Lagrange’s M. V. Theorem

Q.1. Use lagrange’s mean value theorem to determine a point P on the curve $y = \sqrt{(x - 2)}$ defined in the interval [2, 3] , where the tangent is parallel to the chord joining the end points on the curve.

Solution : 1

$y = \sqrt{(x - 2)}$ defined in [2, 3]

$dy/dx = 1/\{2\sqrt{(x - 2)}\}$which exists for all x ε (2, 3) .

Therefore, (1) y is continuous in [2, 3] and (2) y is differentiable in (2, 3).

Therefore , $1/\{2(x - 2)\} = [f(3) - f(2)]/(3 - 2) = (1 - 0)/(3 - 2) = 1$

Or, $1/2 = \sqrt{(x - 2)}$

Or, $1/4 = x - 2$

Or, $x = 2 + 1/4 = 9/4$ ε (2, 3)

Therefore , $y = \sqrt{(9/4 - 2)} = 1/2$.

Therefore , P is (9/4, 1/2) .

Q.2. Verify Lagrange’s mean value theorem for the function $f(x) = 3x^2 - 5x + 1$ defined in interval [2, 5] .

Solution : 2

We have , $f(x) = 3x^{\mathbf{2}} - 5x + 1$ where , x ε [2, 5] .

(1) f(x) is a polynomial function , hence continuous in the interval [2, 5] .

(2) f(x) is a polynomial function , hence differentiable in the interval (2, 5) .

(3) $f(5) = 3(5)^{\mathbf{2}} - 5 \times 5 + 1 = 51$, $f(2) = 3(2) 2 - 5 \times 2 + 1 = 3$.

Also , $f'(x) = 6x - 5 \Rightarrow f'(c) = 6c - 5$.

Now , $f'(c) = [f(b) - f(a)]/(b - a)$

Or , $6c - 5 = [f(5) - f(2)]/(5 - 2) = (51 - 3)(5 - 2) = 48/3 = 16$

Or , $6c = 16 + 5 = 21 \Rightarrow c = 21/5 \in (2, 5)$.

Hence , **Lagrange's mean value theorem is verified.**

Q.3. Find a point on the parabola $y = (x - 3)^2$, where the tangent is parallel to the chord joining (3, 0) and (4, 1).

Solution : 3

We have $y = (x - 3)^{\mathbf{2}}$ where $x \in [3, 4]$.

(1) f(x) is a polynomial function and is continuous in [3, 4] .

(2) $f'(x) = 2(x - 3)$, which exist for all $x \in (3, 4)$.

Thus both the conditions of Lagrange's mean value theorem is satisfied.

Hence , there must exist a point $c \in (3, 4)$ such that $f'(c) = [f(4) - f(3)]/(4 - 3) = 1$.

Now , $f'(c) = 1 \Rightarrow 2(c - 3) = 1 \Rightarrow c = 7/2 \in (3, 4)$.

Thus when $x = 7/2$, $y = (7/2 - 3)2 = 1/4$.

Hence , the required point is (7/2, 1/4) on the parabola , the tangent is parallel to the chord joining (3, 0) and (4, 1) .

Q.4. Examine the validity and conclusion of Lagrange's mean value theorem for the function $f(x) = x(x - 1)(x - 2)$, for every $x \in [0, 1/2]$.

Solution : 4

We have $f(x) = x(x - 1)(x - 2) = x^3 - 3x^{\mathbf{2}} + 2x$.

(1) f(x) is polynomial function and is continuous in [0, 1/2] .

(2) $f'(x) = 3x^{\mathbf{2}} - 6x + 2$, which exists and hence is differentiable in (0, 1/2) .

Now $f'(c) = 3c2 - 6c + 2$, $f(0) = 0$, $f(1/2) = 1/2(1/2 - 1)(1/2 - 2) = 3/8$.

$[f(b) - f(a)]/(b - a) = f'(c)$

Or, $(3/8 - 0)/(1/2 - 0) = 3c^2 - 6c + 2$

Or, $3/4 = 3c^2 - 6c + 2$

Or, $12c^2 - 24c + 5 = 0$

Or, $c = [24 \pm \sqrt{(576 - 240)}]/24 = [24 \pm \sqrt{336}]/24 = 1 \pm [(\sqrt{21})/6]$.

Or, $c = 1 - \sqrt{21}/6 = 0.236$ (approx).

And $c = 1 + \sqrt{21}/6 > 1/2$, is not acceptable.

Q.5. Is Lagrange's mean value theorem applicable to : $f(x) = 4 - (6 - x)^{2/3}$ in [5, 7] .

Solution : 5

We have $f(x) = 4 - (6 - x)^{2/3}$.

Therefore , $f'(x) = (-2/3) \times (6 - x)^{-1/3} \times (-1) = 2/[3(6 - x)^{1/3}]$, which does not exist at $x = 6$, and $6 \in (5, 7)$.

Therefore f(x) is not differentiable in the interval (5, 7) .

Hence, Lagrange's mean value theorem is not applicable.

Q.6. Show that the function $f(x) = x^2 - 6x + 1$ satisfies the Lagrange's Mean Value Theorem. Also find the co-ordinate of a point atwhich the tangent to the curve represented by the above function is parallel to the chord joining A(1, – 4) and B(3, – 8).

Solution : 6

We have $f(x) = x^2 - 6x + 1$,

(1) f(x) being a polynomial function is continuous.

(2) $f'(x) = 2x - 6$, i.e. f(x) is differentiable.

Slope of line AB = $f'(x) = \{-8 - (-4)\}/(3 - 1) = -2$.

Or, $2x - 6 = -2$

Or, $x = 2$ and $y = f(x) = (2)^2 - 6 \times 2 + 1 = -7$.

Hence , point is (2, – 7).