

26. Scalar Triple Product

Exercise 26.1

1 A. Question

Evaluate the following :

$$[\hat{i} \hat{j} \hat{k}] + [\hat{j} \hat{k} \hat{i}] + [\hat{k} \hat{i} \hat{j}]$$

Answer

Formula: -

$$(i) [\vec{a}\vec{b}\vec{c}] = \vec{a}(\vec{b} \times \vec{c}) = \vec{b}(\vec{c} \times \vec{a}) = \vec{c}(\vec{a} \times \vec{b})$$

$$(ii) \hat{i} \cdot \hat{i} = 1, \hat{j} \cdot \hat{j} = 1, \hat{k} \cdot \hat{k} = 1$$

$$(iii) \hat{i} \times \hat{j} = \vec{k}, \hat{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j}$$

we have

$$[\hat{i}\hat{j}\hat{k}] + [\hat{j}\hat{k}\hat{i}] + [\hat{k}\hat{i}\hat{j}] = (\hat{i} \times \hat{j}) \cdot \hat{k} + (\hat{j} \times \hat{k}) \cdot \hat{i} + (\hat{k} \times \hat{i}) \cdot \hat{j}$$

using Formula(i) and (iii)

$$\Rightarrow [\hat{i}\hat{j}\hat{k}] + [\hat{j}\hat{k}\hat{i}] + [\hat{k}\hat{i}\hat{j}] = \hat{k} \cdot \hat{k} + \hat{i} \cdot \hat{i} + \hat{j} \cdot \hat{j}$$

$$\Rightarrow [\hat{i}\hat{j}\hat{k}] + [\hat{j}\hat{k}\hat{i}] + [\hat{k}\hat{i}\hat{j}] = 1 + 1 + 1 = 3$$

therefore, using Formula (ii)

$$[\hat{i}\hat{j}\hat{k}] + [\hat{j}\hat{k}\hat{i}] + [\hat{k}\hat{i}\hat{j}] = 3$$

1 B. Question

Evaluate the following :

$$[2\hat{i} \hat{j} \hat{k}] + [\hat{i} \hat{k} \hat{j}] + [\hat{k} \hat{j} 2\hat{i}]$$

Answer

Formula: -

$$(i) [\vec{a}\vec{b}\vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$(ii) \hat{i} \cdot \hat{i} = 1, \hat{j} \cdot \hat{j} = 1, \hat{k} \cdot \hat{k} = 1$$

$$(iii) \hat{i} \times \hat{j} = \vec{k}, \hat{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j}$$

Given: -

we have

$$[2\hat{i}\hat{j}\hat{k}] + [\hat{i}\hat{k}\hat{j}] + [\hat{k}\hat{j}2\hat{i}] = (2\hat{i} \times \hat{j}) \cdot \hat{k} + (\hat{i} \times \hat{j}) \cdot \hat{j} + (\hat{k} \times \hat{j}) \cdot 2\hat{i}$$

using Formula (i)

$$\Rightarrow [2\hat{i}\hat{j}\hat{k}] + [\hat{i}\hat{k}\hat{j}] + [\hat{k}\hat{j}2\hat{i}] = 2\hat{k} \cdot \hat{k} + (-\hat{j}) \cdot \hat{j} + (-\hat{i}) \cdot 2\hat{i}$$

using Formula (ii)

$$\Rightarrow [2\hat{i}\hat{j}\hat{k}] + [\hat{i}\hat{k}\hat{j}] + [\hat{k}\hat{j}2\hat{i}] = 2 - 1 - 2$$

$$\Rightarrow [2\hat{i}\hat{j}\hat{k}] + [\hat{i}\hat{k}\hat{j}] + [\hat{k}\hat{i}\hat{j}] = -1$$

therefore,

$$[2\hat{i}\hat{j}\hat{k}] + [\hat{i}\hat{k}\hat{j}] + [\hat{k}\hat{i}\hat{j}] = -1$$

2 A. Question

Find $[\vec{a} \vec{b} \vec{c}]$, when

$$\vec{a} = 2\hat{i} - 3\hat{j}, \vec{b} = \hat{i} + \hat{j} - \hat{k} \text{ and } \vec{c} = 3\hat{i} - \hat{k}$$

Answer

Formula: -

$$\text{if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} +$$

$$(i) \quad c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Given: -

$$\vec{a} = 2\hat{i} - 3\hat{j}, \vec{b} = \hat{i} + \hat{j} - \hat{k} \text{ and } \vec{c} = 3\hat{i} - \hat{k}$$

using Formula(i)

$$[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 2(-1 - 0) + 3(-1 + 3)$$

$$= -2 + 6$$

$$= 4$$

therefore,

$$[\vec{a}\vec{b}\vec{c}] = 4$$

2 B. Question

Find $[\vec{a} \vec{b} \vec{c}]$, when

$$\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{b} = 2\hat{i} + \hat{j} - \hat{k} \text{ and } \vec{c} = \hat{j} + \hat{k}$$

Answer

Formula: -

$$(i) \text{ If } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Given: -

$$\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{b} = 2\hat{i} + \hat{j} - \hat{k} \text{ and } \vec{c} = \hat{j} + \hat{k}$$

$$[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 1(1 + 1) + 2(2 + 0) + 3(2 - 0)$$

$$= 2 + 4 + 6 = 12$$

therefore,

$$[\vec{a}\vec{b}\vec{c}] = 12$$

3 A. Question

Find the volume of the parallelepiped whose coterminal edges are represented by the vectors:

$$\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}, \vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$$

Answer

Formula : -

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Given: -

$$\vec{a} = 2\hat{i} + 3\hat{j}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}, \vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$$

we know that the volume of parallelepiped whose three adjacent edges are

$$\vec{a}, \vec{b}, \vec{c} \text{ is equal to } |[\vec{a}\vec{b}\vec{c}]|.$$

we have

$$[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 2(4 - 1) - 3(2 + 3) + 4(-1 - 6)$$

$$= -37$$

therefore, the volume of the parallelepiped is $[\vec{a}\vec{b}\vec{c}] = |-37| = 37$ cubic unit.

3 B. Question

Find the volume of the parallelepiped whose coterminal edges are represented by the vectors:

$$\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}, \vec{c} = 3\hat{i} - \hat{j} - 2\hat{k}$$

Answer

Formula :-

$$(i) \text{ if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Given :-

$$\vec{a} = 2\hat{i} - 3\hat{j}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}, \vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$$

we know that the volume of parallelepiped whose three adjacent edges are

$\vec{a}, \vec{b}, \vec{c}$ is equal to $|[\vec{a}\vec{b}\vec{c}]|$.

we have

$$[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & -2 \end{vmatrix}$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 2(-4 - 1) - 3(-2 + 3) + 4(-1 - 6)$$

$$= -35$$

therefore, the volume of the parallelepiped is $[\vec{a}\vec{b}\vec{c}] = |-35| = 35$ cubic unit.

3 C. Question

Find the volume of the parallelepiped whose coterminous edges are represented by the vectors:

$$\vec{a} = 11\hat{i}, \vec{b} = 2\hat{j}, \vec{c} = 13\hat{k}$$

Answer

Formula :-

$$\begin{aligned} \text{(i) if } \vec{a} &= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} \\ &= c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned}$$

$$\begin{aligned} \text{(ii) } &\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ &+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

Given: -

$$\vec{a} = 11\hat{i}, \vec{b} = 2\hat{j}, \vec{c} = 13\hat{k}$$

we know that the volume of parallelepiped whose three adjacent edges are

$$\vec{a}, \vec{b}, \vec{c} \text{ is equal to } |[\vec{a}\vec{b}\vec{c}]|.$$

we have

$$[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} 11 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 13 \end{vmatrix}$$

now, using

$$\begin{aligned} &\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ &+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

$$= 11(26 - 0) + 0 + 0 = 286$$

therefore, the volume of the parallelepiped is $[\vec{a} \vec{b} \vec{c}] = |286| = 286$ cubic unit.

