Chapter 8

Binomial Theorem

Miscellaneous Exercise

Q. 1 Find a, b and n in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

Answer:

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is $T_{r+1} = {}^n c_r a^{n-r} b^r$

The first three terms of the expansion are given as 729, 7290 and 30375 respect

Therefore, we obtain

$T_1 = {}^n C_0 a^{n-0} b^0 = a^n = 729$ … (1)

$T_2 = {}^n C_1 a^{n-2} b^2 = {}^n a^{n-1} b = 7290$ … (2)

$T_3 = {}^n C_2 a^{n-2} b^2 = \frac{n(n-1)}{2} a^{n-2} b_2 = 30375$ … (3)

Dividing (2) by (1), we obtain

$\frac{na^{n-1}b}{a^n} = \frac{7290}{729}$

$= \frac{nb}{a} = 10$ … (4)

Dividing (3) by 92), we obtain

$\frac{n(n-1)a^{n-2}b^2}{2na^{n-1}b} = \frac{30375}{7290}$

$= \frac{(n-1)b}{2a} = \frac{30375}{7290}$

$= \frac{(n-1)b}{a} = \frac{30375 \times 2}{7290} = \frac{2}{3}$

$= \frac{nb}{a} - \frac{b}{a} = \frac{25}{3}$

$= 10 - \frac{b}{a} = \frac{25}{3}$ [using (4)]

$= \frac{b}{a} - 10 - \frac{25}{3} = \frac{5}{3}$ … (5)

From (4) and (5), we obtain

n, $\frac{5}{3} = 10$

= n = 6

Substituting n = 6 in equation (1), we obtain a 6

= 729

= a = $\sqrt[6]{729}$ = 3

From (5), we obtain

$\frac{b}{3} = \frac{5}{3}$ b = 5

Thus, a = 3, b = 5, and n = 6

Q. 2 Find a if the coefficients of x_2 and x_3 in the expansion of $(3 + ax)^9$ are equal.

Answer:

It is known that (r + 1) th term, (T_{r+1}), in the binomial expansion of 9a + b) ^{n} is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Assuming that x2 occurs in the (r + 1) th term in the expansion of 93 + ax) ^{9}, we obtain

$T_{r+1} = {}^9C_r (3)^{9-r} (ax)^r = {}^9C_r (3)^{9-r} a^r x^r$

Comparing the indices of x in x^2 and in T_{r+2}, we obtain

$r = 2$

thus, the coefficient of x^2 is

$^9C_2 (3)^{9-2} a^2 = \frac{9!}{2!7!} (3)^7 a^2 = 36 (3)^7 a^2$

Assuming that x^2 occurs in the $(k + 1)^{th}$ term in the expansion of $(3 + ax)^9$, we obtain

$T_{k+1} = {}^9C_k (3)^{9-k} (ax)^k = {}^9C_k (3)^{9-k} a^k x^k$

Comparing the indices of x in x^3 and in T_{k+1}, we obtain $k = 3$

Thus, the coefficient of x^3 is

$^9C_3 (3)^{9-3} a^3 = \frac{9!}{3!6!} (3)^6 a^3 = 84(3)^6 a^3$

It is given that the coefficient of x^2 and x^3 are the same.

$84(3)^6 a^3 = 36 (3)^7 a^2$

$= a = \frac{36\times3}{84} = \frac{104}{84}$

$= a = \frac{9}{7}$

Thus, the required value of is $\frac{9}{7}$.

