Chapter 6

Numerical Methods

CHAPTER HIGHLIGHTS

- *Numerical methods*
- *Accuracy and precision*
- *Curve fitting*
- *Numerical integration*
- *Numerical solutions of ordinary differential equations*
- *Multi-step methods*
- *Runge–Kutta methods*
- *Predictor-corrector methods*

Numerical Methods

We encounter problems in Engineering mathematics for which analytical methods are not available to find solutions. Further, it may be sufficient in engineering applications to find approximate solutions. The numerical methods offer us approximate solutions.

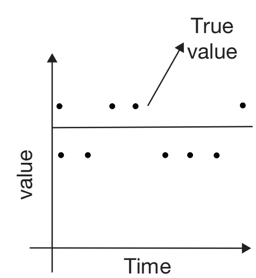
1. Methods for finding roots of algebraic or transdental equations
2. Solutions to system of linear equation
3. Numerical Integration
4. Numerical solutions of ordinary differential equation.

Accuracy and Precision

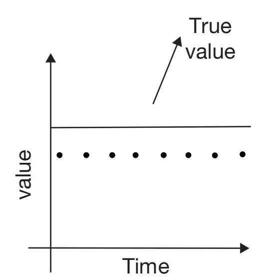
Solutions of problems computed by numerical methods are approximate. Errors associated with calculations can be characterized with reference to accuracy and precision.

Accuracy: Accuracy refers to how closely a computed value agrees with the true value.

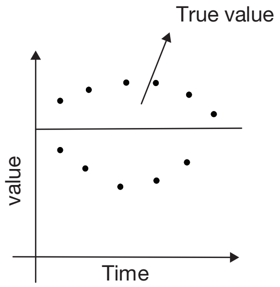
Precision: Precision refers to how closely computed values agree with each other after repeated iterations. The following figures illustrate the difference between accuracy and precision where the horizontal line denote the true value (or) actual value of the solution where as the dots denote the values computed by a numerical method.



Accurate but not precise



Precise but not accurate



Neither accurate nor precise

Errors in the solutions obtained by numerical methods: As the numerical methods give approximate solutions, these solutions contain errors.

Let x denote the actual value (or) true value and let $\bar{x}$ denote an approximate value of the solution obtained by a numerical method.

Error $= \in = x - \bar{x}$

Absolute error $= |\in| = |x - \bar{x}|$

Relative error $= \in_r = \dfrac{|\in|}{|x|} = \dfrac{|x-\bar{x}|}{|x|}$

Percentage error $= \in_p = \in_r \times 100 = \dfrac{|x-\bar{x}|}{x} \times 100.$

Types of Errors

Inherent Error

The error which is already present in the statement of the problem before its solution, is called the inherent error.

This type of errors arise due to any one or more of the following reasons.

- Wrong formulation of the problem
- Unsuitable solution procedure
- Invalid assumptions in the formulation
- Inaccurate data

Round off Error

Real numbers such as $\frac{5}{6}, \sqrt{2}, \pi$, etc., contain an infinite number of digits when expressed in decimal form. In general, in scientific and engineering computations, a real number x is represented as $x = \pm\, 0.\, d_1 d_2 d_3 \ldots d_n \ldots \times 10^k$, known as floating point form of x.

(where $d_1, d_2, \ldots d_n \ldots$ are all digits from 0 to 9 and k is a non zero integer). Each digit $d_1, d_2 \ldots$ other than the leading zeros (the zeros that occur before the first non-zero digit) is called a significant digit. As its not possible to retain infinite number of digits in a number, we round off the number to, say n significant digits.

To round off a number to n significant digits, proceed as follows:

Possibility	Procedure to Follow
The $(n+1)$th digit is less than 5 (OR) The $(n+1)$th digit is equal to 5 and the nth digit is even	Discard all the digits to the right of nth digit and leave the nth digit as it is
The $(n+1)$th digit is greater than 5 (OR) The $(n+1)$th digit is equal to 5 and the nth digit is odd.	Discard all the digits to the right of nth digit and increase the nth digit by 1.

For example,

Consider the number 25.31465.

When written in floating point form

$25.31465 = 0.2531465 \times 10^2$

$\approx 0.253146 \times 10^2$ (Rounded off to six significant digits)

$\approx 0.25315 \times 10^2$ (Rounded off to five significant digits)

$\approx 0.2532 \times 10^2$ (Rounded off to four significant digits)

$\approx 0.253 \times 10^2$ (Rounded off to three significant digits)

Definition The difference between the true value and its rounded off value is called the rounded off error.

- If x is the true value and x^* is its rounded off value such that $|x - x^*| \le 0.5 \times 10^{-m}$ (OR) $|x - x^*| \le 5 \times 10^{-(m+1)}$ then x^* is said to denote x correct to m significant digits.

Truncation Error The error in a method, which occurred because some series (finite or infinite) is truncated to a fewer (and finite) number of terms is called the truncation error.

For instance,

Let $f(x) = f(x_0) + (x - x_0) f'(x_0) + \dfrac{(x-x_0)^2}{2!} f''(x_0) + \cdots +$

$$\frac{(x-x_0)^{m-1}}{(m-1)} f(m-1)(x_0) + \frac{(x-x_0)^m}{m} f^{(m)}(x_0) + \cdots \infty \tag{1}$$

denote the Taylor's series expansion of $f(x)$ about $x = x_0$.

If we retain the first m terms, we get

$f(x) \approx$

$$f(x_0) + (x-x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots + \frac{(x-x_0)^{m-1}}{(m-1)!} f^{(m-1)}(x_0) \tag{2}$$

where the series of infinite terms $\dfrac{(x-x_0)^m}{m!}$

$$f^{(m)}(x_0) + \frac{(x-x_0)^{m-1}}{(m+1)!} f^{(m+1)}(x_0) + \cdots \infty \tag{3}$$

is neglected.

The first term in this neglected part of the series is called the principal part of the truncation error or simply the truncation error.

$\therefore$ Truncation error $= TE = \dfrac{(x-x_0)^m}{m!} f^{(m)}(\xi);\ x_0 < \xi < x$

As ξ is an unknown function of x, we have

$|TE| \le \dfrac{1}{m!} \max[|(x-x_0)^m| \, M_m]$

where $M_m = [a, b] \max |f^{(m)}(x)|$

SOLVED EXAMPLES

Example 1

If the number $\frac{\pi}{4} = 0.785398163$ is approximated by $\frac{11}{14}$, then

(i) Find the number of digits upto which, this approximation is accurate.

(ii) Find the absolute and the percentage errors.

Solution

Given $\frac{\pi}{4} = 0.785398163$

Let $x = \frac{\pi}{4} = 0.785398163$ (Exact value) and $\bar{x} = \frac{11}{14}$

$= 0.785714285 \left(\text{Approximate value of } \frac{\pi}{4}\right)$

(i) $|x - \bar{x}| = \left|\frac{\pi}{4} - \frac{11}{14}\right|$

$= |0.785398163 - 0.785714285|$

$= 3.16122 \times 10^{-4}$

$\leq 5 \times 10^{-4}$

$\therefore$ The approximation $\frac{11}{14}$ to $\frac{\pi}{4}$ is accurate upto three significant digits

(ii) Absolute error $= |x - \bar{x}|$

$$= \frac{11}{14} - \frac{\pi}{4} = 3.16122 \times 10^{-4}$$

Percentage error $= \frac{|x - \bar{x}|}{x} \times 100 = 0.04\%.$

Example 2

Using the Taylor's series expansion about $x = 0$, find a second degree polynomial approximation to $f(x) = \sqrt{1+3x}$. Also find the maximum error for this approximation when $x \in [0, 1]$.

Solution

We know that the Taylor's series expansion of $f(x)$ about $x = 0$ is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \cdots \infty \quad (1)$$

$\therefore$ Considering the terms upto second degree, we have

$$f(x) \approx f(0) + x f'(0) + \frac{x^2}{2!} f''(0) \quad (2)$$

Here $f(x) = \sqrt{1+3x} \Rightarrow f(0) = 1$

$$f'(x) = \frac{3}{2\sqrt{1+3x}} \Rightarrow f'(0) = \frac{3}{2}$$

$$f''(x) = \frac{-9}{4(1-3x)^{\frac{3}{2}}} \Rightarrow f''(0) = \frac{-9}{4}$$

and $f'''(x) = \frac{27}{8} \times \frac{1}{(1+3x)^{\frac{5}{2}}}$

$\therefore$ Substituting these in (2), we get

$$f(x) = \sqrt{1+3x} \approx 1 + \frac{3}{2} \times -\frac{9}{4} \frac{x^2}{2!}$$

$$= 1 + \frac{3}{2} \times -\frac{9}{8} x^2$$

Truncation error $= \frac{x^3}{3!} f'''(0)$

$$\leq \frac{1}{3!}\left(\underset{0 \leq x \leq 1}{\text{Max}} \quad x^3\right)$$

$$\left(\underset{0 \leq x \leq 1}{\text{Max}} \frac{27}{8(1+3x)^{5/2}}\right)$$

$$= \frac{1}{3!}(1)\left(\frac{27}{8}\right)$$

$= 0.5625.$

Methods for Finding the Real Roots (Zeros) of $f(x) = 0$

The equation of the form $f(x) = 0$ is called an Algebraic or Transcendental according as $f(x)$ is purely a polynomial in x or contains some other functions such as exponential, logarithmic and trigonometric functions etc.

Examples:

1. $x^9 + 8x^5 - 4x^3 - 11x + 3 = 0 \rightarrow$ Algebraic equation
2. $10x^4 - \log(x^2 - 3) + e^{-x}\sin x + \tan^2 x = 0 \rightarrow$ Transcendental equation

In this chapter, we obtain the solution of an equation $f(x) = 0$, i.e., we mean to find the zeros of $f(x)$.

We shall now discuss few methods to find the real roots of both algebraic (with numerical coefficients) and transcendental equations.

We first find an approximate value of the root of the given equation and then successively improve it to some desired degree of accuracy.

We start with an initial approximate value, say x_0, and then find the better approximations successively $x_1, x_2, x_3 \ldots, x_n$ by repeating the same method.

If the successive approximations at each step of a method approach the root more and more closely, we say that the method converges.

The Intermediate Value Theorem

If a function $f(x)$ is continuous between a and b and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one root say α between a and b of the equation

$f(x) = 0$, i.e., $f(\alpha) = 0$

NOTE

Root 'α' of $f(x) = 0$, will be unique in (a, b) if $f'(x)$ has the same sign in (a, b) (i.e., $f'(x) > 0$ or $f'(x) < 0$ in $a < x < b$)

Relations between Roots and Coefficients

An nth order equation has n roots. Corresponding to every root, there is a factor. If α is a root of $f(x) = 0$, then $x - \alpha$ is a factor of $f(x)$. Sometimes $(x - \alpha)^2$ may also be a factor. In such a case, α is said to be a double root. Similarly equations can have triple roots, quadruple roots and roots of multiplicity m. If m is the greatest value of k, for which $(x - \alpha)^k$ is a factor of $f(a)$, then α is said to be a root of multiplicity m. If all the roots are counted by taking their multiplicity into account, the number of roots is equal to n, the degree of the equation.

