20 Limits

Basics of Limits and Methods of Evaluation of Limits

1. What is $\lim_{x\to 0}\frac{(1+x)^n-1}{x}$ equal to? *[2014-I]*

(a) 0 (b) 1
(c) n (d) $n-1$

2. What is $\lim_{x\to 0}\frac{x}{\sqrt{1-\cos x}}$ equal to? *[2014-I]*

(a) $\sqrt{2}$ (b) $-\sqrt{2}$
(c) $\frac{1}{\sqrt{2}}$ (d) Limit does not exist

3. What is $\lim_{x\to 0}\frac{\log_5(1+x)}{x}$ equal to? *[2014-II]*

(a) 1 (b) $\log_5 e$
(c) $\log_e 5$ (d) 5

4. What is $\lim_{x\to 0}\frac{5^x-1}{x}$ equal to? *[2014-II]*

(a) $\log_e 5$ (b) $\log_5 e$
(c) 5 (d) 1

5. What is $\lim_{n\to\infty}\frac{1+2+3+...+n}{1^2+2^2+3^2+...n^2}$ equal to? *[2014-II]*

(a) 5 (b) 2
(c) 1 (d) 0

Consider the following for the next two (02) items that follow: [2015-I]

Given that $\lim_{x\to\infty}\left(\frac{2+x^2}{1+x}-Ax-B\right)=3$.

6. What is the value of A? *[2015-I]*

(a) -1 (b) 1
(c) 2 (d) 3

7. What is the value of B? *[2015-I]*

(a) -1 (b) 3
(c) -4 (d) -3

8. If $f(x)=\frac{\sin(e^{x-2}-1)}{\ln(x-1)}$, then $\lim_{x\to} f(x)$ is equal to *[2015-II]*

(a) -2 (b) -1
(c) 0 (d) 1

Consider the following for the next two (02) items that follow:

Consider the function $f(x)=\frac{a^{[x]+x}-1}{[x]+x}$ where [.] denotes the greatest integer function. *[2016-I]*

9. What is $\lim_{x\to 0^+} f(x)$ equal to?

(a) 1 (b) ln a
(c) $1-a^{-1}$ (d) Limit does not exist

10. What is $\lim_{x\to 0^-} f(x)$ equal to?

(a) 0 (b) ln a
(c) $1-a^{-1}$ (d) Limit does not exist

11. If $\lim_{x\to 0}\phi(x)=a^2$, where $a\neq 0$, then what is $\lim_{x\to 0}\phi\left(\frac{x}{a}\right)$ equal to? *[2016-I]*

(a) a^2 (b) a^{-2}
(c) $-a^2$ (d) $-a$

12. What is $\lim_{x\to 0} e^{-1/x^2}$ equal to? *[2016-I]*

(a) 0 (b) 1
(c) -1 (d) Limit does not exist

13. What is $\lim_{x\to 0}\frac{e^x-(1+x)}{x^2}$ equal to? *[2017-I]*

(a) 0 (b) $\frac{1}{2}$
(c) 1 (d) 2

14. Consider the following statements: *[2017-I]*

1. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a}\{f(x)\,g(x)\}$ exists.
2. If $\lim_{x\to a}\{f(x)\,g(x)\}$ exists, then both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ must exist.

Which of the above statements is/are correct?

(a) 1 only (b) 2 only
(c) Both 1 and 2 (d) Neither 1 nor 2

15. If $\lim_{x\to\frac{\pi}{2}} \frac{\sin x}{x} = l$ and $\lim_{x\to\infty} \frac{\cos x}{x} = m$, then which one of the following is correct? **[2017-II]**

(a) $l = 1, m = 1$
(b) $l = \frac{2}{\pi}, m = \infty$
(c) $l = \frac{2}{\pi}, m = 0$
(d) $l = 1, m = \infty$

16. Consider the function $f(x) = \begin{cases} x^2 \ln|x| & x \neq 0 \\ 0 & x = 0 \end{cases}$. What is $f'(0)$ equal to? **[2018-I]**

(a) 0
(b) 1
(c) –1
(d) It does not exist

17. What is $\lim_{x\to 0} \frac{\tan x}{\sin 2x}$ equal to? **[2018-I]**

(a) $\frac{1}{2}$
(b) 1
(c) 2
(d) Limit does not exist

18. What is $\lim_{h\to 0} \frac{\sqrt{2x+3h}-\sqrt{2x}}{2h}$ equal to? **[2018-I]**

(a) $\frac{1}{2\sqrt{2x}}$
(b) $\frac{3}{\sqrt{2x}}$
(c) $\frac{3}{2\sqrt{2x}}$
(d) $\frac{3}{4\sqrt{2x}}$

