Mathematics

TARGET : JEE (Main + Advanced)

NURTURE COURSE

Logarithm

CONTENTS

Logarithm

JEE (Advanced) Syllabus :

Logarithms and their properties.

LOGARITHM

1. DEFINITION :

Every positive real number N can be expressed in exponential form as $a^x = N$ where 'a' is also a positive real number different than unity and is called the base and 'x' is called an exponent.

We can write the relation $a^x = N$ in logarithmic form as $\log_a N = x$. Hence $a^x = N \Leftrightarrow \log_a N = x$.

Hence logarithm of a number to some base is the exponent by which the base must be raised in order to get that number.

Limitations of logarithm: $\log_a N$ is defined only when

(i) $N > 0$ (ii) $a > 0$ (iii) $a \neq 1$

Note :

(i) For a given value of N, $\log_a N$ will give us a unique value.

(ii) Logarithm of zero does not exist.

(iii) Logarithm of negative reals are not defined in the system of real numbers.

Illustration 1 : If $\log_4 m = 1.5$, then find the value of m.

Solution : $\log_4 m = 1.5 \Rightarrow m = 4^{3/2} \Rightarrow m = 8$

Illustration 2 : If $\log_5 p = a$ and $\log_2 q = a$, then prove that $\frac{p^4 q^4}{100} = 100^{2a-1}$

Solution : $\log_5 p = a \Rightarrow p = 5^a$

$\log_2 q = a \Rightarrow q = 2^a$

$$\Rightarrow \frac{p^4q^4}{100} = \frac{5^{4a}.2^{4a}}{100} = \frac{(10)^{4a}}{100} = \frac{(100)^{2a}}{100} = 100^{2a-1}$$

Illustration 3 : The value of N, satisfying $\log_a[1 + \log_b\{1 + \log_c(1 + \log_p N)\}] = 0$ is -

(A) 4 (B) 3 (C) 2 (D) 1

Solution : $1 + \log_b\{1 + \log_c(1 + \log_p N)\} = a^0 = 1$

$\Rightarrow \log_b\{1 + \log_c(1 + \log_p N)\} = 0 \quad \Rightarrow \quad 1 + \log_c(1 + \log_p N) = 1$

$\Rightarrow \log_c(1 + \log_p N) = 0 \quad \Rightarrow \quad 1 + \log_p N = 1$

$\Rightarrow \log_p N = 0 \quad \Rightarrow \quad N = 1$ **Ans. (D)**

Do yourself - 1 :

(i) Express the following in logarithmic form :

(a) $81 = 3^4$ (b) $0.001 = 10^{-3}$ (c) $2 = 128^{1/7}$

(ii) Express the following in exponential form :

(a) $\log_2 32 = 5$ (b) $\log_{\sqrt{2}} 4 = 4$ (c) $\log_{10} 0.01 = -2$

(iii) If $\log_{2\sqrt{3}} 1728 = x$, then find x.

2. FUNDAMENTAL IDENTITIES :

Using the basic definition of logarithm we have 3 important deductions :

(a) $\log_a 1 = 0$ i.e. logarithm of unity to any base is zero.

(b) $\log_N N = 1$ i.e. logarithm of a number to the same base is 1.

(c) $\log_{\frac{1}{N}} N = -1 = \log_N \frac{1}{N}$ i.e. logarithm of a number to the base as its reciprocal is –1.

Note : $N = (a)^{\log_a N}$ e.g. $2^{\log_2 7} = 7$

Do yourself - 2 :

(i) Find the value of the following :

(a) $\log_{1.4\bar{3}} \frac{43}{30}$ **(b)** $\left(\frac{1}{2}\right)^{\log_2 5}$

(ii) If $4^{\log_2 2x} = 36$, then find x.

3. THE PRINCIPAL PROPERTIES OF LOGARITHMS :

If m,n are arbitrary positive numbers where a > 0, a ≠ 1 and x is any real number, then-

(a) $\log_a mn = \log_a m + \log_a n$

(b) $\log_a \frac{m}{n} = \log_a m - \log_a n$

(c) $\log_a m^x = x \log_a m$

Illustration 4 : Find the value of $2\log\frac{2}{5} + 3\log\frac{25}{8} - \log\frac{625}{128}$

Solution :

$$2\log\frac{2}{5} + 3\log\frac{25}{8} + \log\frac{128}{625}$$

$$= \log\frac{2^2}{5^2} + \log\left(\frac{5^2}{2^3}\right)^3 + \log\frac{2^7}{5^4}$$

$$= \log\frac{2^2}{5^2}.\frac{5^6}{2^9}.\frac{2^7}{5^4} = \log 1 = 0$$

Illustration 5 : If $\log_e x - \log_e y = a$, $\log_e y - \log_e z = b$ & $\log_e z - \log_e x = c$, then find the value of $\left(\frac{x}{y}\right)^{b-c} \times \left(\frac{y}{z}\right)^{c-a} \times \left(\frac{z}{x}\right)^{a-b}$

Solution :

$$\log_e x - \log_e y = a \Rightarrow \log_e \frac{x}{y} = a \Rightarrow \frac{x}{y} = e^a$$

$$\log_e y - \log_e z = b \Rightarrow \log_e \frac{y}{z} = b \Rightarrow \frac{y}{z} = e^b$$

$$\log_e z - \log_e x = c \Rightarrow \log_e \frac{z}{x} = c \Rightarrow \frac{z}{x} = e^c$$

$$\therefore \left(e^a\right)^{b-c} \times \left(e^b\right)^{c-a} \times \left(e^c\right)^{a-b}$$

$$= e^{a(b-c)+b(c-a)+c(a-b)} = e^0 = 1$$

Illustration 6 : If $a^2 + b^2 = 23ab$, then prove that $\log\frac{(a+b)}{5} = \frac{1}{2}(\log a + \log b)$.

