CHAPTER XI.

PERMUTATIONS AND COMBINATIONS.

139. EACH of the *arrangements* which can be made by taking some or all of a number of things is called a **permutation.**

Each of the *groups* or *selections* which can be made by taking some or all of a number of things is called a **combination.**

Thus the *permutations* which can be made by taking the letters a, b, c, d two at a time are twelve in number, namely,

$$\begin{array}{cccccc} ab, & ac, & ad, & bc, & bd, & cd, \\ ba, & ca, & da, & cb, & db, & dc; \end{array}$$

each of these presenting a different *arrangement* of two letters.

The *combinations* which can be made by taking the letters a, b, c, d two at a time are six in number: namely,

$$ab, \quad ac, \quad ad, \quad bc, \quad bd, \quad cd;$$

each of these presenting a different *selection* of two letters.

From this it appears that in forming *combinations* we are only concerned with the number of things each selection contains; whereas in forming *permutations* we have also to consider the order of the things which make up each arrangement; for instance, if from four letters a, b, c, d we make a selection of three, such as abc, this single combination admits of being arranged in the following ways:

$$abc, \quad acb, \quad bca, \quad bac, \quad cab, \quad cba,$$

and so gives rise to six different permutations.

140. Before discussing the general propositions of this chapter there is an important principle which we proceed to explain and illustrate by a few numerical examples.

If one operation can be performed in m *ways, and (when it has been performed in any one of these ways) a second operation can then be performed in* n *ways; the number of ways of performing the two operations will be* $m \times n$.

If the first operation be performed in *any one* way, we can associate with this any of the n ways of performing the second operation: and thus we shall have n ways of performing the two operations without considering more than *one* way of performing the first; and so, corresponding to *each* of the m ways of performing the first operation, we shall have n ways of performing the two; hence altogether the number of ways in which the two operations can be performed is represented by the product $m \times n$.

Example 1. There are 10 steamers plying between Liverpool and Dublin; in how many ways can a man go from Liverpool to Dublin and return by a different steamer?

There are *ten* ways of making the first passage; and with each of these there is a choice of *nine* ways of returning (since the man is not to come back by the same steamer); hence the number of ways of making the two journeys is 10×9, or 90.

This principle may easily be extended to the case in which there are more than two operations each of which can be performed in a given number of ways.

Example 2. Three travellers arrive at a town where there are four hotels; in how many ways can they take up their quarters, each at a different hotel?

The first traveller has choice of four hotels, and when he has made his selection in any one way, the second traveller has a choice of three; therefore the first two can make their choice in 4×3 ways; and with any one such choice the third traveller can select his hotel in 2 ways; hence the required number of ways is $4 \times 3 \times 2$, or 24.

141. *To find the number of permutations of* n *dissimilar things taken* r *at a time.*

This is the same thing as finding the number of ways in which we can fill up r places when we have n different things at our disposal.

The first place may be filled up in n ways, for any one of the n things may be taken; when it has been filled up in any one of these ways, the second place can then be filled up in $n-1$ ways; and since each way of filling up the first place can be associated with each way of filling up the second, the number of ways in which the first two places can be filled up is given by the product $n(n-1)$. And when the first two places have been filled up in any way, the third place can be filled up in $n-2$ ways. And reasoning as before, the number of ways in which three places can be filled up is $n(n-1)(n-2)$.

Proceeding thus, and noticing that a new factor is introduced with each new place filled up, and that at any stage the number of factors is the same as the number of places filled up, we shall have the number of ways in which r places can be filled up equal to

$$n(n-1)(n-2)\ldots\ldots \text{to } r \text{ factors};$$

and the r^{th} factor is

$$n-(r-1), \quad \text{or} \quad n-r+1.$$

Therefore the number of permutations of n things taken r at a time is

$$n(n-1)(n-2)\ldots\ldots(n-r+1).$$

Cor. The number of permutations of n things taken all at a time is

$$n(n-1)(n-2)\ldots\ldots \text{to } n \text{ factors},$$

or

$$n(n-1)(n-2)\ldots\ldots 3 \,.\, 2 \,.\, 1.$$

It is usual to denote this product by the symbol $\underline{|n}$, which is read "factorial n." Also $n!$ is sometimes used for $\underline{|n}$.