19. What is $\lim_{\theta\to 0} \frac{\sqrt{1-\cos\theta}}{\theta}$ equal to? **[2018-II]**

(a) $\sqrt{2}$
(b) $2\sqrt{2}$
(c) $\frac{1}{\sqrt{2}}$
(d) $-\frac{1}{2\sqrt{2}}$

20. What is $\lim_{x\to\frac{\pi}{6}} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1}$ to? **[2018-II]**

(a) $-\frac{1}{2}$
(b) $-\frac{1}{3}$
(c) –2
(d) –3

21. $\lim_{x\to 0} \frac{1-\cos^3 4x}{x^2}$ is equal to **[2019-I]**

(a) 0 (b) 12 (c) 24 (d) 36

22. What is the value of $\lim_{x\to 0} \frac{\sin x^\circ}{\tan 3x^\circ}$? **[2019-II]**

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 1

23. What is $\lim_{x\to 1} \frac{x+x^2+x^3-3}{x-1}$ equal to? **[2020-I & II]**

(a) 1
(b) 2
(c) 3
(d) 6

24. If $\lim_{x\to 1} \frac{x^4-1}{x-1} = \lim_{x\to k} \frac{x^3-k^3}{x^2-k^2}$, where $k \neq 0$, then what is the value of k? **[2020-I & II]**

(a) $\frac{2}{3}$ (b) $\frac{4}{3}$ (c) $\frac{8}{3}$ (d) 4

25. What is $\lim_{x\to 0} \frac{\sin x \log(1-x)}{x^2}$ equal to? **[2020-I & II]**

(a) –1
(b) Zero
(c) $-e$
(d) $-\frac{1}{e}$

26. What is $\lim_{x\to 0} \frac{3^x + 3^{-x} - 2}{x}$ equal to? **[2020-I & II]**

(a) 0
(b) –1
(c) 1
(d) Limit does not exist

27. If $\lim_{x\to a} \frac{a^x - x^a}{x^a - a^a} = -1$ then what is the value of a? **[2021-I]**

(a) –1 (b) 0 (c) 1 (d) 2

28. What is $\lim_{x\to -1} \frac{x^3+x^2}{x^2+3x+2}$ equal to? **[2021-I]**

(a) 0 (b) 1 (c) 2 (d) 3

29. If a differentiable function $f(x)$ satisfies $\lim_{x\to -1} \frac{f(x)+1}{x^2-1} = -\frac{3}{2}$, then what is $\lim_{x\to -1} f(x)$ equal to? **[2021-I]**

(a) $-\frac{3}{2}$ (b) –1 (c) 0 (d) 1

30. What is $\lim_{n\to\infty} \frac{a^n+b^n}{a^n-b^n}$ where $a > b > 1$, equal to? **[2021-II]**

(a) –1
(b) 0
(c) 1
(d) Limit does not exist

31. Let $f(x) = \begin{cases} 1+\frac{x}{2k}, & 0 < x < 2 \\ kx, & 2 \leq x < 4 \end{cases}$

If $\lim_{x\to 2} f(x)$ exists, then what is the value of k? **[2021-II]**

(a) –2 (b) –1 (c) 0 (d) 1

32. If $f(x) = \frac{[x]}{|x|}, x \neq 0$, where $[\cdot]$ denotes the greatest integer function, then what is the right-hand limit of $f(x)$ at $x = 1$? **[2021-II]**

(a) –1
(b) 0
(c) 1
(d) Right-hand limit of $f(x)$ at $x = 1$ does not exist

33. What is $\lim_{x\to 0} x^3 (\text{cosec } x)^2$ equal to? **[2022-I]**

(a) 0
(b) $\frac{1}{2}$
(c) 1
(d) Limit does not exist

34. What is $\lim_{x\to 1} \frac{x^3-1}{\sqrt{x}-1}$ equal to? **[2022-I]**

(a) 0
(b) 3
(c) 6
(d) Limit does not exist

35. What is $\lim_{x\to 0}\frac{x}{\sqrt{1-\cos 4x}}$ equal to? **[2022-II]**

(a) $\frac{1}{2\sqrt{2}}$ (b) $-\frac{1}{2\sqrt{2}}$

(c) $\sqrt{2}$ (d) Limit does not exist

36. What is $\lim_{x\to \frac{\pi}{2}}\frac{4x-2\pi}{\cos x}$ equal to? **[2022-II]**

(a) –4 (b) –2 (c) 2 (d) 4

37. If $f(x)=\frac{x^2+x^2+|x|}{x}$, then what is $\lim_{x\to 0}f(x)$ equal to? **[2022-II]**

(a) 0 (b) 1

(c) 2 (d) $\lim_{x\to 0}f(x)$ does not exist

38. What is $\lim_{h\to 0}\frac{\sin^2(x+h)-\sin^2 x}{h}$ equal to? **[2022-II]**

(a) $\sin^2 x$ (b) $\cos^2 x$

(c) $\sin 2x$ (d) $\cos 2x$

Consider the following for the next three (03) items that follow:

$$f(x)=\begin{vmatrix}\cos x & x & 1\\ 2\sin x & x^2 & 2x\\ \tan x & x & 1\end{vmatrix}s$$