Q. 3 Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 - x)^7$ using binomial theorem.

Answer:

Using binomial theorem, the expressions, $(1 + 2x)^6$ and $(1 - x)^7$, can be expanded as

$(1 + 2x)^6 = {}^6C_0 + {}^6C_1(2x) + {}^6C_2 (2x)^2 + {}^6C_3 (2x)^3 + {}^6C_4 (2x)^4 + {}^6C_5 (2x)^5 + {}^6C_6 (2x)^6$

$= 1 + 6(2x) + 15(2x)^2 + 20(2x)^3 + 15(2x)^4 + 6(2x)^5 + (2x)^6$

$= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$

$(1 - x)^7 = {}^7C_0 - {}^7C_1(x) + {}^7C_2 (x)^2 - {}^7C_3(x)^3 + {}^7C_4(x)^4 - {}^7C_5 (x)^5 + {}^7C_6 (x)^6 - {}^7C_7 (x)^7$

$= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$

$\therefore (1 + 2x)^6 (1 - x)^7$

$= \{1+ 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6\} \{1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7\}$

The complete multiplication of the two brackets is not required to be carried out. Only those terms, which involve x^5, are required.

The terms containing x^5 are

$1 (-21x^5) + (12x) (32x^4) + (60x^2) (-35x^3) + (160x^3) (21x^3) + (240x^4) (-7x) + (192x^5) (1) = 171x^5$

Thus, the coefficient of x5 in the given product is 171.

Q. 4 If a and b are distinct integers, prove that a – b is a factor of $a^n - b^n$, whenever n is a positive integer. [Hint write an = $(a - b + b)^n$ and expand]

Answer:

In order to prove that (a – b) is a factor of $(a^n - b^n)$, it has to be prove that

$a^n - b^n = k (a - b)$, where k is some natural formula

It can be written that, a = a – b + b

$\therefore$ an = $(a - b + b)^n = [(a - b) + b]^n$

$= {}^nC_0 (a - b)^n + {}^nC_2 (a - b)^{n-1} b + \dots + {}^nC_{n-1} (a - b) b^{n-1} + {}^nC_n b_n$

$= (a - b)^n + {}^nC_2 (a - b)^{n-1} b + \dots + {}^nC^{n-1} (a - b) b^{n-1} + b^n$

$= a^n - b^n = (a - b) [(a - b)^{n-1} + {}^nC_2 (a - b)^{n-2} b + \dots + {}^nC^{n-1} b^{n-1}]$

$= a^n - b^n = k(a - b)$

Where, $k = [(a - b)^{n-1} + {}^nC_2 (a - b)^{n-2} b + \ldots {}^nC_{n-1} b^{n-1}$ is a natural number.

This, shows that $(a - b)$ is a factor of $(a^n - b^n)$, where n is a positive integer.

Q. 5 Evaluate $\left(\sqrt{3} - \sqrt{2}\right)^6 - \left(\sqrt{3} - \sqrt{2}\right)^6$

Answer:

Firstly, the expression $(a + b)^6 - (a - b)^6$ is simplified by using binomial theorem. This, can be done as

$(a + b)^6 = {}^6C_0a^6 + {}^6C_1a^5b + {}^6C_2a^4b^2 + {}^6C_3a^3b^3 + {}^6C_4a^2b^4 + {}^6C_5a^1b^5 + {}^6C_6b^6$

$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

$(a - b)^6 = {}^6C_0a^6 - {}^6C_1a^5b + {}^6C_2a^4b^2 - {}^6C_3a^3b^3 + {}^6C_4a^2b^4 - {}^6C_5ab^5 + {}^6C^6b^6$

$= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$

$\therefore (a + b)^6 - (a - b)^6 = 2[6a^5b + 20a^3b^3 + 6ab^5]$

Putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$\left(\sqrt{3} - \sqrt{2}\right)^6 - \left(\sqrt{3} - \sqrt{2}\right)^6 = 2\left[6\left(\sqrt{3}\right)^5\left(\sqrt{2}\right) + 20\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3 + 6\left(\sqrt{3}\right)\left(\sqrt{2}\right)^5\right]$

$= 2[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}]$

$= 2 \times 198\sqrt{6}$

$= 396\sqrt{6}$

Q. 6 Find the value of $\left(a^2 + \sqrt{a^2 - 1}\right)^4 + \left(a^2 - \sqrt{a^2 + 1}\right)^4$

Answer:

Firstly, the expression $(x + y)^4 + (x - y)^4$ is simplified by using binomial theorem

