\\ -4 & 8\end{bmatrix} + \begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix} = \begin{bmatrix}1-8+7 & 12-12+0\\ -4+4+0 & 1-8+7\end{bmatrix} = 0.$

Now $f(A) = 0 \Rightarrow A^2 - 4A + 7I_2 = 0$

$\Rightarrow A^2 = 4A - 7I_2$

$\therefore\ A^3 = A^2.\,A = (4A - 7I_2)\,A = 4A^2 - 7I_2A = 4A^2 - 7A$ $(\because I_2A = A)$

$\Rightarrow A^3 = 4(4A - 7I_2) - 7A = 9A - 28I_2$ (Replacing A^2 by $4A - 7I_2$)

$\therefore\ A^4 = (9A - 28I_2)\,A = 9A^2 - 28I_2A = 9(4A - 7I_2) - 28A = 36A - 63I_2 - 28A = 8A - 63I_2$

$\therefore\ A^5 = (8A - 63I_2)A = 8A^2 - 63I_2A = 8(4A - 7I_2) - 63A = 32A - 56I_2 - 63A = -31A - 56I_2$

$\therefore\ A^5 = -31\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix} - 56\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-62 & -93\\ 31 & -62\end{bmatrix} - \begin{bmatrix}56 & 0\\ 0 & 56\end{bmatrix}$

$$= \begin{bmatrix} -62-56 & -93-0 \\ 31-0 & -62-56 \end{bmatrix} = \begin{bmatrix} \mathbf{-118} & \mathbf{-93} \\ \mathbf{31} & \mathbf{-118} \end{bmatrix}$$

Ex. 9. If $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$, verify that $(AB)^T = B^T A^T$.

Sol. $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 5 & -2 \\ 3 & 7 & 1 \\ 4 & 9 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 3 \\ 5 & 0 & 1 \end{bmatrix}$

Now, $AB = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 5 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 8+3+0 & 0+6+12 & 10+0+4 \\ 20+7+0 & 0+14+27 & 25+0+9 \\ -8+1+0 & 0+2+3 & -10+0+1 \end{bmatrix} = \begin{bmatrix} 11 & 18 & 14 \\ 27 & 41 & 34 \\ -7 & 5 & -9 \end{bmatrix}$

$(AB)^T = \begin{bmatrix} 11 & 27 & -7 \\ 18 & 41 & 5 \\ 14 & 34 & -9 \end{bmatrix}$

Now, $B^T A^T = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 3 \\ 5 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 & -2 \\ 3 & 7 & 1 \\ 4 & 9 & 1 \end{bmatrix} = \begin{bmatrix} 8+3+0 & 20+7+0 & -8+1+0 \\ 0+6+12 & 0+14+27 & 0+2+3 \\ 10+0+4 & 25+0+9 & -10+0+1 \end{bmatrix} = \begin{bmatrix} 11 & 27 & -7 \\ 18 & 41 & 5 \\ 14 & 34 & -9 \end{bmatrix}$

$\therefore \quad (AB)^T = B^T A^T.$

Ex. 10. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, show that $A - A^T$ is a skew-symmetric matrix.

Sol. A square matrix $A = [a_{ij}]$ is said to be a skew-symmetric matrix if $A^T = -A$.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$\therefore \quad A - A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$(A - A^T)^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -(A - A^T).$

Hence, $A - A^T$ is a skew-symmetric matrix.

Ex. 11. Express the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 4 \\ -1 & 6 & 2 \end{bmatrix}$ as a sum of a symmetric and a skew-symmetric matrix.

Sol. We know that symmetric part of the matrix A is $\frac{1}{2}(A + A^T)$ and the skew-symmetric part is $\frac{1}{2}(A - A^T)$.

Here $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 4 \\ -1 & 6 & 2 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 6 \\ 3 & 4 & 2 \end{bmatrix}$

$$\text{Symmetric part} = \frac{1}{2}(A+A^T) = \frac{1}{2}\left\{\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 4 \\ -1 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 6 \\ 3 & 4 & 2 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 10 \\ 2 & 10 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 5 \\ 1 & 5 & 2 \end{bmatrix}$$

$$\text{Skew-symmetric part} = \frac{1}{2}(A-A^T) = \frac{1}{2}\left\{\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 4 \\ -1 & 6 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 6 \\ 3 & 4 & 2 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & -2 \\ -4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$\therefore \quad A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 5 \\ 1 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

Ex. 12. Find the adjoint of A, where $A = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} & \mathbf{5} \\ \mathbf{-2} & \mathbf{0} & \mathbf{1} \end{bmatrix}$.

Sol. Adj A is the transpose of the matrix obtained by replacing the elements of A by their corresponding cofactors.

$$A = [a_{ij}] = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A_{11} = \text{cofactor of } a_{11}\,(1) = (-1)^{1+1}\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$$

$$A_{12} = \text{cofactor of } a_{12}(-1) = (-1)^{1+2}\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -(2+10) = -12$$

$$A_{13} = \text{cofactor of } a_{13}(2) = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6$$

$$A_{21} = \text{cofactor of } a_{21}(2) = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1-0) = 1$$

$$A_{22} = \text{cofactor of } a_{22}(3) = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 + 4 = 5$$

$$A_{23} = \text{cofactor of } a_{23}(5) = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -(0-2) = 2$$

$$A_{31} = \text{cofactor of } a_{31}(-2) = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5 - 6 = -11$$

$$A_{32} = \text{cofactor of } a_{32}(0) = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5-4) = -1$$

$$A_{33} = \text{cofactor of } a_{33}(1) = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 + 2 = 5$$

$$\therefore \text{ adj. } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{3} & \mathbf{1} & \mathbf{-11} \\ \mathbf{-12} & \mathbf{5} & \mathbf{-1} \\ \mathbf{6} & \mathbf{2} & \mathbf{5} \end{bmatrix}.$$

Ex. 13. If $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 3 \end{bmatrix}$, find the value of (adj. A) A without finding adj. A.

Sol. We know that A(adj. A) = (adj. A) A = $|A|\, I$

$\therefore$ Here (adj. A) $A = |A|\, I_3$

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 3 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 3 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 0 \\ 3 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 1(3 - 0) - 3\,(6 - 0) + 1\,(4 - 3) = 3 - 18 + 1 = -14$$

$$\therefore \quad (\text{adj. } A)\, A = |A|\, I_3 = -14 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \mathbf{-14} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{-14} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-14} \end{vmatrix}.$$

Ex. 14. Find the inverse of the matrix $\begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

Then $|A| = -2 \times 4 - 5 \times 3 = -23 \neq 0$

$\therefore$ $|A| \neq 0 \Rightarrow A$ is a non-singular matrix $\Rightarrow A^{-1}$ exists.

$\therefore$ $A_{11} = (-1)^{1+1}\,|4| = 4 \qquad A_{21} = (-1)^{2+1}\,|5| = -5$

$A_{12} = (-1)^{1+2}\,|3| = -3 \quad A_{22} = (-1)^{2+2}\,|-2| = -2$

$$\therefore \text{ adj. } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ -3 & -2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{adj.\,A}{|A|} = \frac{1}{-23}\begin{bmatrix} 4 & -5 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} \mathbf{-4/23} & \mathbf{5/23} \\ \mathbf{3/23} & \mathbf{2/23} \end{bmatrix}.$$

Ex. 15. Let $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $G(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$, show that $[F(\alpha).\,G(\beta)]^{-1} = G(-\beta).\,F(-\alpha)$.

Sol. $$F(\alpha), F(-\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 & 0 \\ 0 & \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore$ $F(\alpha).\,F(-\alpha) = I \Rightarrow [F(\alpha)]^{-1} = F(-\alpha)$

Similarly, $$G(\beta).\,G(-\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}\begin{bmatrix} \cos(-\beta) & 0 & \sin(-\beta) \\ 0 & 1 & 0 \\ -\sin(-\beta) & 0 & \cos(-\beta) \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\beta + 0 + \sin^2\beta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2\beta + 0 + \cos^2\beta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore\ G(\beta).\ G(-\beta) = I \Rightarrow [G(\beta)]^{-1} = G(-\beta)$ $\qquad (\because (AB)^{-1} = B^{-1}A^{-1})$

$\therefore\ \{F(\alpha).\ G(\beta)\}^{-1} = \{G(\beta)\}^{-1}.\ \{F(\alpha)\}^{-1}$

$\qquad = G(-\beta).\ F(-\alpha)$

Ex. 16. For the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ show that $A^2 - 4A + 5I = 0$. Hence obtain A^{-1}.

Sol. $A^2 - 4A + 5I = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-2 & -1-3 \\ 2+6 & -2+9 \end{bmatrix} - \begin{bmatrix} 4 & -4 \\ 8 & 12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix} + \begin{bmatrix} -4 & +4 \\ -8 & -12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -1-4+5 & -4+4+0 \\ 8-8+0 & 7-12+5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Now premultiplying both the sides of $A^2 - 4A + 5I$ by A^{-1}, we get

$A^{-1}A^2 - 4A^{-1}A + 5A^{-1}I = 0$

$\Rightarrow (A^{-1}A)A - 4I + 5A^{-1} = 0$ $\qquad (\because A^{-1}A = I \text{ and } A^{-1}I = A^{-1})$

$\Rightarrow IA - 4I + 5A^{-1} = 0$

$\Rightarrow A - 4I + 5A^{-1} = 0 \Rightarrow 5A^{-1} = 4I - A$

$\Rightarrow 5A^{-1} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

$\therefore\ \boldsymbol{A^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}}$.

Ex. 17. Find the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$ and show that $aA^{-1} = (a^2 + bc + 1)\, I_2 - aA$.