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ (not necessarily distinct) are the roots of $f(x) = 0$, then

$$f(x) = a_n(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$$
$$= a_n [x^n - S_1 x^{n-1} + S_2 x^{n-2} + \cdots + (-1)^n S_n]$$

Where

S_1 = The sum of the roots

S_2 = The sum of the products of the roots taken 2 at a time

S_3 = The sum of the product of the roots taken 3 at a time and so on.

S_n = The 'sum' of the product of the roots taken n (or all) at a time. Thus, S_n is a single term.

$S_n = \alpha_1 \alpha_2 \ldots \alpha_n$

Let us write down the polynomial $f(x)$ in two forms:

The standard form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$$

In terms of the roots of the corresponding equation.

$$f(x) = a_n [x_n - S_1 x^{n-1} + S_2 x^{n-2} + \cdots + (-1)^{x-1} S_{n-1} x + (-1)^{n S_n}]$$

These polynomials are identically equal, i.e., equal for all values of x. Therefore the corresponding coefficients are equal. The sum of the roots $S_1 = -\dfrac{a_{n-1}}{a_n}$.

The sum of the products of the roots, taken two at a time, $S_2 = \dfrac{a_{n-2}}{a_n}$.

The sum of the products of the roots, taken three at a time, $S_3 = -\dfrac{a_{n-3}}{a_n}$ and so on.

The 'sum' of the 'products' of the roots taken $m (m \le n)$ at a time $S_m = \Sigma \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_m = (-1)^m \dfrac{a_{n-m}}{a_n}$.

$$\therefore \quad S_n = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n = (-1)^n \frac{a_0}{a_n}$$

For example, consider the polynomial equation

$$(x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6 = 0$$

(We can see immediately that the roots are 1, 2, 3)

The sum of roots $= (1 + 2 + 3) = -\dfrac{(-6)}{1}$

The sum of the products of the roots, taken two at a time

$S_2 = 1(2) + 1(3) + 2(3) = 11 = \dfrac{11}{1}$

We can drop the word 'sum' and 'products' for the last relation, because there is only one term (only one product).

The product $= 1(2)(3) = 6 = -\dfrac{(-6)}{1}$.

Roots of Equations and Descartes' Rule

If the coefficients are all real and the complex number z_1, is a root of $f(x) = 0$, then the conjugate of z_1, viz, $\bar{z}_1$ is also a root of $f(x) = 0$. Thus, for equations with real, coefficients, complex roots occur in pairs.

A consequence of this is that any equation of an odd degree must have at least one real root.

The number of roots is related to very simple properties of the equation as illustrated below.

Let α_1 be a positive root, i.e., $x - \alpha_1$, is a factor.

Let α_2 be another positive root, i.e., $x^2 - (\alpha_1 + \alpha_2)x + \alpha_1 \alpha_2$ is a factor.

Let α_3 be another positive root i.e., $x^3 - (\alpha_1 + \alpha_2 + \alpha_3)x_2 + (\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1)x - \alpha_1 \alpha_2 \alpha_3$ is a factor.

We note that every positive root introduces a sign change in the polynomial. For 1 root, there is 1 sign change (the coefficient of x is positive and $-\alpha_1$ is negative)

The second root results in a second sign change [$x^2 - (\alpha_1 + \alpha_2)x + \alpha_1 \alpha_2$ has 2 sign changes] and so on.

But every sign change need not correspond to a real positive root. (For example, $x^2 - 2x + 4$ has two sign changes but the corresponding equation $x^2 - 2x + 4 = 0$ has no real roots.

The number of positive roots is at the most equal to the number of sign changes. It could also be less than that by 2, 4…, i.e., if there are k sign changes in $f(x)$, the number of positive roots could be $k, k - 2, k - 4, \ldots$

This is called **Descartes' Rule of Signs**. This rule can be extended to negative roots as follows. The number of negative roots of $f(x) = 0$ is equal to the number of positive roots of $g(x) = f(-x) = 0$

For example, consider $f(x) = x^5 - 3x^3 + 6x^2 - 28x + 24$. There are 4 sign changes in $f(x)$

$\therefore$ The number of positive roots could be 4, 2 or 0.

Consider $g(x) = f(-x)(-x)^5 - 3(-x)^3 + 6(-x)^2 - 28(-x) + 24$
$= -x^5 + 3x^3 + 6x^2 + 28x + 24$

There is only one sign change in $f(-x)$.

$\therefore$ The number of negative roots of $f(x) = 0$ is 1. (It can't be $-1, -3, \ldots$).

The following table shows the various possibilities for the roots.

Negative	Positive	Complex
1	4	0
1	2	2
1	0	4

We have considered one specific equation and this specific equation has 5 specific roots. We can use more advanced techniques to find the actual roots. But even without that, using only Descartes rule, we expect exactly one of the 3 situations shown in the table above.

Example 3

Find the nature of roots of the equation, $f(x) = x^3 + x - 2 = 0$.

Solution

There is only 1 change of sign in $f(x)$.

We know that when $f(x)$ has r changes of sign then $f(x)$ has $r, r-2, r-4, \ldots$ positive roots.

$\therefore$ $f(x) = 0$ has one positive root.

Now $f(-x) \equiv -x^3 - x - 2 = 0$. $q = 0$

Since there is no change of sign in $f(-x)$, $f(x)$ has no negative roots. The number of complex roots is even.

$\therefore$ The equation has one positive root, and two complex roots.

Hence $f(x) = 0$ has 1 real root and two complex roots.

Example 4

How many non real-roots does the equation $x^4 - 2x^2 + 3x - 2 = 0$ have?

Solution

Let $f(x) = x^4 - 2x^2 + 3x - 2$

$f(x) = 0$ has 3 sign changes

$\therefore$ $f(x)$ has 3 or 1 positive roots.

$f(-x) = x^4 - 2x^2 - 3x - 2$

$\therefore$ $f(-x)$ has one sign change

$\therefore$ $f(x)$ has exactly one negative root.

As the sum of the co-efficient of $f(x)$ is zero,

$x = 1$ is a root of $f(x) = 0$

$\therefore$ $f(x) = (x-1)(x^3 + x^2 - x + 2) = (x-1)\,f_1(x)$. By trial, $f_1(-2) = 0$

$\therefore$ $f_1(x) = (x+2)(x^2 - x + 1)$

We can see that $x^2 - x + 1 = 0$ has two non-real roots.

$\therefore$ $f(x)$ has one positive, one negative and two non-real roots.

Example 3

If $p - q, p, p + q$ are the roots of the equation $x^3 - 18x^2 + 99x - 162 = 0$, then find the values of p and q.

Solution

Given $p - q, p, p + q$ are the roots of the equation.

$\therefore$ The sum of the roots is $(p - q) + p + (p + q) = 18$

$\Rightarrow \quad 3p = 18 \quad \Rightarrow \quad p = 6$

and the product of the roots is $(p - q)\,p(p + q) = 162$

$$p^2 - q^2 = \frac{162}{6} = 27 \quad \Rightarrow \quad 36 - q^2 = 27$$

$\Rightarrow q = \pm 3 \quad \therefore \quad p = 6$ and $q = \pm 3$.

Bisection Method (Bolzano Method) or (Halving Method)

Consider the equation $f(x) = 0$ (1)

If $f(x)$ is continuous between a and b and $f(a)\,f(b) > 0$, then there exists one root between a and b. Let $f(a)$ be negative and $f(b)$ be positive. The bisection method isolates the root in $[a, b]$ by halving process, approximately dividing the given interval $[a, b]$ into two, four, eight, etc. equal parts.

Thus, the first approximation to the root is given by:

$$x_0 = \frac{a+b}{2}$$

a — — —•— — —b

$\frac{a+b}{2}$

If $f(x_0) = 0$, then x_0 is a root, otherwise the root lies either between a and x_0 or x_0 and b depending on whether $f(x_0)$ is positive or negative. We again bisect the interval and repeat the process until the root is obtained to desired accuracy.

Example 4

Find a real root of the equation $f(x) = x^3 - 2x^2 + 3x - 1$ on the interval (0, 1) using bisection method with four iterations.

Solution

We have $f(0) = -1 < 0$ and

$$f(1) = 1 - 2 + 3 - 1 = 1 > 0$$

$\therefore$ A root lies between 0 and 1

$\therefore$ The first approximation to the root is $\frac{0+1}{2} = 0.5$. Now $f(0.5) = (0.5)^3 - 2(0.5)^2 + 3(0.5) - 1 = 0.125 > 0$ and $f(0) < 0$

$\therefore$ The root lies between 0 and 0.5. The second approximation to the root is $\frac{0 + 0.5}{2} = 0.25$.

Now $f(0.25) = (0.25)^3 - 2(0.25)^2 + 3(0.25) - 1$
$= -0.359 < 0$ and $f(0.5) > 0$

$\therefore$ The root lies between 0.25 and 0.5.

$\therefore$ The third approximation to the root is $\frac{0.25 + 0.5}{2} = \frac{0.75}{2} = 0.375$.

Now $f(0.375) = (0.375)^3 - 2(0.375)^2 + 3(0.375) - 1$

$$= -0.103 < 0 \text{ and } f(0.5) > 0$$

∴ The root lies between 0.375 and 0.5.

∴ The fourth approximation to the root is $\frac{0.375 + 0.5}{2}$

$$= \frac{0.875}{2} = 0.4375.$$

Convergence of Bisection Method

If $x_1, x_2, x_3, \ldots, x_n$ is the sequence of midpoints obtained by bisection method, then $|c - x_n| \le \frac{b-a}{2^n}$, $n = 1, 2, 3 \ldots$ where 'c' is between a and b.

NOTE

In bisection method, the convergence is very slow but definite. The order of convergence is linear or of first order.

Regula Falsi Method or (The Method of False Position)

In this method, to find the real root of the equation $f(x) = 0$, we consider a sufficiently small interval (a, b), $a < b$ such that $f(a)$ and $f(b)$ will have opposite signs. This implies a root lies between a and b according to intermediate value theorem. Also the curve $y = f(x)$ will meet the X-axis at some point between $A\,[a, f(a)]$ and $B\,[b, f(b)]$. The equation of the chord joining $A\,[a, f(a)]$ and $B\,[b, f(b)]$ is given by:

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a) \quad (1)$$

The point of intersection of the chord (1) with X-axis is given by $y = 0$ in Eq. (1)

$$-f(a) = \frac{f(b) - f(a)}{b - a}(x - a),$$

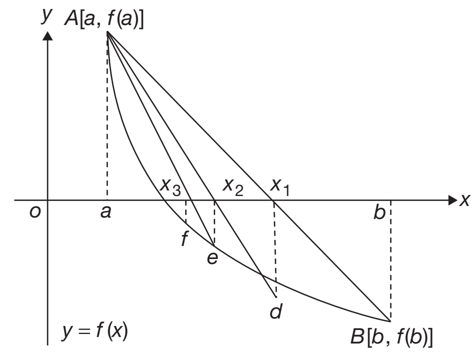
$$\Rightarrow \quad x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

∴ The first approximation $x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$ (2)

If $f(x_1) = 0$, then x_1 is the root. If $f(x_1) \ne 0$ and if $f(x_1)$ and $f(a)$ have opposite signs, the second approximation

$$x_2 = \frac{af(x_1) - x_1 f(a)}{f(x_1) - f(a)} \quad (3)$$

Proceeding in the same way, we get x_3, x_4 and so on. Geometrically, the required root is shown in the figure below.