142. We shall in future denote the number of permutations of n things taken r at a time by the symbol ${}^nP_r$, so that

$${}^nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1);$$

also

$${}^nP_n = \underline{|n}.$$

In working numerical examples it is useful to notice that the suffix in the symbol ${}^nP_r$ always denotes the number of factors in the formula we are using.

143. The number of permutations of n things taken r at a time may also be found in the following manner.

Let ${}^nP_r$ represent the number of permutations of n things taken r at a time.

Suppose we form all the permutations of n things taken $r-1$ at a time; the number of these will be ${}^nP_{r-1}$.

With *each of these* put one of the remaining $n-r+1$ things. Each time we do this we shall get one permutation of n things r at a time; and therefore the whole number of the permutations of n things r at a time is ${}^nP_{r-1} \times (n-r+1)$; that is,

$$ {}^nP_r = {}^nP_{r-1} \times (n-r+1). $$

By writing $r-1$ for r in this formula, we obtain

$$ {}^nP_{r-1} = {}^nP_{r-2} \times (n-r+2), $$

similarly,

$$ {}^nP_{r-2} = {}^nP_{r-3} \times (n-r+3), $$

....

$$ {}^nP_3 = {}^nP_2 \times (n-2), $$

$$ {}^nP_2 = {}^nP_1 \times (n-1), $$

$$ {}^nP_1 = n. $$

Multiply together the vertical columns and cancel like factors from each side, and we obtain

$$ {}^nP_r = n(n-1)(n-2) \ldots\ldots (n-r+1). $$

Example 1. Four persons enter a railway carriage in which there are six seats; in how many ways can they take their places?

The first person may seat himself in 6 ways; and then the second person in 5; the third in 4; and the fourth in 3; and since each of these ways may be associated with each of the others, the required answer is $6 \times 5 \times 4 \times 3$, or 360.

Example 2. How many different numbers can be formed by using six out of the nine digits 1, 2, 3,...9?

Here we have 9 different things and we have to find the number of permutations of them taken 6 at a time;

$$ \therefore \text{ the required result} = {}^9P_6 $$
$$ = 9 \times 8 \times 7 \times 6 \times 5 \times 4 $$
$$ = 60480. $$

144. *To find the number of combinations of* n *dissimilar things taken* r *at a time.*

Let ${}^nC_r$ denote the required number of combinations.

Then each of these combinations consists of a group of r dissimilar things which can be arranged among themselves in $\underline{|r}$ ways. [Art. 142.]

Hence ${}^nC_r \times \underline{|r}$ is equal to the number of *arrangements* of n things taken r at a time; that is,

$$ {}^nC_r \times \underline{|r} = {}^nP_r $$

$$ = n(n-1)(n-2)\ldots(n-r+1); $$

$$ \therefore\ {}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}} \quad \ldots\ldots\ldots (1). $$

Cor. This formula for ${}^nC_r$ may also be written in a different form; for if we multiply the numerator and the denominator by $\underline{|n-r}$ we obtain

$$ \frac{n(n-1)(n-2)\ldots(n-r+1)\times\underline{|n-r}}{\underline{|r}\,\underline{|n-r}}. $$

The numerator now consists of the product of all the natural numbers from n to 1;

$$ \therefore\ {}^nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2). $$

It will be convenient to remember both these expressions for ${}^nC_r$, using (1) in all cases where a numerical result is required, and (2) when it is sufficient to leave it in an algebraical shape.

Note. If in formula (2) we put $r=n$, we have

$$ {}^nC_n = \frac{\underline{|n}}{\underline{|n}\,\underline{|0}} = \frac{1}{\underline{|0}}; $$

but ${}^nC_n=1$, so that if the formula is to be true for $r=n$, the symbol $\underline{|0}$ must be considered as equivalent to 1.

Example. From 12 books in how many ways can a selection of 5 be made, (1) when one specified book is always included, (2) when one specified book is always excluded?

(1) Since the specified book is to be included in every selection, we have only to choose 4 out of the remaining 11.

Hence the number of ways $= {}^{11}C_4$

$$ = \frac{11\times10\times9\times8}{1\times2\times3\times4} $$

$$ = 330. $$

(2) Since the specified book is always to be excluded, we have to select the 5 books out of the remaining 11.