Sol. For A^{-1} to exist, $|A| \neq 0$.

$$|A| = \begin{vmatrix} a & b \\ c & \frac{1+bc}{a} \end{vmatrix} = a\left(\frac{1+bc}{a}\right) - bc = 1 + bc - bc = 1 \neq 0$$

$\Rightarrow A^{-1}$ exists and $A^{-1} = \dfrac{1}{|A|} \times adj.\ A.$

$A_{11} = \dfrac{1+bc}{a},\ A_{12} = -c,\ A_{21} = -b,\ A_{22} = a$

$\therefore$ adj. $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

$\therefore\ A^{-1} = \dfrac{1}{1}\begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

LHS $= aA^{-1} = a\begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1+bc & -ab \\ -ac & a^2 \end{bmatrix}$

$$\text{RHS} = (a^2 + bc + 1)\, I_2 - aA$$

$$= (a^2+bc+1)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - a\begin{bmatrix}a & b\\ c & \dfrac{1+bc}{a}\end{bmatrix} = \begin{bmatrix}a^2+bc+1 & 0\\ 0 & a^2+bc+1\end{bmatrix} - \begin{bmatrix}a^2 & ab\\ ac & 1+bc\end{bmatrix}$$

$$= \begin{bmatrix}a^2+bc+1-a^2 & 0-ab\\ 0-ac & a^2+bc+1-1-bc\end{bmatrix} = \begin{bmatrix}bc+1 & -ab\\ -ac & a^2\end{bmatrix} = \text{LHS.}$$

Ex. 18. Solve the following system of equations by matrix method:

$5x + 3y + z = 16$

$2x + y + 3z = 19$

$x + 2y + 4z = 25.$

Sol. Writing the given equations in the matrix form $AX = B$, we have $\begin{bmatrix}5 & 3 & 1\\2 & 1 & 3\\1 & 2 & 4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}16\\19\\25\end{bmatrix}$, where

$$A = \begin{bmatrix}5 & 3 & 1\\2 & 1 & 3\\1 & 2 & 4\end{bmatrix},\ X = \begin{bmatrix}x\\y\\z\end{bmatrix},\ B = \begin{bmatrix}16\\19\\25\end{bmatrix}$$

$$|A| = \begin{vmatrix}5 & 3 & 1\\2 & 1 & 3\\1 & 2 & 4\end{vmatrix} = 5(4-6) - 3(8-3) + 1(4-1) = -10 - 15 + 3 = -22 \neq 0$$

$\therefore$ A is non – singular.

$\therefore$ The given system of equations has a unique solution $X = A^{-1}B$.

$$A_{11} = \begin{vmatrix}1 & 3\\2 & 4\end{vmatrix} = 4 - 6 = -2,\ A_{12} = -\begin{vmatrix}2 & 3\\1 & 4\end{vmatrix} = -(8-3) = -5,\ A_{13} = \begin{vmatrix}2 & 1\\1 & 2\end{vmatrix} = 4 - 1 = 3$$

$$A_{21} = -\begin{vmatrix}3 & 1\\2 & 4\end{vmatrix} = -(12-2) = -10,\ A_{22} = \begin{vmatrix}5 & 1\\1 & 4\end{vmatrix} = 20 - 1 = 19,\ A_{23} = -\begin{vmatrix}5 & 3\\1 & 2\end{vmatrix} = -(10-3) = -7$$

$$A_{31} = \begin{vmatrix}3 & 1\\1 & 3\end{vmatrix} = (9-1) = 8,\ A_{32} = -\begin{vmatrix}5 & 1\\2 & 3\end{vmatrix} = -(15-2) = -13,\ A_{33} = \begin{vmatrix}5 & 3\\2 & 1\end{vmatrix} = 5 - 6 = -1$$

$$\therefore \quad A^{-1} = \frac{adj.\,A}{|A|} = \frac{1}{|A|}\begin{bmatrix}A_{11} & A_{21} & A_{31}\\A_{12} & A_{22} & A_{32}\\A_{13} & A_{23} & {}_{33}\end{bmatrix} = \frac{1}{-22}\begin{bmatrix}-2 & -10 & 8\\-5 & 19 & -13\\3 & -7 & -1\end{bmatrix} = \begin{bmatrix}2/22 & 10/22 & -8/22\\5/22 & -19/22 & 13/22\\-3/22 & 7/22 & 1/22\end{bmatrix}$$

$$\therefore \quad X = A^{-1}B$$

$$\Rightarrow \quad \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2/22 & 10/22 & -8/22\\5/22 & -19/22 & 13/22\\-3/22 & 7/22 & 1/22\end{bmatrix}\begin{bmatrix}16\\19\\25\end{bmatrix} = \begin{bmatrix}\dfrac{32+190-200}{22}\\[2mm]\dfrac{80-361+325}{22}\\[2mm]\dfrac{-48+133+25}{22}\end{bmatrix} = \begin{bmatrix}\dfrac{22}{22}\\[2mm]\dfrac{44}{22}\\[2mm]\dfrac{110}{22}\end{bmatrix} = \begin{bmatrix}1\\2\\5\end{bmatrix}$$

$\therefore$ $x = 1, y = 2, z = 5.$

Ex. 19. Use matrix method to examine the following system of equations for consistency or inconsistency.

$2x + 5y = 7,\ 6x + 15y = 13$

Sol. Writing the given equations in the matrix form, we have $\begin{bmatrix}2 & 5\\6 & 15\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}7\\13\end{bmatrix}$ or $AX = B$, where

$$A = \begin{bmatrix} 2 & 5 \\ 6 & 15 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B \begin{bmatrix} 7 \\ 13 \end{bmatrix}$$

Now $\quad |A| = \begin{vmatrix} 2 & 5 \\ 6 & 15 \end{vmatrix} = 30 - 30 = 0 \Rightarrow A$ is singular

$\Rightarrow$ Further the system has no solution or infinite number of solutions. So now we find (adj. A) B.

$A_{11} = 15, A_{12} = -6, A_{21} = -5, A_{22} = 2$

$\therefore$ adj. $A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 15 & -5 \\ -6 & 2 \end{bmatrix}$

$\therefore$ (adj. A) $B = \begin{bmatrix} 15 & -5 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 13 \end{bmatrix} = \begin{bmatrix} 105-65 \\ -42+26 \end{bmatrix} = \begin{bmatrix} 40 \\ -16 \end{bmatrix} \neq 0$

$\therefore$ The given system has no solution and is therefore inconsistent.

Note: If $|A| = 0$ and (adj. A) $B = 0$, then the given system is consistent and has infinite number of solutions.

Based on Properties of Determinants

Ex. 20. Without expanding, *i.e.*, using properties of determinants, show that :

(*a*) $\begin{vmatrix} 1/a & a & bc \\ 1/b & b & ca \\ 1/b & c & ab \end{vmatrix} = 0$ $\qquad$ **(*b*)** $\begin{vmatrix} 3x+y & 2x & x \\ 4x+3y & 3x & 3x \\ 5x+6y & 4x & 6x \end{vmatrix} = x^3$

Sol. (*a*) Given, $\Delta = \begin{vmatrix} 1/a & a & bc \\ 1/b & b & ca \\ 1/c & c & ab \end{vmatrix}$

Multiply R_1 by a, R_2 by b, R_3 by c. Then,

$$\Delta = \frac{1}{abc}\begin{vmatrix} 1 & a^2 & abc \\ 1 & b^2 & abc \\ 1 & c^2 & abc \end{vmatrix} = \frac{1}{abc} \times abc \begin{vmatrix} 1 & a^2 & 1 \\ 1 & b^2 & 1 \\ 1 & c^2 & 1 \end{vmatrix} \text{ (Taking out } abc \text{ common from } C_3)$$

$= 1 \times 0 = \mathbf{0}$ (Two columns being identical)

(*b*) $$\Delta = \begin{vmatrix} 3x+y & 2x & x \\ 4x+3y & 3x & 3x \\ 5x+6y & 4x & 6x \end{vmatrix} = \begin{vmatrix} 3x & 2x & x \\ 4x & 3x & 3x \\ 5x & 4x & 6x \end{vmatrix} + \begin{vmatrix} y & 2x & x \\ 3y & 3x & 3x \\ 6y & 4x & 6x \end{vmatrix}$$

$$= x^3 \begin{vmatrix} 3 & 2 & 1 \\ 4 & 3 & 3 \\ 5 & 4 & 6 \end{vmatrix} + x^2y \begin{vmatrix} 1 & 2 & 1 \\ 3 & 3 & 3 \\ 6 & 4 & 6 \end{vmatrix} = x^3 \begin{vmatrix} 3 & 2 & 1 \\ 4 & 3 & 3 \\ 5 & 4 & 6 \end{vmatrix} + x^2y \times 0 \text{ } (\because \text{ Two columns } C_1 \text{ and } C_3 \text{ are identical})$$

$= x^3 [3 (18 - 12) -2 (24 - 15) +1 (16 - 15)] = x^3 (18 - 18 + 1) = \mathbf{x^3}.$

Ex. 21. For positive number x, y and z, show that the numerical value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} = 0$

(IIT 1993)

Sol. Let $\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} = \begin{vmatrix} 1 & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & 1 & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & 1 \end{vmatrix}$ (Using $\log_n m = \frac{\log m}{\log n}$)

$$= \frac{1}{\log x . \log y . \log z} \begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix}$$

(Multiplying R_1, R_2, R_3 by $\log x$, $\log y$ and $\log z$ respectively)

$$= \frac{1}{\log x . \log y . \log z} \times 0 = 0$$ [$\because$ R_1, R_2 and R_3 are identical].

Ex. 22. Prove that $\begin{vmatrix} \boldsymbol{a+b+2c} & \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{b+c+2a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{a} & \boldsymbol{c+a+2b} \end{vmatrix} = \boldsymbol{2(a+b+c)^3}$**.**

Sol. Given determinant $\Delta = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Applying $C_1 \longrightarrow C_1 + (C_2 + C_3)$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix} = 2\,(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Applying $R_2 \longrightarrow R_2 - R_1$ and $R_3 \longrightarrow R_3 - R_1$

$$\Delta = 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix} = 2(a+b+c) \times 1 \begin{vmatrix} b+c+a & 0 \\ 0 & c+a+b \end{vmatrix}$$

$= 2\,(a + b + c) \times \{(b + c + a).\,(c + a + b) - 0 \times 0\} = \boldsymbol{2(a + b + c)^3}$**.**

Ex. 23. Solve the following equations: $\begin{vmatrix} \boldsymbol{x} & \boldsymbol{3} & \boldsymbol{7} \\ \boldsymbol{2} & \boldsymbol{x} & \boldsymbol{2} \\ \boldsymbol{7} & \boldsymbol{6} & \boldsymbol{x} \end{vmatrix} = \boldsymbol{0}$ ***(EAMCET 1992, IIT***

Sol. Given, $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$ (Applying $R_1 \longrightarrow R_1 + R_2 + R_3$)

$$\Rightarrow (x+9) \begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

$$\Rightarrow (x+9)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x-2 & 0 \\ 7 & -1 & x-7 \end{vmatrix} = 0$$ (Applying $C_2 \longrightarrow C_2 - C_1$, $C_3 \longrightarrow C_3 - C_1$)

$$\Rightarrow (x+9)\begin{vmatrix} (x-2) & 0 \\ -1 & (x-7) \end{vmatrix} = 0$$ (Expanding along R_1)

$\Rightarrow (x + 9)\,(x - 2)\,(x - 7) = 0$

$\Rightarrow x = -9$ or 2 or 7.