3 D. Question

Find the volume of the parallelepiped whose coterminous edges are represented by the vectors:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i} - \hat{j} + \hat{k}, \vec{c} = \hat{i} + 2\hat{j} - \hat{k}$$

Answer

Formula: -

$$(i) \text{ if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Given: -

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}, \vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}.$$

we know that the volume of parallelepiped whose three adjacent edges are

$$\vec{a}, \vec{b}, \vec{c} \text{ is equal to } |[\vec{a}\vec{b}\vec{c}]|.$$

we have

$$[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 1(1 - 2) - 1(-1 - 1) + 1(2 + 1)$$

$$= 4$$

$$\text{therefore, the volume of the parallelepiped is } [\vec{a}\vec{b}\vec{c}] = |4|$$

$$= 4 \text{ cubic unit.}$$

4 A. Question

Show that each of the following triads of vectors is coplanar :

$$\vec{a} = \hat{i} + 2\hat{j} - \hat{k}, \vec{b} = 3\hat{i} + 2\hat{j} + 7\hat{k}, \vec{c} = 5\hat{i} + 6\hat{j} + 5\hat{k}$$

Answer

Formula : -

$$(i) \text{ if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} \\ = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(iii) Three vectors \vec{a} , \vec{b} , and \vec{c} are coplanar if and only if

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$$

Given: -

$$\vec{a} = \hat{i} + 2\hat{j} - \hat{k}, \vec{b} = 3\hat{i} + 2\hat{j} + 7\hat{k}, \vec{c} = 5\hat{i} + 6\hat{j} + 5\hat{k}$$

we know that three vector $\vec{a}, \vec{b}, \vec{c}$ are coplanar if their scalar triple product is zero

$$[\vec{a}\vec{b}\vec{c}] = 0.$$

we have

$$[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 2 & 7 \\ 5 & 6 & 5 \end{vmatrix}$$

using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 1(10 - 42) - 2(15 - 35) - 1(18 - 10)$$

$$= 0.$$

Hence, the Given vector are coplanar.

4 B. Question

Show that each of the following triads of vectors is coplanar :

$$\vec{a} = -4\hat{i} - 6\hat{j} - 2\hat{k}, \vec{b} = -\hat{i} + 4\hat{j} + 3\hat{k}, \vec{c} = -8\hat{i} - \hat{j} + 3\hat{k}$$

Answer

Formula : -

$$(i) \text{ if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(iii) Three vectors \vec{a}, \vec{b} , and \vec{c} are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

Given: -

$$\vec{a} = -4\hat{i} - 6\hat{j} - 2\hat{k}, \vec{b} = -\hat{i} + 4\hat{j} + 3\hat{k}, \vec{c} = -8\hat{i} - \hat{j} + 3\hat{k}$$

we know that three vector $\vec{a}, \vec{b}, \vec{c}$ are coplanar if their scalar triple product is zero

$$[\vec{a}\vec{b}\vec{c}] = 0.$$

we have

$$[\vec{abc}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -4(12 + 13) + 6(-3 + 24) - 2(1 + 32)$$

$$= 0$$

hence, the Given vector are coplanar.

4 C. Question

Show that each of the following triads of vectors is coplanar :

$$\hat{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \hat{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}, \hat{c} = \hat{i} - 3\hat{j} + 5\hat{k}$$

Answer

Formula : -

$$(i) \text{ if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{abc}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(iii) Three vectors \vec{a}, \vec{b} and \vec{c} are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

Given: -

$$\hat{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \hat{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}, \hat{c} = \hat{i} - 3\hat{j} + 5\hat{k}$$

we know that three vector $\vec{a}, \vec{b}, \vec{c}$ are coplanar if their scalar triple product is zero

$$[\vec{abc}] = 0.$$

we have

$$[\vec{abc}] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\
+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
= 1(15 - 12) + 2(-10 + 4) + 3(6 - 3) \\
= 3 - 12 + 9 = 0$$

5 A. Question

Find the value of λ so that the following vectors are coplanar.

$$\vec{a} = \hat{i} - \hat{j} + \hat{k}, \vec{b} = 2\hat{i} + \hat{j} - \hat{k}, \vec{c} = \lambda\hat{i} - \hat{j} + \lambda\hat{k}$$

Answer

Formula :-

$$\begin{aligned}
\text{(i) if } \vec{a} &= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} \\
&= c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\
+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\end{aligned}$$

$$\text{(iii) Three vectors } \vec{a}, \vec{b}, \text{ and } \vec{c} \text{ are coplanar if and only if } \vec{a} \cdot (\vec{b} \times \vec{c}) = 0$$

Given :-

$$\vec{a} = \hat{i} - \hat{j} + \hat{k}, \vec{b} = 2\hat{i} + \hat{j} - \hat{k}, \vec{c} = \lambda\hat{i} - \hat{j} + \lambda\hat{k}$$

we know that vector $\vec{a}, \vec{b}, \vec{c}$ are coplanar if their scalar triple product is zero

$$[\vec{a}\vec{b}\vec{c}] = 0.$$

we have

$$[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ \lambda & -1 & \lambda \end{vmatrix}$$

now, using

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\
+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\end{aligned}$$

$$\Rightarrow 0 = 1(\lambda - 1) + 1(2\lambda + \lambda) + 1(-2 - \lambda)$$

$$\Rightarrow 0 = \lambda - 1 + 3\lambda - 2 - \lambda$$

$$\Rightarrow 0 = 3\lambda - 3$$

$$\Rightarrow \lambda = 1$$

5 B. Question

Find the value of λ so that the following vectors are coplanar.

$$\vec{a} = 2\hat{i} - \hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}, \vec{c} = \lambda\hat{i} + \lambda\hat{j} + 5\hat{k}$$

Answer

Formula : -

$$(i) \text{ if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(iii) Three vectors \vec{a} , \vec{b} and \vec{c} are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

Given: -

$$\vec{a} = 2\hat{i} - \hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}, \vec{c} = \lambda\hat{i} + \lambda\hat{j} + 5\hat{k}$$

we know that vector $\vec{a}, \vec{b}, \vec{c}$ are coplanar if their scalar triple product is zero

$$[\vec{a}\vec{b}\vec{c}] = 0.$$

we have

$$[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ \lambda & \lambda & 5 \end{vmatrix}$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Rightarrow 0 = 2(10 + 3\lambda) + 1(5 + 3\lambda) + 1(\lambda - 2\lambda)$$

$$\Rightarrow 0 = 8\lambda + 25$$

$$\Rightarrow \lambda = \frac{-25}{8}$$

5 C. Question

Find the value of λ so that the following vectors are coplanar.

$$\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}, \vec{b} = 3\hat{i} + \lambda\hat{j} + \hat{k}, \vec{c} = \hat{i} + 2\hat{j} + 2\hat{k}$$

Answer

Formula : -

$$(i) \text{ if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(iii) Three vectors \vec{a} , \vec{b} and \vec{c} are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

Given: -

$$\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}, \vec{b} = 3\hat{i} + \lambda\hat{j} + \hat{k}, \vec{c} = \hat{i} + 2\hat{j} + 2\hat{k}$$

we know that vector $\vec{a}, \vec{b}, \vec{c}$ are coplanar if their scalar triple product is zero

$$[\vec{a}\vec{b}\vec{c}] = 0.$$

we have

$$[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} 1 & 2 & -3 \\ 3 & \lambda & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Rightarrow 0 = 1(2\lambda - 2) - 2(6 - 1) - 3(6 - \lambda)$$

$$\Rightarrow 0 = 5\lambda - 30$$

$$\Rightarrow \lambda = 6$$

5 D. Question

Find the value of λ so that the following vectors are coplanar.

$$\vec{a} = \hat{i} + 3\hat{j}, \vec{b} = 5\hat{k}, \vec{c} = \lambda\hat{i} - \hat{j}$$

Answer

Formula: -

$$(i) \text{ if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(iii) Three vectors \vec{a} , \vec{b} , and \vec{c} are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

Given: -

$$\vec{a} = \hat{i} + 3\hat{j}, \vec{b} = 5\hat{k}, \vec{c} = \lambda\hat{i} - \hat{j}$$

we know that vector \vec{a} , \vec{b} , \vec{c} are coplanar if their scalar triple product is zero

$$[\vec{a}\vec{b}\vec{c}] = 0.$$

we have

$$[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & 5 \\ \lambda & -1 & 0 \end{vmatrix}$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Rightarrow 0 = 1(0 + 5) - 3(0 - 5\lambda) + 0$$

$$\Rightarrow 0 = 5 + 15\lambda$$

$$\Rightarrow \lambda = \frac{-1}{3}$$

6. Question

Show that the four points having position vectors $6\hat{i} - 7\hat{j}$, $16\hat{i} - 19\hat{j} - 4\hat{k}$, $3\hat{j} - 6\hat{k}$, $2\hat{i} + 5\hat{j} + 10\hat{k}$ are not coplanar.