39. What is $f(0)$ equal to? **[2023-I]**

(a) –1 (b) 0 (c) 1 (d) 2

40. What is $\lim_{x\to 0}\frac{f(x)}{x}$ equal to? **[2023-I]**

(a) –1 (b) 0 (c) 1 (d) 2

41. What is $\lim_{x\to 0}\frac{f(x)}{x^2}$ equal to? **[2023-I]**

(a) –1 (b) 0 (c) 1 (d) 2

42. What is $\lim_{x\to 5}\frac{5-x}{|x-5|}$ equal to? **[2023-I]**

(a) –1 (b) 0

(c) 1 (d) Limit does not exist

43. What is $\lim_{x\to 1}\frac{x^9-1}{x^3-1}$ equal to? **[2023-I]**

(a) –1 (b) –3

(c) 3 (d) Limit does not exist

Consider the following for the next two (02) items that follow: [2023-II]

Let $f(x)=\frac{a^{x-1}+b^{x-1}}{2}$ and $g(x)=x-1$.

44. What is $\lim_{x\to 1}\frac{f(x)-1}{g(x)}$ equal to? **[2023-II]**

(a) $\frac{\ln(ab)}{4}$ (b) $\frac{\ln(ab)}{2}$ (c) $\ln(ab)$ (d) $2\ln(ab)$

45. What is $\lim_{x\to 1}f(x)^{\frac{1}{g(x)}}$ equal to? **[2023-II]**

(a) $\sqrt{ab}$ (b) ab (c) $2ab$ (b) $\frac{\sqrt{ab}}{2}$

Consider the following for the next two (02) items that follow:

Let $f(x)=|x|$ and $g(x)=[x]-1$, where [.] is the greatest integer function.

Let $h(x)=\frac{f(g(x))}{g(f(x))}$

46. What is $\lim_{x\to 0+}h(x)$ equal to ? **[2023-II]**

(a) –2 (b) –1 (c) 0 (d) 1

47. What is $\lim_{x\to 0-}h(x)$ equal to ? **[2023-II]**

(a) –2 (b) –1 (c) 0 (d) 2

ANSWER KEY

1. (c)	2. (d)	3. (b)	4. (a)	5. (d)	6. (b)	7. (c)	8. (d)	9. (b)	10. (c)
11. (a)	12. (a)	13. (b)	14. (a)	15. (c)	16. (a)	17. (a)	18. (d)	19. (c)	20. (d)
21. (c)	22. (b)	23. (d)	24. (c)	25. (a)	26. (a)	27. (c)	28. (b)	29. (b)	30. (c)
31. (d)	32. (c)	33. (a)	34. (c)	35. (d)	36. (a)	37. (d)	38. (c)	39. (b)	40. (b)
41. (a)	42. (d)	43. (c)	44. (b)	45. (a)	46. (b)	47. (a)			

EXPLANATIONS

1. (*c*) $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{x}$

$= \lim\limits_{x\to 0}\dfrac{{}^nC_0+{}^nC_1x+{}^nC_2x^2\ldots+{}^nC_nx^n-1}{x}$

$= \lim\limits_{x\to 0}\dfrac{x({}^nC_1+{}^nC_2x+\ldots+{}^nC_nx^{n-1})}{x}$

$= \lim\limits_{x\to 0}{}^nC_1+{}^nC_2x+\ldots{}^nC_nx^{n-1}$

Put $x=0 \Rightarrow {}^nC_1=n$

2. (*d*) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1-\cos x}}$

$= \lim\limits_{x\to 0}\dfrac{x}{\sqrt{1-\left(1-2\sin^2\dfrac{x}{2}\right)}}$

$= \lim\limits_{x\to 0}\dfrac{x}{\sqrt{2\sin^2\dfrac{x}{2}}}=\dfrac{1}{\sqrt{2}}\lim\limits_{x\to 0}\dfrac{x}{\left|\sin\dfrac{x}{2}\right|}$

L.H.L $= f(0-0) = \lim\limits_{h\to 0}\dfrac{x}{\left|\sin\dfrac{x}{2}\right|}$

$= -\dfrac{1}{\sqrt{2}}\lim\limits_{x\to 0}\dfrac{2\left(\dfrac{h}{2}\right)}{\sin\dfrac{h}{2}}$