Solution : $a^2 + b^2 = (a + b)^2 - 2ab = 23ab$

$\Rightarrow (a + b)^2 = 25ab \Rightarrow a+b = 5\sqrt{ab}$ ....(i)

Using (i)

$$\text{L.H.S.} = \log\frac{(a+b)}{5} = \log\frac{5\sqrt{ab}}{5} = \frac{1}{2}\log ab = \frac{1}{2}(\log a + \log b) = \text{R.H.S.}$$

Illustration 7 : If $\log_a x = p$ and $\log_b x^2 = q$, then $\log_x \sqrt{ab}$ is equal to (where $a, b, x \in R^+ - \{1\}$)-

(A) $\frac{1}{p}+\frac{1}{q}$ (B) $\frac{1}{2p}+\frac{1}{q}$ (C) $\frac{1}{p}+\frac{1}{2q}$ (D) $\frac{1}{2p}+\frac{1}{2q}$

Solution : $\log_a x = p \Rightarrow a^p = x \Rightarrow a = x^{1/p}$.

similarly $b^q = x^2 \Rightarrow b = x^{2/q}$

$$\text{Now, } \log_x \sqrt{ab} = \log_x \sqrt{x^{1/p}x^{2/q}} = \log_x x^{\left(\frac{1}{p}+\frac{2}{q}\right)\cdot\frac{1}{2}} = \frac{1}{2p}+\frac{1}{q}$$

Do yourself - 3 :

(i) Show that $\frac{1}{2}\log 9 + 2\log 6 + \frac{1}{4}\log 81 - \log 12 = 3\log 3$

4. BASE CHANGING THEOREM :

Can be stated as "quotient of the logarithm of two numbers is independent of their common base."

Symbolically, $\log_b m = \frac{\log_a m}{\log_a b}$, where $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$

Note :

(i) $\log_b a.\ \log_a b = \frac{\log a}{\log b}.\frac{\log b}{\log a} = 1$; hence $\boxed{\log_b a = \frac{1}{\log_a b}}$.

(ii) $a^{\log_b c} = c^{\log_b a}$

(iii) **Base power formula :** $\log_{a^k} m = \frac{1}{k}\log_a m$

(iv) The base of the logarithm can be any positive number other than 1, but in normal practice, only two bases are popular, these are 10 and e(=2.718 approx). Logarithms of numbers to the base 10 are named as 'common logarithm' and the logarithms of numbers to the base e are called Natural or Napierian logarithm. **We will consider logx as $\log_e x$ or $\ell n x$.**

(v) Conversion of base e to base 10 & viceversa :

$$\log_e a = \frac{\log_{10} a}{\log_{10} e} = 2.303 \times \log_{10} a\ ;\quad \log_{10} a = \frac{\log_e a}{\log_e 10} = \log_{10} e \times \log_e a = 0.434\log_e a$$

Illustration 8 : If a, b, c are distinct positive real numbers different from 1 such that

$(\log_b a \,.\, \log_c a - \log_a a) + (\log_a b \,.\, \log_c b - \log_b b) + (\log_a c \,.\, \log_b c - \log_c c) = 0$, then abc is equal to -

(A) 0 (B) e (C) 1 (D) none of these

Solution : $(\log_b a \log_c a - 1) + (\log_a b \,.\, \log_c b - 1) + (\log_a c \log_b c - 1) = 0$

$$\Rightarrow \frac{\log a}{\log b} \cdot \frac{\log a}{\log c} + \frac{\log b}{\log a} \cdot \frac{\log b}{\log c} + \frac{\log c}{\log a} \cdot \frac{\log c}{\log b} = 3$$

$\Rightarrow (\log a)^3 + (\log b)^3 + (\log c)^3 = 3\log a \log b \log c$

$\Rightarrow (\log a + \log b + \log c) = 0 \quad [\because$ If $a^3 + b^3 + c^3 - 3abc = 0$, then $a + b + c = 0$ if $a \neq b \neq c]$

$\Rightarrow \log abc = \log 1 \Rightarrow abc = 1$

Illustration 9 : Evaluate : $81^{1/\log_5 3} + 27^{\log_9 36} + 3^{4/\log_7 9}$

Solution : $81^{\log_3 5} + 3^{3\log_9 36} + 3^{4\log_9 7}$

$= 3^{4\log_3 5} + 3^{\log_3 (36)^{3/2}} + 3^{\log_3 7^2}$

$= 625 + 216 + 49 = 890.$

Do yourself - 4 :

(i) Evaluate : $\dfrac{\log_3 135}{\log_{15} 3} - \dfrac{\log_3 5}{\log_{405} 3}$

(ii) Evaluate : $\log_9 27 - \log_{27} 9$

(iii) Evaluate : $2^{\log_3 5} - 5^{\log_3 2}$

(iv) Evaluate : $\log_3 4 \,.\, \log_4 5 \,.\, \log_5 6 \,.\, \log_6 7 \,.\, \log_7 8 \,.\, \log_8 9$

