4 Continuity

Review of Key Notes and Formulae

1. Continuity of a Function at a Point :

A function $f(x)$ is said to be continuous at $x = a$, if

$$\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

Otherwise, it is said to be discontinuous.

2. Continuity of a Function is an Interval :

(i) A function $f(x)$ is said to be continuous in an open interval (a, b), if $f(x)$ is continuous at every point of the interval.

(ii) A function $f(x)$ is said to be continuous in a closed interval $[a, b]$, if $f(x)$ is continuous in (a, b). In addition, $f(x)$ is continuous at $x = a$ from right and $f(x)$ is continuous at $x = b$ from left.

★ **Note :** To determine function's continuity in an interval, the best way is to draw graph.

⇒ If graph of a function has no break or gap, then it is continuous, otherwise it will be discontinuous function.

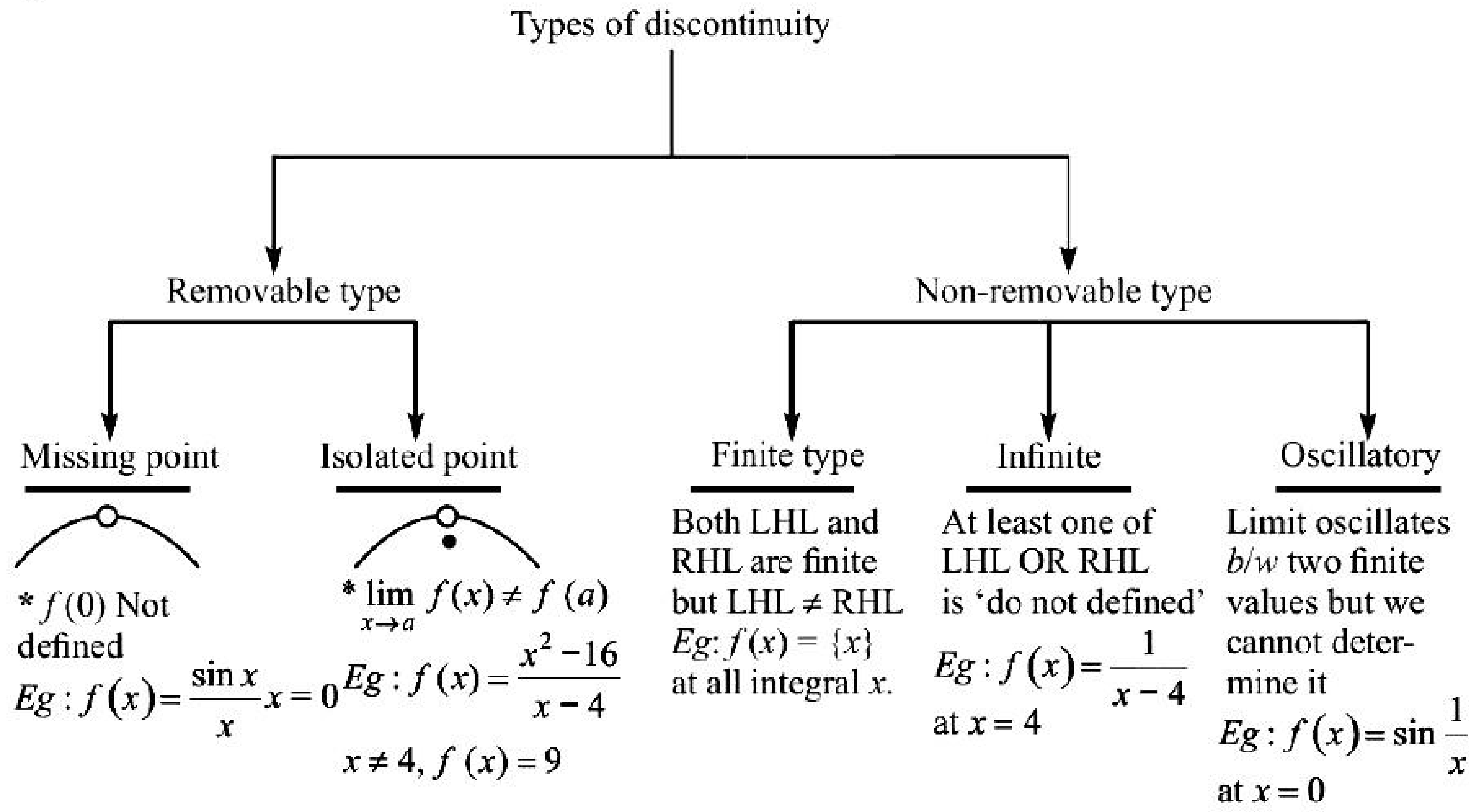
3. Reason's of Discontinuity:

(i) $\lim_{x \to a} f(x)$ does not exist.

(ii) $\lim_{x \to a} f(x)$ exists $\neq f(a)$

(iii) $f(a)$ is not defined.

4.



5. **Jump of Discontinuity :**

Jump = | LHL – RHL|, provided both LHL and RHL are finite.