$\therefore$ The solution set is $\{-9, 2, 7\}$.

Ex. 24. Without expanding prove that $\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix} = 0$

Sol. We know that, $\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}\begin{vmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & 0 \end{vmatrix} = 0 \times 0 = 0$ (Using row by row multiplication of determinants)

$$\Rightarrow \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + 0 & a_1\alpha_2 + b_1\beta_2 + 0 & a_1\alpha_3 + b_1\beta_3 + 0 \\ a_2\alpha_1 + b_2\beta_1 + 0 & a_2\alpha_2 + b_2\beta_2 + 0 & a_2\alpha_3 + b_2\beta_3 + 0 \\ a_3\alpha_1 + b_3\beta_1 + 0 & a_3\alpha_2 + b_3\beta_2 + 0 & a_3\alpha_3 + b_3\beta_3 + 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix} = 0.$$

Ex. 25. Express $\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$ **as a product of two determinants and hence factorize it.**

Sol. $\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix} = \begin{vmatrix} a^2 - 2ax + x^2 & a^2 - 2ay + y^2 & a^2 - 2az + z^2 \\ b^2 - 2bx + x^2 & b^2 - 2by + y^2 & b^2 - 2bz + z^2 \\ c^2 - 2cx + x^2 & c^2 - 2cy + y^2 & c^2 - 2cz + z^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}.$$ (Row by row multiplication)

$$= -2\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 2\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$ (Interchanging C_1 and C_3 of first determinant)

Now $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & a-b & a^2 - b^2 \\ 0 & b-c & b^2 - c^2 \\ 1 & c & c^2 \end{vmatrix}$ (Applying $R_1 \longrightarrow R_1 - R_2$; $R_2 \longrightarrow R_2 - R_3$)

$$= \begin{vmatrix} 0 & (a-b) & (a-b)(a+b) \\ 0 & (b-c) & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} \quad \text{[Taking } (a-b) \text{ common from } R_1, (b-c) \text{ common from } R_2]$$

$$= (a-b)(b-c).1.\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix} \quad \text{(Expanding along } C_1)$$

$$= (a-b)(b-c)(b+c-a-b) = (a-b)(b-c)(c-a)$$

Similarly, we have $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)\ (y-z)\ (z-x)$

$$\therefore\ 2\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 2\ (a-b)(b-c)(c-a)(x-y)(y-z)(z-x).$$

PRACTICE SHEET

1. If $a_{ij} = \left[\dfrac{i}{j}\right]$, where $[x]$ stands for the greatest integer function, then a matrix $A_{2\times 2} = [a_{ij}]$ will be

(*a*) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ (*b*) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ (*c*) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ (*d*) $\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$

2. If $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, then A^n is equal to

(*a*) $\begin{bmatrix} n & nk \\ 0 & n \end{bmatrix}$ (*b*) $\begin{bmatrix} 1 & nk \\ 0 & 1 \end{bmatrix}$ (*c*) $\begin{bmatrix} n & k^n \\ 0 & n \end{bmatrix}$ (*d*) $\begin{bmatrix} 1 & k^n \\ 0 & 1 \end{bmatrix}$

3. If $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ and A^2 is the unit matrix, then the value of $x^3 + x - 2$ is equal to

(*a*) –8 (*b*) –2 (*c*) 0 (*d*) 6

(*Kerala PET 2011*)

4. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$ and I is the unit matrix of order 3, then $A^2 + 2A^4 + 4A^6$ is equal to:

(*a*) $7I$ (*b*) $8A^7$ (*c*) $8A^8$ (*d*) $7A^8$

5. Matrix A is such that $A^2 = 2A - I$, where I is the identity matrix, then for $n \geq 2$, A^n is equal to

(*a*) $nA - (n-1)I$ (*b*) $nA - I$

(*c*) $2^{n-1}A - I$ (*d*) $2^{n-1}A - (n-1)I$

6. If $[1 \quad x \quad 1]\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix} = 0$, then x is equal to

(*a*) $\dfrac{1}{4}$ (*b*) $\dfrac{3}{4}$ (*c*) 1 (*d*) $\dfrac{5}{4}$

(*Odisha JEE 2008*)

7. Let M be a 3 × 3 matrix satisfying

$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, $M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$. Then the sum of the diagonal entries of M is

(*a*) 1 (*b*) 3 (*c*) 6 (*d*) 9

(*IIT 2011*)

8. If $A = \text{diag}\ (1 \quad -4 \quad 8)$, $B = \text{diag}\ (-2 \quad 3 \quad 5)$ and $C = \text{diag}\ (-3 \quad 7 \quad 10)$, find $B + 2C - A$

(*a*) diag $(-4 \quad 1 \quad 12)$ (*b*) diag $(-9 \quad 21 \quad 17)$

(*c*) diag $(-7 \quad 13 \quad 30)$ (*d*) diag $(-4 \quad -9 \quad -7)$

9. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $A^n =$

(*a*) $2^{n+1}.A$ (*b*) $2^{n-1}.A$ (*c*) $2^{n+2}.A$ (*d*) $2^{n-2}.A$

10. If the matrix $\begin{bmatrix} 4 & -2 \\ k & -4 \end{bmatrix}$ is nilpotent of order 2, then k equals

(*a*) 2 (*b*) 8 (*c*) –1 (*d*) 0

11. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then

(*a*) $A^2 - 5A - 7I_2 = 0$ (*b*) $A^2 + 5A - 7I_2 = 0$
(*c*) $A^2 - 5A + 6I_2 = 0$ (*d*) $A^2 - 5A + 7I_2 = 0$

12. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then $A^T. A$ is a

(*a*) Null matrix (*b*) Identity matrix
(*c*) Diagonal matrix (*d*) None of these

13. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying $AA^T = 9I_3$, then the values of ***a*** and ***b*** are respectively

(*a*) –2, –1 (*b*) –1, 2 (*c*) 1, –2 (*d*) 2, –1

(*Kerala PET 2011*)

14. If the orders of the matrices *A*, *B* and *C* are 5 × 4, 5 × 6 and 7 × 4 respectively, then the order of $(A^T \times B)^T \times C^T$ is of order

(*a*) 4 × 6 (*b*) 6 × 5 (*c*) 6 × 7 (*d*) 4 × 4

15. If *A* is 3 × 4 matrix and *B* is a matrix such that $A^T B$ and $B^T A$ are both defined, then *B* is of the order

(*a*) 3 × 3 (*b*) 3 × 4 (*c*) 4 × 3 (*d*) 4 × 4

16. If matrix *A* is symmetric as well as skew-symmetric, then *A* is a

(*a*) Unit matrix (*b*) Null matrix
(*c*) Triangular matrix (*d*) Diagonal matrix

17. Let *A* and *B* be symmetric matrices of the same order. Then,

(*a*) *A* + *B* is a symmetric matrix
(*b*) *AB* – *BA* is a skew-symmetric matrix
(*c*) *AB* + *BA* is a symmetric matrix
(*d*) All of these

18. If matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, where *a*, *b* and *c* are real positive numbers, $abc = 1$ and $A^T. A = I$, then find the value of $a^3 + b^3 + c^3$.

(*a*) 0 (*b*) 1 (*c*) 3 (*d*) 4

(*IIT 2003*)

19. The matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is which of the following:

(*a*) Nilpotent (*b*) Orthogonal
(*c*) Idempotent (*d*) Involuntary

20. If *A* and *B* are square matrices of the same order such that *AB* = *A* and *BA* = *B*, then A and *B* are both

(*a*) Singular (*b*) Idempotent
(*c*) Involuntary (*d*) Non-singular

21. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then det. *A* is equal to

(*a*) 0 (*b*) 1 (*c*) –1 (*d*) 2

22. Find *x* if $\begin{bmatrix} x & 0 & 1 \\ 2 & -1 & 4 \\ 1 & 2 & 0 \end{bmatrix}$ is a singular matrix?

(*a*) $\frac{3}{4}$ (*b*) $\frac{2}{3}$ (*c*) $\frac{5}{8}$ (*d*) $\frac{1}{8}$

23. For the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}$, *A* . (adj . *A*) is equal to

(*a*) $|A|\, I_3$ (*b*) I_3
(*c*) Null matrix (*d*) None of these

24. Let $A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. Then the only correct statement about the matrix *A* is

(*a*) *A* is a zero matrix
(*b*) $A^2 = I$
(*c*) *A* = (–1) *I*, where *I* is a unit matrix
(*d*) A^{-1} does not exist.

25. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, then A^{-1} equals

(*a*) *A* (*b*) $\frac{1}{11}A$ (*c*) $\frac{1}{19}A$ (*d*) A^T

26. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, then $(A^T)^{-1}$ equals

(*a*) $\begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ (*b*) $\begin{bmatrix} -9 & 8 & 2 \\ -8 & 7 & -2 \\ -5 & 4 & 1 \end{bmatrix}$

(*c*) $\begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$ (*d*) $\begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

27. The value of determinant $\begin{bmatrix} \log_y x & 1 \\ 1 & \log_x y \end{bmatrix}$ is equal to

(*a*) –1 (*b*) 0 (*c*) 1 (*d*) 3

28. If $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ and $f(x) = 1 + x + x^2 + ... + x^{20}$, then $f(A) =$

(*a*) $\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ (*b*) $\begin{bmatrix} 0 & 3 \\ 1 & 3 \end{bmatrix}$ (*c*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (*d*) $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

29. If $\begin{vmatrix} 2a & 1 \\ bc + ab & c \end{vmatrix} = 0$, then *a*, *b*, *c* are in

(*a*) A.P. (*b*) G.P.
(*c*) H.P. (*d*) None of these

30. If $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, then $I + A + A^2 + ... + \infty = ...$

(*a*) $\begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$ (*b*) $\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$

(*c*) $\begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 2/3 \end{bmatrix}$ (*d*) $\begin{bmatrix} 2/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$

31. If $A = \dfrac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$ then A^{-1} is equal to

(*a*) A^2 (*b*) I (*c*) A^T (*d*) 0

32. The value of the determinant $\begin{vmatrix} 0 & \tan\theta & 1 \\ 1 & -\sec\theta & 0 \\ \sec\theta & \tan\theta & 1 \end{vmatrix}$ is

(*a*) $\tan^2\theta$ (*b*) $\cos^2\theta$ (*c*) $\sec^2\theta$ (*d*) 1

33. $\begin{vmatrix} 9 & 9 & 12 \\ 1 & -3 & -4 \\ 1 & 9 & 12 \end{vmatrix}$ is equal to

(*a*) –121 (*b*) 136 (*c*) 0 (*d*) 10

34. The determinant $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ is equal to

(*a*) $(a + b)(b + c)(c - a)$ (*b*) $(a + b)(b + c)(c + a)$

(*c*) $(a - b)(b - c)(c - a)$ (*d*) $(a + b)(b - c)(c + a)$

35. If $a + b + c = 0$, then the determinant

$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ is equal to

(*a*) 0 (*b*) abc

(*c*) $2(a + b + c)$ (*d*) $a^2 + b^2 + c^2$

(***IAS 2001***)

36. The value of the determinant $\begin{vmatrix} \log e & \log e^2 & \log e^3 \\ \log e^2 & \log e^3 & \log e^4 \\ \log e^3 & \log e^4 & \log e^5 \end{vmatrix}$ is

(*a*) 0 (*b*) 1 (*c*) $4 \log e$ (*d*) $5 \log e$

(***EAMCET 2006***)

37. The value of the determinant $\begin{vmatrix} 1 & ab & c(a+b) \\ 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \end{vmatrix}$ is equal to

(*a*) 0 (*b*) abc (*c*) $a + b + c$ (*d*) $ab+bc+ca$

38. One root of the equation $\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$ is

(*a*) $-(ab + bc + ca)$ (*b*) $-(a + b + c)$

(*c*) $-abc$ (*d*) $-(a^2 + b^2 + c^2)$

39. The value of the determinant $\begin{vmatrix} 1+\log a & \log b & \log c \\ \log a & 1+\log b & \log c \\ \log a & \log b & 1+\log c \end{vmatrix}$ is

(*a*) $\log(abc)$ (*b*) $1 - \log(abc)$

(*c*) $\log(a + b + c)$ (*d*) $1 + \log(abc)$

40. If A is a 2×2 matrix and $|A| = 2$, then the matrix represented by A (adj. A) is equal to

(*a*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (*b*) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (*c*) $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ (*d*) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(***J&K CET 2011***)

41. If l, m, n are the pth, qth and rth terms of a GP, then $\begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix}$ is equal to

(*a*) 0 (*b*) $l + m + n$ (*c*) pqr (*d*) lmn

(***EAMCET 2009***)

42. If A is an invertible matrix which satisfies the relation $A^2 + A - I = 0$, then A^{-1} equals

(*a*) A^2 (*b*) $I + A$ (*c*) $I - A$ (*d*) $I - A^2$

(***MPPET 2009***)

43. For non-singular square matrices A, B and C of same order, $(AB^{-1}C)^{-1}$ is equal to

(*a*) $C^{-1}BA$ (*b*) $C^{-1}BA^{-1}$ (*c*) $A^{-1}BC^{-1}$ (*d*) CBA^{-1}

44. If $A = \begin{bmatrix} 2x & 0 \\ x & x \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, then x equals

(*a*) $-\dfrac{1}{2}$ (*b*) $\dfrac{1}{2}$ (*c*) 1 (*d*) 2

(***UPSEE 2008***)

45. If the system of equations $x + ay = 0$, $az + y = 0$ and $ax + z = 0$ has infinite solutions, then the value of a is

(*a*) –1 (*b*) 0

(*c*) 1 (*d*) no real values

(***IIT 2003***)

46. If the system of linear equations

$x + 2ay + az = 0$, $x + 3by + bz = 0$, $x + 4cy + cz = 0$ has a non-zero solution, then a, b, c.

(*a*) are in A.P. (*b*) are in G.P.

(*c*) are in H.P. (*d*) satisfy $a + 2b + 3c = 0$.

(***AIEEE 2003***)

47. If $3x - 2y = 5$ and $6x - 4y = 9$, then the system of equations has

(*a*) Unique solution (*b*) No solution

(*c*) Infinitely many solutions (*d*) None of these

48. If $x + 5y = 3$, $2x + 10y = 6$, then the system of equations has

(*a*) Unique solution (*b*) No solution

(*c*) Infinitely many solutions (*d*) None of these

49. The system of equations $5x + 3y + z = 16$, $2x + y + 3z = 19$ and $x + 2y + 4z = 25$ has

(*a*) No solution (*b*) Unique solution

(*c*) Infinitely many solutions (*d*) None of these.

50. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of matrix Q is

(*a*) 2^{10} (*b*) 2^{11} (*c*) 2^{12} (*d*) 2^{13}

(*IIT JEE 2012*)

ANSWERS

1. (*c*)	**2.** (*b*)	**3.** (*b*)	**4.** (*d*)	**5.** (*a*)	**6.** (*d*)	**7.** (*d*)	**8.** (*b*)	**9.** (*b*)	**10.** (*b*)
11. (*d*)	**12.** (*b*)	**13.** (*a*)	**14.** (*c*)	**15.** (*b*)	**16.** (*b*)	**17.** (*d*)	**18.** (*d*)	**19.** (*b*)	**20.** (*b*)
21. (*b*)	**22.** (*c*)	**23.** (*c*)	**24.** (*b*)	**25.** (*c*)	**26.** (*d*)	**27.** (*b*)	**28.** (*d*)	**29.** (*c*)	**30.** (*a*)
31. (*c*)	**32.** (*c*)	**33.** (*c*)	**34.** (*c*)	**35.** (*a*)	**36.** (*a*)	**37.** (*a*)	**38.** (*b*)	**39.** (*d*)	**40.** (*b*)
41. (*a*)	**42.** (*b*)	**43.** (*b*)	**44.** (*b*)	**45.** (*a*)	**46.** (*c*)	**47.** (*b*)	**48.** (*c*)	**49.** (*b*)	**50.** (*d*)

HINTS AND SOLUTIONS

1. Given, $a_{ij} = \left[\frac{i}{j}\right]$ where $[x]$ stands for greatest integer function.

$$\therefore \quad A_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$a_{11} = \left[\frac{1}{1}\right] = 1,\ a_{12} = \left[\frac{1}{2}\right] = [0.5] = 0$$

$$a_{21} = \left[\frac{2}{1}\right] = 2,\ a_{22} = \left[\frac{2}{2}\right] = 1$$

$$\therefore \quad A_{2\times 2} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}.$$

2. $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

$$\therefore \quad A^2 = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & k+k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 . A = \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & k+2k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3k \\ 0 & 1 \end{bmatrix}$$

$\therefore$ On generalisation, $\mathbf{A^n = \begin{bmatrix} 1 & nk \\ 0 & 1 \end{bmatrix}}$.

3. Given, $A^2 = I$

$$\Rightarrow \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x^2+1 & x+0 \\ x+0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

($\because$ Two matrices are equal if their corresponding elements are equal)

$\Rightarrow$ $x^2 + 1 = 1$ and $x = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$.

$\therefore$ $x^3 + x - 2 = \mathbf{-2}$.

4. Given, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$

$$\therefore \quad A^2 = A . A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ a+0-a & 0+b-b & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

$\therefore$ $A^4 = A^2 . A^2 = I . I = I$

$A^6 = A^2 . A^2 . A^2 = I . I . I = I$

$A^8 = A^2 . A^2 . A^2 . A^2 = I$

$\Rightarrow$ $A^2 + 2A^4 + 4A^6 = I + 2I + 4I = 7I = \mathbf{7A^8}$.

5. $A^2 = 2A - I$ ...(*i*)

$\therefore$ $A^3 = A . A^2 = A . (2A - I)$

$= 2A^2 - AI = 2A^2 - A$

$= 2(2A - I) - A = 3A - 2I$ [Using (*i*)]

$A^4 = A . A^3 = A(3A - 2I)$

$= 3A^2 - 2AI = 3(2A - I) - 2A$

$= 6A - 3I - 2A = 4A - 3I$

$\Rightarrow$ $A^n = nA - (n-1)I$

6. $$[1 \quad x \quad 1]_{1\times 3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix}_{3\times 3} \begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix}_{3\times 1} = 0$$

Using associative law and multiplying the first two matrices, we have

$$[1+0+0 \quad 2+5x+3 \quad 3+x+2]_{1\times 3} \begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix}_{3\times 1} = 0$$

$$\Rightarrow \begin{bmatrix}1 & 5+5x & 5+x\end{bmatrix}_{1\times 3}\begin{bmatrix}x\\1\\-2\end{bmatrix}_{3\times 1} = 0$$

$\Rightarrow$ $[x + 5 + 5x - 10 - 2x]_{1\times 1} = 0$

$\Rightarrow$ $[4x - 5] = [0]$

$\Rightarrow$ $4x - 5 = 0 \Rightarrow x = \mathbf{\frac{5}{4}}.$

7. Let $M = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ be the required 3 × 3 matrix.

Then, according to the first condition,

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\2\\3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a_{12}\\a_{22}\\a_{32}\end{bmatrix} = \begin{bmatrix}-1\\2\\3\end{bmatrix} \Rightarrow \left.\begin{matrix}a_{12} = -1\\ a_{22} = 2\\ a_{32} = 3\end{matrix}\right\} \quad ...(i)$$

According to second condition,

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a_{11}-a_{12}\\a_{21}-a_{22}\\a_{31}-a_{32}\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix} \Rightarrow \begin{matrix}a_{11}-a_{12} = 1\\ a_{21}-a_{22} = 1\\ a_{31}-a_{32} = -1\end{matrix}$$