(v) If $\dfrac{1}{\log_3 \pi} + \dfrac{1}{\log_4 \pi} > x$, then x can be -

(A) 2 (B) 3 (C) 3.5 (D) π

(vi) If $\log_a 3 = 2$ and $\log_b 8 = 3$, then $\log_a b$ is -

(A) $\log_3 2$ (B) $\log_2 3$ (C) $\log_3 4$ (D) $\log_4 3$

5. POINTS TO REMEMBER :

(i) If base of logarithm is greater than 1 then logarithm of greater number is greater. i.e. $\log_2 8 = 3$, $\log_2 4 = 2$ etc. and if base of logarithm is between 0 and 1 then logarithm of greater number is smaller. i.e. $\log_{1/2} 8 = -3$, $\log_{1/2} 4 = -2$ etc.

$$\log_a x < \log_a y \Leftrightarrow \begin{bmatrix} x < y & \text{if} & a > 1 \\ x > y & \text{if} & 0 < a < 1 \end{bmatrix}$$

It must be noted that whenever the number and the base are on the same side of unity then logarithm of that number to that base is positive, however if the number and the base are located on different side of unity then logarithm of that number to that base is negative.

e.g. $\log_{10}\sqrt[3]{10} = \frac{1}{3}$; $\log_{\sqrt{7}} 49 = 4$; $\log_{\frac{1}{2}}\left(\frac{1}{8}\right) = 3$; $\log_2\left(\frac{1}{32}\right) = -5$; $\log_{10}(0.001) = -3$

(iii) $x + \frac{1}{x} \geq 2$ if x is positive real number and $x + \frac{1}{x} \leq -2$ if x is negative real number

(iv) $n \geq 2,\ n \in N$

$\sqrt[n]{a} = a^{1/n} \Rightarrow$ n^{th} root of 'a' ('a' is a non negative number)

Some important values : $\log_{10}2 \approx 0.3010$; $\log_{10}3 \approx 0.4771$; $\ell n2 \approx 0.693$, $\ell n10 \approx 2.303$

6. CHARACTERISTIC AND MANTISSA :

For any given number N, logarithm can be expressed as $\log_a N$ = Integer + Fraction

The integer part is called characteristic and the fractional part is called mantissa. When the value of log n is given, then to find digits of 'n' we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part (if $n \geq 1$) or the number of zeros after decimal & before first non-zero digit in the number (if $0 < n < 1$).

Note :

(i) The mantissa part of logarithm of a number is always non-negative $(0 \leq m < 1)$

(ii) If the characteristic of $\log_{10}N$ be n, then the number of digits in N is (n + 1)

(iii) If the characteristic of $\log_{10}N$ be (–n), then there exist (n – 1) zeros after decimal in N.

7. ANTILOGARITHM :

The positive real number 'n' is called the antilogarithm of a number 'm' if log n = m

Thus, **log n = m ⇔ n = antilog m**

Do yourself - 5 :

(i) Evaluate : $\log_{10}(0.06)^6$

(ii) Find number of digits in 18^{20}

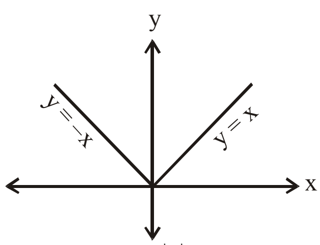
(iii) Determine number of cyphers (zeros) between decimal & first significant digit in $\left(\frac{1}{6}\right)^{200}$

(iv) Find antilog of $\frac{5}{6}$ to the base 64.

8. ABSOLUTE VALUE FUNCTION/MODULUS FUNCTION :

The symbol of modulus function is |x|

and is defined as : $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



y = |x|

Properties of Modulus :

For any a, b $\in$ R

(a) $|a| \geq 0$

(b) $|a| = |-a|$

(c) $|ab| = |a||b|$

(d) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

(e) $\sqrt{a^2} = |a|$

Illustration 10 : If $||x-1| - 2| = 5$, then find x.

Solution : $|x - 1| - 2 = \pm 5$

$|x - 1| = 7, -3$

Case-I : When $|x - 1| = 7 \Rightarrow x - 1 = \pm 7 \Rightarrow x = 8, -6$

Case-II : When $|x - 1| = -3$ (reject)

Illustration 11 : If $|x - 1| + |x + 1| = 2$, then find x.

Solution : **Case-I :** If $x \leq -1$

$-(x - 1) - (x + 1) = 2$

$\Rightarrow -x + 1 - x - 1 = 2$

$\Rightarrow -2x = 2 \Rightarrow x = -1$ ....(i)

Case-II : If $-1 < x < 1$

$-(x - 1) + (x + 1) = 2$

$\Rightarrow -x + 1 + x + 1 = 2$

$\Rightarrow 2 = 2 \Rightarrow -1 < x < 1$ ....(ii)

Case-III : If $x \geq 1$

$x - 1 + x + 1 = 2$

$\Rightarrow x = 1$ ....(iii)

Thus from (i), (ii) and (iii) $-1 \leq x \leq 1$

Do yourself - 6 :

(i) Solve : $|x + 3| = 2(5 - x)$

(ii) Solve : $x|x| + 7x - 8 = 0$

Miscellaneous Illustrations :

Illustration 12 : Show that $\log_4 18$ is an irrational number.

Solution : $\log_4 18 = \log_4(3^2 \times 2) = 2\log_4 3 + \log_4 2 = 2\dfrac{\log_2 3}{\log_2 4} + \dfrac{1}{\log_2 4} = \log_2 3 + \dfrac{1}{2}$

assume the contrary, that this number $\log_2 3$ is rational number.