$= -\dfrac{1}{\sqrt{2}}\times 2\times 1 \qquad \left(\because \lim\limits_{\theta\to 0}\dfrac{\theta}{\sin\theta}=1\right)$

$= -\sqrt{2}$

RHL $= f(0+0) = \lim\limits_{h\to 0} f(0+h)$

$= \dfrac{1}{\sqrt{2}}\lim\limits_{h\to 0}\dfrac{2\left(\dfrac{h}{2}\right)}{\sin\dfrac{h}{2}}=\dfrac{1}{\sqrt{2}}\times 2\times 1$

$=$ LHL $\neq$ RHL $= \sqrt{2}$

Therefore limit does not exist.

3. (*b*) Given equation $\lim\limits_{x\to 0}\dfrac{\log_5(1+x)}{x}$

$\lim\limits_{x\to 0}\dfrac{\log_e(1+x)}{x\log_e 5}\left[\because \log_x y=\dfrac{\log_e y}{\log_e x}\right]$

$= \dfrac{1}{\log_e 5}\lim\limits_{x\to 0}\dfrac{\log_e(1+x)}{x}=\log_5 e$

$\left[\because \lim\limits_{x\to 0}\dfrac{\log_e(1+x)}{x}=1, \log_x y=\dfrac{1}{\log_y x}\right]$

4. (*a*) $\because \lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log_e a$

$\therefore \lim\limits_{x\to 0}\dfrac{5^x-1}{x}=\log_e 5$

5. (*d*) $\lim\limits_{n\to\infty}\dfrac{1+2+3+\ldots+n}{1^2+2^2+3^2+\ldots+n^2}$

$= \lim\limits_{n\to\infty}\dfrac{\dfrac{n(n+1)}{2}}{\dfrac{n(n+1)(2n+1)}{6}}$

$\therefore \lim\limits_{n\to\infty}\dfrac{3}{2n+1}=0$

Sol. (6-7):

$\lim\limits_{x\to\infty}\left(\dfrac{2+x^2}{1+x}-Ax-B\right)=3$

$\Rightarrow \lim\limits_{x\to\infty}\left(\dfrac{2+x^2-Ax-B-Ax^2-Bx}{1+x}\right)=3$

$\Rightarrow \lim\limits_{x\to\infty}\left[\dfrac{(1-A)x^2-(A+B)x+2-B}{1+x}\right]=3$

Applying L'Hospital rule,

$2x(1-A)-(A+B)=3$

Comparing coefficients

$2(1-A)=0$

$\therefore A=1$ and

$-(A+B)=3$

$A+B=-3$

$\therefore B=-3-1=-4$

$A=1, B=-4$

6. (*b*)

7. (*c*)

8. (*d*) $f(x)=\dfrac{\sin(e^{x-2}-1)}{\ln(x-1)}$

$\lim\limits_{x\to 2}\dfrac{\sin(e^{x-2}-1)}{\ln(x-1)}=L$ (let)

It is $\dfrac{0}{0}$ (undefined) condition so using L' hospital's rule

$\Rightarrow L=\lim\limits_{x\to 2}\dfrac{\cos(e^{x-2}-1)\cdot e^{(x-2)}}{1/(x-1)}$

$\Rightarrow L=\lim\limits_{x\to 2}\cos(e^{2-2}-1)\,e^{2-2}\cdot(2-1)$

$\Rightarrow L=\cos(0)\,e^0\cdot 1$

$\Rightarrow L=1$

9. (*b*) Given $f(x)=\dfrac{a^{[x]+x}-1}{[x]+x}$

$\therefore \lim\limits_{x\to 0^+}f(x)=\lim\limits_{h\to 0}\dfrac{a^{[0+h]+(0+h)}-1}{[0+h]+(0+h)}$

$= \lim\limits_{h\to 0}\dfrac{a^{[h]+(h)}-1}{[h]+h}$

$= \lim\limits_{h\to 0}\dfrac{(a^h-1)}{h}=\log_e a$

10. (*c*) $\lim\limits_{x\to 0^-}f(x)=\lim\limits_{h\to 0}\left[\dfrac{a^{[0-h]+(0-h)}-1}{[0-h]+(0-h)}\right]$

$= \lim\limits_{h\to 0}\dfrac{a^{[-h]-h}-1}{[-h]+(-h)}=\lim\limits_{h\to 0}\dfrac{a^{-1-h}-1}{-1-h}$

$= \dfrac{a^{-1-0}-1}{-1-0}=\dfrac{a^{-1}-1}{-1}=\lim\limits_{h\to 0^-}f(x)=(1-a^{-1})$

11. (*a*) $\lim\limits_{x\to 0}\phi(x)=a^2, a\neq 0$

$\Rightarrow \lim\limits_{x\to 0}\phi\left(\dfrac{x}{a}\right)=a^2$

[because function value is constant]

12. (*a*) $\lim\limits_{x\to 0}e^{-\frac{1}{x^2}}=e^{\frac{-1}{0}}=e^{-\infty}=0$

13. (*b*) $\lim\limits_{x\to 0}\dfrac{e^x-(1+x)}{x^2}\left(\dfrac{0}{0}\text{form}\right)$

So, applying L'Hospital role.