$\therefore$ Using (i), we have

$a_{11} - (-1) = 1 \Rightarrow a_{11} = 0$

$a_{21} - 2 = 1 \Rightarrow a_{21} = 3$

$a_{31} - 3 = -1 \Rightarrow a_{31} = 2$

According to the third condition,

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\12\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a_{11}+a_{12}+a_{13}\\a_{21}+a_{22}+a_{23}\\a_{31}+a_{32}+a_{33}\end{bmatrix} = \begin{bmatrix}0\\0\\12\end{bmatrix}$$

$\Rightarrow$ $a_{11} + a_{12} + a_{13} = 0,\ a_{21} + a_{22} + a_{23} = 0,$
$a_{31} + a_{32} + a_{33} = 12$

$\Rightarrow$ $0 - 1 + a_{13} = 0,\ 3 + 2 + a_{23} = 0,$
$2 + 3 + a_{33} = 12$

$\Rightarrow$ $a_{13} = 1,\ a_{23} = -5,\ a_{33} = 7$

$\therefore$ Sum of diagonal elements of

$M = a_{11} + a_{22} + a_{33} = 0 + 2 + 7 = \mathbf{9}.$

8. Given, $A = \text{diag.}\,[1\ \ -4\ \ 8]$, $B = \text{diag}\,[-2\ \ 3\ \ 5]$ and $C = \text{diag}\,[-3\ \ 7\ \ 10]$. Then,

$$B + 2C - A = \begin{bmatrix}-2 & 0 & 0\\0 & 3 & 0\\0 & 0 & 5\end{bmatrix} + 2\begin{bmatrix}-3 & 0 & 0\\0 & 7 & 0\\0 & 0 & 10\end{bmatrix} - \begin{bmatrix}1 & 0 & 0\\0 & -4 & 0\\0 & 0 & 8\end{bmatrix}$$

$$= \begin{bmatrix}-2-6-1 & 0 & 0\\0 & 3+14+4 & 0\\0 & 0 & 5+20-8\end{bmatrix}$$

$$= \begin{bmatrix}-9 & 0 & 0\\0 & 21 & 0\\0 & 0 & 17\end{bmatrix} = \text{diag}\,[-9\ \ 21\ \ 17]\,.$$

9. $A = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

$$\Rightarrow \quad A^2 = A\,.\,A = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1+1 & 1+1\\1+1 & 1+1\end{bmatrix} = \begin{bmatrix}2 & 2\\2 & 2\end{bmatrix} = 2\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

$$= 2^{2-1}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = 2^{2-1}\,.\,A$$

$$A^3 = A\,.\,A^2 = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}$$

$$= \begin{bmatrix}2+2 & 2+2\\2+2 & 2+2\end{bmatrix} = \begin{bmatrix}4 & 4\\4 & 4\end{bmatrix}$$

$$= 4\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = 2^2\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

$$= 2^{3-1}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = 2^{3-1}\,.\,A$$

$\therefore$ $\mathbf{A^n = 2^{n-1} A.}$

10. Given, a matrix A is nilpotent of order 2 $\Rightarrow A^2 = 0.$

$$\Rightarrow \begin{bmatrix}4 & -2\\k & -4\end{bmatrix}\begin{bmatrix}4 & -2\\k & -4\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}16-2k & -8+8\\4k-4k & -2k+16\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$\Rightarrow$ $16 - 2k = 0 \Rightarrow k = \mathbf{8}.$

11. Given, $A = \begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$, then

$$A^2 = \begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$$

$$= \begin{bmatrix}9-1 & 3+2\\-3-2 & -1+4\end{bmatrix} = \begin{bmatrix}8 & 5\\-5 & 3\end{bmatrix}$$

$$A^2 - 5A = \begin{bmatrix}8 & 5\\-5 & 3\end{bmatrix} - 5\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$$

$$= \begin{bmatrix}8 & 5\\-5 & 3\end{bmatrix} - \begin{bmatrix}15 & 5\\-5 & 10\end{bmatrix} = \begin{bmatrix}8-15 & 5-5\\-5+5 & 3-10\end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} = -7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -7I$$

$\therefore$ $\mathbf{A^2 - 5A + 7I = 0.}$

12. $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$\Rightarrow$ $A^T = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$\Rightarrow$ $A^T . A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

13. Given, $AA^T = 9I_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+4+4 & 2+2-4 & a+4+2b \\ 2+2-4 & 4+1+4 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$\therefore$ Equating the corresponding elements of both the matrices,

$a + 4 + 2b = 0$ ...(*i*)

$2a + 2 - 2b = 0$ ...(*ii*)

$a^2 + 4 + b^2 = 9$ ...(*iii*)

$\Rightarrow$ $a + 2b = -4$ ...(*i*)

$2a - 2b = -2$ ...(*ii*)

On adding (*i*) and (*ii*)

$3a = -6 \Rightarrow a = -2$

Substituting $a = -2$ in (*i*), we get

$-2 + 2b = -4 \Rightarrow 2b = -2 \Rightarrow b = -1$

$\therefore$ $\mathbf{a = -2, b = -1.}$

14. A is of order $5 \times 4 \Rightarrow A^T$ is of order 4×5

B is of order $5 \times 6 \Rightarrow B^T$ is of order 6×5

C is of order $7 \times 4 \Rightarrow C^T$ is of order 4×7

$\therefore$ $(A^T \times B)$ is of order $\boxed{\overset{A^T}{4 \times \boxed{5 \quad 5}} \overset{B}{\times 6}}$ 4×6

$\therefore$ $(A^T \times B)^T$ is of order 6×4

$\Rightarrow$ $(A^T \times B)^T \times C^T$ is of order $6 \times \boxed{4 \quad 4} \times 7 = \mathbf{6 \times 7.}$

15. A is of order $3 \times 4 \Rightarrow A^T$ is of order 4×3.

Let the order of B be $p \times q$.

Since, A^TB is defined. So,

No. of rows of B = No. of columns of $A^T \Rightarrow p = 3$

Also, BA^T is defined. So,

No. of columns of B = No. of rows of $A^T \Rightarrow q = 4$

$\therefore$ B is of order $\mathbf{3 \times 4.}$

16. A is a symmetric matrix $\Rightarrow A^T = A$ ...(*i*)

A is a skew-symmetric matrix $\Rightarrow A^T = -A$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*),

$A = -A \Rightarrow A + A = 0 \Rightarrow 2A = 0 \Rightarrow A = 0$

$\Rightarrow$ A is null matrix.

17. Given, A and B are symmetric matrices of same order.

$\therefore$ $A^T = A$ and $B^T = B$.

$\therefore$ $(A + B)^T = A^T + B^T$

$= A + B \Rightarrow (A + B)$ is a symmetric matrix

$(AB - BA)^T = (AB)^T - (BA)^T$

$= B^TA^T - A^TB^T = BA - AB = -(AB - BA)$

$\Rightarrow$ $AB - BA$ is a skew-symmetric matrix

$(AB + BA)^T = (AB)^T + (BA)^T$

$= B^TA^T + A^TB^T = BA + AB = AB + BA$

$\Rightarrow$ $(AB + BA)$ is a symmetric matrix.

$\therefore$ All the given options hold.

18. $A^T. A = I$

$$\Rightarrow \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+b^2+c^2 & ab+bc+ca & ac+ab+cb \\ ba+cb+ac & b^2+c^2+a^2 & cb+ca+ab \\ ca+ab+bc & cb+ac+ba & c^2+a^2+b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ac \\ ab+bc+ac & a^2+b^2+c^2 & ab+bc+ac \\ ab+bc+ac & ab+bc+ac & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $a^2 + b^2 + c^2 = 1$, $ab + bc + ca = 0$

(On equating corresponding elements of equal matrices)

Now, we know that

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$= 1 + 0 = 1$

$\Rightarrow$ $a + b + c = 1$

Also, $a^3 + b^3 + c^3 - 3abc$

$= (a + b + c)\{a^2 + b^2 + c^2 - (ab + bc + ca)\}$

$= 1\{1 - 0\} = 1$

$\therefore$ $a^3 + b^3 + c^3 = 1 + 3abc = 1 + 3 \times 1 = \mathbf{4}$

($\because$ Given, $abc = 1$)

19. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$A^T = \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A \,.\, A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is an orthogonal matrix.

20. $AB = A$

$\Rightarrow$ $(AB)\,A = A\,.\,A$ (Multiplying by A on both the sides)

$\Rightarrow$ $A\,(BA) = A^2$ (Associative law)

$\Rightarrow$ $AB = A^2$ $(\because BA = B)$

$\Rightarrow$ $A = A^2$ $(\because AB = A)$

$\Rightarrow$ A is idempotent.

Similarly,

$$BA = B \;\Rightarrow\; (BA)B = B.\,B \;\Rightarrow\; B(AB) = B^2$$

$$\Rightarrow\; BA = B^2 \;\Rightarrow\; B = B^2 \;\Rightarrow\; B \text{ is idempotent.}$$

21. Given, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Then, det. $A = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

$$= \cos\theta \,.\, \cos\theta - (-\sin\theta) \,.\, \sin\theta$$

$$= \cos^2\theta + \sin^2\theta = 1$$

$$\left[\begin{array}{l} \because\; A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \Rightarrow\; |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \end{array}\right]$$

22. Let, $A = \begin{bmatrix} x & 0 & 1 \\ 2 & -1 & 4 \\ 1 & 2 & 0 \end{bmatrix}$

Then, det. $A = \begin{vmatrix} x & 0 & 1 \\ 2 & -1 & 4 \\ 1 & 2 & 0 \end{vmatrix}$

Now expanding along the first row, we have

$$\text{det. } A = x\begin{vmatrix} -1 & 4 \\ 2 & 0 \end{vmatrix} - 0\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= x\,(-1 \times 0 - 4 \times 2) + 1\,(2 \times 2 - (-1) \times 1)$$

$$= x \times (-8) + 5 = -8x + 5$$

For matrix A to be a singular matrix det. $A = 0$

$$\therefore \quad -8x + 5 = 0 \;\Rightarrow\; x = \mathbf{\frac{5}{8}}.$$

23. We know that $A^{-1} = \dfrac{\text{adj}\,.\,A}{|A|}$ where $|A| = \text{det. } A$

$\Rightarrow$ $A^{-1}\,.\,|A| = \text{adj}\,.\,A$

$\Rightarrow$ $A\,.\,(\text{adj}\,.\,A) = A\,.\,(A^{-1}\,.\,|A|)$

$= AA^{-1}\,.\,|A|$

$= I\,|A| = |A|\,I;$

where I is the identity matrix $(AA^{-1} = I)$

So to find A . adj . A. we need to find det. A.

$$\therefore \qquad A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}$$

$$\therefore \qquad \text{det. } A = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{vmatrix}$$

Expanding along the first row, we have

$$\text{det. } A = |A| = 1\begin{vmatrix} 3 & 0 \\ 2 & 10 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 0 \\ 18 & 10 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 18 & 2 \end{vmatrix}$$

$$= 1 \times 30 + 1 \times 20 + 1 \times (4 - 54)$$

$$= 30 + 20 - 50 = 0$$

$\therefore$ $A\,.\text{adj}\,A = |A|\,I = 0 \times I = 0$

$\Rightarrow$ $A\,.\text{adj}\,A$ is a null matrix.