$\Rightarrow \log_2 3 = \dfrac{p}{q}$. Since $\log_2 3 > 0$ both numbers p and q may be regarded as natural number

$\Rightarrow 3 = 2^{p/q} \Rightarrow 2^p = 3^q$

But this is not possible for any natural number p and q. The resulting contradiction completes the proof.

Illustration 13 : If in a right angled triangle, a and b are the lengths of sides and c is the length of hypotenuse and $c - b \neq 1$, $c + b \neq 1$, then show that

$\log_{c+b} a + \log_{c-b} a = 2\log_{c+b} a \,.\, \log_{c-b} a$.

Solution : We know that in a right angled triangle

$c^2 = a^2 + b^2$

$c^2 - b^2 = a^2$ .... (i)

$$\text{LHS} = \frac{1}{\log_a(c+b)} + \frac{1}{\log_a(c-b)} = \frac{\log_a(c-b) + \log_a(c+b)}{\log_a(c+b).\log_a(c-b)}$$

$$= \frac{\log_a(c^2-b^2)}{\log_a(c+b).\log_a(c-b)} = \frac{\log_a a^2}{\log_a(c+b).\log_a(c-b)} \quad \text{(using (i))}$$

$$= \frac{2}{\log_a(c+b).\log_a(c-b)} = 2\log_{(c+b)} a \,.\, \log_{(c-b)} a = \text{RHS}$$

ANSWERS FOR DO YOURSELF

1 :	**(i)** **(a)** $\log_3 81 = 4$	**(b)** $\log_{10}(0.001) = -3$		**(c)** $\log_{128} 2 = 1/7$		
	(ii) **(a)** $32 = 2^5$	**(b)** $4 = (\sqrt{2})^4$		**(c)** $0.01 = 10^{-2}$		
	(iii) 6					
2 :	**(i)** **(a)** 1	**(b)** $\dfrac{1}{5}$	**(ii)** 3			
4 :	**(i)** 3	**(ii)** 5/6	**(iii)** 0	**(iv)** 2	**(v)** (A)	**(vi)** (C)
5 :	**(i)** $\bar{8}.6686$	**(ii)** 26	**(iii)** 155	**(iv)** 32		
6 :	**(i)** $\dfrac{7}{3}$	**(ii)** x = 1				

ELEMENTARY EXERCISE

Solve for x ∈ R (Q.1 to Q.7)

1. $4^x - 10.2^{x-1} = 24$

2. $4.2^{2x} - 6^x = 18.3^{2x}$

3. $3^{2x-3} - 9^{x-1} + 27^{2x/3} = 675.$

4. $7^{x+2} - \frac{1}{7}.7^{x+1} - 14.7^{x-1} + 2.7^x = 48$

5. $\left(\frac{5}{3}\right)^{x+1} \cdot \left(\frac{9}{25}\right)^{x^2+2x-11} = \left(\frac{5}{3}\right)^9$

6. $\left(3^{x^2-7.2x+3.9} - 9\sqrt{3}\right)\log(7-x) = 0$

7. $5^{2x} = 3^{2x} + 2.5^x + 2.3^x$

8. Find the logarithms of the following numbers to the base 2.

(i) 1 (ii) 2 (iii) 4 (iv) 8 (v) $\frac{1}{2}$

(vi) $\frac{1}{32}$ (vii) $\frac{1}{16}$ (viii) $\sqrt{2}$ (ix) $\sqrt[3]{8}$ (x) $2\sqrt{2}$

(xi) $\frac{1}{\sqrt[5]{2}}$ (xii) $\frac{1}{\sqrt[7]{8}}$

9. Find the logarithms of the following numbers to the base $\frac{1}{3}$.

(i) 1 (ii) $\frac{1}{3}$ (iii) $\frac{1}{9}$ (iv) 3 (v) 9

(vi) 81 (vii) $\sqrt[3]{3}$ (viii) $\frac{1}{\sqrt[7]{3}}$ (ix) $9\sqrt{3}$ (x) $\frac{1}{9\sqrt[4]{3}}$

10. Find all values of 'a' for which each of the following equalities hold true.

(i) $\log_2 a = 2$ (ii) $\log_a 2 = 1$ (iii) $\log_a 1 = 0$

(iv) $\log_{10}(a(a+3)) = 1$ (v) $\log_{1/3}(a^2 - 1) = -1$ (vi) $\log_2(a^2 - 5) = 2$

11. Find the logarithms of the following numbers to the base 3.

(i) 1 (ii) 3 (iii) 9 (iv) 81 (v) $\frac{1}{3}$

(vi) $\sqrt{3}$ (vii) $\frac{1}{3\sqrt{3}}$ (viii) $27\sqrt{3}$ (ix) $\sqrt[7]{9}$

12. Find the logarithms of the following numbers to the base $\frac{1}{2}$.

(i) 1 (ii) $\frac{1}{2}$ (iii) $\frac{1}{8}$ (iv) 16

(v) $\sqrt{2}$ (vi) $\frac{1}{\sqrt{2}}$ (vii) $2\sqrt{2}$ (viii) $\frac{1}{4\sqrt[4]{2}}$