$= \dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=\dfrac{1}{2}\times 1=\dfrac{1}{2}.$

14. (*a*) If $\lim\limits_{x\to a}f(x)$ and $\lim\limits_{x\to a}g(x)$ both exists, then $\lim\limits_{x\to a}f(x)\cdot g(x)$ exists. But if $\lim\limits_{x\to a}f(x)\cdot g(x)$ exists, then it is not necessary that both $\lim\limits_{x\to a}f(x)$ and $\lim\limits_{x\to a}g(x)$ exists.

15. (*c*) Given, $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sin x}{x}=l$ and $\lim\limits_{x\to\infty}\dfrac{\cos x}{x}=m$

$\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sin x}{x}=\dfrac{1}{\dfrac{\pi}{2}}=\dfrac{2}{\pi}$ and $\lim\limits_{x\to\infty}\dfrac{\cos x}{x}=0$

$\therefore l=\dfrac{2}{\pi}, m=0$

16. (*a*) Given, $f(x)=\begin{cases}x^2\ln|x| & x\neq 0\\ 0 & x=0\end{cases}$

$f'(0)=\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h}$

$= \lim\limits_{h\to 0}\dfrac{h^2\ln|h|-0}{h}=\lim\limits_{h\to 0}h\ln|h|=0$

17. (*a*) $\lim\limits_{x\to 0}\dfrac{\tan x}{\sin 2x}$

Since $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$ and $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=1$

$= \lim\limits_{x\to 0}\dfrac{\dfrac{\tan x}{x}}{2\left(\dfrac{\sin 2x}{2x}\right)}=\dfrac{1}{2}$

18. (*d*) $\lim\limits_{h\to 0}\dfrac{\sqrt{2x+3h}-\sqrt{2x}}{2h}$

Rationalise the numerator.

$$\lim_{h\to 0}\left(\frac{\sqrt{2x+3h}-\sqrt{2x}}{2h}\times\frac{\sqrt{2x+3h}+\sqrt{2x}}{\sqrt{2x+3h}+\sqrt{2x}}\right)$$

$$=\lim_{h\to 0}\frac{2x+3h-2x}{2h(\sqrt{2x+3h}+\sqrt{2x})}$$

$$=\frac{3}{2(\sqrt{2x+0}+\sqrt{2x})}=\frac{3}{4\sqrt{2x}}$$

19. (*c*) $\lim\limits_{\theta\to 0}\dfrac{\sqrt{1-\cos\theta}}{\theta}=\lim\limits_{\theta\to 0}\dfrac{\sqrt{2}\sin(\theta/2)}{\theta}$

$$=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$$

20. (*d*) $\lim\limits_{x\to\frac{\pi}{6}}\dfrac{(\sin x+1)(2\sin x-1)}{(\sin x-1)(2\sin x-1)}=\dfrac{\frac{1}{2}+1}{\frac{1}{2}-1}=-3$

21. (*c*) Given:

$$\lim_{x\to 0}\frac{1-\cos^3 4x}{x^2}$$

$$=\lim_{x\to x}\frac{-3\cos^2 4x(-\sin 4x)4}{2x}$$

$$=\lim_{x\to x}\frac{12\cos^2 4x\sin 4x}{(2x)}\times 2$$

$$=24\,(1)=24$$

22. (*b*) It is given that: $\lim\limits_{x\to 0}\dfrac{\sin x^\circ}{\tan 3x^\circ}$

We know that: $180^\circ=\pi$ radian

Then, we get: $1^\circ=\dfrac{\pi}{160^\circ}$ radian

$x^\circ=\dfrac{\pi x}{180^\circ}$ radian

The value then obtained is:

$$\lim_{x\to 0}\frac{\sin x^\circ}{\tan 3x^\circ}=\lim_{x\to 0}\frac{\sin\frac{\pi x}{180^\circ}}{\tan\frac{3\pi x}{180^\circ}}$$

$$=\lim_{x\to 0}\frac{\dfrac{\sin\frac{\pi x}{180^\circ}}{\frac{\pi x}{180^\circ}}\times\frac{\pi x}{180^\circ}}{\dfrac{\tan\frac{3\pi x}{180^\circ}}{\frac{3\pi x}{180^\circ}}\times\frac{3\pi x}{180^\circ}}=\frac{1}{3}$$

$$\left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\ \&\ \lim_{x\to 0}\frac{\tan x}{x}=1\right]$$