NOTE

This method is faster than the first order fixed point iteration.

Convergence of Regula Falsi Method

The order of convergence of the method of false position is greater than 1.

The Secant Method

This method is quite similar to that of Regula–Falsi method except for the condition $f(a)\,f(b) < 0$. The interval at each iteration may not contain the root. Let the initial limits of the interval be 'a' and 'b'.

The formula for successive approximation general form is

$$x_{n+1} = x_n + \frac{(x_n - x_{n-1})f(x_n)}{f(x_{n-1}) - f(x_n)}$$

In case at any stage $f(x_n) = f(x_{n-1})$ the method fails.

NOTES

1. This method does not converge always, but Regula–Falsi method always converges.
2. If it converges, it converges with order 1.62 approximately, which is more rapidly than the Regula–Falsi method.

Example 7

Find a root for $2e^x \sin x = 3$ using Regula–Falsi method and correct to three decimal places with three iterations.

Solution

Let $f(x) = 2e^x \sin x - 3$

$$f(0) = -3 < 0, f(1) = 2e' \sin 1 - 3$$
$$= 1.5747 > 0$$

∴ A root lies between 0 and 1.

Here $a = 0$ and $b = 1$

∴ The first approximation

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0(1.5747) - 1(-3)}{1.5747 - (-3)}$$

$$= \frac{3}{4.5747} = 0.6557.$$

Now $f(0.6557) = 2e^{0.6557}\sin(0.6557) - 3$
$= -0.6507 < 0$ and $f(1) > 0$

∴ The root lies between 0.6557 and 1.

The second approximation x_2

$$= \frac{(0.6557)(1.5747) - 1(-0.6507)}{1.5747 - (-0.6507)}$$

$$= \frac{1.0325 + 0.6507}{2.2254} = \frac{1.6832}{2.2254} = 0.7563$$

Now $f(0.7563) = -0.0761 < 0$ and $f(1) > 0$

∴ The root lies between 0.7563 and 1

∴ The third approximation to the root x_3

$$= \frac{(0.7653)(1.5747) - 1(-0.0761)}{1.5747 - (-0.0761)}$$

$$= \frac{1.1909 + 0.0761}{1.6508} = 0.7675$$

∴ The best approximation to the root upto three decimal places is 0.768

Newton–Raphson Method

Let x_0 be the approximate root of $f(x) = 0$ and let $x_1 = x_0 + h$ be the correct root. Then $f(x_1) = 0$

$\Rightarrow \quad f(x_0 + h) = 0 \qquad (1)$

Expanding Eq. (1) using Taylor's theorem,
We get $f(x_0) + hf'(x_0) + \cdots = 0$

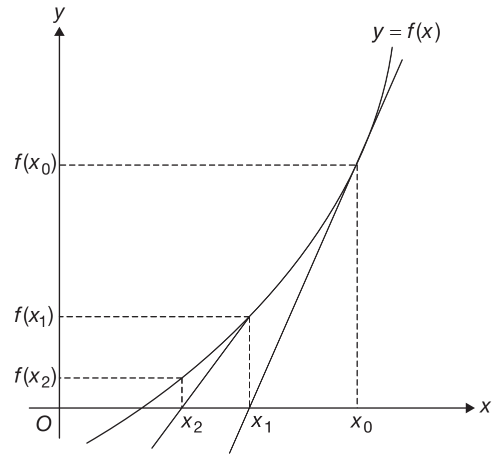
$$\Rightarrow \quad h = \frac{-f(x_0)}{f'(x_0)},$$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now x_1 is the better approximation than x_0. Proceeding this way, the successive approximations $x_2, x_3, \ldots, x_{n+1}$ are given by $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

This is called Newton–Raphson formula.

Geometrical Interpretation of Newton–Raphson Formula



The geometrical meaning of Newton–Raphson method is a tangent is drawn at the point $[x_0, f(x_0)]$ to the curve $y = f(x)$. It cuts the x-axis at x_1 which will be a better approximation of the root. Now drawing another tangent at $[x_1, f(x_1)]$ which cuts the x-axis at x_2 which is a still better approximation and the process can be continued till the desired accuracy has been achieved.

Convergence of Newton–Raphson Method

The order of convergence of Newton–Raphson method is 2 or the convergence is quadratic. It converges if $|f(x).f''(x)| < |f'(x)|^2$. Also this method fails if $f'(x) = 0$

Newton's Iterative Formula to Find bth Root of a Positive Real Number a

The iterative formula is given by x_{n+1}

$$= \frac{1}{b}\left\{(b-1)x_n + \frac{a}{x_n^{\,b-1}}\right\}$$

Newton's Iterative Formula to Find a Reciprocal of a Number N

The iterative formula is given by

$$x_{i+1} = x_i(2 - x_i N)$$

Example 8

Find a real root of the equation $-4x + \cos x + 2 = 0$, by Newton–Raphson method upto four decimal places assuming $x_0 = 0.5$

Solution

Let $f(x) = -4x + \cos x + 2$ and

$$f'(x) = -4 - \sin x$$

Also $f(0) = 1 + 2 = 3 > 0$ and

$f(1) = -4 + \cos 1 + 2 = -1.4596 < 0$

So, a root lies between 0 and 1.

Given $x_0 = 0.5$

∴ The first approximation

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{[-4(0.5) + \cos(0.5) + 2}{-4 - \sin(0.5)}$$

$$= (0.5) - \frac{[-2 + 2 + \cos(0.5)}{-4 - \sin 0.5}$$

$$= 0.5 - \frac{0.8775}{-4.4794}$$

$$= 0.5 + 0.1958 = 0.6958.$$

Example 9

Obtain the cube root of 14 using Newton–Raphson method, with the initial approximation as 2.5.

Solution

We know that, the iterative formula to find $\sqrt[b]{a}$ is

$$x_{n+1} = \frac{1}{b}\left\{(b-1)x_n + \frac{a}{x_n^{b-1}}\right\}$$

Here $b = 3$ and $a = 14$ and let $x_0 = 2.5$

$$\therefore \quad x_1 = \frac{1}{3}\left\{2x_0 + \frac{14}{x_0^{\ 2}}\right\}$$

$$x_1 = \frac{1}{3}\left\{2(2.5) + \frac{14}{(2.5)^2}\right\}$$

$$= \frac{1}{3}\left\{5 + \frac{14}{6.25}\right\} = \frac{1}{3}\{5 + 2.24\} = 2.413$$

$$x_2 = \frac{1}{3}\left\{2(2.413) + \frac{14}{(2.413)^2}\right\}$$

$$= \frac{1}{3}\left\{4.826 + \frac{14}{5.822569}\right\}.$$

$$= \frac{1}{3}\{4.826 + 2.4044\} = 2.410$$

∴ The approximate cube root of 14 is 2.41.

Example 10

Find the reciprocal of 24 using Newton–Raphson method with the initial approximation as 0.041.

Solution

The iterative formula to find $\frac{1}{N}$ is,

$$x_{n+1} = x_n(2 - x_n N)$$

Let $x_0 = 0.041$. Then $x_1 = x_0(2 - x_0\,(24))$

$\Rightarrow \quad x_1 = (0.041)\,(2 - (24)\,(0.041))$

$= 0.04165$

$x_2 = (0.0416)\ \{2 - (24)\ (0.04165)\} = 0.04161$, similarly proceeding we get $x_3 = 0.041666$

∴ The reciprocal of 24 is 0.04166.

Curve Fitting

In engineering applications, many a times, we need to find a suitable relation or law that may exist between the variables x and y from a given set of observed values (x_1, y_1), (x_2, y_2), …, (x_n, y_n), The relation connecting x and y is called as empirical law.

The process of finding the equation of the curve of best fit which may be most suitable for predicting the value of y for a given value of x is known as curve fitting.

Least Squares Approximation

Least squares approximation method is one of the best methods available for curve fitting.

Let (x_1, y_1), (x_2, y_2), …, (x_n, y_n) be the pairs of observed set of values of x and y. Let $y = f(x)$ be the functional relationship sought between the variables x and y where $f(x)$ consists of some unknown parameters. We need to find the relationship $y = f(x)$ by using the observed values.

Procedure

1. Find the residual $d_i = y_i - f(x_i)$ $(i = 1, 2, \ldots, n)$ for every pair of observed value y_i and $f(x_i)$, the value of the functional relation $f(x)$ at $x = x_i$
2. Find the sum of the squares of residuals corresponding to all pairs of values of y_i and $f(x_i)$ and let it be S

   i.e., $S = \sum_{i=1}^{n} (y_i - f(x))^2.$
3. Find the values of the parameters in $f(x)$ such that S is minimum.

Fitting a Straight Line Let $y = a + bx$ be a straight line to be fitted to the data $(x_1\ y_1)$, $(x_2\ y_2)$, …, $(x_n\ y_n)$.

∴ Residual $= d_i = y_i - (a + bx_i)$, $i = 1, 2, \ldots, n$

Sum of the squares of the residuals $= S = \Sigma(y_i - (a + bx_i))^2$

Now we have to find the parameters a and b such that S is minimum

$$\frac{\partial S}{\partial a} = \sum 2[y_i - (a + bx_i)(-1)] \text{ and}$$

$$\frac{\partial S}{\partial b} = \sum [2(y_i - (a + bx_i))(-x_i)]$$

For S to be minimum,

$$\frac{\partial S}{\partial a} = 0 \text{ and } \frac{\partial S}{\partial a} = 0$$

$$\Rightarrow \sum[-2(y_i-(a+bx_i))]=0 \text{ and}$$
$$\sum[2(y_i-(a+bx_i))(-x_i)]=0$$
$$\Rightarrow \sum y_i = na+b\sum x_i \quad (1)$$
$$\text{and } \sum x_i y_i = a\sum x_i + b\sum x_i^2 \quad (2)$$

Eqs. (1) and (2) are known as normal equations.

By solving these equations, we get the values of 'a' and b

Fitting a Parabola (Quadratic Equation)

To fit a parabola of the type $y = a + bx + cx^2$ to the set of data points (x_1, y_1), (x_2, y_2), …, (x_n, y_n), by proceeding as above, we get the normal equations as

$$\sum y_i = na + b\sum x_i + c\sum x_i^2 \quad (1)$$
$$\sum x_i y_i = a\sum x_i + b\sum x_i^2 + c\sum x_i^3 \quad (2)$$
$$\sum x_i^2 y_i = a\sum x_i^2 + b\sum x_i^3 + c\sum x_i^4 \quad (3)$$

By solving Eqs. (1), (2) and (3), we can get the values of the parameters a, b and c

Fitting of various exponential curves that can be brought into the form of a straight line: Exponential curves of the type $y = ax^b$, $y = ab^x$ and $y = ae^{bx}$ can be fitted to the given data by transforming it into the form of a straight line by applying logarithm as follows.

Equation of the curve to be fitted	Equation obtained after applying $\log_e(= ln)$	Transformed equation into the form of a straight line
$y = ax^b$	$\ln y = \ln a + b\ln x$	$Y = A + bX$ where $Y = \ln y$; $A = \ln a$ and $X = \ln x$
$y = ab^x$	$\ln y = \ln a + x\ln b$	$Y = A + Bx$ where $Y = \ln y$; $A = \ln a$ and $B = \ln b$
$y = ae^{bx}$	$\ln y = \ln a + bx$	$Y = A + bx$ where $Y = \ln y$; $A = \ln a$

Example 11

Using the method of least squares, fit a straight line $y = a + bx$ to the following data.

x	1	2	3	4
y	4	11	35	100

Hence find the value of y at $x = 5$.