13. Find all values of 'a' for which the following equalities hold true.

(i) $\log_3 a = 2$ (ii) $\log_{1/3}(a) = 4$ (iii) $\log_{1/3}(a) = 0$

(iv) $\log_a 1 = 0$ (v) $\log_a(a+2) = 2$ (vi) $\log_3(a^2+1) = 1$

14. Find all values of x for which the following equalities hold true.

(i) $\log_2 x^2 = 1$ (ii) $\log_3 x = \log_3(2 - x)$ (iii) $\log_4 x^2 = \log_4 x$

(iv) $\log_{1/2}(2x + 1) = \log_{1/2}(x + 1)$ (v) $\log_{1/3}(x^2 + 8) = -2$

15. Find all the values of x for which the following equalities hold true.

(i) $\log_2 x^2 = 2$ (ii) $\log_{1/4} x^2 = 1$

(iii) $\log_{1/2} x - \log_{1/2}(3 - x) = 0$ (iv) $\log_2(x + 1) - \log_2(2x - 3) = 0$

16. Find the logarithms of the following numbers to the base 5.

(i) 1 (ii) 5 (iii) 25 (iv) 625 (v) $\frac{1}{5}$ (vi) $\frac{1}{25}$

(vii) $\frac{1}{\sqrt{5}}$ (viii) $\sqrt{\sqrt{5}}$ (ix) $5^{\frac{1}{2}}$ (x) $5^{\frac{1}{3}}$ (xi) $\sqrt[4]{5\sqrt[3]{5}}$

17. Find value of following

(i) $\log_{\sqrt{3}-\sqrt{2}}\left(\sqrt{5-2\sqrt{6}}\right)$ (ii) $\log_{\sqrt{5}+1}(6+2\sqrt{5})$

(iii) $\log_7 3.\log_5 2.\log_3 7.\log_2(125)$ (iv) $25^{\log_5 3}$

(v) $6^{\log_6 5} + 3^{\log_9 16}$ (vi) $\log_6 4 + \frac{1}{\log_9 6}$

18. Find the value of following

(i) $\log_5\left(\frac{1}{\sqrt{5}}\right)$ (ii) $\log_3\left(4\sin^2(x) + 4\cos^2(x) - 1\right)$

(iii) $\log_{12} 8 + \log_{12} 3 + \log_{12} 6$ (iv) $\log_5 \frac{500}{3} - \log_5 \frac{4}{3}$

(v) $\log_{39}\frac{15}{7} + \log_{39}\frac{13}{3} - \log_{39}\frac{5}{21}$ (vi) $2\log_6 2 + 3\log_6 3 + \log_6 12$

19. If $2\left(\sqrt{3+\sqrt{5-\sqrt{13+\sqrt{48}}}}\right) = \sqrt{a} + \sqrt{b}$ where a and b are natural number find (a + b).

EXERCISE (O-1)

1. If $2^a = 3$ and $9^b = 4$ then value of (ab) is-

(A) 1 (B) 2 (C) 3 (D) 4

2. If $\log_2\left(4 + \log_3(x)\right) = 3$, then sum of digits of x is -

(A) 3 (B) 6 (C) 9 (D) 18

3. Sum of all the solution(s) of the equation $\log_{10}(x) + \log_{10}(x + 2) - \log_{10}(5x + 4) = 0$ is-

(A) –1 (B) 3 (C) 4 (D) 5

4. The product of all the solutions of the equation $x^{1+\log_{10} x} = 100000x$ is-

(A) 10 (B) 10^5 (C) 10^{-5} (D) 1

5. If x_1 and x_2 are the roots of equation $e^{3/2}.\ x^{2\ell nx} = x^4$, then the product of the roots of the equation is -

(A) e^2 (B) e (C) $e^{3/2}$ (D) e^{-2}

6. If $\log_2(x^2+1) + \log_{13}(x^2+1) = \log_2(x^2+1)\log_{13}(x^2+1)$, $(x \neq 0)$, then $\log_7(x^2+24)$ is equal to
(A) 1 (B) 2 (C) 3 (D) 4

7. Given $\log_3 a = p = \log_b c$ and $\log_b 9 = \frac{2}{p^2}$. If $\log_9\left(\frac{a^4b^3}{c}\right) = \alpha p^3 + \beta p^2 + \gamma p + \delta$ $(\forall\ p \in R - \{0\})$, then $(\alpha+\beta+\gamma+\delta)$ equals
(A) 1 (B) 2 (C) 3 (D) 4

8. If $\log_a\left(1-\sqrt{1+x}\right) = \log_{a^2}\left(3-\sqrt{1+x}\right)$, then number of solutions of the equation is-
(A) 0 (B) 1 (C) 2 (D) infinitely many

9. The number of solution(s) of $\sqrt{\log_3(3x^2).\log_9(81x)} = \log_9 x^3$ is-
(A) 0 (B) 1 (C) 2 (D) 3

10. If x_1 & x_2 are the two values of x satisfying the equation $7^{2x^2} - 2\left(7^{x^2+x+12}\right) + 7^{2x+24} = 0$, then $(x_1 + x_2)$ equals-
(A) 0 (B) 1 (C) –1 (D) 7

11. If $x, y \in 2^n$ when $n \in I$ and $1 + \log_x y = \log_2 y$, then the value of $(x + y)$ is
(A) 2 (B) 4 (C) 6 (D) 8

12. If $n \in N$ such that characteristic of n^2 to the base 8 is 2, then number of possible values of n is-
(A) 14 (B) 15 (C) 448 (D) infinite

EXERCISE (O-2)