23. (*d*) $\lim\limits_{x\to 1}\dfrac{x+x^2+x^3-3}{x-1}$

$$\lim_{x\to 1}\frac{x+x^2+x^3-3}{x-1}$$

$$=\lim_{x\to 1}\frac{(x-1)+(x^2-1)+(x^3-1)}{x-1}$$

$$=\lim_{x\to 1}\frac{(x-1)+(x+1)(x-1)+(x-1)(x^2+x+1)}{x-1}$$

$$=\lim_{x\to 1}1+(x+1)+(x^2+x+1)=6$$

24. (*c*) Given:-

$$\lim_{x\to 1}\frac{x^4-1}{x-1}=\lim_{x\to k}\frac{x^3-k^3}{x^2-k^2}$$

First, we we solve the left hand side part

$$\lim_{x\to 1}\frac{x^4-1}{x-1}=\lim_{x\to 1}\frac{(x^2-1)(x^2+1)}{x-1}$$

$$=\lim_{x\to 1}(x+1)(x^2+1)=4$$

R.H.S

$$\lim_{x\to k}\frac{x^3-k^3}{x^2-k^2}=\lim_{x\to k}\frac{(x-k)(x^2+xk+k^2)}{(x+k)(x-k)}$$

$$=\lim_{x\to k}\frac{(x^2+xk+k^2)}{(x+k)}=\frac{(k^2+k^2+k^2)}{(k+k)}=\frac{3k}{2}$$

Since, L.H.S = R.H.S

$$4=\frac{3k}{2}\Rightarrow k=\frac{8}{3}$$

25. (*a*) Given,

$$\lim_{x\to 0}\frac{\sin x\log(1-x)}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\log(1-x)}{x}$$

$$=1\cdot\lim_{x\to 0}\frac{\log(1-x)}{x}=\lim_{x\to 0}\frac{\log(1-x)}{x}$$

Since on applying the limit, the value is undefined,

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{\log(1-x)}{x}$$

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{\frac{-1}{(1-x)}}{1}=-1$$

26. (*a*) It is given that -

$$\lim_{x\to 0}\frac{3^x+3^{-x}-2}{x}$$

Apply L'Hospital's rule -

$$=\lim_{x\to 0}\left(\frac{\ln(3)\cdot 3^x-\ln(3)\cdot 3^{(-x)}}{1}\right)$$

Substitute the value of $x=0$.

$$=\frac{\ln(3)\cdot 3^0-\ln(3)\cdot 3^{(-0)}}{1}$$

$$=3^0\ln(3)-3^{-0}\ln(3)$$

$$=\ln(3)-\ln(3)=0$$

27. (*c*) $\lim\limits_{x\to a}\dfrac{a^x-x^a}{x^a-a^a}=-1$

$$\Rightarrow\frac{a^a\log_e a-a\cdot a^{a-1}}{a\cdot a^{a-1}}=-1$$

(L' Hospital' rule)

$$\Rightarrow\log_e a=0\Rightarrow a=e^0=1$$

28. (*b*) **Hint:** Use L'Hospital's rule

29. (*b*) We have, $\lim\limits_{x\to -1}\dfrac{f(x)+1}{x^2-1}=\dfrac{-3}{2}$

$\because\lim\limits_{x\to -1}\dfrac{f(x)+1}{x^2-1}$ has denominator equal to 0 at $x=-1$

$$\Rightarrow\lim_{x\to -1}f(x)+1=0\Rightarrow\lim_{x\to -1}f(x)=-1$$

30. (*c*) Given,

$$\lim_{n\to\infty}\frac{a^n+b^n}{a^n-b^n}\text{ where }a>b>1$$

$$\Rightarrow\lim_{n\to\infty}\frac{a^n\left[1+\left(\frac{b}{a}\right)^n\right]}{a^n\left[1-\left(\frac{b}{a}\right)^n\right]}=1$$

$$\left[\because\frac{b}{a}<1\Rightarrow\left(\frac{b}{a}\right)^\infty=0\right]$$