Solution

We have to fit the line $y = a + bx$ to the given data.

The normal equations are

$$\sum y_i = na + b\sum x_i \quad (1)$$
$$\sum x_i y_i = a\sum x_i + b\sum x_i^2 \quad (2)$$

The required values in the normal equations can be found using the following table

x_i	y_i	x_iy_i	x_i^2
1	4	4	1
2	11	22	4
3	35	105	9
4	100	400	16

$$\sum x_i = 10; \sum y_i = 150; \sum x_i y_i = 531; \sum x_i^2 = 30$$

Substituting these values in Eqn. (1) and (2), we get

$150 = 4a + 10b$ and $531 = 10a + 30b$

$\Rightarrow 4a + 10b = 150$

$10a + 30b = 531$

Solving these linear equations, we get

$a = -40.5$ and $b = 31.2$

$\therefore$ The straight line that fits to the given data is $y = a + bx$

$\Rightarrow y = -40.5 + 31.2x$

The value of y at $x = 5$ is

$y = -40.5 + 31.2 \times 5 \Rightarrow y = 115.5.$

Example 12

Fit a quadratic equation $y = a + bx + cx^2$ to the following data by the method of least squares.

x	–2	–1	0	1	2
y	1	5	10	22	38

Solution

We have to fit the curve $y = a + bx + cx^2$ to the given data.

Here the normal equations are

$$\sum y_i = na + b\sum x_i + c\sum x_i^2 \quad (1)$$
$$\sum x_i y_i = a\sum x_i + b\sum x_i^2 + c\sum x_i^3 \quad (2)$$
$$\sum x_i^2 y_i = a\sum x_i^2 + b\sum x_i^3 + c\sum x_i^4 \quad (3)$$

The values required in the normal equations can be obtained by the following table:

x_i	y_i	x_iy_i	x_i^2	x_i^3	x_i^4	$x_i^2y_i$
–2	1	–2	4	–8	16	4
–1	5	–5	1	–1	1	5
0	10	0	0	0	0	0
1	22	22	1	1	1	22
2	38	76	4	8	16	152

$\sum x_i = 0$; $\sum y_i = 76$; $\sum x_i y_i = 91$; $\sum x_i^2 = 10$; $\sum x_i^3 = 0$; $\sum x_i^4 = 34$; $\sum x_i^2 y_i = 183$

Substituting these values in the normal equation we have

$$76 = 5a + b \times 0 + c \times 10$$
$$91 = a \times 0 + b \times 10 + c \times 0$$

$$183 = a \times 10 + b \times 0 + c \times 34$$

$$\Rightarrow 5a + 10c = 76$$

$$10b = 91$$

$$10a + 34c = 183$$

Solving these equations for a, b and c we get

$a = 10.77$, $b = 9.1$ and $c = 2.21$

Substituting these in $y = a + bx + cx^2$, we get the required parabola as

$y = 10.77 + 9.1x + 2.21x^2$.

Interpolation

The process of finding the most appropriate estimate for the unknown values of a function $y = f(x)$ at some values of x by using the given pairs of values $(x, f(x))$ is called interpolation.

Assumptions in Interpolation

1. The frequency distribution is normal and not marked by sudden ups and downs.
2. The changes in the series are uniform within a period.

   Before looking into interpolation, let us get familiarity with the finite differences which we use in interpolation.

Finite Differences

1. **Forward differences:** Consider a function $y = f(x)$. Let we were given the following table representing the values of $y = f(x)$ corresponding to the values x_1, x_2, …, x_n of x that are equally spaced (i.e., $x_i = x_0 + ih$; $i = 1, 2, \ldots, n$).

$x = x_i$	x_1	x_2	x_3	…	x_n
$y = f(x_i)$	y_1	y_2	y_3		y_n

The forward difference of $f(x)$ denoted by $\Delta y = \Delta[f(x)]$ can be defined as

$$\Delta y = \Delta[f(x)] = f(x + h) - f(x)$$

$$\therefore\ \Delta y_0 = f(x_0 + h) - f(x_0) = y_1 - y_0$$

$$\Delta y_1 = y_2 - y_1$$

$$\Delta y_2 = y_3 - y_2$$

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$$\Delta y_{n-1} = y_n - y_{n-1}$$

where Δ is called the forward difference operator and Δy_0, Δy_1, …, Δy_{n-1} are called the first order forward differences of $y = f(x)$

Similarly, $\Delta^2 y = \Delta[f(x + h)] - \Delta[f(x)]$

$$\therefore\ \Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

$$\Delta^2 y_{n-2} = \Delta y_{n-1} - \Delta y_{n-2}$$

where $\Delta^2 y_0$, $\Delta^2 y_1$, …, $\Delta^2 y_{n-2}$ are called the second order forward differences.

And in general, the nth order forward differences are given by

$$\Delta^n y = \Delta^n[f(x)] = \Delta^{n-1}[f(x + h)] - \Delta^{n-1}[f(x)]$$

$$\therefore\ \Delta^n y_0 = \Delta^{n-1} y_1 - \Delta^{n-1} y_0$$

$$\Delta^n y_1 = \Delta^{n-1} y_2 - \Delta^{n-1} y_1$$

.

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These forward differences of various orders can be found and represented in a table called the forward difference table as shown below

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
$x_1 = x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_2 = x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_3 = x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_4 = x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_5 = x_0 + 5h$	y_5					

2. **Backward differences:** Consider a function $y = f(x)$. Let we were given the following table representing the values of $y = f(x)$ corresponding to the values x_1, x_2, …, x_n of x that are equally spaced (i.e; $x_i = x_0 + ih$, $i = 1, 2, \ldots n$)

$x = x_i$	x_1	x_2	x_3	…	x_n
$y_i = f(x_i)$	y_1	y_2	y_3	…	y_n

The backward difference of $f(x)$ denoted by ∇y (or) $\nabla[f(x)]$ can be defined as

$$\nabla y = \nabla[f(x)] = f(x) - f(x - h)$$

$$\therefore\ \nabla y_1 = y_1 - y_0$$

$$\nabla y_2 = y_2 - y_1$$

$$\nabla y_3 = y_3 - y_2$$

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.

$$\nabla y_n = y_n - y_{n-1}$$

where ∇ is called the backward difference operator and ∇y_1, ∇y_2, …, ∇y_n are called the first order backward differences of $y = f(x)$

Similarly, $\nabla^2 y = \nabla^2[f(x)] = \nabla[f(x) - \nabla[f(x - h)]$

$\therefore\ \nabla^2 y_2 = \nabla y_2 - \nabla y_1$

$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$

.

.

.

$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$

where $\nabla^2 y_2, \nabla^2 y_3, \ldots, \nabla^2 y_n$ are called the second order backward differences.

And in general, the nth order backward differences are given by

$\nabla^n y = \nabla^n[f(x)] = \nabla^{n-1}[f(x)] - \nabla^{n-1}[f(x-h)]$

$\therefore\ \nabla^n y_n = \nabla^{n-1} y_n - \nabla^{n-1} y_{n-1}$

These backward differences of various orders can be found and represented in a table called the backward difference table as shown below

x	**y = f(x)**	**∇y**	**∇²y**	**∇³y**	**∇⁴y**	**∇⁵y**
x_0	y_0					
		∇y_1				
$x_1 = x_0 + h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
$x_2 = x_0 + 2h$	y_2		$\nabla^2 y_3$			
		∇y_3		$\nabla^3 y_4$	$\nabla^4 y_4$	
$x_3 = x_0 + 3h$	y_3		$\nabla^2 y_4$			
		∇y_4		$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$
$x_4 = x_0 + 4h$	y_4		$\nabla^2 y_5$			
		∇y_5				
$x_5 = x_0 + 5h$	Y_5					

Relation between forward and backward differences:
First order: $\nabla[f(x+h)] = \Delta[f(x)]$

Second order: $\nabla^2[f(x+2h)] = \Delta^2[f(x)]$

Third order: $\nabla^3[f(x+3h)] = \Delta^3[f(x)]$

In general, the nth order forward and backward differences are connected by the relation.

$\nabla^n[f(x+nh)] = \Delta^n[f(x)]$

3. **Divided differences:** Consider a function $y = f(x)$. Let we were given a table of values of $y = f(x)$ at $x_1, x_2, x_3, \ldots, x_n$, (need not be equally spaced) as shown below.

$x = x_i$	x_1	x_2	x_3	…	x_n
$y_i = f(x_i)$	y_1	y_2	y_3	…	y_n

Then the first order divided differences are given by:

$$[x_0 x_x] = \frac{y_1 - y_0}{x_1 - x_0}$$

$$[x_1 x_2] = \frac{y_2 - y_1}{x_2 - x_1}$$

.

.

.

$$[x_{n-1}, x_n] = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$

The second order divided differences are given by:

$$[x_0, x_1, x_2] = \frac{|x_1, x_2| - |x_0, x_1|}{x_3 - x_0}$$

$$[x_1, x_2, x_3] = \frac{[x_2, x_3] - [x_1, x_2]}{x_3 - x_1}$$

Similarly, the third order divided differences are given by

$$[x_0, x_2, x_3, x_4] = \frac{|x_1, x_2, x_3| - |x_0, x_1, x_2|}{x_3 - x_0}$$

$$[x_1, x_2, x_3, x_4] = \frac{|x_2, x_3, x_4| - |x_1, x_2, \cdot|}{x_4 - x_1}$$

Note that $[x_0, x_1] = [x_1, x_0]$

And $[x_0, x_1, x_2] = [x_1, x_2, x_0] = [x_2, x_0, x_1]$

Interpolation Formulae

1. **Newton's forward interpolation formula:** If the function $y = f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ respectively at the equally spaced points $x_0, x_1, x_2, \ldots, x_n$ (i.e., $x_{i+1} = x_0 + ih$ (OR) $x_{i+1} = x_i + h$, then the Newton's forward interpolation formula is given by:

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}$$

$$\Delta^3 y_0 + \cdots + \frac{p(p-1)(p-2)\cdots(p-(n-1))}{n!}\Delta^n y_0$$

where $p = \dfrac{x - x_0}{h}$ (OR) $x = x_0 + ph$.

2. **Newton's backward interpolation formula:** If the function $y = f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ respectively at the equally spaced points $x_0, x_1, x_2, \ldots, x_n$ (i.e., $x_{i+1} = x_0 + ih$ (OR) $x_{i+1} = x_i + h$), then the Newton's backward interpolation formula is given by

$$Y_p = \nabla y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}$$

$$\nabla^3 y_n + \cdots + \frac{p(p+1)(p+2)\ldots(p+(n-1))}{n!}\nabla^n y_n$$

where $p = \dfrac{x - x_n}{h}$ (OR) $x = x_n + ph$.