1. If $x = \log_2\left(\sqrt{56+\sqrt{56+\sqrt{56+\sqrt{56+.......\infty}}}}\right)$, then which of the following statements holds good ?
(A) $x < 0$ (B) $0 < x < 2$ (C) $2 < x < 4$ (D) $3 < x < 4$

2. The greatest value of $(4\log_{10}x - \log_x(.0001))$ for $0 < x < 1$ is-
(A) 4 (B) –4 (C) 8 (D) –8

3. The number of integral solutions of $|\log_5 x^2 - 4| = 2 + |\log_5 x - 3|$ is-
(A) 1 (B) 2 (C) 3 (D) 0

4. If $60^a = 3$ and $60^b = 5$ then the value of $12^{\frac{1-a-b}{2(1-b)}}$ equals
(A) 2 (B) 3 (C) $\sqrt{3}$ (D) $\sqrt{12}$

5. Let ABC be a triangle right angled at C. The value of $\frac{\log_{b+c}a + \log_{c-b}a}{\log_{b+c}a \cdot \log_{c-b}a}$ $(b+c \neq 1, c-b \neq 1)$ equals
(A) 1 (B) 2 (C) 3 (D) 1/2

6. If α and β are the roots of the equation $(\log_2 x)^2 + 4(\log_2 x) - 1 = 0$ then the value of $\log_\beta\alpha + \log_\alpha\beta$ equals
(A) 18 (B) – 16 (C) 14 (D) – 18

7. The number of solution of the equation $e^{2x} + e^x + e^{-2x} + e^{-x} = 3(e^{-2x} + e^x)$ is
(A) 0 (B) 2 (C) 1 (D) more than 2

8. If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$, then x lies in the interval
(A) $(2, \infty)$ (B) $(1, 2)$ (C) $(1, \infty)$ (D) none of these

[MULTIPLE CORRECT CHOICE TYPE]

9. If $\left(\log_{\beta}\alpha\right)^2+\left(\log_{\alpha}\beta\right)^2=79$, ($\alpha > 0$, $\beta > 0$, $\alpha \neq 1$, $\beta \neq 1$) then value of $(\log_{\beta}\alpha) + (\log_{\alpha}\beta)$ can be-

(A) 7 (B) –9 (C) 9 (D) –7

10. Which of the following statements is(are) correct ?

(A) $7^{1/7} > (42)^{1/14} > 1$

(B) $\log_3(5)\log_7(9)\log_{11}(13) > -2$

(C) $\sqrt{99+70\sqrt{2}}+\sqrt{99-70\sqrt{2}}$ is rational

(D) $\dfrac{1}{\log_4 3}+\dfrac{1}{\log_7 3} > 3$

EXERCISE (S-1)

1. Let A denotes the value of $\log_{10}\left(\dfrac{ab+\sqrt{(ab)^2-4(a+b)}}{2}\right)+\log_{10}\left(\dfrac{ab-\sqrt{(ab)^2-4(a+b)}}{2}\right)$

when a = 43 and b = 57 and B denotes the value of the expression $\left(2^{\log_6 18}\right).\left(3^{\log_6 3}\right)$.

Find the value of (A.B).

2. Compute the following : (a) $\log_{1/3}\sqrt[4]{729.\sqrt[3]{9^{-1}}.27^{-4/3}}$ (b) $a^{\frac{\log_b(\log_b N)}{\log_b a}}$

3. Find the square of the sum of the roots of the equation

$\log_3 x.\log_4 x.\log_5 x = \log_3 x.\log_4 x + \log_4 x.\log_5 x + \log_5 x.\log_3 x.$

4. Calculate : $4^{5\log_{4\sqrt{2}}\left(3-\sqrt{6}\right)-6\log_8\left(\sqrt{3}-\sqrt{2}\right)}$

5. Simplify : $\dfrac{81^{\frac{1}{\log_5 9}}+3^{\frac{3}{\log_{\sqrt{6}} 3}}}{409}.\left(\left(\sqrt{7}\right)^{\frac{2}{\log_{25} 7}}-(125)^{\log_{25} 6}\right)$

6. Simplify : $5^{\log_{1/5}\left(\frac{1}{2}\right)}+\log_{\sqrt{2}}\dfrac{4}{\sqrt{7}+\sqrt{3}}+\log_{1/2}\dfrac{1}{10+2\sqrt{21}}$

7. Given that $\log_2 a = s$, $\log_4 b = s^2$ and $\log_{c^2}(8)=\dfrac{2}{s^3+1}$. Write $\log_2\dfrac{a^2b^5}{c^4}$ as a function of 's'

(a,b,c>0, c≠1).

8. Find the value of $49^{(1-\log_7 2)}+5^{-\log_5 4}$.

9. Prove that $\dfrac{\log_2 24}{\log_{96} 2}-\dfrac{\log_2 192}{\log_{12} 2}=3$.

10. Prove that $a^x - b^y = 0$ where $x=\sqrt{\log_a b}$ & $y=\sqrt{\log_b a}$, a >0 , b > 0 & a , b ≠ 1.

11. Solve the following equations :

i. $\log_{x-1}3 = 2$ **ii.** $\log_4(2\log_3(1+\log_2(1+3\log_3 x))) = \frac{1}{2}$