31. (*d*) **Hint:** $\lim\limits_{x\to 2^-}f(x)=\lim\limits_{x\to 2^+}f(x)$

32. (*c*) Given that, $f(x)=\dfrac{[x]}{|x|},x\neq 0$

$$\text{RHL}=\lim_{x\to 1^+}\frac{[x]}{|x|}$$

$x=1+h$, where $h\to 0$

$$\text{RHL}=\lim_{h\to 0}=\frac{[1+h]}{|1+h|}=\frac{1}{|1+0|}=1$$

33. (*a*) $\lim\limits_{x\to 0}x^3(\operatorname{cosec}x)^2=\lim\limits_{x\to 0}\dfrac{x^3}{\sin^2 x}$

$$=\lim_{x\to 0}\frac{x^2}{\sin^2 x}\cdot x=1\times 0=0$$

34. (*c*) $\lim\limits_{x\to 1}\dfrac{x^3-1}{\sqrt{x}-1}=\lim\limits_{x\to 1}\dfrac{3x^2}{\frac{1}{2\sqrt{x}}}$ [Using L' Hospital's rule]

$$=3\times 2=6$$

35. (*d*) L.H.L. =

$$\lim_{x\to 0^-}\frac{x}{\sqrt{1-\cos 4x}}=\lim_{x\to 0^-}\frac{x}{\sqrt{2}\,|\sin 2x|}$$

$$=\lim_{x\to 0^-}\frac{2x}{2\sqrt{2}\sin 2x}=\frac{-1}{2\sqrt{2}}$$

R.H.L. =

$$\lim_{x\to 0^+}\frac{x}{\sqrt{1-\cos 4x}}=\lim_{x\to 0^+}\frac{x}{\sqrt{2}\,|\sin 2x|}$$

$$= \lim_{x\to 0^+} \frac{2x}{2\sqrt{2}\sin 2x} = \frac{1}{2\sqrt{2}}$$

Thus, L.H.L. ≠ R.H.S. Hence, limit does not exist.

36. (*a*) [**Hint:** Use L-Hospital rule.]

37. (*d*) $f(x) = \frac{x^2+x+x}{x} = x+2$, when $x > 0$

and $f(x) = \frac{x^2+x-x}{x} = x$, when $x < 0$

$\therefore \lim_{x\to 0^+} f(x) = 2$ and $\lim_{x\to 0^-} f(x) = 0$

Hence, limit doesn't exist

38. (*c*) Use *L*-Hospital, differentiate w.r.t *h*

39. (*b*) $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$

$$\Rightarrow f(x) = x\begin{vmatrix} \cos x & 1 & 1 \\ 2\sin x & x & 2x \\ \tan x & 1 & 1 \end{vmatrix}$$

$$\Rightarrow f(x) = x\begin{vmatrix} \cos x & 1 & 0 \\ 2\sin x & x & x \\ \tan x & 1 & 0 \end{vmatrix} \{C_3 = C_3 - C_2\}$$

$$\Rightarrow f(x) = x^2\{-x(\cos x - \tan x)\}$$

$$\Rightarrow f(x) = x^2(\tan x - \cos x)$$

$$\Rightarrow x = 0 \Rightarrow f(0) = 0$$

40. (*b*) $\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{x^2(\tan x - \cos x)}{x}$

$$= \lim_{x\to 0} x(\tan x - \cos x)$$

$$= 0$$

41. (*a*) $\lim_{x\to 0} \frac{f(x)}{x^2} = \lim_{x\to 0} \frac{x^2(\tan x - \cos x)}{x^2} = -1$

42. (*d*) We have, $f(x) = \frac{5-x}{|x-5|}$

L.H.L

$$= \lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} \frac{5-x}{|x-5|} = \lim_{x\to 5^-}\left(-\frac{5-x}{x-5}\right)$$

$$= \lim_{x\to 5^-}(-(-1)) = \lim_{x\to 5^-} 1 = 1$$

R.H.L

$$= \lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} \frac{5-x}{|x-5|} = \lim_{x\to 5^+} \frac{5-x}{x-5}$$

$$= \lim_{x\to 5^+}(-1) = -1$$

Thus, L.H.L ≠ R.H.L at $x = 5$

∴ Limit does not exist at $x = 5$.

43. (*c*) $\lim_{x\to 1} \frac{x^9-1}{x^3-1} = \lim_{x\to 1} \frac{(x^3-1)\left((x^3)^2 + x^3\times 1 + 1^2\right)}{(x^3-1)}$

$$= \lim_{x\to 1}(x^6 + x^3 + 1) = 1+1+1 = 3$$

44. (*b*) Given, $f(x) = \frac{a^{x-1}+b^{x-1}}{2}$ and $g(x) = x-1$

to find the value of $\lim_{x\to 1} \frac{f(x)-1}{g(x)}$ we can take the derivate of both numerator and denominator

$$= \lim_{x\to 1} \frac{\frac{d}{dx}(f(x)-1)}{\frac{d}{dx}(g(x))} = \lim_{x\to 1} \frac{\frac{d}{dx}(a^{x-1}+b^{x-1})}{\frac{d}{dx}(x-1)}$$