NOTES

1. Newton's forward interpolation formula is used to interpolate (estimate) the values $y = f(x)$ near the beginning of the set of tabulated values given or for estimating the value of $y = f(x)$ to the left of the beginning.
2. Newton's backward interpolation formula is used to interpolate (estimate) the values $y = f(x)$ near the end of the set of tabulated values or for estimating the values of $y = f(x)$ to the right of the last tabulated value y_n.

When the values $x_0, x_1, x_2, \ldots, x_n$ of x are not equally spaced, then we can't make use of Newton's forward as well as backward interpolation formulae. In such situations, the following two interpolation formulae will be helpful.

3. **Newton's divided difference formula:** If the function $y = f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to the values $x_0, x_1, x_2, \ldots, x_n$ (need not be equally spaced) of x, then the Newton's divided difference interpolation polynomial is given by
$y(x) = f(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})[x_0, x_1, \ldots x_n]$
4. **Lagrange's interpolation formula:** If the function $y = f(x)$ takes the values $y_0, y_1, \ldots, y_n$ respectively at the points $x_0, x_1, x_2, \ldots, x_n$ (need not be equally spaced) of x, then the Lagrange's interpolation polynomial is given by $y(x) = f(x) =$

$$\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n.$$

Example 13

If Δ denotes the forward difference operator, then show that $\Delta^4 y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$

Hence find the general expression for $\Delta^n y_0$ in terms of $y_0, y_1, \ldots, y_n$ that does not involve the difference operators.

Solution

We know that $\Delta y_0 = y_1 - y_0$

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$

$= (y_2 - y_1) - (y_1 - y_0)$

$= y_2 - 2y_1 + y_0$

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$

$= (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$

and $\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$

$= (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0)$

$\therefore \Delta^4 y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$

From the above discussion, one can observe that the coefficients of y_i s are nothing but the binomial coefficients with positive and negative signs alternatively.

$\therefore \Delta^n y_0 = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} + \cdots + (-1)^n y_0.$

Example 14

A function $y = f(x)$ is given by the following table

x	5	10	15	20	25
y = f(x)	31	42	51	62	76

Using Newton's forward interpolation formula, find the value of y at $x = 7$.

Solution

First let us form the forward difference table:

x	y = f(x)	Δy	Δ²y	Δ³y	Δ⁴y
5	31				
		11			
10	42		–2		
		9		4	
15	51		2		–3
		11		–1	
20	62		3		
		14			
25	76				

By Newton's forward difference formula,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots \quad (1)$$

Here $x_0 = 5$; $h = 5$ and $x = 7$

$$\therefore p = \frac{x - x_0}{h} = \frac{7-5}{5} = 0.4$$

Substituting these in Eq. (1), we have

$$y(7) = 31 + (0.4) \times 11 + \frac{(0.4)(0.4-1)}{2!} \times (-2) + \frac{(0.4)(0.4-1)(0.4-2)}{3!} \times 4 + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{4!} \times (-3)$$

$= 31 + 4.4 + 0.24 + 0.256 + 0.1248 = 36.0208$

$\therefore$ The value of y at $x = 7$ is 36.0208.

Example 15

Following table shows the values of a function $y = f(x)$ at 0, 2, 5 and 9

x	0	2	5	9
y = f(x)	6	15	27	40

Using the Lagrange's interpolation formula, find $y(6)$.

Solution

Given values of $y = f(x)$ are

x	0	2	5	9
Y = f(x)	6	15	27	40

By Lagrange's interpolation formula, we have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0$$

$$+ \frac{(x-x_0(x-x_2)(x-c_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$\therefore f(x) = \frac{(x-2)(x-5)(x-9)}{(0-2)(0-5)(0-9)} \times 6 + \frac{(x-0)(x-5)(x-9)}{(2-0)(2-5)(2-9)} \times$$

$$15 + \frac{(x-0)(x-2)(x-9)}{(5-0)(5-2)(5-9)} \times 27 + \frac{(x-0)(x-2)(x-5)}{(9-0)(9-2)(9-5)} \times 40$$

Now, $y(6) = f(6) = \dfrac{(6-2)(6-5)(6-9)}{(-2)\times(-5)\times(-9)}$

$$\times 6 + \frac{(6-0)(6-5)(6-9)}{2x(-3)\times(-6)} \times 15 + \frac{(6-0)(6-2)(6-9)}{-5x-3x(-4)} \times$$

$$2 + \frac{(6-0)(6-2)(6-5)}{9\times7\times4} \times 40 = \frac{4}{5} - \frac{45}{7} + \frac{162}{5} + \frac{80}{21} = 30.5807.$$

Numerical Differentiation

In numerical differentiation, we find the derivatives by using the interpolation formulae.

1. **Derivatives using newton's forward difference interpolation formula:** We know that the Newton's forward difference interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}$$
$\Delta^3 y_0 + \ldots$

Differentiating both sides wrt p,

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \cdots$$

$$\text{As } p = \frac{x-x_0}{h} = \frac{dp}{dx} = \frac{1}{h}$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx} = [\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0$$

$$+ \frac{3p^2-6p+2}{3!}\Delta y_0 + \cdots]\frac{1}{h}$$

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{(3p^2-6p+2)}{3!}\Delta^3 y_0 + \cdots\right]$$

$$\therefore \left(\frac{dy}{dx}\right) \text{ at } x = x_0$$

$$= \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5{}_0 \ldots\right]$$

($\because$ At $i = x_0$; $p = 0$)

$$\text{And } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx} = \frac{1}{h}$$

$$\left[\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p+22}{4!}\Delta^4 y_0 \cdots\right]\frac{1}{h}$$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p+22}{1!}\Delta^4 y_0 \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{\text{at } x=x_0}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$\therefore$ By using Newton's forward interpolation formula, the first and second derivatives of $y = f(x)$ at $x = x_0$ are given by $\left(\dfrac{dy}{dx}\right)_{x=x_0}$

$$= \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 \ldots\right]$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\right]$$

2. **Derivatives using newton's backward difference interpolation formula:** We know that the Newton's backward difference interpolation formula is $y = y_n +$

$$p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} + \nabla^3 y_n \ldots$$

Differentiating on both sides wrt p,

$$\frac{dy}{dx} = \nabla y_n + \frac{2p+1}{2}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \cdots$$

$$\text{As } p = \frac{x-x_n}{h}\frac{dp}{dx} = \frac{1}{h}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx}$$

$$= \left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \cdots\right]\frac{1}{h}$$

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \ldots\right]$$

($\because$ At $x = x_n$; $p = 0$)

And $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx}$

$$= \frac{1}{h}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{12p^2+36p+22}{4!}\nabla^4 y_n + \cdots\right]\frac{1}{h}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{12p^2+36p+22}{4!}\nabla^4 y_n + \cdots\right]$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n\right]$$

$\therefore$ By using Newton's backward difference interpolation formula, the first and second derivatives of $y = f(x)$ at $x = x_n$ are given by $\left(\frac{dy}{dx}\right)_{x=x_n}$

$$= \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \cdots\right]$$

and $\left(\frac{d^2y}{dx^2}\right)_{x=x_n}$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right]$$

Example 16

Using the values of a function $y = f(x)$ given in the following table, find the first two derivatives of $f(x)$ at $x = 7$.

x	2	3	4	5	6	7
y = f(x)	4	8	15	27	36	42

Solution

As we have to find the first two derivatives of $y = f(x)$ at $x = 7$, (end point of the given data), we use the derivatives' formulae obtained from Newton's backward interpolation formula.

The backward difference table for the given data is

X	y = f(x)	Δy	Δ²y	Δ³y	Δ⁴y	Δ⁵y
2	4					
		4				
3	8		3			
		7		–2		
4	15		5		6	
		12		–8		2
5	27		–3		8	
		9		0		
6	36		–3			
		6				
7	42					

By Newton's backward interpolation formula, we have $\left(\frac{dy}{dx}\right)_{x=x_n=7}$

$$= \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \nabla^4 y_n + \frac{1}{5}\nabla^5 y_n\right]$$

$$= \frac{1}{1}\left[6 + \frac{1}{2}\times(-3) + \frac{1}{3}\times 0 + \frac{1}{4}\times 8 + \frac{1}{5}\times 2\right] = \frac{69}{10} = 6.9$$

And $\left(\frac{d^2y}{dx^2}\right)_{x=x_n=7}$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} + \nabla^4 y_n + \frac{5}{6}\nabla^5 y_n\right]$$

$$= \frac{1}{1^2}\left[-3 + 0 + \frac{11}{12}\times 8 + \frac{5}{6}\times 2\right] = 6.$$

Example 17

Find the first derivative at $x = 6$ for a function $y = f(x)$ with the following data.

x	5	7	10	11	13
y = f(x)	100	294	900	1210	2028

Solution

As the given values of x are not equally spaced, to find the first derivative of $f(x)$, we will make use of the Newton's divided difference interpolation formula, which is given by

$$y = f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\,[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)(x - x_2)(x - x_3)][x_0, x_1, x_2, x_3, x_4] + \cdots \quad (1)$$

The divided differences of various orders for the given data can be represented as shown below.

x	y = f(x)	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
5	100				
		$\frac{294-100}{7-5}=97$			
			$\frac{202-97}{10-5}=21$		
7	294			$\frac{27-21}{11-5}=1$	
		$\frac{900-294}{10-7}=202$			
			$\frac{310-202}{11-7}=27$		$\frac{1-1}{13-5}=0$
10	900				
		$\frac{1210-900}{11-10}=310$		$\frac{33-27}{13-7}=1$	
11	1,210		$\frac{409-310}{13-10}=33$		
		$\frac{2028-1210}{13-11}=409$			
13	2,028				

Substituting these in Eq. (1), we get

$y = f(x) = 100 + (x - 5) \times 97 + (x - 5)(x - 7)\,21 + (x - 5)(x - 7)(x - 10) \times 1 + (x - 5)(x - 7)(x - 10)(x - 11) \times 0$

$\therefore f(x) = 100 + 97(x - 5) + 21(x^2 - 12x + 35) + (x^3 - 22x^2 + 155x - 350)$

$$\therefore \frac{dy}{dx} = 97 + 21(2x - 12) + (3x^2 - 44x + 155)$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=6} = 97 + 21(2\times 6 - 12) + (3\times 6^2 - 44\times 6 + 155)$$

$= 97 + 0 + (-1) = 96.$

Numerical Integration

The numerical integration can be stated as follows: Given a set of $(n + 1)$ data points (x_i, y_i), $i = 0, 1, 2, 3, \ldots, n$ of the function $y = f(x)$, where $f(x)$ is not known explicitly, it is required to find $\int_{x_0}^{x_n} f(x)dx$.

NOTE

Numerical integration is also known as Numerical quadrature.

Newton–Cote's Quadrature Formula

[General Quadrature formula]

Consider the integral $I = \int_a^b f(x)dx$

Let the interval $[a, b]$ be divided into 'n' equal subintervals of width h so that $a = x_0, x_1 = x_0 + h, x_2 = x_0 + 2h \ldots b = x_0 + nh$

$$\therefore I = \int_{x_0}^{x_0+nh} f(x)dx$$

Put $x = x_0 + mh \Rightarrow dx = h \cdot dm$ as $x \to x_0$, $m \to 0$ and $x \to x_0 + nh$, $m \to n$

$$I = h\int_0^n f(x_0 + mh)dm$$

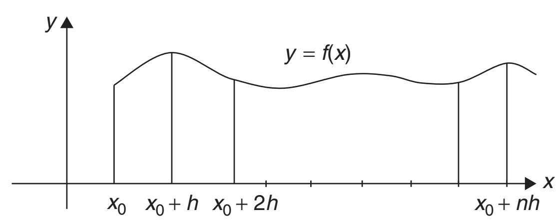
Applying Newton's forward interpolation formula

$$I = h\int_0^n \left(y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \cdots\right)dm$$

Integrating term by term and applying the limits, we get

$$\int_{x_0}^{x_0+nh} f(x)dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

(Newton–Cotes quadrature formula)



On substituting $n = 1, 2, 3, \ldots$ in Newton-Cote's quadrature formula, we get various quadrature formulae.