Answer

Formula: -

$$(i) \text{ if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$(iii) \text{ if } \vec{OA} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \text{ and } \vec{OB} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, \text{ then } \vec{OB} - \vec{OA} = (b_1 - a_1)\hat{i} + (b_2 - a_2)\hat{j} + (b_3 - a_3)\hat{k}$$

(iv) Three vectors \vec{a} , \vec{b} , and \vec{c} are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

Given: -

$$\vec{OA} = 6\hat{i} - 7\hat{j}, \vec{OB} = 16\hat{i} - 19\hat{j} - 4\hat{k}, \vec{OC} = 3\hat{j} - 6\hat{k}, \vec{OD} = 2\hat{i} + 5\hat{j} + 10\hat{k}$$

$$\vec{AB} = \vec{OB} - \vec{OA} = 10\hat{i} - 12\hat{j} - 4\hat{k}$$

$$\vec{AC} = \vec{OC} - \vec{OA} = -6\hat{i} + 10\hat{j} - 6\hat{k}$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = -4\hat{i} + 12\hat{j} + 10\hat{k}$$

The four points are coplaner if vector AB, AC, AD are coplanar.

$$[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = \begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 12 & 10 \end{vmatrix}$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 10(100 + 72) + 12(-60 - 24) - 4(-72 + 40) = 840$$

$\neq 0$.

hence the point are not coplanar

7. Question

Show that the points A(-1, 4, -3), B(3, 2, -5), C(-3, 8, -5) and D(-3, 2, 1) are coplanar.

Answer

Formula: -

$$(i) \text{ if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$(iii) \text{ Three vectors } \vec{a}, \vec{b}, \text{ and } \vec{c} \text{ are coplanar if and only if } \vec{a} \cdot (\vec{b} \times \vec{c}) = 0$$

$$(iv) \text{ if } \overrightarrow{OA} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \text{ and } \overrightarrow{OB} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, \text{ then } \overrightarrow{OB} - \overrightarrow{OA} = (b_1 - a_1)\hat{i} + (b_2 - a_2)\hat{j} + (b_3 - a_3)\hat{k}$$

Given: -

AB = position vector of B - position vector of A

$$= 4\hat{i} - 2\hat{j} - 2\hat{k}$$

AC = position vector of c - position vector of A

$$= -2\hat{i} + 4\hat{j} - 2\hat{k}$$

AD = position vector of c - position vector of A

$$= -2\hat{i} - 2\hat{j} + 4\hat{k}$$

The four pint are coplanar if the vector are coplanar.

thus,

$$[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = \begin{vmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{vmatrix}$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 4(16 - 4) + 2(-8 - 4) - 2(-4 + 8) = 0$$

hence proved.

8. Question

Show that four points whose position vectors are $6\hat{i} - 7\hat{j}, 16\hat{i} - 19\hat{j} - 4\hat{k}, 3\hat{i} - 6\hat{k}, 2\hat{i} - 5\hat{j} + 10\hat{k}$ are coplanar.

Answer

Formula :-

(i) if $\overrightarrow{OA} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\overrightarrow{OB} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\overrightarrow{OB} - \overrightarrow{OA} = (b_1 - a_1)\hat{i} + (b_2 - a_2)\hat{j} + (b_3 - a_3)\hat{k}$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(iii) Three vectors $\vec{a}, \vec{b},$ and \vec{c} are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

(iv) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then, $[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

let

$$\overrightarrow{OA} = 6\hat{i} - 7\hat{j}, \overrightarrow{OB} = 16\hat{i} - 19\hat{j} - 4\hat{k}, \overrightarrow{OC} = 3\hat{i} - 6\hat{k}, \overrightarrow{OD} = 2\hat{i} - 5\hat{j} + 10\hat{k}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 10\hat{i} - 12\hat{j} - 4\hat{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -6\hat{i} + 10\hat{j} - 6\hat{k}$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = -4\hat{i} + 2\hat{j} + 10\hat{k}$$

The four points are coplanar if the vector $\overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}$ are coplanar.

$$[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = \begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 2 & 10 \end{vmatrix}$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 10(100 + 12) + 12(-60 - 24) - 4(-12 + 40) = 0.$$

hence the point are coplanar

9. Question

Find the value of λ for which the four points with position vectors $-\hat{j} - \hat{k}$, $4\hat{i} + 5\hat{j} + \lambda\hat{k}$, $3\hat{i} + 9\hat{j} + 4\hat{k}$ and $-4\hat{i} + 4\hat{j} + 4\hat{k}$ are coplanar.

Answer

Formula :-

(i) if $\vec{OA} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{OB} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then
 $\vec{OB} - \vec{OA} = (b_1 - a_1)\hat{i} + (b_2 - a_2)\hat{j} + (b_3 - a_3)\hat{k}$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(iii) Three vectors \vec{a} , \vec{b} , and \vec{c} are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

(iv) if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then, $[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Given :-

$$OA = -\hat{j} - \hat{k}, OB = 4\hat{i} + 5\hat{j} - \lambda\hat{k}, OC = 3\hat{i} + 9\hat{j} + 4\hat{k}, OD = -4\hat{i} - 4\hat{j} + 4\hat{k}$$

$$AB = OB - OA = 4\hat{i} + 6\hat{j} + (\lambda + 1)\hat{k}$$

$$AC = OC - OA = 3\hat{i} + 10\hat{j} + 5\hat{k}$$

$$AD = OD - OA = -4\hat{i} + 5\hat{j} + 5\hat{k}$$

The four points are coplaner if vector AB, AC, AD are coplanar.

$$[AB^{\rightarrow}, AC^{\rightarrow}, AD^{\rightarrow}] = \begin{vmatrix} 4 & 6 & (\lambda + 1) \\ 3 & 10 & 5 \\ -4 & 5 & 5 \end{vmatrix}$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Rightarrow 4(50 - 25) - 6(15 + 20) + (\lambda + 1)(15 + 40) = 0.$$

$$\Rightarrow \lambda = 1$$

hence the point are coplanar

10. Question

Prove that : -

$$(\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\} = 0$$

Answer

Formula: -

$$(i) [\vec{a}\vec{b}\vec{c}] = \vec{a}(\vec{b} \times \vec{c}) = \vec{b}(\vec{c} \times \vec{a}) = \vec{c}(\vec{a} \times \vec{b})$$

$$(ii) \vec{a} \times \vec{a} = \vec{b} \times \vec{b} = \vec{c} \times \vec{c} = 0$$

taking L.H.S

$$(\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\} = [(\vec{a} - \vec{b})(\vec{b} - \vec{c})(\vec{c} - \vec{a})]$$

using Formula (i)

$$\Rightarrow (\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\} = (\vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{b} \times \vec{b} + \vec{b} \times \vec{c}) \cdot (\vec{c} - \vec{a})$$

using Formula(ii)

$$\Rightarrow (\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\} = (\vec{a} \times \vec{b} - \vec{a} \times \vec{c} - 0 + \vec{b} \times \vec{c}) \cdot (\vec{c} - \vec{a})$$