$$= \lim_{x\to 1} \frac{\frac{1}{2}[\ln(a).a^{x-1} + \ln(b).b^{x-1}]}{1}$$

(because $\frac{d}{dx}\ a^x = \ln(a).\ a^x$)

$$= \frac{1}{2}\lim_{x\to 1}\ [\ln(a)\cdot ax^{-1} + \ln(b)\cdot bx^{-1}]$$

now we can put $x = 1$

$$= \frac{1}{2}\ [\ln(a)\cdot a^0 + \ln(b)\cdot b^0] = \frac{1}{2}\ [\ln(a) + \ln(b)]$$

$$= \frac{1}{2}\ \ln(ab) = \frac{\ln(ab)}{2}$$

45. (*a*) Given, $f(x) = \frac{a^{x-1}+b^{x-1}}{2}$

and $g(x) = x - 1$

we have to find $\lim_{x\to 1}[f(x)^{\frac{1}{g(x)}}]$

now, $\lim_{x\to 1} e^{[\ln(f(x)^{\frac{1}{g(x)}})]}$

(because $x = e^{\ln(x)}$)

$$= e^{\lim_{x\to 1}[\ln(f(x)^{\frac{1}{g(x)}})]}$$

if we solve the exponent then,

$$\lim_{x\to 1}[\ln(f(x)^{\frac{1}{g(x)}})] = \lim_{x\to 1}\left[\frac{1}{g(x)}\ln(f(x))\right]$$

$$\lim_{x\to 1}\left[\frac{\ln(f(x))}{g(x)}\right]$$

now we can take the derivative of numerator and denominator

$$\lim_{x\to 1}\left[\frac{\frac{1}{f(x)}\cdot f'(x)}{g'(x)}\right] \quad \left(\because \frac{d}{dx}\ \ln(x) = \frac{1}{x}\frac{d}{dx}\right)$$

$$= \lim_{x\to 1}\left[\frac{f'(x)}{f(x)\cdot g'(x)}\right]$$

$$= \lim_{x\to 1}\left[\frac{\frac{1}{2}(\ln(a)\cdot a^{x-1} + \ln(b)\cdot b^{x-1})}{\left(\frac{a^{x-1}+b^{x-1}}{2}\right)\cdot 1}\right]$$

$(\because g'(x)=1)$

now, put $x = 1$ then,

$$= \left[\frac{\frac{1}{2}(\ln(a)\cdot a^0 + \ln(b)\cdot b^0)}{\left(\frac{a^0+b^0}{2}\right)}\right]$$

$$= \left[\frac{\frac{1}{2}(\ln(a) + \ln(b))}{\left(\frac{2}{2}\right)}\right] = \frac{\ln(a\cdot b)}{2}$$

now this value we can put in the exponent of *e*

$$e^{\lim_{x\to 1}[\ln(f(x)^{\frac{1}{g(x)}})]} = e^{\frac{\ln(ab)}{2}}$$

$$(ab)^{\frac{1}{2}} = \sqrt{ab}$$

$\lim_{x\to 1} f(x)^{\frac{1}{g(x)}}$ ie equal to $\sqrt{ab}$

Hence, option (*a*) is correct.

46. (*b*) Given, $f(x) = |x|$ and $g(x) = [x] - 1$

now, $f(g(x)) = f([x] - 1)$

$= |[x] - 1|$

$= [x - 1]$

and $g(f(x)) = g(|x|)$

$= |[x]| - 1$

now, $\lim_{x\to 0^+} h = \lim_{x\to 0^+} \frac{f(g(x))}{g(f(x))}$

$$= \lim_{x\to 0^+} \frac{|[x]-1|}{[|x|]-1}$$

as we know that, for positive value close to 0 $[x] = 0$ and $|x| = 0$

therefore $\lim_{x\to 0^+} \frac{|0-1|}{|0|-1} = \frac{|-1|}{-1}$

$$\frac{1}{-1} = -1$$

47. (*a*) given, $f(x) = |x|$ and $g(x) = [x]$

now, $f(g(x)) = f([x]-1) = |[x]-1| = [x-1]$

and $g(f(x)) = g(|x|)$

$= |[x]| - 1$

now, $\lim_{x\to 0^-} h = \lim_{x\to 0^-} \frac{f(g(x))}{g(f(x))}$

$$= \lim_{x\to 0^-} \frac{|[x]-1|}{[|x|]-1}$$

as we know that, for negative value close to 0 , $[x] = -1$ and $|x| = 0$

$$= \frac{|-1-1|}{[0]-1} = \frac{|-2|}{-1} = \frac{2}{-1} \Rightarrow -2$$