Trapezoidal Rule [Two-point Quadrature]

Substituting $n = 1$ in the Newton–Cotes formula and taking the curve $y = f(x)$ through (x_0, y_0) and (x_1, y_1) as a straight line so that differences of order higher than one becomes zero, we get

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_0+h} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}[y_0 + y_1]$$

Similarly,

$$\int_{x_1}^{x_2} f(x)dx = \int_{x_0+h}^{x_0+2h} f(x)dx = h\left[y_1 + \frac{1}{2}\Delta y_1\right] = \frac{h}{2}[y_1 + y_2]$$

$$\int_{x_2}^{x_3} f(x)dx = \int_{x_0+2h}^{x_0+3h} f(x)dx = \frac{h}{2}(y_2 + y_3$$

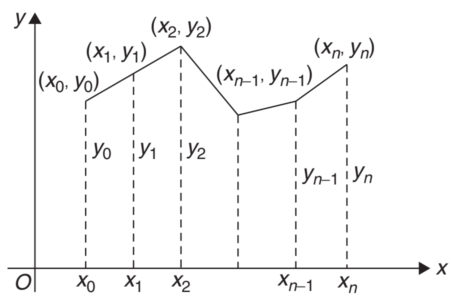
Proceeding, $$\int_{x_0+(n-1)h}^{(x_0+nh)} f(x)dx = \frac{h}{2}(y_{n-1} + y_n)$$

Hence, $$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

Thus, $$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[\begin{array}{l}(\text{sum of the first and last ordinates})\\ +2\,(\text{sum of remaining ordinates})\end{array}\right]$$

The above rule is known as Trapezoidal rule.

Geometrical Interpretation of Trapezoidal Rule



Geometrically, the curve $y = f(x)$ is replaced by n straight line segments joining the points (x_0, y_0) and (x_1, y_1); (x_1, y_1) and (x_2, y_2); …, (x_{n-1}, y_{n-1}) and (x_n, y_n). The area bounded by the curve $y = f(x)$ is then approximately equal to the sum of the areas of n trapeziums as shown in the figure.

Simpson's One-third Rule [Three-point Quadrature]

Substituting $n = 2$ in the Newton–Cotes quadrature formula taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola, so that the differences of order higher than 2 becomes zero, we get

$$\int_{x_0}^{x_0+2h} f(x)dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right)$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Therefore adding all these we get when 'n' is even,

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[\begin{array}{l}(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1})\\ + 2(y_2 + y_4 + \cdots + y_{n-2})\end{array}\right]$$

$= \frac{h}{3}$ [(sum of the first and last ordinates + 4 (sum of the odd ordinates) + 2 (sum of the even ordinates)]

This is known as Simpson's $\frac{1}{3}$ rule.

Simpson's Three-eighth Rule

Substituting $n = 3$ in the Newton Cotes quadrature formula and taking curve through (x_0, y_0), (x_1, y_1), (x_2, y_2) and (x_3, y_3) so that the differences of order higher than three becomes zero, we get

$$\int_{x_0}^{x_3} f(x)dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{2}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly,

$$\int_{x_3}^{x_6} f(x)dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all these integrals from x_0 to x_n where 'n' is a multiple of 3, we get

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-2})$$
$$+ 2(y_3 + y_6 + y_9 + \cdots + y_{n-3})]$$

The above rule is called Simpson's $\frac{3}{8}$ rule which is applicable only when 'n' is a multiple of 3.

Example 18

Evaluate: $\int_0^2 \sqrt{1+x^2}\,dx$ taking $h = 0.2$ using

(i) Trapezoidal rule and

(ii) Simpson's $\frac{1}{3}$rd rule

Solution

Here, $a = 0$, $b = 2$, $h = 0.2$

So, $n = \dfrac{b-a}{h} = \dfrac{2-0}{0.2} = 10$

The values of x and y are tabulated as follows:

x	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
$y = \sqrt{1+x^2}$	1 y_0	1.0198 y_1	1.077 y_2	1.1661 y_3	1.2806 y_4	1.414 y_5	1.562 y_6	1.7204 y_7	1.8867 y_8	2.059 y_9	2.236 y_{10}

(i) By Trapezoidal rule,

$$\int_0^2 \sqrt{1+x^2}\,dx = \frac{h}{2}[(y_0+y_{10})+2(y_1+y_2+\cdots+y_9)]$$

$$= \frac{0.2}{2}[(1 + 2.236) + 2(1.0198 + 1.077 + 1.1661 + 1.2806 + 1.414 + 1.562 + 1.7204 + 1.8867 + 2.059)]$$

$$= 0.1\ [(3.236) + 2(13.1856)]$$

$$= 0.1\ [29.6072] = 2.96072.$$

(ii) By Simpson's $\frac{1}{3}$ rule,

$$\int_0^2 \sqrt{1+x^2}\,dx$$

$$= \frac{h}{3}\left[\begin{matrix}(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9)\\ +2(y_2+y_4+y_6+y_8)\end{matrix}\right]$$

$$= \frac{0.2}{3}[(1 + 2.236) + 4(1.0198 + 1.1661 + 1.414 + 1.7204 + 2.059) + 2(1.077 + 1.2806 + 1.562 + 1.8867)]$$

$$= \frac{0.2}{3}[(3.236) + 29.5172 + 11.6126]$$

$$= 2.95772.$$

Example 19

Evaluate $\int_0^{\pi/2} e^{\cos x}\,dx$ by Simpson's three-eighth rule.

Solution

Taking $h = \frac{\pi}{6}$, the range can be divided into three equal, sub intervals with the division points. The values of x and y are tabulated as below.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y = e^{\cos x}$	2.718(y_0)	2.3774(y_1)	1.6487(y_2)	1(y_3)

By Simpon's three-eighth rule,

$$\int_0^{\pi/2} e^{\cos x}\,dx = \frac{3h}{8}\ [(y_0+y_3)+3(y_1+y_2)]$$

$$= \frac{3}{8}\times\frac{\pi}{6}\ [(2.718+1)+3(2.3774+1.6487)]$$

$$= \frac{\pi}{16}[(3.718)+(12.0783)] = 3.10159.$$

Numerical Solutions of Ordinary Differential Equations

The following methods are discussed on the numerical solutions of ordinary differential equations.

Single-step Methods

1. Taylor's series method
2. Picard's method of successive approximation

Multi-step Methods

1. Euler's method
2. Modified euler's method
3. Runge–Kutta method
4. Predictor–Corrector methods [Milne's and Adam's]

Taylor's Series Method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \text{ with initial condition } y(x_0) = y_0 \qquad (1)$$

Let $y = f(x)$ be the solution of Eq. (1)

Writing the Taylor's series expansion of $f(x)$ at x_0

$$f(x) = f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\cdots$$

$$\Rightarrow y = y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}y_0''+\cdots$$

Put $x = x_1$, we get

$$y_1 = y_0+(x_1-x_0)y_0'+\frac{(x_1-x_0)^2}{2!}y_0''+\cdots$$

If we take $h = x_1 - x_0$

$$\Rightarrow y_1 = y_0+hy_0'+\frac{h^2}{2!}y_0''+\cdots$$

∴ In general,

$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!} y_n'' + \ldots$ will be the iterative formula.

Example 22

Given $\frac{dy}{dx} = x - y^2$ with the initial condition $y(0) = 1$

Find $y(0.1)$ using Taylor series method with step size 0.1.

Solution

$$f(x, y) = x - y^2$$
$$x = 0.1, x_0 = 0, y_0 = 1, h = 0.1$$
$$y' = x - y^2 \Rightarrow y'(0) = x_0 - y_0{}^2 = -1;$$
$$y'' = 1 - 2yy' \Rightarrow y''(0) = 1 - 2y_0y_0'$$
$$= 1 - 2\,(1)\,(-1) = 3$$
$$y''' = -2yy'' - 2(y')^2$$
$$\Rightarrow y'''\,(0) = -2\,(1)\,(3) - 2\,(-1)^2$$
$$= -6 - 2 = -8$$

By Taylor's formula,

$$y\,(0.1) = y_1 = y_0 + hy'\,(0) + \frac{h^2}{2!} y''\,(0) + \frac{h^3}{3!} y'''\,(0) + \cdots$$

$$\Rightarrow y_1 = 1 + (0.1)\,(-1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(-8) + \cdots$$

$= 1 - 0.1 + 0.015 - 0.0013 + \cdots$

$y_1 = 0.9137.$

Picard's Method of Successive Approximation

Given the differential equation $\frac{dy}{dx} = f(x, y)$ (1)

Integrate Eq. (1) from x_0 to x, we get

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x, y)dx$$

$$\Rightarrow y(x) - y(x_0) = \int_{x_0}^{x} f(x, y)dx$$

$$\Rightarrow y(x) = y(x_0) + \int_{x_0}^{x} f(x, y)dx \quad (2)$$

Put $y = y_0$, we get the first approximation,

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx.$$

Example 23

Given $\frac{dy}{dx} = 1 + xy$ and $y\,(0) = 1$. Evaluate $y\,(0.1)$ by Picard's method upto three approximations.

Solution

$$f(x, y) = 1 + xy$$
$$x_0 = 0, y_0 = 1$$

The first approximation $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$

$$1 + \int_{x_0}^{x} 1 + xy_0 dx = 1 + \int_{0}^{x} 1 + xdx$$

$$y_1 = 1 + x + \frac{x^2}{2}$$

At $x = 0.1$, $y_1 = 1 + (0.1) + \frac{(0.1)^2}{2} = 1.105$

The second approximation y_2,

$$= y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

$$\Rightarrow y_2 = 1 + \int_{0}^{x} 1 + xy_1 dx$$

$$\Rightarrow y_2 = 1 + \int_{0}^{x} \left[1 + x\left(1 + x + \frac{x^2}{2}\right)\right] dx$$

$$= 1 + \int_{0}^{x} \left(1 + x + x^2 + \frac{x^3}{2}\right) dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

At $x = 0.1$, $y_2 = 1 + (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8}$

$y(0.1) = 1.10534$

The third approximation $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)dx$

$$\Rightarrow y_3 = 1 + \int_{0}^{x} (1 + xy_2)dx.$$

$$1 + \int_{0}^{x} \left(1 + x\left[1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right]\right) dx$$

$$1 + \int_{0}^{x} \left(1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{8}\right) dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

At $x = 0.1$,

$$y_3 = 1 + (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{15} + \frac{(0.1)^6}{48}$$

$= 1 + 0.1 + 0.005 + 0.0003 + 0.0000125 + 0.0000006 + 0.00000002$

$y_3 = 1.105313$

Multi-step Methods

Euler's Method

For the differential equation $\frac{dy}{dx} = f(x, y)$ with initial condition $y(x_0) = y_0$, the Euler's iteration formula is

$$y_n = y_{n-1} + h\,f(x_{n-1}, y_{n-1}),\ n = 1, 2, 3, \ldots$$

NOTE

The process is very slow and to obtain accuracy, h must be very small, i.e., we have to divide $[x_0, x_n]$ into a more number of subintervals of length 'h'.