This can be done as

$(x + y)^4 = {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4$

$= x^4 + 4x_3y + 6x^2y^2 + 4xy^3 + y^4$

$(x - y)^4 = {}^4C_0x^4 - {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4$

$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

$\therefore (x + y)^4 + (x - y)^4 = 2\ (x^4 + 6x^2y^2 + y^4)$

Putting $x = a^2$ and $y = \sqrt{a^2 + 1}$, we obtain

$$\left(a^2 + \sqrt{a^2} + 1\right)^4 + \left(a^2 - \sqrt{a^2} - 1\right)^4 = 2\left[(a)^{2^4} + 6(a)^{2^2}\left(\sqrt{a^2 + 1}\right)^2\left(\sqrt{a^2 - 1}\right)^4\right]$$

$= 2[a^8 + 6a^4(a^2 - 1) + (a^2 - 1)^2]$

$= 2\ [a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1]$

$= 2\ [a^8 + 6a^6 - 5a^4 - 2a^2 + 1]$

$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$

Q. 7 Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Answer:

$0.99 = 1 - 0.01$

$\therefore (0.99)^5 = (1 - 0.01)^5$

$= {}^5C_0(1)^5 - {}^5C_2(1)^4\ (0.01) + {}^5C_2(1)^3\ (0.01)^2$ [Approximately]

$= 1 - 5\ (0.01) + 10\ (0.01)^2$

$= 1 - 0.05 + 0.001$

= 1.001 – 0.05

= 0.951

Thus, the value of (0.99)5 is approximately 0.951.

Q. 8 Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left\{\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right\}^n$ is $\sqrt{6}$: 1

Answer:

In the expansion, $(a + b)^n = {}^nC_0\, a^{n-2}\, b^2 + \ldots + {}^nC_1\, ab^{n-2} + {}^nC_n\, b^n$

Fifth term from the beginning = ${}^nC_4\, a^{n-4}\, b^4$

Fifth term from the end = ${}^nC_4\, a^4\, b^{n-4}$

Therefore, it is evident that in the expansion of $\left\{\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right\}^n$ are fifth term from the beginning is

${}^nC_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4$ and the fifth term from the end is ${}^nC_{n-4}$ $\left(\sqrt[4]{2}\right)^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$

$${}^nC_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4 = {}^nC_4\,\frac{\left(\sqrt[4]{2}\right)^n}{\left(\sqrt[4]{2}\right)^4}\cdot\frac{1}{3} = \frac{n!}{6.4!(n-4)!}\left(\sqrt[4]{2}\right)^n \ldots (1)$$

$${}^nC_{n-4}\left(\sqrt[4]{2}\right)^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} = {}^nC_{n-4}.2.\frac{3}{\left(\sqrt[4]{3}\right)^n} = \frac{6n!}{(n-4)!4!}\cdot\frac{1}{\left(\sqrt[4]{3}\right)^n} \ldots (2)$$

It is given that the ratio of the fifth term from the beginning to the fifth term from the end is $\sqrt{6}$: 1 therefore, from (1) and (2), we obtain

$$\frac{n!}{6.4!(n-4)!}\left(\sqrt[4]{2}\right)^n : \frac{6n!}{(n-4)!4!}\cdot\frac{1}{\left(\sqrt[4]{3}\right)^n} = \sqrt{6} : 1$$

$$= \frac{\left(\sqrt[4]{2}\right)^n}{6} : \frac{6}{\left(\sqrt[4]{3}\right)^n} = \sqrt{6} : 1$$

$= \frac{(\sqrt[4]{2})^n}{6} \times \frac{(\sqrt[4]{3})^n}{6} = \sqrt{6}$

$= (\sqrt[4]{6})^n = 36\sqrt{6}$

$= 6^{n/4} = 6^{5/2}$

$= \frac{n}{4} = \frac{5}{2}$

$= n = 4 \times \frac{5}{2} = 10$

Thus, the value of n is 10.