24. Let us examine each statement separately.

- A is not a zero matrix

- $$A^2 = A\,.\,A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+1 & 0 & 0 \\ 0 & 0+1+0 & 0 \\ 0 & 0 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

- $$(-1)\,I = (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \neq A$$

- For A^{-1} to exist, det $A = |A| \neq 0$ (Expanding $|A|$ along the first row R_1), we have

$$\therefore \quad |A| = -1\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -1 \times -1 = 1 \neq 0$$

$\Rightarrow$ A^{-1} exists.

$\therefore$ The only correct statement is (b).

25. $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$

$$|A| = (2 \times -2) - (3 \times 5) = -4 - 15 = -19 \neq 0$$

As $|A| \neq 0$, therefore A is non-singular and hence A^{-1} exists.

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$$

$\therefore$ $A_{11} = (-1)^{1+1}\,|-2| = -2,\quad A_{12} = (-1)^{1+2}\,|5| = -5$

$A_{21} = (-1)^{2+1}\,|3| = -3,\quad A_{22} = (-1)^{2+2}\,|2| = 2$

$$\therefore \qquad A^{-1} = -\frac{1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} = \frac{1}{19}A.$$

> **Note:** Do not confuse here | |, *i.e.*, determinant sign with absolute value sign.

26. $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ 2 & 2 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 0 & 2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$

$$\therefore \text{ Let } \quad B = \text{det. } A^T = \begin{vmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} - 0\begin{vmatrix} -2 & 2 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix}$$

$$= 1(-1-8) - 2(-8+3)$$

$$= -9 - (2 \times -5) = -9 + 10 = 1 \neq 0.$$

$|B| \neq 0 \Rightarrow B$ is non-singular $\Rightarrow B^{-1}$, *i.e*, $(A^T)^{-1}$ exists

$$\therefore \qquad B^{-1} = \frac{\text{adj } B}{|B|}$$

$$\text{adj } B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}^T = \begin{bmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \\ B_{13} & B_{23} & B_{33} \end{bmatrix}$$

$$B_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} = (-1-8) = -9$$

$$B_{12} = (-1)^{1+2}\begin{vmatrix} -2 & 2 \\ 3 & 1 \end{vmatrix} = -(-2-6) = 8$$

$$B_{13} = (-1)^{1+3}\begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix} = -8+3 = -5$$

$$B_{21} = (-1)^{2+1}\begin{vmatrix} 0 & -2 \\ 4 & 1 \end{vmatrix} = -(0+8) = -8$$

$$B_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = 1+6 = 7$$

$$B_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = -(4-0) = -4$$

$$B_{31} = (-1)^{3+1}\begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} = (0-2) = -2$$

$$B_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} = -(2-4) = +2$$

$$B_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix} = -1$$

$$\therefore \qquad (A^T)^{-1} = B^{-1} = \mathbf{\frac{1}{1}\begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}}.$$

27. $\begin{vmatrix} \log_y x & 1 \\ 1 & \log_x y \end{vmatrix} = \log_y x \times \log_x y - 1 \times 1$

$$= \frac{\log x}{\log y} \cdot \frac{\log y}{\log x} - 1 = 1 - 1 = \mathbf{0.}$$

28. $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \quad A^2 = A \, . \, A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$$\Rightarrow \quad A^3 = A^2 . A = O . A = O$$

$$\Rightarrow \quad A^2 = A^3 = A^4 = A^5 = ... = O$$

$$\therefore \quad f(A) = I + A + A^2 + ... + A^{20}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} + O + ... + O = \mathbf{\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}}.$$

29. $\begin{vmatrix} 2a & 1 \\ bc + ab & c \end{vmatrix} = 0$

$\Rightarrow \; 2ac - (bc + ab) = 0$

$\Rightarrow \; 2ac = ab + bc \Rightarrow 2ac = b(a + c)$

$\Rightarrow \; b = \dfrac{2ac}{a+c} \Rightarrow \dfrac{1}{b} = \dfrac{a+c}{2ac} = \dfrac{1}{2}\left[\dfrac{1}{c} + \dfrac{1}{a}\right]$

$\Rightarrow \; a, b, c$ are in *H.P.*

30. $I + A + A^2 + ... + \infty$ is the sum of an infinite *G.P.* with first term $(a) = I$ and common ration $(r) = A$.

As we know that, sum of an infinite *G.P.* $= \dfrac{a}{1-r}$

$$\therefore \; I + A + A^2 + ... + \infty = \frac{I}{I-A} = I \, . \, (I-A)^{-1} = (I-A)^{-1}$$

$$\text{Now,} \quad I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\therefore \quad (I-A)^{-1} = \frac{\text{adj.}(I-A)}{\text{det.}(I-A)}$$

$$det.\,(I-A) = 4 - 1 = 3 \neq 0.$$

$$adj.\,(I-A) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\therefore \quad (I-A)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}.$$

31. If $A^{-1} = A^T$ then,

$A . A^{-1} = I \Rightarrow AA^T = I$. Where I = unit matrix $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A . A^T = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} . \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+1+16 & -32+4+28 & -8-8+16 \\ -32+4+28 & 16+16+49 & 4-32+28 \\ -8-8+16 & 4-32+28 & 1+64+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

$\therefore \; \mathbf{A^T = A^{-1}.}$

32. Let $\Delta = \begin{vmatrix} 0 & \tan\theta & 1 \\ 1 & -\sec\theta & 0 \\ \sec\theta & \tan\theta & 1 \end{vmatrix}$

Expanding along Row 1 (R_1)

Then, $\Delta = 0\begin{vmatrix} -\sec\theta & 0 \\ \tan\theta & 1 \end{vmatrix} - \tan\theta\begin{vmatrix} 1 & 0 \\ \sec\theta & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & -\sec\theta \\ \sec\theta & \tan\theta \end{vmatrix}$

$= -\tan\theta \times 1 + (\tan\theta + \sec^2\theta) = \mathbf{\sec^2\theta}$.

33. Let $\Delta = \begin{vmatrix} 9 & 9 & 12 \\ 1 & -3 & -4 \\ 1 & 9 & 12 \end{vmatrix}$.

Taking out 3 common from C_2 and 4 common from C_3, we have

$\Delta = (3\times4)\begin{vmatrix} 9 & 3 & 3 \\ 1 & -1 & -1 \\ 1 & 3 & 3 \end{vmatrix} = 12 \times 0 = \mathbf{0}.$ $\left(\because C_2 \text{ and } C_3 \text{ are now identical}\right)$

34. Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Operating $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we have

$\Delta = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)\begin{vmatrix} 0 & 1 & (a+b) \\ 0 & 1 & (b+c) \\ 1 & c & c^2 \end{vmatrix}$

[Taking out $(a-b)$ common from R_1 and $(b-c)$ common from R_2]

$= (a-b)(b-c).1\begin{vmatrix} 1 & (a+b) \\ 1 & (b+c) \end{vmatrix}$ (Expanding along C_1)

$= (a-b)(b-c)(b+c-a-b)$

$= \mathbf{(a-b)(b-c)(c-a)}$.

35. Let $\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Operating $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we have

$\Delta = \begin{vmatrix} -a-b-c & 0 & 2a \\ 0 & -b-c-a & 2b \\ c+a+b & c+a+b & c-a-b \end{vmatrix}$

$= \begin{vmatrix} -(a+b+c) & 0 & 2a \\ 0 & -(a+b+c) & 2b \\ a+b+c & a+b+c & c-a-b \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 2a \\ 0 & 0 & 2b \\ 0 & 0 & c-a-b \end{vmatrix} = 0$ ($\because$ $a+b+c=0$ is given)

36. Let $\Delta = \begin{vmatrix} \log e & \log e^2 & \log e^3 \\ \log e^2 & \log e^3 & \log e^4 \\ \log e^3 & \log e^4 & \log e^5 \end{vmatrix}$

$= \begin{vmatrix} \log e & 2\log e & 3\log e \\ 2\log e & 3\log e & 4\log e \\ 3\log e & 4\log e & 5\log e \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

($\because$ $\log e = 1$)

Operating $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we have

$\Delta = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix}$

(Taking out 2 common from C_3)

$= 2 \times 0 = 0$ ($\because$ Two columns are identical)

37. Let $\Delta = \begin{vmatrix} 1 & ab & c(a+b) \\ 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \end{vmatrix}$

Operating $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we have

$\Delta = \begin{vmatrix} 0 & ab-bc & ca+cb-ab-ac \\ 0 & bc-ca & ab+ac-bc-ba \\ 1 & ca & b(c+a) \end{vmatrix}$

$= \begin{vmatrix} 0 & b(a-c) & -b(a-c) \\ 0 & c(b-a) & -c(b-a) \\ 1 & ca & b(c+a) \end{vmatrix}$

Taking out $b(a-c)$ common from R_1 and $c(b-a)$ common from R_2, we have

$\Delta = b(a-c).c(b-a)\begin{vmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & ca & b(c+a) \end{vmatrix} = 0$

($\because$ Two rows are identical)

38. Let $\Delta = \begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$

$\Delta = \begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & a \\ x+a+b+c & b & x+c \end{vmatrix}$

$= (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & a \\ 1 & b & x+c \end{vmatrix}$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we have

$\Delta = (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & x & a-c \\ 0 & -x & x+c-a \end{vmatrix}$

$= (x + a + b + c)\,[x\,(x + c - a) - x\,(c - a)]$ (On expanding by C_1)

$= (x + a + b + c)\,x^2$

$$\therefore \begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0 \Rightarrow x^2(x + a + b + c) = 0$$

$\Rightarrow$ $x = \mathbf{0}$ or $x = \mathbf{-(a + b + c)}$.