Example 24

Solve $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$, find $y(0.5)$ by Euler's method choosing $h = 0.25$.

Solution

$$f(x, y) = \frac{y-x}{y+x}$$

$$x_0 = 0, y_0 = 1, h = 0.25$$

Euler's iteration formula,

$$y_n = y_{n-1} + h\,f(x_{n-1}, y_{n-1})$$

Put $n = 1$,

$$x_1 = 0.25 \Rightarrow y_1 = y(0.25) = y_0 + h\,f(x_0, y_0)$$

$$= 1 + (0.25)\left(\frac{y_0 - x_0}{y_0 + x_0}\right)$$

$$= 1 + (0.25)\,\frac{1-0}{1+0} = 1.25$$

Put $n = 2$

$$x_2 = 0.5 \Rightarrow y_2 = y(0.5) = y_1 + h\,f(x_1, y_1)$$

$$= (1.25) + (0.25)\left[\frac{y_1 - x_1}{y_1 + x_1}\right]$$

$$= 1.25 + (0.25)\left[\frac{1.25 - 0.25}{1.25 + 0.25}\right]$$

$$= 1.25 + 0.166666 = 1.4166$$

$\therefore\ y(0.5) = 1.4166$

Modified Euler's Method

For the differential equation $\frac{dy}{dx} = f(x, y)$ with initial condition $y(x_0) = y_0$, the Modified Euler's iteration formula is

$$y_r^{(n)} = y_{r-1} + \frac{h}{2}\,[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{\,n-1})]$$

NOTE

To find y_n, we proceed to find the approximations $y_n^{(0)}$, $y_n^{(1)}$, $y_n^{(2)}$ … until the two successive approximations are approximately equal.

$y_n^{(0)}$ is found by Euler's method, i.e., $y_n^{(0)} = y_{n-1} + hf(x_{n-1}, y_{n-1})$

Example 25

Find y for $x = 0.1$ using modified Euler's method for the differential equation $\frac{dy}{dx} = \log(x + y)$ with $y(0) = 1$.

Solution

Given $f(x, y) = \log(x + y)$

$$x_0 = 0, y_0 = 1, h = 0.1$$

To find y_1, $x_1 = 0.1$

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0)$$

$$= 1 + (0.1)\log(0 + 1) = 1$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= y_0 + \frac{h}{2}[\log(x_0 + y_0) + \log(x_1 + y_1^{(0)})]$$

$$= 1 + \frac{0.1}{2}[\log(0 + 1) + \log(0.1 + 1)]$$

$$= 1 + \frac{0.1}{2}[\log 1 + \log 1.1] = 1.0047$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= y_0 + \frac{h}{2}[\log(0 + 1) + \log(x_1 + y_1^{(1)})]$$

$$= 1 + \frac{0.1}{2}[\log(0 + 1) + \log(0.1 + 1.0047)]$$

$$= 1.0049$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 1 + \frac{0.1}{2}[\log(0 + 1) + \log(0.1 + 1.0049)]$$

$$= 1.0049$$

$\therefore\quad y_1 = 1.0049.$

Runge–Kutta Methods

First Order Runge–Kutta Method

$$y_1 = y_0 + hy_0^{\,1} \text{ [same as Euler's method]}$$

Second Order Runge–Kutta Method

The formula is $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

where $k_1 = h\,f(x_0, y_0)$ and $k_2 = h\,f(x_0 + h, y_0 + k_1)$

Third Order Runge-Kutta Method

The formula is $y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$

where $k_1 = h\,f(x_0, y_0)$

$$k_2 = hf\left(x_0+\frac{1}{2}h,\ y_0+\frac{1}{2}k_1\right) \text{ and}$$

$k_3 = hf(x_0 + h, y_0 + k')$ where $k' = hf(x_0 + h, y_0 + k_1)$.

Fourth Order Runge–Kutta Method

The formula is $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0+\frac{1}{2}h,\ y_0+\frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0+\frac{1}{2}h,\ y_0+\frac{1}{2}k_2\right)$$

and $k_4 = hf(x_0 + h, y_0 + k_3)$

Example 26

Given $\frac{dy}{dx} = x^2 + y^2$, $y(1) = 1.2$. Find $y(1.05)$ applying fourth order Runge–Kutta method, with $h = 0.05$.

Solution

$f(x, y) = x^2 + y^2$, $x_0 = 1$, $y_0 = 1.2$, $h = 0.05$

$\therefore k_1 = hf(x_0, y_0) = (0.05)\,[x_0^{\,2} + y_0^{\,2}]$

$= (0.05)\,[1^2 + (1.2)^2] = 0.122$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$$

$= (0.05)\,[f(x_0 + 0.025, y_0 + 0.061]$

$= (0.05)\,[f(1.025, 1.261)]$

$= (0.05)\,[(1.025)^2 + (1.261)^2] = 0.1320$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$$

$= (0.05)\,f(1 + 0.025, 1.2 + 0.066)$

$= (0.05)\,f(1.025, 1.266)$

$= (0.05)\,[(1.025)^2 + (1.266)^2] = 0.1326$ and $k_4 = h\,f(x_0 + h, y_0 + k_3)$

$= (0.05)\,f(1 + 0.05, 1.2 + 0.1326)$

$= (0.05)\,f(1.05, 1.3326)$

$= (0.05)\,[(1.05)^2 + (1.3326)^2] = 0.1439$

$$\therefore y_1 = y(1.05) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.2 + \frac{1}{6}[0.122 + 2(0.1320) + 2(0.1326) + 0.1439]$$

$$= 1.2 + \frac{1}{6}[0.7951] = 1.3325$$

Predictor–Corrector Methods

Milne's Predictor Formula

$$y^p_{n+1} = y_{n-3} + \frac{4h}{3}\,(2y_{n-2} - y_{n-1} + 2y_n)$$

Milne's Corrector Formula

$$y^c_{n+1} = y_{n-1} + \frac{h}{3}\,[y_{n-1} + 4y_n + y_{n+1}{}^p]$$

Adams–Bashforth Predictor Formula

$$y^p_{n+1} = y_n + \frac{h}{24}[55y_n - 59y_{n-1} + 37y_{n-2} - 9y_{n-3}]$$

Adams–Moulton Corrector Formula

$$y^c_{n+1} = y_n + \frac{h}{24}\,[9y^p_{n+1} + 19y_n - 5y_{n-1} + y_{n-2}]$$

Exercises

1. Three of the roots of the equation $x^4 + lx^3 + mx^2 + nx + 24 = 0$ are 3, 1 and –2. Which of the following could be the value of $l + m - n$?
(A) 0 (B) 1
(C) 2 (D) 3

2. If one of the roots of the equation $x^3 + 5x^2 - 12x - 36 = 0$ is thrice another root, then the third root is
(A) –6 (B) 3
(C) –2 (D) $-\frac{89}{13}$

3. If the equation $x^6 + 5x^5 + 11x^4 + 34x^2 + 20x + 24 = 0$ has exactly four non-real roots, then the number of negative roots is
(A) 1 (B) 0
(C) 3 (D) 2

4. A student finds, by trial, two negative and one positive root(s) of the equation $x^5 + 5x^4 + 2802x + 3024 = 103x^3 + 329x^2$. How many non–real roots does the equation have?
(A) 0 (B) 1
(C) 2 (D) 4

5. If the equation $3x^4 - 13x^3 + 7x^2 + 17x + a - 10 = 0$ has exactly three positive roots, then a can be

(A) 11 (B) 4
(C) 13 (D) 12

6. If two of the roots of the equation $x^3 + 3x^2 - 10x - 24 = 0$ are such that one is twice the other, then the third root is

(A) –4 (B) –3
(C) –2 (D) 3

7. If 2.236146 is an approximation to $\sqrt{5}$, then the relative error is

(A) 3.4883×10^{-5} (B) 4.8383×10^{-5}
(C) 8.3483×10^{-4} (D) 5.8438×10^{-4}

8. The least number of terms required to be considered in the Taylor's series approximation of $f(x) = \dfrac{1}{(2+x)}$ about $x = 0$ such that the truncation error is at most 5×10^{-4} for $x \in [0, 1]$ is

(A) 3 (B) 5
(C) 6 (D) 8

9. Let $f(x) = x^3 - x - 5 = 0$. By bisection method first two approximations x_0 and x_1 are 1.5 and 2.25 respectively, then x_2 is

(A) 1.625 (B) 1.875
(C) 1.999 (D) None of these

10. Find the fourth approximation of the root of the equation $x^3 + x - 11 = 0$, between 2 and 3, using Bisection method.

(A) 1.925 (B) 2.832
(C) 2.5215 (D) 2.0625

11. The absolute error bisection method is

(A) 2^n (B) $\dfrac{1}{2^n}|b - a|$
(C) $\dfrac{1}{|b-a|}$ (D) $|b - a|2^n$

12. If the first two approximations x_0 and x_1 to a root of $x^3 - x - 4 = 0$ are 1.666 and 1.780 respectively, then find x_2 by Regula–Falsi method.

(A) 1.974 (B) 1.794
(C) 1.896 (D) None of these

13. Find the second approximation to the root of the equation $2x - 5 = 3\sin x$ between (2, 3) using the method of false position.

(A) 2.2523 (B) 2.012
(C) 2.8804 (D) None of these

14. For $N = 28$ and $x_0 = 5.5$, the first approximation to $\sqrt{N}$ by Newton's iteration formula is

(A) 5.295 (B) 5.582
(C) 5.396 (D) None of these

15. The Newton's iterative formula to find the value of $\sqrt[3]{N}$ is

(A) $x_{i+1} = \left(2x_i - \dfrac{N}{x_i^2}\right)$

(B) $x_{i+1} = \dfrac{1}{3}\left(x_i - \dfrac{N}{x_i^2}\right)$

(C) $x_{i+1} = \dfrac{1}{3}\left(2x_i + \dfrac{N}{x_i^2}\right)$

(D) $x_{i+1} = \dfrac{1}{3}\left(2x_i - \dfrac{N}{x_i^2}\right)$

16. Find the second approximation to the cube root of 24 correct to three decimal places using Newton's iterative formula.

(A) 2.695 (B) 2.885
(C) 3.001 (D) None of these

17. The Newton's iterative formula to find the value of $\dfrac{1}{N}$ is

(A) $x_{i+1} = x_i(2 + x_iN)$
(B) $x_{i+1} = x_i(2 - x_iN)$
(C) $x_{i+1} = x_i^2(2 + x_iN)$
(D) None of these

18. Find the reciprocal of 22 using Newton–Raphson method.

(A) 0.0454545 (B) 0.4504504
(C) 0.54054 (D) None of these

19. If the first approximation of the root of $x^3 - 3x - 5 = 0$ is $(x_0 =)$ 2, then find x_1 by Newton–Raphson method.

(A) 2.2806 (B) 2.2790
(C) 2.3333 (D) None of these

20. Find the first approximation of the real root by Newton–Raphson method for $x^4 + x^3 - 7x^2 - x + 5 = 0$ by taking $x_0 = 2$.