$$\begin{aligned} \Rightarrow (\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\} &= (\vec{a} \times \vec{b}) \cdot \vec{c} - (\vec{a} \times \vec{b}) \cdot \vec{a} + (\vec{c} \times \vec{a}) \cdot \vec{c} - (\vec{c} \times \vec{a}) \cdot \vec{a} + (\vec{b} \times \vec{c}) \cdot \vec{c} \\ &\quad - (\vec{b} \times \vec{c}) \cdot \vec{a} \end{aligned}$$

$$\Rightarrow (\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\} = [\vec{a}\vec{b}\vec{c}] - [\vec{a}\vec{b}\vec{a}] + [\vec{c}\vec{a}\vec{c}] - [\vec{c}\vec{a}\vec{a}] + [\vec{b}\vec{c}\vec{c}] - [\vec{b}\vec{c}\vec{a}]$$

$$\Rightarrow (\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\} = [\vec{a}\vec{b}\vec{c}] - [\vec{b}\vec{c}\vec{a}]$$

$$\Rightarrow (\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\} = [\vec{a}\vec{b}\vec{c}] - [\vec{a}\vec{b}\vec{c}]$$

$$\Rightarrow (\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\} = 0$$

$$\Rightarrow (\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\} = 0$$

L.H.S = R.H.S

11. Question

\vec{a} , \vec{b} and \vec{c} are the position vectors of points A, B and C respectively, prove that : $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$ is a vector perpendicular to the plane of triangle ABC.

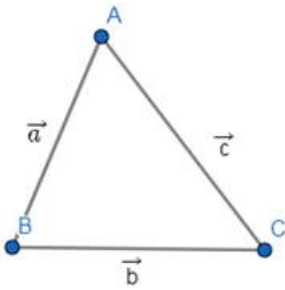
Answer

if \vec{a} represents the sides AB,

if \vec{b} represent the sides BC,

if \vec{c} respresent the sidesAC of triangle ABC

$\vec{a} \times \vec{b}$ is perpendicular to plane of triangle ABC. (i)



$\vec{b} \times \vec{c}$ is perpendicular to plane of triangle ABC. (ii)

$\vec{c} \times \vec{a}$ is perpendicular to plane of triangle ABC. (iii)

adding all the (i) + (ii) + (iii)

hence $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$ is a vector perpendicular to the plane of the triangle ABC

12 A. Question

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then,

If $c_1 = -1$ and $c_2 = 2$, find c_3 which makes \vec{a} , \vec{b} and \vec{c} coplanar.

Answer

Formula: -

$$(i) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$(ii) \text{ if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} \\ = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(iii) Three vectors \vec{a} , \vec{b} , and \vec{c} are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

Given: -

\vec{a} , \vec{b} , \vec{c} are coplanar if

$$[\vec{a}\vec{b}\vec{c}] = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & c_3 \end{vmatrix} = 0$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Rightarrow 0 - 1(c_3) + 1(2) = 0$$

$$\Rightarrow c_3 = 2$$

12 B. Question

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then,

If $c_1 = -1$ and $c_3 = 1$, show that no value of c_2 can make \vec{a} , \vec{b} and \vec{c} coplanar.

Answer

Formula: -

$$(i) \text{ if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(iii) Three vectors \vec{a} , \vec{b} , and \vec{c} are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

we know that \vec{a} , \vec{b} , \vec{c} are coplanar if

$$[\vec{a}\vec{b}\vec{c}] = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & c_2 & 1 \end{vmatrix} = 0$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Rightarrow 0 - 1(c_3) + 1(2) = 0$$

$$\Rightarrow c_3 = 2$$

13. Question

Find for which the points A(3, 2, 1), B(4, λ , 5), C(4, 2, -2) and D(6, 5, -1) are coplanar.

Answer

Formula: -

$$(i) \text{ if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$(iii) \text{ Three vectors } \vec{a}, \vec{b} \text{ and } \vec{c} \text{ are coplanar if and only if } \vec{a} \cdot (\vec{b} \times \vec{c}) = 0$$

$$(iv) \text{ if } \vec{OA} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \text{ and } \vec{OB} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ then } \vec{OB} - \vec{OA} \\ = (b_1 - a_1)\hat{i} + (b_2 - a_2)\hat{j} + (b_3 - a_3)\hat{k}$$

let position vector of

$$\vec{OA} = 3\hat{i} + 2\hat{j} + \hat{k}$$

position vector of

$$\vec{OB} = 4\hat{i} + \lambda\hat{j} + 5\hat{k}$$

position vector of

$$\vec{OC} = 4\hat{i} + 2\hat{j} - 2\hat{k}$$

position vector of

$$\vec{OD} = 6\hat{i} + 5\hat{j} - \hat{k}$$

The four points are coplanar if the vector $\vec{AB}, \vec{AC}, \vec{AD}$ are coplanar

$$\vec{AB} = \hat{i} + (\lambda - 2)\hat{j} + 4\hat{k}$$

$$\vec{AC} = \hat{i} + 0\hat{j} - 3\hat{k}$$

$$\vec{AD} = 3\hat{i} + 3\hat{j} - 2\hat{k}$$

$$\begin{vmatrix} 1 & (\lambda - 2) & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

now, using

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Rightarrow 1(9) - (\lambda - 2)(-2 + 9) + 4(3 - 0) = 0$$

$$\Rightarrow 7\lambda = 35$$

$$\Rightarrow \lambda = 5$$

14. Question

If four points A, B, C and D with position vectors $4\hat{i} + 3\hat{j} + 3\hat{k}$, $5\hat{i} + x\hat{j} + 7\hat{k}$, $5\hat{i} + 3\hat{j}$ and $7\hat{i} + 6\hat{j} + \hat{k}$ respectively are coplanar, then find the value of x.

Answer

Formula: -

$$(i) \text{ if } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \text{ then, } [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(ii) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(iii) Three vectors \vec{a}, \vec{b} , and \vec{c} are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

$$(iv) \text{ if } \vec{OA} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \text{ and } \vec{OB} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ then } \vec{OB} - \vec{OA} = (b_1 - a_1)\hat{i} + (b_2 - a_2)\hat{j} + (b_3 - a_3)\hat{k}$$

let position vector of

$$\vec{OA} = 4\hat{i} + 3\hat{j} + 3\hat{k}$$

position vector of

$$\vec{OB} = 5\hat{i} + x\hat{j} + 7\hat{k}$$

position vector of

$$\vec{OC} = 5\hat{i} + 3\hat{j}$$

position vector of

$$\vec{OD} = 7\hat{i} + 6\hat{j} + \hat{k}$$

The four points are coplanar if the vector $\vec{AB}, \vec{AC}, \vec{AD}$ are coplanar

$$\vec{AB} = \hat{i} + (x-3)\hat{j} + 4\hat{k},$$

$$\vec{AC} = \hat{i} + 0\hat{j} - 3\hat{k},$$

$$\vec{AD} = 3\hat{i} + 3\hat{j} - 2\hat{k}$$

$$\begin{vmatrix} 1 & (x-2) & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\Rightarrow 1(9) - (x-2)(-2+9) + 4(3) = 0$$

$$\Rightarrow 9 - 7x + 14 + 12 = 0$$

$$\Rightarrow 35 = 7x$$

$$\Rightarrow x = 5$$

Very short answer

1. Question

Write the value of $\begin{bmatrix} 2\hat{i} & 3\hat{j} & 4\hat{k} \end{bmatrix}$.

Answer

The meaning of the notation $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix}$ is the scalar triple product of the three vectors; which is computed as $\vec{a} \cdot (\vec{b} \times \vec{c})$

$$\text{So we have } 2\hat{i} \cdot (3\hat{j} \times 4\hat{k}) = 2\hat{i} \cdot 12\hat{i} = 24 \quad (\hat{j} \times \hat{k} = \hat{i})$$

2. Question

Write the value of $\begin{bmatrix} \hat{i} + \hat{j} & \hat{j} + \hat{k} & \hat{k} - \hat{i} \end{bmatrix}$

Answer

$$\text{Here we have } \vec{a} = \hat{i} + \hat{j}, \vec{b} = \hat{j} + \hat{k}, \vec{c} = \hat{k} - \hat{i}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

3. Question

Write the value of $\begin{bmatrix} \hat{i} - \hat{j} & \hat{j} - \hat{k} & \hat{k} - \hat{i} \end{bmatrix}$.