39. Let $\Delta = \begin{vmatrix} 1+\log a & \log b & \log c \\ \log a & 1+\log b & \log c \\ \log a & \log b & 1+\log c \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$, we have

$$\Delta = \begin{vmatrix} 1+\log a+\log b+\log c & \log b & \log c \\ 1+\log a+\log b+\log c & 1+\log b & \log c \\ 1+\log a+\log b+\log c & \log b & 1+\log c \end{vmatrix}$$

$$= \begin{vmatrix} 1+\log(abc) & \log b & \log c \\ 1+\log(abc) & 1+\log b & \log c \\ 1+\log(abc) & \log b & 1+\log c \end{vmatrix}$$

Taking out $1 + \log abc$ common from C_1

$$\Delta = (1+\log(abc)) \begin{vmatrix} 1 & \log b & \log c \\ 1 & 1+\log b & \log c \\ 1 & \log b & 1+\log c \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have

$$\Delta = (1+\log(abc)) \begin{vmatrix} 1 & \log b & \log c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1+\log(abc)\,.1.\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$= \mathbf{1 + \log (abc)}$ (Expanding along C_1)

40. A (adj. A) = $|A|\,I$ where I is the identity matrix, so

$$A\,(\text{adj. } A) = 2\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{vmatrix}.$$

41. Let the first term of the GP be a and common ratio k. As,
T_n (nth term of a GP) $= ak^{n-1}$, here
$l = ak^{p-1}$, $m = ak^{q-1}$, $n = ak^{r-1}$

Let, $$\Delta = \begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} = \begin{vmatrix} \log ak^{p-1} & p & 1 \\ \log ak^{q-1} & q & 1 \\ \log ak^{r-1} & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \log a+(p-1)\log k & p & 1 \\ \log a+(q-1)\log k & q & 1 \\ \log a+(r-1)\log k & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \log a & p & 1 \\ \log a & q & 1 \\ \log a & r & 1 \end{vmatrix} + \begin{vmatrix} (p-1)\log k & p & 1 \\ (q-1)\log k & q & 1 \\ (r-1)\log k & r & 1 \end{vmatrix}$$

Taking out $\log a$ common from C_1 of first determinant and $\log k$ common from C_1 of second determinant we have

$$\Delta = \log a \begin{vmatrix} 1 & p & 1 \\ 1 & q & 1 \\ 1 & r & 1 \end{vmatrix} + \log k \begin{vmatrix} (p-1) & p & 1 \\ (q-1) & q & 1 \\ (r-1) & r & 1 \end{vmatrix}$$

$$= \underset{\begin{pmatrix}\text{Two columns} \\ \text{are identical}\end{pmatrix}}{\log a \times 0} + \log k \begin{vmatrix} p-1-p+1 & p & 1 \\ q-1-q+1 & q & 1 \\ r-1-r+1 & r & 1 \end{vmatrix}$$

Using $C_1 \to C_1 - C_2 + C_3$ in second determinant.

$$= \log k \begin{vmatrix} 0 & p & 1 \\ 0 & q & 1 \\ 0 & r & 1 \end{vmatrix} = \mathbf{0}.$$

42. Given, $A^2 + A - I = O$

Premultiplying by A^{-1} on both the sides, we have

$A^{-1}A^2 + A^{-1}A - A^{-1}\,I = A^{-1}O$

$\Rightarrow$ $(A^{-1}A)\,A + I - A^{-1} = O$ $(\because A^{-1}A = I, A^{-1}O = O)$

$\Rightarrow$ $IA + I - A^{-1} = O$

$\Rightarrow$ $A + I - A^{-1} = O$ $(\because IA = A)$

$\Rightarrow$ $A + I = O + A^{-1}$

$\Rightarrow$ $A + I = A^{-1}$ $(\because O + A^{-1} = A^{-1})$

43. $(AB^{-1}\,C)^{-1}$

$= [(AB^{-1})\,C]^{-1} = C^{-1}.\,(AB^{-1})^{-1}$ $(\because (XY)^{-1} = Y^{-1}X^{-1})$

$= C^{-1}.(B^{-1})^{-1}\,A^{-1}$

$= \mathbf{C^{-1}\,BA^{-1}}.$ $(\because (X^{-1})^{-1} = X)$

44. We know that $(A^{-1})^{-1} = A$

$|A^{-1}| = 2 \neq 0$, Hence $(A^{-1})^{-1}$ exists.

Now, we find the cofactors of the elements of the matrix A^{-1}.

Let $B = A^{-1} = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix}$. Then,

$B_{11} = 2$, $B_{12} = -(-1) = 1$, $B_{21} = -(0) = 0$, $B_{22} = 1$

$$\therefore \text{ adj. } B = \text{adj } (A^{-1}) = \begin{vmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}$$

$$\therefore (A^{-1})^{-1} = \frac{1}{2}\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1/2 & 1/2 \end{vmatrix}$$

$\therefore$ $A = (A^{-1})^{-1}$

$$\Rightarrow \begin{vmatrix} 2x & 0 \\ x & x \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1/2 & 1/2 \end{vmatrix} \Rightarrow \mathbf{x = \frac{1}{2}}.$$

45. The given system of equations can be written as:

$x + ay + 0.z = 0$

$0.x + y + a.z = 0$

$ax + 0.y + z = 0$

These in matrix form can be written as: $AX = O$, where,

$$A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, the given system of equations will have an infinite number of solutions, if $|A| = 0$.

$$|A| = \begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & a \\ 0 & 1 \end{vmatrix} - a\begin{vmatrix} 0 & a \\ a & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ a & 0 \end{vmatrix}$$

(Expanding along R_1)

$$= 1 \times 1 - a(0 - a^2) + 0 = 1 + a^3$$

$\therefore \quad |A| = 0 \Rightarrow 1 + a^3 = 0 \Rightarrow a^3 = -1 \Rightarrow a = -1.$

46. The given system of linear equations are:

$x + 2ay + az = 0$
$x + 3by + bz = 0$
$x + 4cy + cz = 0$

These can be written in the matrix form as $AX = O$, *i.e.*,

$$\begin{bmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $A = \begin{bmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{bmatrix}, X \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

The system will have a non-zero solution if $|A| = 0$

$$|A| = \begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 1\begin{vmatrix} 3b & b \\ 4c & c \end{vmatrix} - 1\begin{vmatrix} 2a & a \\ 4c & c \end{vmatrix} + 1\begin{vmatrix} 2a & a \\ 3b & b \end{vmatrix}$$

$= 3bc - 4cb - (2ac - 4ac) + 2ab - 3ba$
$= -bc + 2ac - ab$

$\therefore \quad |A| = 0 \Rightarrow -bc + 2ac - ab = 0 \Rightarrow 2ac = ab + bc$

$$\Rightarrow \frac{2}{b} = \frac{1}{c} + \frac{1}{a} \Rightarrow a, b, c \text{ are in } H.P.$$

47. The system of equations is:

$3x - 2y = 5$
$6x - 4y = 9$

Writing the system of equations in matrix form, we have

$$\begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

A X = B

where $A = \begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$

To check the consistency of system of equations, find $|A|$

$|A| = -12 - (-12) = 0$

$\Rightarrow$ Either the system of equations has infinitely many solutions or no solution.

Now we find (adj. A) B

$$\text{adj. } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$$

$A_{11} = -4,\ A_{12} = -6,\ A_{21} = 2,\ A_{22} = 3$

$$\therefore \quad (\text{adj } A)\, B = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} -20 + 18 \\ -30 + 27 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \neq 0$$

$\Rightarrow$ No solution (inconsistent).

48. The system of equations is:

$x + 5y = 3$
$2x + 10y = 6$

Writing in matrix form, we have

$$AX = B \Rightarrow \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$|A| = 10 - 10 = 0$

$\Rightarrow$ Either the system of equations have no solution or infinitely many solutions.

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 10 & -5 \\ -2 & 1 \end{bmatrix}$$

$$(\text{adj. } A)\, B = \begin{bmatrix} 10 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 30 - 30 \\ -6 + 6 \end{bmatrix} = 0$$

$\therefore$ The system of equations has infinitely many solutions.

49. The system of equations is:

$5x + 3y + z = 16$
$2x + y + 3z = 19$
$x + 2y + 4z = 25$

Here, $D = \begin{vmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix}$

$$= 5\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$= 5\,(4 - 6) - 3\,(8 - 3) + 1\,(4 - 1)$
$= 5 \times -2 - 3 \times 5 + 1 \times 3 = -10 - 15 + 3 = -22 \neq 0$

$\because \quad D \neq 0$, the system has a unique solution.

Note: $x = \dfrac{Dx}{D}, y = \dfrac{Dy}{D}, z = \dfrac{Dz}{D}$, where

$$Dx = \begin{vmatrix} 16 & 3 & 1 \\ 19 & 1 & 3 \\ 25 & 2 & 4 \end{vmatrix}, Dy \begin{vmatrix} 5 & 16 & 1 \\ 2 & 19 & 3 \\ 1 & 25 & 4 \end{vmatrix}, Dz = \begin{vmatrix} 5 & 3 & 16 \\ 2 & 1 & 19 \\ 1 & 2 & 25 \end{vmatrix}$$

50. Here,

$$P = [a_{ij}]_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$Q = [b_{ij}]_{3\times 3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \text{ where } b_{ij} = 2^{i+j} a_{ij}$$

$$\therefore \quad |Q| = \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix}$$

$$= 2^2.2^3.2^4 \begin{vmatrix} a_{11} & 2a_{12} & 4a_{13} \\ a_{21} & 2a_{22} & 4a_{23} \\ a_{31} & 2a_{32} & 4a_{33} \end{vmatrix}$$

$$= 2^9 \times 2 \times 4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$= 2^{12}.\,|P| = 2^{12}.2 = \mathbf{2^{13}}$.