Q. 9Expand using Binomial Theorem $\left(1 + \frac{x}{2} - \frac{2}{x}\right)^n$

Answer:

$= {}^nC_0\left(1 + \frac{x}{2}\right)^4 - {}^nC_1\left(1 + \frac{x}{2}\right)^3\left(\frac{2}{x}\right) + {}^nC_2\left(1 + \frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2 - {}^nC_3\left(1 + \frac{x}{2}\right)\left(\frac{2}{x}\right)^3 + {}^nC_4\left(\frac{2}{x}\right)^4$

$= \left(1 + \frac{x}{2}\right)^4 - 4\left(1 + \frac{x}{2}\right)^3\left(\frac{2}{x}\right) + 6\left(1 + x + \frac{x^2}{4}\right)\left(\frac{4}{x^2}\right) - 4\left(1 + \frac{x}{2}\right)\left(\frac{8}{x^3}\right) + \frac{16}{x^4}$

$= \left(1 + \frac{x}{2}\right)^4 - \frac{8}{x}\left(1 + \frac{x}{2}\right)^3 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4}$

$= \left(1 + \frac{x}{2}\right)^4 - \frac{8}{x}\left(1 + \frac{x}{2}\right)^3 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \dots (1)$

Again by using binomial theorem, we obtain

$\left(1 + \frac{x}{2}\right)^4 = 4C0(1)4 + 4C1(1)3\left(\frac{x}{2}\right) + 4C2(1)2\frac{x^2}{2} + 4C3(1)\left(\frac{x}{2}\right)^3 + 4C4\frac{x^4}{2}$

$= 1 + 4 \times \frac{x}{2} + 6 \times \frac{x^4}{4} + 4 \times \frac{x^3}{8} + \frac{x^4}{16}$

$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} \ldots (2)$

$= \left(1 + \frac{x}{2}\right)^3 = 3C0(1)3 + 3C1(1)2\left(\frac{x}{2}\right) + 3C2(1)\left(\frac{x}{2}\right)^2 + 3C3\left(\frac{x}{2}\right)^3$

$= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} \ldots (3)$

From (1), (2) and (3), we obtain

$= \left[\left(1 + \frac{x}{2}\right) - \frac{2}{x}\right]^4$

$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x}\left(1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8}\right) + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$

$= 1 + 2x + \frac{3}{2}x^2 + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} - 12 - 6x - x^2 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$

$= \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5$

Q. 10 Find the expansion of $(3x^2 – 2ax + 3a^2)^3$ using binomial theorem.

Answer:

Using binomial theorem, the given expression $(3x^2 – 2ax + 3a^2)^3$ can be expanded as $[(3x^2 – 2ax) + 3a^2]^3$

$= {}^3C_0(3x^2 – 2ax^2)^3 + {}^3C_1(3x^2 – 3ax)^2(3a^2) + {}^3C_2(3x^2 – 2ax)(3a^2)^2 + {}^3C_3(3a^2)^3$

$= (3x^2 – 2ax)^3 + 3(9x^4 – 12ax^3 + 4a^2x^2)(3a^2) + 3(3x^2 – 2ax)(9a^4) + 27a^4$

$= (3x^2 – 2ax)^3 + 81a^2x^4 – 108a^3x^4 + 36a^4x^2 + 81a^4x^2 – 54a^5x + 27a^6$

$= (3x^2 – 2ax)^3 + 81a^2x^4 – 108a^3x^3 + 117a^4x^2 – 54a^5x + 27a^6 \ldots (1)$

Again by using binomial theorem, we obtain

$(3x^2 – 2ax)^3$

$= {}^3C_0(3x^2)^3 – {}^3C_1(3x^2)^2(2ax) + {}^3C_2(3x^2)(2ax)^2 – {}^3C_3(2ax)^3$

$= 27x^5 - 3\ (9x^4)\ (2ax) + 3\ (3x^2)\ (4a^2x^2) - 8a^3x^3$

$= 27x^5 - 54ax^5 + 36a^2x^4 - 5a^3x^3 \ldots (2)$

From (1) and (2), we obtain

$(3x^2 - 2ax + 3a^2)^3$

$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$

$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$