Hence the number of ways $= {}^{11}C_5$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4 \times 5}$$

$$= 462.$$

145. *The number of combinations of* n *things* r *at a time is equal to the number of combinations of* n *things* n − r *at a time.*

In making all the possible combinations of n things, to each group of r things we select, there is left a corresponding group of $n-r$ things; that is, the number of combinations of n things r at a time is the same as the number of combinations of n things $n-r$ at a time;

$$\therefore\ {}^nC_r = {}^nC_{n-r}.$$

The proposition may also be proved as follows:

$${}^nC_{n-r} = \frac{\lfloor n}{\lfloor n-r \,\lfloor n-(n-r)} \qquad \text{[Art. 144.]}$$

$$= \frac{\lfloor n}{\lfloor n-r \,\lfloor r}$$

$$= {}^nC_r.$$

Such combinations are called *complementary*.

NOTE. Put $r=n$, then ${}^nC_0 = {}^nC_n = 1$.

The result we have just proved is useful in enabling us to abridge arithmetical work.

Example. Out of 14 men in how many ways can an eleven be chosen?

The required number $= {}^{14}C_{11}$

$$= {}^{14}C_3$$

$$= \frac{14 \times 13 \times 12}{1 \times 2 \times 3}$$

$$= 364.$$

If we had made use of the formula ${}^{14}C_{11}$, we should have had to reduce an expression whose numerator and denominator each contained 11 factors.

146. *To find the number of ways in which* m + n *things can be divided into two groups containing* m *and* n *things respectively.*

This is clearly equivalent to finding the number of combinations of $m+n$ things m at a time, for every time we select one group of m things we leave a group of n things behind.

Thus the required number $= \dfrac{|\underline{m+n}}{|\underline{m}\ |\underline{n}}$.

NOTE. If $n=m$, the groups are equal, and in this case the number of *different* ways of subdivision is $\dfrac{|\underline{2m}}{|\underline{m}\ |\underline{m}\ |\underline{2}}$; for in any one way it is possible to interchange the two groups without obtaining a new distribution.

147. *To find the number of ways in which* m + n + p *things can be divided into three groups containing* m, n, p *things severally.*

First divide $m+n+p$ things into two groups containing m and $n+p$ things respectively: the number of ways in which this can be done is $\dfrac{|\underline{m+n+p}}{|\underline{m}\ |\underline{n+p}}$.

Then the number of ways in which the group of $n+p$ things can be divided into two groups containing n and p things respectively is $\dfrac{|\underline{n+p}}{|\underline{n}\ |\underline{p}}$.

Hence the number of ways in which the subdivision into three groups containing m, n, p things can be made is

$$\frac{|\underline{m+n+p}}{|\underline{m}\ |\underline{n+p}} \times \frac{|\underline{n+p}}{|\underline{n}\ |\underline{p}}, \text{ or } \frac{|\underline{m+n+p}}{|\underline{m}\ |\underline{n}\ |\underline{p}}.$$

NOTE. If we put $n=p=m$, we obtain $\dfrac{|\underline{3m}}{|\underline{m}\ |\underline{m}\ |\underline{m}}$; but this formula regards as different all the possible orders in which the three groups can occur in any one mode of subdivision. And since there are $|\underline{3}$ such orders corresponding to each mode of subdivision, the number of *different* ways in which subdivision into three *equal* groups can be made is $\dfrac{|\underline{3m}}{|\underline{m}\ |\underline{m}\ |\underline{m}\ |\underline{3}}$.

Example. The number of ways in which 15 recruits can be divided into three equal groups is $\dfrac{|\underline{15}}{|\underline{5}\ |\underline{5}\ |\underline{5}\ |\underline{3}}$; and the number of ways in which they can be drafted into three different regiments, five into each, is $\dfrac{|\underline{15}}{|\underline{5}\ |\underline{5}\ |\underline{5}}$.

148. In the examples which follow it is important to notice that the formula for *permutations* should not be used until the suitable *selections* required by the question have been made.

Example 1. From 7 Englishmen and 4 Americans a committee of 6 is to be formed; in how many ways can this be done, (1) when the committee contains exactly 2 Americans, (2) at least 2 Americans?

(1) We have to choose 2 Americans and 4 Englishmen.