iii. $\log_3(1 + \log_3(2^x - 7)) = 1$ **iv.** $\log_3(3^x - 8) = 2 - x$

v. $\frac{\log_2(9-2^x)}{3-x} = 1$ **vi.** $\log_{5-x}(x^2 - 2x + 65) = 2$

vii. $\log_{10}5 + \log_{10}(x+10) - 1 = \log_{10}(21x-20) - \log_{10}(2x-1)$

viii. $x^{1+\log_{10}x} = 10x$ **ix.** $2(\log_x\sqrt{5})^2 - 3\log_x\sqrt{5} + 1 = 0$

x. $3 + 2\log_{x+1}3 = 2\log_3(x+1)$

12. Solve the following equations :

i. $\log_{1/2}^2(4x) + \log_2\left(\frac{x^2}{8}\right) = 8$ **ii.** $\log_{0.5x}x^2 - 14\log_{16x}x^3 + 40\log_{4x}\sqrt{x} = 0$

iii. $\log_3(4.3^x - 1) = 2x + 1$ **iv.** $\log_5\left(\frac{2+x}{10}\right) = \log_5\left(\frac{2}{x+1}\right)$

v. $1 + 2\log_{(x+2)}5 = \log_5(x+2)$ **vi.** $\log_4 2^{4x} = 2^{\log_2 4}$

vii. $\log_2(4.3^x - 6) - \log_2(9^x - 6) = 1$ **viii.** $2\log_8(2x) + \log_8(x^2+1-2x) = \frac{4}{3}$

ix. $\log_3^2 6 - \log_3^2 2 = (\log_{10}^2 x - 2)\log_3 12$ **x.** $\log_6 2^{x+3} - \log_6(3^x - 2) = x$

EXERCISE (S-2)

1. Let a and b be real numbers greater than 1 for which there exists a positive real number c, different from 1, such that $2(\log_a c + \log_b c) = 9\log_{ab}c$. Find the largest possible value of $\log_a b$.

2. Find the value of the expression $\frac{2}{\log_4(2000)^6} + \frac{3}{\log_5(2000)^6}$

3. Given that $\log_2 3 = a$, $\log_3 5 = b$ and $\log_7 2 = c$, express the logarithm of the number 63 to the base 140 in terms of a,b & c.

4. If a,b and c are positive real numbers such that $a^{\log_3 7} = 27; b^{\log_7 11} = 49$ and $c^{\log_{11}25} = \sqrt{11}$. Find the value of $\left(a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11}25)^2}\right)$.

5. If 'x' and 'y' are real numbers such that $2\log(2y - 3x) = \log x + \log y$, find $\frac{x}{y}$.

6. The real x and y satisfy $\log_8 x + \log_4 y^2 = 5$ and $\log_8 y + \log_4 x^2 = 7$, find xy.

7. If $a = \log_{12}18$ & $b = \log_{24}54$ then find the value of $ab + 5(a - b)$.

8. Solve following equations

(i) $|x-1| = 5$ (ii) $|x+1| = |2x-1|$

(iii) $|x-1| + 2|x+1| = 10$ (iv) $|2x-1| - |x-3| = 7$

(v) $x + |3x-2| = 2$ (vi) $|\log_2 x| = 3$

9. Solve the system of equations :

$\log_a x \log_a(xyz) = 48$

$\log_a y \log_a(xyz) = 12$, $a > 0$, $a \neq 1$

$\log_a z \log_a(xyz) = 84$

10. Let $y = \sqrt{\log_2 3.\log_2 12.\log_2 48.\log_2 192 + 16} - \log_2 12.\log_2 48 + 10$. Find $y \in N$.

11. Let **'L'** denotes the antilog of 0.4 to the base 1024.

and **'M'** denotes the number of digits in 6^{10} (Given $\log_{10}2 = 0.3010$, $\log_{10}3 = 0.4771$)

and **'N'** denotes the number of positive integers which have the characteristic 2, when base of the logarithm is 6.

Find the value of LMN.

12. Find the product of the positive roots of the equation $\sqrt{(2017)}(x)^{\log_{2017} x} = x^2$.

13. If (x_1,y_1) and (x_2,y_2) are the solution of the system of equation.

$\log_{225}(x) + \log_{64}(y) = 4$

$\log_x(225) - \log_y(64) = 1$,

then show that the value of $\log_{30}(x_1y_1x_2y_2) = 12$.

14. (a) Given : $\log_{10}34.56 = 1.5386$, find $\log_{10}3.456$; $\log_{10}0.3456$ & $\log_{10}0.003456$.

(b) Find the number of positive integers which have the characteristic 3, when the base of the logarithm is 7.

(c) If $\log_{10}2 = 0.3010$ & $\log_{10}3 = 0.4771$, find the value of $\log_{10}(2.25)$

(d) Find the antilogarithm of 0.75, if the base of the logarithm is 2401.

15. (a) If $x,y > 0$, $\log_y x + \log_x y = \frac{10}{3}$ and $xy = 144$, then $\frac{x+y}{2} = \sqrt{N}$ where N is a natural number, find the value of N.

(b) If $x = 1 + \log_a bc$, $y = 1 + \log_b ca$ and $z = 1 + \log_c ab$, then prove that $xyz = xy + yz + zx$.

(c) If $\frac{\log_a N}{\log_c N} = \frac{\log_a N - \log_b N}{\log_b N - \log_c N}$ where $N > 0$ & $N \neq 1$, $a,b,c > 0$ & not equal to 1, then prove that $b^2 = ac$.