6. **Theorems Continuity :** At $x = a$.

$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x). g(n)$	$\frac{f(x)}{g(x)}; g(a) \neq 0$
C	C	C	C	C
C	D	D	C or D	C or D
D	C	D	C or D	C or D
D	D	C or D	C or D	C or D

Where C → Continuous, D → Discontinuous function.

7. **Continuity of Composite Function:**

If $f(x)$ is continuous at $x = a$ and $g(x)$ is continuous at $x = f(a)$, then $g(f(x))$ is continuous at $x = a$.

TIPS AND TRICKS: (T-1)

If $f(x)$ is continuous in $x \in [a, b]$ and $f(a), f(b)$ are opposite in sign, then there exists at least one root of equation $f(x) = 0$ in $x \in (a, b)$

Illustration 1

Show that $x = a \sin x + b$, where $0 < a < 1, b > 0$ has at least one positive root which doesn't exceed $b + a$.

Short-cut solution :

Using T-1 Let $f(x) = x - a \sin x - b$ is continuous function.

Now, $f(0) = -b < 0$

$f(a + b) = a - a \sin (a + b) = a(1 - \sin (a + b)) \geq 0$

Hence, one positive root in $[0, a + b]$

Illustration 2

Show that $f(x) = x^3 + 2x - 1$ has root in the interval $x \in [0, 1]$

Short-cut solution :

Using T-1 $f(x)$ is continous in $x \in [0, 1]$

Since, $\left.\begin{array}{r} f(0) = -1 < 0 \\ f(1) = 2 > 0 \end{array}\right\} \Rightarrow$ There exists at least one c in $(0, 1)$ such that $f(c) = 0$.

TIPS AND TRICKS: (T-2)

$f : R \rightarrow R$, if $f(x)$ is even degree polynomial whose leading coefficient and absolute constant term are of opposite in sign, then $f(x) = 0$ has at least 2 real roots.

Illustration 3

If $f : R \rightarrow R, f(x) = 2x^6 - 3x^5 + 4x^3 - x - 7$. Then prove that $f(x) = 0$ must have at least two real roots.

Short-cut solution :

Using T-2 $\because$ Leading coefficient > 0 (2)

Absolute term < 0 (-7)

Hence, $f(x) = 0$ must have at least 2 real roots.

TIPS AND TRICKS: (T-3)

If $y = (x)$ is continuous function and takes rational values for all x then $f(x)$ will be a constant function.

Illustration 4

Let $f(x)$ be a continuous functions for $x \in [1, 3]$. If $f(x)$ takes rational values for all x and $f(2) = 10$, then value of $f(1.5)$ is

(a) 7.5 (b) 10

(c) 8 (d) 9

Short-cut solution :

Using T-3 $\because f(x)$ is continuous and takes rational values.

$\Rightarrow \boxed{f(x)=10} \Rightarrow f(1.5)=10$ **Ans. (b)**

TIPS AND TRICKS: (T-4)

If $y = f(x)$ is monotonically increasing and continuous on an interval $x \in$ (a, b) then $f^{-1}(y)$ exists and continuous and monotonically increasing

Illustration 5

Check whether inverse of $f(x) = \sin x + 1\ \forall\ x \in (-\pi^2, \pi^2)$ is continuous or not.

Short-cut solution :

Using T-4 $\because f(x)$ is continuous in $x \in \left(\ \pi^2, \pi^2\right)$

$\Rightarrow f^{-1}(x)$ is also continuous

TECHNIQUE

Intermediate value theorem (IVT)

Let $f(x)$ be continuous in closed interval $[a, b]$ then $f(x)$ will attain the least value (say m) and the greatest value (say M) for $x \in [a, b]$ then there exists at least one $c \in [a, b]$ such that $f(c)=\dfrac{\lambda_1 m+\lambda_2 M}{\lambda_1+\lambda_2}; \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1+\lambda_2 \neq 0$

Illustration 6

Does function $f(x)=\dfrac{x^3}{4}-\sin(\pi x)+3$, take on value $2\dfrac{1}{3}$ within the interval $[-2, 2]$

Short-cut solution :

Using Tech. Since, $f(x)$ is continuous in $[-2, 2]$ and $f(-2) = 1$, $f(2) = 5$

Hence, by IVT, it takes the value $2\dfrac{1}{3}$

Concept Booster Exercise

1. The equation $2x^3 - 6x + 1 = 0$ on $x \in (1, 2)$ has
   (a) no solution
   (b) at least one real solution
   (c) infinite solution
   (d) None of these

2. The equation $2 \cos x + 6x - 3$ has
   (a) no solution
   (b) at least one real solution in $x \in [0, \pi^3]$
   (c) infinite solution
   (d) None of these

3. The equation $\frac{a_1}{x-\lambda_1} + \frac{a_2}{x-\lambda_2} + \frac{a_3}{x-\lambda_3} = 0$ where $a_1, a_2\, a_3 > 0$ and $\lambda_1 < \lambda_2 < \lambda_1$ has
   (a) no solution
   (b) one real solution
   (c) two real roots
   (d) infinite roots