(A) 2.066 (B) 2.981
(C) 2.819 (D) None of these

21. If $y = 2.6 + 0.7x$ is a line that fits the data:

x	–2	–1	0	1	2
y	1	2	3	K	4

Then the value of K is

(A) 3 (B) 5
(C) 6 (D) 7

22. If a curve $y = ab^x$ is fitted to the following data, then the value of 'b' is

x	–2	–1	0	1	2
y	11	13	20	25	34

(A) 0.2911 (B) 0.9845
(C) 1.3379 (D) 2.0034

23. For a set of 5 pairs of values (x_0, y_0), (x_1, y_1), (x_2, y_2) (x_3, y_3) and (x_4, y_4) of (x, y), if $\Delta^3 y_0 = 4$ and $\Delta^3 y_1 = 10$, then the value of $\nabla^4 y_4$ is
(A) 4 (B) 6
(C) 10 (D) 14

24. The value of $y(1.5)$ computed from the following data using Newton's forward interpolation formula is

x	1	2	3	4	5
y	6	7	12	21	34

(A) 6 (B) 6.5
(C) 7 (D) 7.5

25. The Lagrange's interpolation polynomial corresponding to the pairs of values of x and y given in the following table is

x	1	3	4	7
y	36	16	9	72

(A) $x^3 - 6x^2 + 9x + 36$
(B) $x^3 - 6x^2 + 18x - 45$
(C) $3x^3 + 4x^2 - 5x + 27$
(D) $x^3 - 7x^2 + 5x + 37$

26. The values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 5$ from the following table respectively are

x	0	1	2	3	4	5
y = f(x)	1	4	9	16	21	28

(A) 15.00 and 26.45
(B) 13.73 and 23.33
(C) 17.13 and 31.42
(D) 21.64 and 43.00

27. The value of $\frac{dy}{dx}$ at $x = 12$ from the table given below is

x	10	15	20	25
y = f(x)	354	332	291	260

(A) –3.9427 (B) 4.6125
(C) –0.4652 (D) 1.3549

28. The magnitude of error when $\frac{dy}{dx}$ at $x = 2$ is found by Newton's forward interpolation formula for $y = \; y = \frac{1}{x}$ using the following data is

x	2	4	6	8
y = 1/x	0.5000	0.2500	0.1667	0.1250

(A) 0.0005 (B) 0.0025
(C) 0.0125 (D) 0.0625

29. Find the value of $\int_2^3 \frac{1}{1+x^2}dx$ taking four intervals by trapezoidal rule and also find the error when compared to its exact value.
(A) 0.1759, 0.000004 (B) 0.1826, 0.04
(C) 0.1953, 0.004 (D) 0.1423, –0.0004

30. Find $\int_0^1 \frac{x^2}{1+8x^3}dx$ using Trapezoidal rule by taking 4 strips.
(A) 0.0911 (B) 0.9011
(C) 0.1901 (D) None of these

31. The estimate of $\int_{0.5}^{1.5} \frac{dx}{x}$ obtained using Simpson's rule with three point function evaluation exceeds the exact value by
(A) 0.235 (B) 0.068
(C) 0.024 (D) 0.012

32. Find the value of $\int_2^6 x \log x dx$ taking 4 strips by Simpson's $\frac{1}{3}$ rd rule upto four decimals.
(A) 21.8901 (B) 22.8661
(C) 23.6581 (D) None of these

33. Evaluate $\int_0^{\pi/2} \sin x \, dx$ by Simpson's $\frac{1}{3}$ rule using six intervals.
(A) 0.97768 (B) 0.98869
(C) 0.99968 (D) None of these

34. Find the maximum error in evaluating the above when compared to its exact value.
(A) 0.000032 (B) 0.00032
(C) 0.00000032 (D) 0.0032

35. Evaluate $\int_0^3 \frac{1}{2+x^2}dx$ by using Simpson's $\frac{3}{8}$ rule by taking 3 strips.
(A) 0.507 (B) 0.5007
(C) 0.7839 (D) None of these

36. If $\frac{dy}{dx} = 1 - 3xy^2$, y (0) = 0, then by Taylor's method y (0.1) =
(A) 0.02 (B) 0.001
(C) 0.05 (D) 0.1

37. If $\frac{dy}{dx} = 2x + y$, $y(0) = 1$, the Picard's approximate of y upto second degree terms is
(A) $1 + x + x^2$ (B) $1 + x + \frac{x^2}{2}$
(C) $1 - x - \frac{x^2}{2}$ (D) None of these

38. If $y_0 = 1, f(x_0, y_0) = 1.2, f(x_1, y_1^{(0)}) = 1.9312, h = 0.3$, by modified Euler's formula $y_1^{(1)} =$

(A) 1.4696 (B) 1.2015
(C) 1.325 (D) 1.525

39. Using Euler's modified method, find a solution of the equation $\frac{dy}{dx} = x + |\sqrt{y}|$ with $y(0) = 1$ at $y(0.2)$.

(A) 1.3902 (B) 1.2309
(C) 1.3092 (D) None of these

40. Find k_1, by Runge–Kutta method of fourth order if $\frac{dy}{dx} = 2x + 3y^2$ and $y(0.1) = 1.1165$, $h = 0.1$.

(A) 0.3993 (B) 0.9393
(C) 0.3939 (D) None of these

Direction for questions 41 and 42:

41. Find $y(0.8)$ by Milne's predictor formula, given $\frac{dy}{dx} = x - y^2$, $y_2 = 0.0795$, $y(0.6) = 0.1762$, $y_0 = 0.0000$, $y_1^1 = 0.1996$, $y_2^1 = 0.3937$, $y_3^1 = 0.5689$, $h = 0.2$

(A) 0.9304 (B) 0.4930
(C) 0.3049 (D) None of these

42. For the above problem find $y(0.8)$ using Milne's corrector formula.

(A) 0.3046 (B) 0.4036
(C) 0.436 (D) None of these

Direction for questions 43 and 44:

43. Find using the Adams–Bashforth corrector formula $y(0.4)$, for the differential equation $\frac{dy}{dx} = \frac{1}{2}xy$, given $y(0.1) = 1.01$, $y(0.2) = 1.022$, $y(0.3) = 1.023$, $y_0^1 = 0$, $y_1^1 = 0.0505$, $y_2^1 = 0.1022$, $y_3^1 = 0.1535$.

(A) 1.5418 (B) 1.0410
(C) 1.4100 (D) None of these

44. For the above differential equation find $y(0.5)$ using Adams–Bashforth predictor formula.

(A) 1.00463 (B) 1.06463
(C) 1.00599 (D) None of these

45. The Runge–Kutta methods has the error of order ______.

(A) 1 (B) 3
(C) 5 (D) 2

Previous Years' Questions

1. Given that one root of the equation $x^3 - 10x^2 + 31x - 30 = 0$, is 5 the other two roots are **[GATE, 2007]**

(A) 2 and 3 (B) 2 and 4
(C) 3 and 4 (D) −2 and −3

2. The following equation needs to be numerically solved using the Newton–Raphson method $x^3 + 4x - 9 = 0$

The iterative equation for this purpose is (k indicates the iteration level) **[GATE, 2007]**

(A) $x_{k+1} = \dfrac{2x_k^3 + 9}{3x_k^2 + 4}$ (B) $x_{k+1} = \dfrac{3x_k^2 + 4}{2x_k^2 + 9}$

(C) $x_{k+1} = x_k - 3x_k^2 + 4$ (D) $x_{k+1} = \dfrac{4x_k^2 + 3}{9x_k^2 + 2}$

3. Three values of x and y are to be fitted in a straight line in the form $y = a + bx$ by the method of least squares. Given : $\Sigma x = 6$, $\Sigma y = 21$, $\Sigma x^2 = 14$, and $\Sigma xy = 46$ the values of a and b are respectively. **[GATE, 2008]**

(A) 2 and 3 (B) 1 and 2
(C) 2 and 1 (D) 3 and 2

4. The table below gives values of a function $F(x)$ obtained for values of x at intervals of 0.25.

X	0	0.25	0.5	0.75	1.0
***F*(*x*)**	1	0.9412	0.8	0.64	0.50

The value of the integral of the function between the limits 0 to 1 using Simpon's rule is **[GATE, 2010]**

(A) 0.7854 (B) 2.3562
(C) 3.1416 (D) 7.5000

5. The square root of a number N is to be obtained by applying the Newton–Raphson iterations to the equation $x^2 - N = 0$. If i denotes the iteration index, the correct iterative scheme will be **[GATE, 2011]**

(A) $x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right)$

(B) $x_{i+1} = \frac{1}{2}\left(x_i^2 + \frac{N}{x_i^2}\right)$

(C) $x_{i+1} = \frac{1}{2}\left(x_i + \frac{N^2}{x_i}\right)$

(D) $x_{i+1} = \frac{1}{2}\left(x_i - \frac{N}{x_i}\right)$

6. Find the magnitude of the error (correct to two decimal places) in the estimation of following integral using Simpson's $\frac{1}{3}$ rule. (Take the step length as 1)

$\int_0^4 (x^4 + 10)dx$ **[GATE, 2013]**

7. In Newton–Raphson iterative method, the initial guess value (x_{inj}) is considered as zero while finding the roots of the equation: $f(x) = -2 + 6x - 4x^2 + 0.5x^3$. The correction, Δx, to be added to x_{inj} in the first iteration is ______. **[GATE, 2015]**

8. The quadratic equation $x^2 - 4x + 4 = 0$ is to be solved numerically, starting with the initial guess $x_0 = 3$. The Newton–Raphson method is applied once to get a new estimate and then the Secant method is applied once using the initial guess and this new estimate. The estimated value of the root after the application of the Secant method is ______. **[GATE, 2015]**

9. For step-size, $\Delta x = 0.4$, the value of following integral using Simpson's $\frac{1}{3}$ rule is ________.

$$\int_0^{0.8} (0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5)\,dx$$

[GATE, 2015]

10. Newton–Raphson method is to be used to find root of equation $3x - e^x + \sin x = 0$. If the initial trial value for the root is taken as 0.333, the next approximation for the root would be ______. (note: answer up to three decimal) **[GATE, 2016]**

Answer Keys

Exercises

1. D	**2.** B	**3.** D	**4.** A	**5.** B	**6.** D	**7.** A	**8.** C	**9.** B	**10.** D
11. B	**12.** B	**13.** C	**14.** A	**15.** C	**16.** B	**17.** B	**18.** A	**19.** C	**20.** A
21. A	**22.** C	**23.** B	**24.** A	**25.** D	**26.** B	**27.** A	**28.** D	**29.** D	**30.** A
31. D	**32.** B	**33.** C	**34.** B	**35.** C	**36.** D	**37.** A	**38.** A	**39.** B	**40.** C
41. C	**42.** A	**43.** B	**44.** B	**45.** C					

Previous Years' Questions

1. A **2.** A **3.** D **4.** A **5.** A **6.** 0.52 to 0.55 **7.** 0.3 to 0.4
8. 2.32 to 2.34 **9.** 1.36 to 1.37 **10.** 0.360