EXERCISE (JA)

1. $\log_{3/4}\log_8(x^2+7)+\log_{1/2}\log_{1/4}(x^2+7)^{-1}=-2.$ **[REE 2000, 5 out of 100]**

2. Number of solutions of $\log_4(x-1)=\log_2(x-3)$ is **[JEE 2001 (Screening)]**

(A) 3 (B) 1 (C) 2 (D) 0

3. Let (x_0, y_0) be the solution of the following equations **[JEE 2011, 3 (–1)]**

$$(2x)^{\ln 2}=(3y)^{\ln 3}$$
$$3^{\ln x}=2^{\ln y}$$

Then x_0 is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 6

4. The value of $6+\log_{\frac{3}{2}}\left(\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}......}}}\right)$ is **[JEE 2012, 4M]**

5. If $3^x=4^{x-1}$, then x = **[JEE-Advanced 2013, 4, (–1)]**

(A) $\frac{2\log_3 2}{2\log_3 2-1}$ (B) $\frac{2}{2-\log_2 3}$ (C) $\frac{1}{1-\log_4 3}$ (D) $\frac{2\log_2 3}{2\log_2 3-1}$

ANSWERS

ELEMENTARY EXERCISE

1. x = 3 **2.** x = –2 **3.** x = 3 **4.** x = 0 **5.** $x = \frac{-7}{2}, 2$ **6.** $x = \frac{1}{5}, 6$ **7.** x = 1

8. (i) 0 (ii) 1 (iii) 2 (iv) 3 (v) –1 (vi) –5 (vii) –4 (viii) 1/2
(ix) 1 (x) 3/2 (xi) –1/5 (xii) –3/7

9. (i) 0 (ii) 1 (iii) 2 (iv) –1 (v) –2 (vi) –4 (vii) –1/3 (viii) 1/7 (ix) –5/2 (x) 9/4

10. (i) 4 (ii) 2 (iii) a > 0, a ≠ 1 (iv) –5,2 (v) –2,2 (vi) –3,3

11. (i) 0 (ii) 1 (iii) 2 (iv) 4 (v) –1 (vi) 1/2 (vii) –3/2 (viii) 7/2 (ix) 2/7

12. (i) 0 (ii) 1 (iii) 3 (iv) –4 (v) –1/2 (vi) 1/2 (vii) –3/2 (viii) 9/4

13. (i) 9 (ii) 1/81 (iii) 1 (iv) a > 0, a ≠ 1 (v) 2 (vi) $\sqrt{2}, -\sqrt{2}$

14. (i) $\sqrt{2}, -\sqrt{2}$ (ii) 1 (iii) 1 (iv) 0 (v) 1,–1

15. (i) x = ±2 (ii) $x = \pm\frac{1}{2}$ (iii) $x = \frac{3}{2}$ (iv) x = 4

16. (i) 0 (ii) 1 (iii) 2 (iv) 4 (v) –1 (vi) –2 (vii) –1/2
(viii) 1/4 (ix) 1/2 (x) 1/3 (xi) 1/3

17. (i) 1 (ii) 2 (iii) 3 (iv) 9 (v) 9 (vi) 2

18. (i) $\frac{-1}{2}$ (ii) 1 (iii) 2 (iv) 3 (v) 1 (vi) 4 **19.** 8

EXERCISE (O-1)

1. A **2.** C **3.** C **4.** D **5.** A **6.** B **7.** C **8.** A **9.** B **10.** B **11.** D **12.** B

EXERCISE (O-2)

1. C **2.** D **3.** A **4.** A **5.** B **6.** D **7.** **C** **8.** A **9.** B,C **10.** A,B,D

EXERCISE (S-1)

1. 12 **2.** **(a)** –1 **(b)** $\log_b N$ **3.** 3721 **4.** 9 **5.** 1 **6.** 6

7. $2s + 10s^2 - 3(s^3 + 1)$ **8.** $\frac{25}{2}$

11. i. $\{1+\sqrt{3}\}$ ii. {3} iii. {4} iv. {2} v. {0} vi. {–5}
vii. {3/2, 10} viii. $\{10^{-1}, 10\}$ ix. $\{\sqrt{5}, 5\}$ x. $\{-(3-\sqrt{3})/3, 8\}$

12. i. $\{2^{-7}, 2\}$ ii. $\{1/\sqrt{2}, 1, 4\}$ iii. $\{-1, 0\}$ iv. $\{3\}$ v. $\{-9/5, 23\}$

vi. $\{2\}$ vii. $\{1\}$ viii. $\{2\}$ ix. $\{10^{-\sqrt{3}}, 10^{\sqrt{3}}\}$ x. $\{\log_3 4\}$

EXERCISE (S-2)

1. 2 **2.** 1/6 **3.** $\frac{1+2ac}{2c+abc+1}$ **4.** 469 **5.** 4/9 **6.** $xy = 2^9$ **7.** 1

8. (i) $x = 6, -4$ (ii) $x = 2, 0$ (iii) $x = 3, \frac{-11}{3}$ (iv) $x = 5, -9$ (v) $x = 1, 0$ (vi) $x = 8, \frac{1}{8}$

9. (a^4, a, a^7) or $\left(\frac{1}{a^4}, \frac{1}{a}, \frac{1}{a^7}\right)$ **10.** $y = 6$ **11.** 23040 **12.** $(2017)^2$

14. **(a)** 0.5386; $\overline{1}.5386$; $\overline{3}.5386$ **(b)** 2058 **(c)** 0.3522 **(d)** 343 **15.** (a) 507

EXERCISE (JA)

1. x = 3 or –3 **2.** B **3.** C **4.** 4 **5.** A,B,C

Important Notes