Answer

$$\text{The value of the above product is the value of the matrix } \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

4. Question

Find the values of 'a' for which the vectors $\vec{\alpha} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{\beta} = a\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{\gamma} = \hat{i} + 2\hat{j} + a\hat{k}$ are coplanar.

Answer

Three vectors are coplanar iff (if and only if) $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

$$\text{Hence we have value of the matrix } \begin{vmatrix} 1 & 2 & 1 \\ a & 1 & 2 \\ 1 & 2 & a \end{vmatrix} = 0$$

$$\text{We have } 2a^2 - 3a + 1 = 0$$

$$2a^2 - 2a - a + 1 = 0$$

$$\text{Solving this quadratic equation we get } a = 1, a = \frac{1}{2}$$

5. Question

Find the volume of the parallelepiped with its edges represented by the vectors $\hat{i} + \hat{j}$, $\hat{i} + 2\hat{j}$ and $\hat{i} + \hat{j} + \pi\hat{k}$.

Answer

Volume of the parallelepiped with its edges represented by the vectors $\vec{a}, \vec{b}, \vec{c}$ is $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \vec{a} \cdot (\vec{b} \times \vec{c})$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \pi \end{vmatrix} = \pi$$

6. Question

If \vec{a}, \vec{b} are non-collinear vectors, then find the value of $[\vec{a} \vec{b} \hat{i}] \hat{i} + [\vec{a} \vec{b} \hat{j}] \hat{j} + [\vec{a} \vec{b} \hat{k}] \hat{k}$.

Answer

for any vector \vec{r}

$$\text{We have } \vec{r} = (\vec{r} \cdot \hat{i})\hat{i} + (\vec{r} \cdot \hat{j})\hat{j} + (\vec{r} \cdot \hat{k})\hat{k}$$

$$\text{Replacing } (\vec{r}) = \vec{a} \times \vec{b}$$

$$\vec{a} \times \vec{b} = (\vec{a} \times \vec{b} \cdot \hat{i})\hat{i} + (\vec{a} \times \vec{b} \cdot \hat{j})\hat{j} + (\vec{a} \times \vec{b} \cdot \hat{k})\hat{k}$$

$$\vec{a} \times \vec{b} = [\vec{a} \vec{b} \hat{i}]\hat{i} + [\vec{a} \vec{b} \hat{j}]\hat{j} + [\vec{a} \vec{b} \hat{k}]\hat{k}$$

7. Question

If the vectors $(\sec^2 A) \hat{i} + \hat{j} + \hat{k}, \hat{i} + (\sec^2 B) \hat{j} + \hat{k}, \hat{i} + \hat{j} + (\sec^2 C) \hat{k}$ are coplanar, then find the value of $\operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$.

Answer

$$\text{For three vectors to be coplanar we have } \begin{vmatrix} \sec^2 A & 1 & 1 \\ 1 & \sec^2 B & 1 \\ 1 & 1 & \sec^2 C \end{vmatrix} = 0$$

$$\text{Which gives } \sec^2 A \sec^2 B \sec^2 C - \sec^2 A - \sec^2 B - \sec^2 C + 2 = 0 \dots\dots\dots(1)$$

$$\sec^2 \theta = \frac{\operatorname{cosec}^2 \theta - 2}{\operatorname{cosec}^2 \theta - 1} \dots\dots\dots(2)$$

Substituting equation 2 in 1 we have

$$\frac{(\operatorname{cosec}^2 A - 2)(\operatorname{cosec}^2 B - 2)(\operatorname{cosec}^2 C - 2)}{(\operatorname{cosec}^2 A - 1)(\operatorname{cosec}^2 B - 1)(\operatorname{cosec}^2 C - 1)} - \frac{\operatorname{cosec}^2 A - 2}{\operatorname{cosec}^2 A - 1} - \frac{\operatorname{cosec}^2 B - 2}{\operatorname{cosec}^2 B - 1} - \frac{\operatorname{cosec}^2 C - 2}{\operatorname{cosec}^2 C - 1} + 2 = 0$$

$$\text{Let } \operatorname{cosec}^2 A = x \operatorname{cosec}^2 B = y \text{ and } \operatorname{cosec}^2 C = z$$

$$\text{So we have } \frac{(x-2)(y-2)(z-2)}{(x-1)(y-1)(z-1)} - \frac{x-2}{x-1} - \frac{y-2}{y-1} - \frac{z-2}{z-1} + 2 = 0$$

$$= (x-2)(y-2)(z-2) - (x-2)(y-1)(z-1) - (x-1)(y-2)(z-1) - (x-1)(y-1)(z-2) + 2(x-1)(y-1)(z-1) = 0$$

$$\text{Solving we have } x+y+z=4$$

$$\text{Hence } \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C = 4$$

8. Question

For any two vectors of \vec{a} and \vec{b} of magnitudes 3 and 4 respectively, write the value of $[\vec{a} \vec{b} \vec{a} \times \vec{b}] + (\vec{a} \cdot \vec{b})^2$.

Answer

$$[\vec{a} \vec{b} \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \times (\vec{b} \cdot \vec{c}) \text{ the dot and cross can be interchanged in scalar triple product.}$$

Let the angle between \vec{a} and \vec{b} vector be θ

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta = 12 \cos \theta$$

$$\begin{aligned}
& [\vec{a} \vec{b} \vec{a} \times \vec{b}] + (\vec{a} \cdot \vec{b})^2 = \vec{a} \cdot (\vec{b} \times (\vec{a} \times \vec{b})) \\
& = \vec{a} \times (\vec{b} \cdot (\vec{a} \times \vec{b})) \\
& = (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) \\
& = |(\vec{a} \times \vec{b})| |(\vec{a} \times \vec{b})| \cos 0 \\
& = (|\vec{a}| |\vec{b}| \sin \theta)^2 \\
& = 144 \sin^2 \theta + 144 \cos^2 \theta \\
& = 144(1) \\
& = 144
\end{aligned}$$

9. Question

If $[3\vec{a} \ 7\vec{b} \ \vec{c} \ \vec{d}] = \lambda [\vec{a} \ \vec{b} \ \vec{c}] + \mu [\vec{b} \ \vec{c} \ \vec{d}]$, then find the value of $\lambda + \mu$.

Answer

$$[3\vec{a} \ 7\vec{b} \ \vec{c} \ \vec{d}] = \lambda [\vec{a} \ \vec{b} \ \vec{c}] + \mu [\vec{b} \ \vec{c} \ \vec{d}]$$

$$3\vec{a} = \lambda \vec{a}$$

$$\lambda = 3$$

$$\vec{c} = \mu \vec{c}$$

$$\mu = 1$$

$$\text{So, } \lambda + \mu = 3 + 1$$

$$= 4$$

10. Question

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then find the value of $\frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{(\vec{c} \times \vec{a}) \cdot \vec{b}} + \frac{\vec{b} \cdot (\vec{a} \times \vec{c})}{\vec{c} \cdot (\vec{a} \times \vec{b})}$.

Answer

$$[\vec{a} \ \vec{b} \ \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \times (\vec{b} \cdot \vec{c}) \text{ the dot and cross can be interchanged in scalar triple product.}$$

Also $[\vec{a} \ \vec{b} \ \vec{c}] = [\vec{c} \ \vec{a} \ \vec{b}] = [\vec{b} \ \vec{c} \ \vec{a}]$ (cyclic permutation of three vectors does not change the value of the scalar triple product)

$$[\vec{a} \ \vec{b} \ \vec{c}] = -[\vec{a} \ \vec{c} \ \vec{b}]$$

$$\text{Using these results } \frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{(\vec{c} \times \vec{a}) \cdot \vec{b}} + \frac{\vec{b} \cdot (\vec{a} \times \vec{c})}{\vec{c} \cdot (\vec{a} \times \vec{b})} = \frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{(\vec{c} \times \vec{a}) \cdot \vec{b}} + \frac{-\vec{a} \cdot (\vec{b} \times \vec{c})}{(\vec{c} \times \vec{a}) \cdot \vec{b}} = 0$$

11. Question

Find $\vec{a} \cdot (\vec{b} \times \vec{c})$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$.