SELF ASSESSMENT SHEET

1. If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 - xA + yI = 0$

(*a*) $x = 9, y = -14$ (*b*) $x = 14, y = 9$
(*c*) $x = 9, y = 14$ (*d*) $x = -9, y = 14$

2. If A and B are two square matrices of the same order such that $AB = B$ and $BA = A$, then $A^2 + B^2$ is always equal to

(*a*) I (*b*) $A + B$ (*c*) $2AB$ (*d*) $2BA$

3. If $\begin{bmatrix} a & -1 & 5 \\ b & 3 & -4 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 0 \\ 7 & -6 \end{bmatrix} = \begin{bmatrix} 34 & -30 \\ -1 & 42 \end{bmatrix}$, then (a, b) is equal to

(*a*) (1, 3) (*b*) (–2, 4) (*c*) (0, 6) (*d*) (2, –3)

4. If $A = \begin{bmatrix} 0 & -3 & -4/3 \\ 3 & 0 & -1/4 \\ 4/3 & 1/4 & 0 \end{bmatrix}$, then det. $(A + A^T)$ is equal to

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

5. If every element of a determinant of order 3 of value Δ is multiplied by 5, then the value of the new determinant is

(*a*) Δ (*b*) 5Δ (*c*) 25Δ (*d*) 125Δ

6. If $M = \begin{vmatrix} a & l & p \\ b & m & q \\ c & n & r \end{vmatrix}$ and $N = \begin{vmatrix} p & q & r \\ a & b & c \\ l & m & n \end{vmatrix}$, then

(*a*) $M' = N$ (*b*) $M = -N$ (*c*) $M = N^2$ (*d*) $M = N^3$

7. If $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = k\,a^2b^2c^2$, then k is equal to

(*a*) –4 (*b*) 2 (*c*) 4 (*d*) 8

8. The matrix $\begin{bmatrix} \lambda & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ is invertible if

(*a*) $\lambda \neq -20$ (*b*) $\lambda \neq -19$ (*c*) $\lambda \neq -18$ (*d*) $\lambda \neq -17$

9. The element in the first row and third column of the inverse of the matrix $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ is

(*a*) –2 (*b*) 0 (*c*) 1 (*d*) 7

10. The system of linear equations $2x + 2y = 5$, $5x + ky = 9$ has a unique solution if

(*a*) $k \neq 5$ (*b*) $k = 0$ (*c*) $k \neq -1$ (*d*) $k \neq 2$

11. The system of equations $3x + 4y = 2$ and $6x + 8y = 4$ has

(*a*) No solution (*b*) a unique solution
(*c*) two distinct solutions (*d*) infinitely many solutions.

12. The system of equations $x + 3y = 5$, $2x + 6y = 8$ has

(*a*) No solution (*b*) a unique solution
(*c*) infinitely many solutions (*d*) None of these

ANSWERS

1. (*c*) **2.** (*b*) **3.** (*c*) **4.** (*a*) **5.** (*d*) **6.** (*a*) **7.** (*c*) **8.** (*d*) **9.** (*d*) **10.** (*a*)
11. (*d*) **12.** (*a*)

HINTS AND SOLUTIONS

1. $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \Rightarrow A^2 = A\,.\,A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 16+6 & 12+15 \\ 8+10 & 6+25 \end{bmatrix} = \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix}$$

Given $A^2 - xA + yI = 0 \Rightarrow A^2 - xA = -yI$

$$\Rightarrow \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} - x\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = -y\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 22-4x & 27-3x \\ 18-2x & 31-5x \end{bmatrix} = \begin{bmatrix} -y & 0 \\ 0 & -y \end{bmatrix}$$

$\Rightarrow 22 - 4x = -y,\ 27 - 3x = 0$

Now $27 - 3x = 0 \Rightarrow 3x = 27 \Rightarrow \boldsymbol{x = 9}$

$\therefore\ 22 - 4x = -y \Rightarrow 22 - 36 = -y$

$\Rightarrow -y = -14 \Rightarrow \boldsymbol{y = 14.}$

2. We know that if $AB = B$ and $BA = A$, then A and B are idempotent matrices, *i.e*, $A^2 = A$ and $B^2 = B$.

$A^2 + B^2 = A + B$

Alternatively,

$$\begin{aligned} A^2 + B^2 &= A.A + B.B = A.(BA) + B.(AB) \\ &= (AB)A + (BA)B \\ &\qquad \text{(Matrix multiplication is associative)} \\ &= BA + AB \qquad (\because AB = B, BA = A) \\ &= \boldsymbol{A + B} \qquad (\because BA = A, AB = B) \end{aligned}$$

3. $\begin{bmatrix} a & -1 & 5 \\ b & 3 & -4 \end{bmatrix}_{2\times 3}\begin{bmatrix} 4 & 3 \\ 1 & 0 \\ 7 & -6 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 34 & -30 \\ -1 & 42 \end{bmatrix}_{2\times 2}$

LHS $= \begin{bmatrix} 4a-1+35 & 3a+0-30 \\ 4b+3-28 & 3b+0+24 \end{bmatrix} = \begin{bmatrix} 4a+34 & 3a-30 \\ 4b-25 & 3b+24 \end{bmatrix}$

RHS $= \begin{bmatrix} 34 & -30 \\ -1 & 42 \end{bmatrix}$.

Equating corresponding elements we have

$4a + 34 = 34 \Rightarrow a = 0$ and $4b - 25 = -1 \Rightarrow 4b = 24 \Rightarrow b = 6$
$3a - 30 = -30 \Rightarrow a = 0$ $\quad$ $3b + 24 = 42 \Rightarrow 3b = 18 \Rightarrow b = 6$

$\therefore\ \boldsymbol{a = 0, b = 6.}$

4. $A + A^T = \begin{bmatrix} 0 & -3 & -4/3 \\ 3 & 0 & -1/4 \\ 4/3 & 1/4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 4/3 \\ -3 & 0 & 1/4 \\ -4/3 & -1/4 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ $\det (A + A^T) = 0$.

5. Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Since each element of Δ is multiplied by 5, the new determinant

$$\Delta' = \begin{vmatrix} 5a_1 & 5a_2 & 5a_3 \\ 5b_1 & 5b_2 & 5b_3 \\ 5c_1 & 5c_2 & 5c_3 \end{vmatrix}$$

$$= (5 \times 5 \times 5) \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(Taking out 5, common from each of R_1, R_2, R_3 or C_1, C_2, C_3)

$= 125\,\Delta$.

6. $M = \begin{vmatrix} a & l & p \\ b & m & q \\ c & n & r \end{vmatrix} \Rightarrow M' = \begin{vmatrix} a & b & c \\ l & m & n \\ p & q & r \end{vmatrix}$

(Interchanging R_1 and R_3)

$= -\begin{vmatrix} p & q & r \\ l & m & n \\ a & b & c \end{vmatrix} = \begin{vmatrix} p & q & r \\ a & b & c \\ l & m & n \end{vmatrix}$ (Interchanging R_2 and R_3)

$= N$.

7. Let $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$

(Taking out a, b, c common from R_1, R_2, R_3 respectively)

$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

(Taking out a, b, c common from C_1, C_2, C_3 respectively)

$= a^2b^2c^2 \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$ (Operating $C_1 \to C_1 + C_2$)

$= a^2b^2c^2 \times 2 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = a^2b^2c^2 \times 2\,(1 + 1)$

$= 4a^2b^2c^2$ (Expanding along C_1)

$\therefore$ $k = \mathbf{4}$

8. Let $A = \begin{bmatrix} \lambda & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$, then det. $A = |A| = \begin{vmatrix} \lambda & -1 & 4 \\ -3 & 0 & 1 \\ -1 & 1 & 2 \end{vmatrix}$

The matrix A is invertible if $|A| \neq 0$

$|A| = \lambda \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -3 & 1 \\ -1 & 2 \end{vmatrix} + 4 \begin{vmatrix} -3 & 0 \\ -1 & 1 \end{vmatrix}$

$= \lambda\,(0 - 1) + 1\,(-6 + 1) + 4\,(-3 - 0)$

$= -\lambda - 5 - 12 = -\lambda - 17$

$\therefore$ $|A| \neq 0 \Rightarrow -\lambda - 17 \neq 0 \Rightarrow \lambda \neq \mathbf{-17}$.

9. Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$

$= 1 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow A^{-1}$ exists.

Now we find the cofactor matrix of A. So,

$A_{11} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1,\ A_{12} = -\begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 0,\ A_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$

$A_{21} = -\begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} = -2,\ A_{22} = \begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix} = 1,\ A_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$

$A_{31} = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 7,\ A_{32} = -\begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} = -2,\ A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$

$\therefore$ adj $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

$= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ $A^{-1} = \dfrac{1}{|A|}\,adj\ A = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ The element in the first row and third column of A^{-1} is 7.

10. The given system of equations can be written in the matrix form as:

$$\underset{A}{\begin{bmatrix} 2 & 2 \\ 5 & k \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B}{\begin{bmatrix} 5 \\ 9 \end{bmatrix}}$$

Now there will be a unique solution for $AX = B$ if $|A| \neq 0$

i.e, $2k - 10 \neq 0$

$\Rightarrow$ $k \neq 5$

11. The given system of equations can be written in the matrix form as:

$$\underset{A}{\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B}{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}$$

Now, $|A| = 24 - 24 = 0$

Hence the given system of equations either have infinite solutions or no solution.

Now we find (adj A) B.

Cofactors of A are:

$$A_{11} = 8, A_{12} = -6, A_{21} = -4, A_{22} = 3$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix}$$

$$(\text{adj } A)\, B = \begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 16 - 16 \\ -12 + 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

Hence (adj A) $B = 0 \Rightarrow$ Infinitely many solutions.

12. $$\left.\begin{matrix} x + 3y = 5 \\ 2x + 6y = 8 \end{matrix}\right] \rightarrow \underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B}{\begin{bmatrix} 5 \\ 8 \end{bmatrix}} \qquad \text{(in matrix form)}$$

$$|A| = 6 - 6 = 0$$

$\Rightarrow$ Either the system of equations has no solution or infinite solutions.

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$$

$$\therefore \quad (\text{adj } A)\, B = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 30 - 24 \\ -10 + 8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq 0$$

Hence, the given system of equations is inconsistent with no solution.