4. If $f(x) = \begin{cases} x;\ x \in Q \\ -x;\ x \notin Q \end{cases}$, then $f(x)$ is continous at -

   [AIEEE 2002]

   (a) Only at zero
   (b) Only at 0, 1
   (c) All real numbers
   (d) All rational numbers

5. If $f(x) = \frac{x}{2} - 1$, then on the interval $[0, \pi]$, then
   (a) $f^{-1}(x)$ is continuous
   (b) $f^{-1}(0) = 0$
   (c) $f^{-1}(x)$ is discontinuous
   (d) None of these

Numerical Value Problems

6. If $f(x) = \begin{cases} x^2 + cx + 1; & x \in Q \\ ax^2 + 2x + b; & x \notin Q \end{cases}$ is continuous at $x = 1, 2, 3$, then find $a + b + c$

   [AIEEE 2010]

7. Let $f : R \to R$ be a continuous function defined by

   $f(x) = \frac{1}{e^x + 2e^{-x}}$, there $0 < f(x) < \frac{1}{C\sqrt{C}}, \forall x \in R$

   Therefore 'C' is equal to —

Solutions

1. (b) Using T-1 Let $f(x) = 2x^3 - 6x + 1$ $\because f(x)$ is continuous

Hence, $f(1) = -3 < 0$ and $f(2) = 5 > 0$

$\Rightarrow \exists$ at least one root in $x \in (0, 2)$

2. (b) Using T-1 Let $f(x) = 2 \cos x + 6x - 3$ $\because f(x)$ is continuous function

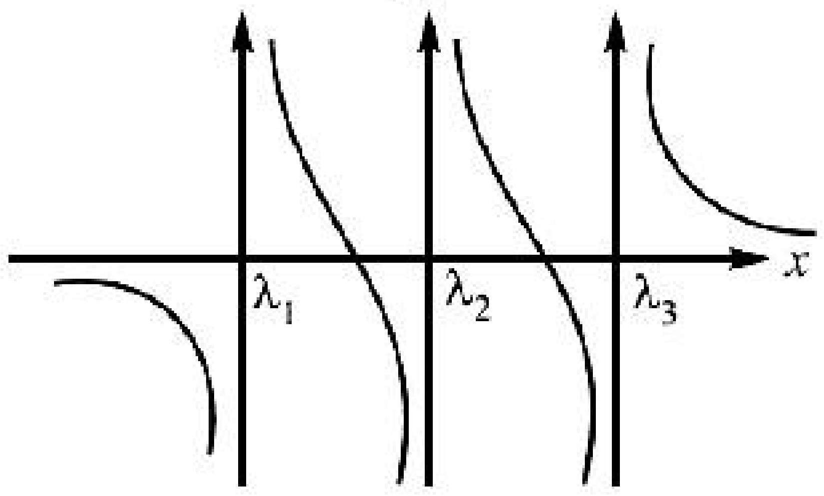
And, $f(0) = -1 < 0$ and $f\left(\frac{\pi}{3}\right) = 2(\pi - 1) > 0$

$\Rightarrow \exists$ at least one root in $x \in \left(0, \frac{\pi}{3}\right)$

3. (c) Let $f(x) = \frac{a_1}{x - \lambda_1} + \frac{a_2}{x - \lambda_2} + \frac{a_3}{x - \lambda_3}$

Now, $f'(x) = -\frac{a_1}{(x - \lambda_1)^2} - \frac{a_2}{(x - \lambda_2)^2} - \frac{a_3}{(x - \lambda_3)^2} < 0$

$\Rightarrow$ Decreasing function



Hence, two real roots.

4. (a) Using T-3 One point is common *i.e.* 0

Hence, $f(x)$ is continous at $x = 0$

5. (a) Using T-4 $\because f(x)$ is continuous is $x \in [0, \pi]$

$\Rightarrow f^{-1}(x)$ will also be continuous

6. (4) Since, $f(x)$ is continuous

$$\Rightarrow (a-1)x^2 + (2-c)x + (b-1) = 0 \begin{matrix} \nearrow 1 \\ \rightarrow 2 \\ \searrow 3 \end{matrix} \quad \text{(on subtracting)}$$

1, 2, 3, must be roots of above equation

$\Rightarrow$ Equation becomes identity (Because more than one root)

$\Rightarrow a = 1, b = 1, c = 2$; Hence, $a + b + c = 4$

7. (2) As we know that AM $\geq$ GM $\Rightarrow \frac{e^x + \frac{2}{e^x}}{2} \geq (2)^{1/2} \Rightarrow \frac{e^x}{e^{2x} + 2} \leq \frac{1}{2\sqrt{2}}$

Hence, $0 < \frac{1}{e^x + 2e^{-x}} < \frac{1}{2\sqrt{2}} \Rightarrow C = 2$