Answer

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = -10$$

MCQ

1. Question

Mark the correct alternative in each of the following:

If \vec{a} lies in the plane of vectors \vec{b} and \vec{c} , then which of the following is correct?

A. $[\vec{a} \vec{b} \vec{c}] = 0$

B. $[\vec{a} \vec{b} \vec{c}] = 1$

C. $[\vec{a} \vec{b} \vec{c}] = 3$

D. $[\vec{b} \vec{c} \vec{a}] = 1$

Answer

Here, \vec{a} lies in the plane of vectors \vec{b} and \vec{c} , which means \vec{a} , \vec{b} and \vec{c} are coplanar.

We know that $\vec{b} \times \vec{c}$ is perpendicular to \vec{b} and \vec{c} .

Also dot product of two perpendicular vector is zero.

Since, \vec{a} , \vec{b} , \vec{c} are coplanar, $\vec{b} \times \vec{c}$ is perpendicular to \vec{a} .

$$\text{So, } \vec{a} \cdot (\vec{b} \times \vec{c}) = 0$$

$$\Rightarrow [\vec{a} \vec{b} \vec{c}] = 0$$

2. Question

Mark the correct alternative in each of the following:

The value of $[\vec{a} - \vec{b} \vec{b} - \vec{c} \vec{c} - \vec{a}]$, where $|\vec{a}| = 1, |\vec{b}| = 5, |\vec{c}| = 3$ is

A. 0

B. 1

C. 6

D. none of these

Answer

$$[\vec{a} - \vec{b} \vec{b} - \vec{c} \vec{c} - \vec{a}] = [\vec{a} \vec{b} - \vec{c} \vec{c} - \vec{a}] - [\vec{b} \vec{b} - \vec{c} \vec{c} - \vec{a}]$$

$$= [\vec{a} \vec{b} \vec{c} - \vec{a}] - [\vec{b} \vec{c} \vec{c} - \vec{a}] - [\vec{b} \vec{b} \vec{c} - \vec{a}] + [\vec{b} \vec{b} \vec{c} - \vec{a}]$$

$$= [\vec{a} \vec{b} \vec{c}] - [\vec{a} \vec{b} \vec{a}] - [\vec{b} \vec{c} \vec{c}] - [\vec{b} \vec{c} \vec{a}] - 0 + 0$$

$$= [\vec{a} \vec{b} \vec{c}] - 0 - 0 - [\vec{b} \vec{c} \vec{a}]$$

$$= [\vec{a} \vec{b} \vec{c}] - [\vec{a} \vec{b} \vec{c}]$$

$$= 0$$

3. Question

Mark the correct alternative in each of the following:

If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar mutually perpendicular unit vectors, then $[\vec{a} \vec{b} \vec{c}]$ is

A. ± 1

- B. 0
C. -2
D. 2

Answer

Here, $\vec{a} \perp \vec{b} \perp \vec{c}$ and $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$.

$$\Rightarrow \vec{a} \times \vec{b} \parallel \vec{c}$$

\Rightarrow angle between $\vec{a} \times \vec{b}$ and \vec{c} is $\theta = 0^\circ$ or $\theta = 180^\circ$.

$$[\vec{a} \vec{b} \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$= |\vec{a}| |\vec{b}| \sin\theta \hat{n} \cdot \vec{c}$$

$$= 1 \cdot 1 \cdot 1 \hat{n} \cdot \vec{c}$$

$$= |\hat{n}| |\vec{c}| \cos\theta$$

$$= 1 \cdot 1 \cos\theta$$

$$= \pm 1$$

4. Question

Mark the correct alternative in each of the following:

If $\vec{r} \cdot \vec{a} = \vec{r} \cdot \vec{b} = \vec{r} \cdot \vec{c} = 0$ for some non-zero vector \vec{r} , then the value of $[\vec{a} \vec{b} \vec{c}]$, is

- A. 2
B. 3
C. 0
D. none of these

Answer

$$\text{Here, } \vec{r} \cdot \vec{a} = \vec{r} \cdot \vec{b} = \vec{r} \cdot \vec{c} = 0$$

$$\Rightarrow \vec{r} \perp \vec{a}, \vec{r} \perp \vec{b}, \vec{r} \perp \vec{c}$$

$\Rightarrow \vec{a}, \vec{b}$ and \vec{c} are coplanar.

$$\Rightarrow [\vec{a} \vec{b} \vec{c}] = 0$$

5. Question

Mark the correct alternative in each of the following:

For any three vector $\vec{a}, \vec{b}, \vec{c}$ the expression $(\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\}$ equals

- A. $[\vec{a} \vec{b} \vec{c}]$
B. $2[\vec{a} \vec{b} \vec{c}]$
C. $[\vec{a} \vec{b} \vec{c}]^2$
D. none of these

Answer

$$\begin{aligned}
(\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\} &= [\vec{a} - \vec{b} \vec{b} - \vec{c} \vec{c} - \vec{a}] \\
&= [\vec{a} \vec{b} - \vec{c} \vec{c} - \vec{a}] - [\vec{b} \vec{b} - \vec{c} \vec{c} - \vec{a}] \\
&= [\vec{a} \vec{b} \vec{c} - \vec{a}] - [\vec{b} \vec{c} \vec{c} - \vec{a}] - [\vec{b} \vec{b} \vec{c} - \vec{a}] + [\vec{b} \vec{b} \vec{c} - \vec{a}] \\
&= [\vec{a} \vec{b} \vec{c}] - [\vec{a} \vec{b} \vec{a}] - [\vec{b} \vec{c} \vec{c}] - [\vec{b} \vec{c} \vec{a}] - 0 + 0 \\
&= [\vec{a} \vec{b} \vec{c}] - 0 - 0 - [\vec{b} \vec{c} \vec{a}] \\
&= [\vec{a} \vec{b} \vec{c}] - [\vec{a} \vec{b} \vec{c}] \\
&= 0
\end{aligned}$$

6. Question

Mark the correct alternative in each of the following:

If \vec{a} , \vec{b} , \vec{c} are non-coplanar vectors, then $\frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{(\vec{c} \times \vec{a}) \cdot \vec{b}} + \frac{\vec{b} \cdot (\vec{a} \times \vec{c})}{\vec{c} \cdot (\vec{a} \times \vec{b})}$ is

- A. 0
- B. 2
- C. 1
- D. none of these

Answer

$$\begin{aligned}
\frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{(\vec{c} \times \vec{a}) \cdot \vec{b}} + \frac{\vec{b} \cdot (\vec{a} \times \vec{c})}{\vec{c} \cdot (\vec{a} \times \vec{b})} &= \frac{[\vec{a} \vec{b} \vec{c}]}{[\vec{c} \vec{a} \vec{b}]} + \frac{[\vec{b} \vec{a} \vec{c}]}{[\vec{c} \vec{a} \vec{b}]} \\
&= \frac{[\vec{a} \vec{b} \vec{c}] + [\vec{b} \vec{a} \vec{c}]}{[\vec{c} \vec{a} \vec{b}]} \\
&= \frac{[\vec{a} \vec{b} \vec{c}] - [\vec{a} \vec{b} \vec{c}]}{[\vec{c} \vec{a} \vec{b}]} \\
&= 0
\end{aligned}$$

7. Question

Mark the correct alternative in each of the following:

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three non-zero vectors such that \vec{c} is

a unit vector perpendicular to both \vec{a} and \vec{b} . If the angle between \vec{a} and \vec{b} is $\frac{\pi}{6}$, then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to

- A. 0
- B. 1
- C. $(\frac{1}{4})|\vec{a}|^2|\vec{b}|^2$
- D. $(\frac{3}{4})|\vec{a}|^2|\vec{b}|^2$

Answer

$$\begin{aligned}
& \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^2 = [\vec{a} \vec{b} \vec{c}]^2 \\
& = [(\vec{a} \times \vec{b}) \cdot \vec{c}]^2 \\
& = [|\vec{a}| |\vec{b}| \sin\left(\frac{\pi}{6}\right) \cdot \vec{c}]^2 \\
& = |\vec{a}|^2 |\vec{b}|^2 \left(\frac{1}{4}\right) \cdot \vec{c}^2 \\
& = |\vec{a}|^2 |\vec{b}|^2 \left(\frac{1}{4}\right) \cdot |\vec{c}|^2 \cos 0 \quad (\because \vec{c} \text{ is perpendicular to } \vec{a} \text{ and } \vec{b} \Rightarrow \text{angle is } 0) \\
& = |\vec{a}|^2 |\vec{b}|^2 \left(\frac{1}{4}\right) \cdot |\vec{c}|^2 \\
& = |\vec{a}|^2 |\vec{b}|^2 \left(\frac{1}{4}\right) \quad (\because \vec{c} \text{ is unit vector}) \\
& = \left(\frac{1}{4}\right) |\vec{a}|^2 |\vec{b}|^2
\end{aligned}$$

8. Question

Mark the correct alternative in each of the following:

If $\vec{a} = 2\hat{i} - 3\hat{j} + 5\hat{k}$, $\vec{b} = 3\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 5\hat{i} - 3\hat{j} - 2\hat{k}$, then the volume of the parallelepiped with conterminous edges $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$, $\vec{c} + \vec{a}$ is

- A. 2
- B. 1
- C. -1
- D. 0

Answer

$$\text{Let } \vec{e} = \vec{a} + \vec{b} = 5\hat{i} - 7\hat{j} + 10\hat{k}$$

$$\vec{f} = \vec{b} + \vec{c} = 8\hat{i} - 7\hat{j} + 3\hat{k}$$

$$\vec{g} = \vec{c} + \vec{a} = 7\hat{i} - 6\hat{j} + 3\hat{k}$$

Now, the volume of the parallelepiped with conterminous edges \vec{e} , \vec{f} , \vec{g} is given by

$$\begin{aligned}
V &= [\vec{e} \vec{f} \vec{g}] \\
&= \begin{vmatrix} e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix} = \begin{vmatrix} 5 & -7 & 10 \\ 8 & -7 & 3 \\ 7 & -6 & 3 \end{vmatrix} \\
&= 5 \times (-21 + 18) + 7 \times (24 - 21) + 10 \times (-48 + 49) \times \\
&= 5 \times (-3) + 7 \times 3 + 10 \times 1 \\
&= -15 + 21 + 10 \\
&= 16
\end{aligned}$$

9. Question

Mark the correct alternative in each of the following:

If $[2\vec{a} + 4\vec{b} \vec{c} \vec{d}] = \lambda [\vec{a} \vec{c} \vec{d}] + \mu [\vec{b} \vec{c} \vec{d}]$ then $\lambda + \mu =$

- A. 6
- B. -6
- C. 10
- D. 8

Answer

$$\begin{aligned}\lambda[\vec{a} \times \vec{c} \vec{d}] + \mu[\vec{b} \times \vec{c} \vec{d}] &= [2\vec{a} + 4\vec{b} \times \vec{c} \vec{d}] \\ &= [2\vec{a} \times \vec{c} \vec{d}] + [4\vec{b} \times \vec{c} \vec{d}] \\ &= 2[\vec{a} \times \vec{c} \vec{d}] + 4[\vec{b} \times \vec{c} \vec{d}]\end{aligned}$$

Now, comparing the coefficient of lhs and rhs we get, $\lambda=2$ and $\mu=4$

$$\therefore \lambda + \mu = 2+4$$

$$=6$$

10. Question

Mark the correct alternative in each of the following:

$$[\vec{a} \times \vec{b} \times \vec{a} \times \vec{b}] + (\vec{a} \cdot \vec{b})^2$$

- A. $|\vec{a}|^2 |\vec{b}|^2$
- B. $|\vec{a} + \vec{b}|^2$
- C. $|\vec{a}|^2 + |\vec{b}|^2$
- D. $2|\vec{a}|^2 + |\vec{b}|^2$

Answer

$$\begin{aligned}[\vec{a} \times \vec{b} \times \vec{a} \times \vec{b}] + (\vec{a} \cdot \vec{b})^2 &= (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) + (\vec{a} \cdot \vec{b})^2 \\ &= (\vec{a} \times \vec{b})^2 + (\vec{a} \cdot \vec{b})^2 \\ &= |a|^2 |b|^2 \sin^2 \theta + |a|^2 |b|^2 \cos^2 \theta \\ &= |a|^2 |b|^2 (\sin^2 \theta + \cos^2 \theta) \\ &= |a|^2 |b|^2\end{aligned}$$

11. Question

Mark the correct alternative in each of the following:

If the vectors $4\hat{i} + 11\hat{j} + m\hat{k}$, $7\hat{i} + 2\hat{j} + 6\hat{k}$ and $\hat{i} + 5\hat{j} + 4\hat{k}$ are coplanar, then $m =$

- A. 0
- B. 38
- C. -10
- D. 10

Answer

$$\vec{a} = (4 \ 11 \ m)$$

$$\vec{b} = (7 \ 02 \ 6)$$

$$\vec{c} = (1 \ 05 \ 4)$$

Here, vector a, b, and c are coplanar. So, $[\vec{a} \ \vec{b} \ \vec{c}] = 0$.

$$\therefore \begin{vmatrix} 4 & 11 & m \\ 7 & 2 & 6 \\ 1 & 5 & 4 \end{vmatrix} = 0$$

$$\therefore 4(8-30)-11(28-6)+m(35-2)=0$$

$$\therefore 4(-22)-11(22)+33m=0$$

$$\therefore -88-242+33m=0$$

$$\therefore 33m=330$$

$$\therefore m=10$$

12. Question

Mark the correct alternative in each of the following:

For non-zero vectors \vec{a} , \vec{b} and \vec{c} the relation $|(\vec{a} \times \vec{b}) \cdot \vec{c}| = |\vec{a}| |\vec{b}| |\vec{c}|$ holds good, if

A. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = 0$

B. $\vec{a} \cdot \vec{b} = 0 = \vec{c} \cdot \vec{a}$

C. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$

D. $\vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$

Answer

Let $\vec{e} = \vec{a} \times \vec{b}$

$$|\vec{e}| = |\vec{a}| |\vec{b}| \sin \alpha \text{ -----(1) } (\because \alpha \text{ is angle between } \vec{a} \text{ and } \vec{b})$$

$$\text{Then } |(\vec{a} \times \vec{b}) \cdot \vec{c}| = |\vec{e} \cdot \vec{c}|$$

$$= |\vec{e}| |\vec{c}| \cos \theta \quad (\because \theta \text{ is angle between } \vec{e} \text{ and } \vec{c} \Rightarrow \theta \text{ is angle between } \vec{a} \times \vec{b} \text{ and } \vec{c})$$

$$= |\vec{a}| |\vec{b}| |\vec{c}| \cos \theta \sin \alpha \quad (\because \text{using (1)})$$

$$\text{Hence, } |(\vec{a} \times \vec{b}) \cdot \vec{c}| = |\vec{a}| |\vec{b}| |\vec{c}| \text{ if and only if } \cos \theta \sin \alpha = 1$$

$$\text{if and only if } \cos \theta = 1 \text{ and } \sin \alpha = 1$$

$$\text{if and only if } \theta = 0 \text{ and } \alpha = \frac{\pi}{2}$$

$$\alpha = \frac{\pi}{2} \Rightarrow \vec{a} \text{ and } \vec{b} \text{ are perpendicular.}$$

Also \vec{e} is perpendicular to both \vec{a} and \vec{b} .

$\theta=0 \Rightarrow \vec{c}$ is perpendicular to both \vec{a} and \vec{b}

$\therefore \vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular.

$$\therefore \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$$

13. Question

Mark the correct alternative in each of the following:

$$(\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c}) \times (\vec{a} + \vec{b} + \vec{c}) =$$

- A. 0
- B. $-\left[\vec{a} \vec{b} \vec{c}\right]$
- C. $2\left[\vec{a} \vec{b} \vec{c}\right]$
- D. $\left[\vec{a} \vec{b} \vec{c}\right]$

Answer

$$\begin{aligned} (\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c}) \times (\vec{a} + \vec{b} + \vec{c}) &= [(\vec{a} + \vec{b}) (\vec{b} + \vec{c}) (\vec{a} + \vec{b} + \vec{c})] \\ &= [\vec{a} \vec{b} + \vec{c} \vec{a} + \vec{b} + \vec{c}] + [\vec{b} \vec{b} + \vec{c} \vec{a} + \vec{b} + \vec{c}] \\ &= [\vec{a} \vec{b} \vec{a} + \vec{b} + \vec{c}] + [\vec{a} \vec{c} \vec{a} + \vec{b} + \vec{c}] + [\vec{b} \vec{b} \vec{a} + \vec{b} + \vec{c}] + [\vec{b} \vec{c} \vec{a} + \vec{b} + \vec{c}] \\ &= [\vec{a} \vec{b} \vec{c}] + [\vec{a} \vec{b} \vec{a}] + [\vec{a} \vec{b} \vec{b}] + [\vec{a} \vec{c} \vec{c}] + [\vec{a} \vec{c} \vec{b}] + [\vec{a} \vec{c} \vec{a}] + 0 + [\vec{b} \vec{c} \vec{a}] \\ &\quad + [\vec{b} \vec{c} \vec{b}] + [\vec{b} \vec{c} \vec{c}] \\ &= [\vec{a} \vec{b} \vec{c}] + 0 + 0 + 0 + [\vec{a} \vec{c} \vec{b}] + 0 + [\vec{b} \vec{c} \vec{a}] + 0 + 0 \\ &= [\vec{a} \vec{b} \vec{c}] + [\vec{a} \vec{c} \vec{b}] - [\vec{a} \vec{c} \vec{b}] \\ &= [\vec{a} \vec{b} \vec{c}] \end{aligned}$$

14. Question

Mark the correct alternative in each of the following:

If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors, then $(\vec{a} + \vec{b} + \vec{c}) \cdot [(\vec{a} + \vec{b}) \times (\vec{a} + \vec{c})]$ equal.

- A. 0
- B. $[\vec{a} \vec{b} \vec{c}]$
- C. $2[\vec{a} \vec{b} \vec{c}]$
- D. $-\left[\vec{a} \vec{b} \vec{c}\right]$

Answer

$$\begin{aligned} (\vec{a} + \vec{b} + \vec{c}) \cdot [(\vec{a} + \vec{b}) \times (\vec{a} + \vec{c})] &= [(\vec{a} + \vec{b} + \vec{c}) (\vec{a} + \vec{b}) (\vec{a} + \vec{c})] \\ &= [\vec{a} + \vec{b} + \vec{c} \vec{a} \vec{a} + \vec{c}] + [\vec{a} + \vec{b} + \vec{c} \vec{b} \vec{a} + \vec{c}] \\ &= [\vec{a} + \vec{b} + \vec{c} \vec{a} \vec{a}] + [\vec{a} + \vec{b} + \vec{c} \vec{a} \vec{c}] + [\vec{a} + \vec{b} + \vec{c} \vec{b} \vec{a}] + [\vec{a} + \vec{b} + \vec{c} \vec{b} \vec{c}] \\ &= [\vec{a} \vec{a} \vec{a}] + [\vec{b} \vec{a} \vec{a}] + [\vec{c} \vec{a} \vec{a}] + [\vec{a} \vec{a} \vec{c}] + [\vec{b} \vec{a} \vec{c}] + [\vec{c} \vec{a} \vec{c}] + [\vec{a} \vec{b} \vec{a}] + \quad + [\vec{c} \vec{b} \vec{c}] \\ &\quad + [\vec{b} \vec{b} \vec{a}] + [\vec{c} \vec{b} \vec{a}] + [\vec{a} \vec{b} \vec{c}] + [\vec{b} \vec{b} \vec{c}] \\ &= 0 + 0 + 0 + 0 + [\vec{b} \vec{a} \vec{c}] + 0 + 0 + 0 + [\vec{c} \vec{b} \vec{a}] + [\vec{a} \vec{b} \vec{c}] + 0 + 0 \\ &= 2[\vec{a} \vec{b} \vec{c}] \end{aligned}$$

15. Question

Mark the correct alternative in each of the following:

$(\vec{a} + 2\vec{b} - \vec{c}) \cdot \{(\vec{a} - \vec{b}) \times (\vec{a} - \vec{b} - \vec{c})\}$ is equal to

A. $[\vec{a} \vec{b} \vec{c}]$

B. $2[\vec{a} \vec{b} \vec{c}]$

C. $3[\vec{a} \vec{b} \vec{c}]$

D. 0

Answer

$$\begin{aligned}
 (\vec{a} + 2\vec{b} - \vec{c}) \cdot \{(\vec{a} - \vec{b}) \times (\vec{a} - \vec{b} - \vec{c})\} &= [\vec{a} + 2\vec{b} - \vec{c} \vec{a} - \vec{b} \vec{a} - \vec{b} - \vec{c}] \\
 &= [\vec{a} \vec{a} - \vec{b} \vec{a} - \vec{b} - \vec{c}] + [2\vec{b} \vec{a} - \vec{b} \vec{a} - \vec{b} - \vec{c}] - [\vec{c} \vec{a} - \vec{b} \vec{a} - \vec{b} - \vec{c}] \\
 &= [\vec{a} \vec{a} \vec{a} - \vec{b} - \vec{c}] - [\vec{a} \vec{b} \vec{a} - \vec{b} - \vec{c}] + [2\vec{b} \vec{a} \vec{a} - \vec{b} - \vec{c}] - [2\vec{b} \vec{b} \vec{a} - \vec{b} - \vec{c}] \\
 &\quad - [\vec{c} \vec{a} \vec{a} - \vec{b} - \vec{c}] + [\vec{c} \vec{b} \vec{a} - \vec{b} - \vec{c}] \\
 &= 0 - [\vec{a} \vec{b} \vec{a}] - [\vec{a} \vec{b} \vec{b}] - [\vec{a} \vec{b} \vec{c}] + [2\vec{b} \vec{a} \vec{a}] - [2\vec{b} \vec{a} \vec{b}] - [2\vec{b} \vec{a} \vec{c}] - [2\vec{b} \vec{b} \vec{a}] \\
 &\quad + [2\vec{b} \vec{b} \vec{b}] + [2\vec{b} \vec{b} \vec{c}] - [\vec{c} \vec{a} \vec{a}] + [\vec{c} \vec{a} \vec{b}] + [\vec{c} \vec{a} \vec{c}] + [\vec{c} \vec{b} \vec{a}] \\
 &\quad - [\vec{c} \vec{b} \vec{b}] - [\vec{c} \vec{b} \vec{c}] \\
 &= 0 - 0 - 0 - [\vec{a} \vec{b} \vec{c}] + 0 - 0 - 2[\vec{b} \vec{a} \vec{c}] - 0 + 0 + 0 - 0 + [\vec{c} \vec{a} \vec{b}] + 0 + [\vec{c} \vec{b} \vec{a}] \\
 &\quad - 0 - 0 \\
 &= -[\vec{a} \vec{b} \vec{c}] + 2[\vec{a} \vec{b} \vec{c}] + [\vec{a} \vec{b} \vec{c}] - [\vec{a} \vec{b} \vec{c}] \\
 &= [\vec{a} \vec{b} \vec{c}]
 \end{aligned}$$