The number of ways in which the Americans can be chosen is ${}^4C_2$; and the number of ways in which the Englishmen can be chosen is ${}^7C_4$. Each of the first groups can be associated with each of the second; hence the required number of ways $= {}^4C_2 \times {}^7C_4$

$$= \frac{4!}{2!\,2!} \times \frac{7!}{4!\,3!}$$

$$= \frac{7!}{2!\,2!\,3!} = 210.$$

(2) The committee may contain 2, 3, or 4 Americans.

We shall exhaust all the suitable combinations by forming all the groups containing 2 Americans and 4 Englishmen; then 3 Americans and 3 Englishmen; and lastly 4 Americans and 2 Englishmen.

The *sum* of the three results will give the answer. Hence the required number of ways $= {}^4C_2 \times {}^7C_4 + {}^4C_3 \times {}^7C_3 + {}^4C_4 \times {}^7C_2$

$$= \frac{4!}{2!\,2!} \times \frac{7!}{4!\,3!} + \frac{4!}{3!} \times \frac{7!}{3!\,4!} + 1 \times \frac{7!}{2!\,5!}$$

$$= 210 + 140 + 21 = 371.$$

In this Example we have only to make use of the suitable formulæ for *combinations*, for we are not concerned with the possible arrangements of the members of the committee among themselves.

Example 2. Out of 7 consonants and 4 vowels, how many words can be made each containing 3 consonants and 2 vowels?

The number of ways of choosing the three consonants is ${}^7C_3$, and the number of ways of choosing the 2 vowels is ${}^4C_2$; and since each of the first groups can be associated with each of the second, the number of combined groups, each containing 3 consonants and 2 vowels, is ${}^7C_3 \times {}^4C_2$.

Further, each of these groups contains 5 letters, which may be arranged among themselves in $5!$ ways. Hence

the required number of words $= {}^7C_3 \times {}^4C_2 \times 5!$

$$= \frac{7!}{3!\,4!} \times \frac{4!}{2!\,2!} \times 5!$$

$$= 5 \times 7!$$

$$= 25200.$$

Example 3. How many words can be formed out of the letters *article*, so that the vowels occupy the even places?

Here we have to put the 3 vowels in 3 specified places, and the 4 consonants in the 4 remaining places; the first operation can be done in $\lfloor 3$ ways, and the second in $\lfloor 4$. Hence

the required number of words $= \lfloor 3 \times \lfloor 4$

$= 144.$

In this Example the formula for permutations is immediately applicable, because by the statement of the question there is but one way of choosing the vowels, and one way of choosing the consonants.

EXAMPLES XI. a.

1. In how many ways can a consonant and a vowel be chosen out of the letters of the word *courage?*

2. There are 8 candidates for a Classical, 7 for a Mathematical, and 4 for a Natural Science Scholarship. In how many ways can the Scholarships be awarded?

3. Find the value of ${}^8P_7$, ${}^{25}P_5$, ${}^{24}C_4$, ${}^{19}C_{14}$.

4. How many different arrangements can be made by taking 5 of the letters of the word *equation?*

5. If four times the number of permutations of n things 3 together is equal to five times the number of permutations of $n-1$ things 3 together, find n.

6. How many permutations can be made out of the letters of the word *triangle?* How many of these will begin with t and end with e?

7. How many different selections can be made by taking four of the digits 3, 4, 7, 5, 8, 1? How many different numbers can be formed with four of these digits?

8. If ${}^{2n}C_3 : {}^nC_2 = 44 : 3$, find n.

9. How many changes can be rung with a peal of 5 bells?

10. How many changes can be rung with a peal of 7 bells, the tenor always being last?

11. On how many nights may a watch of 4 men be drafted from a crew of 24, so that no two watches are identical? On how many of these would any one man be taken?

12. How many arrangements can be made out of the letters of the word *draught*, the vowels never being separated?

13. In a town council there are 25 councillors and 10 aldermen; how many committees can be formed each consisting of 5 councillors and 3 aldermen?

14. Out of the letters A, B, C, p, q, r how many arrangements can be made (1) beginning with a capital, (2) beginning and ending with a capital?

15. Find the number of combinations of 50 things 46 at a time.

16. If ${}^nC_{12} = {}^nC_8$, find ${}^nC_{17}$, ${}^{22}C_n$.

17. In how many ways can the letters of the word *vowels* be arranged, if the letters *oe* can only occupy odd places?

18. From 4 officers and 8 privates, in how many ways can 6 be chosen (1) to include exactly one officer, (2) to include at least one officer?

19. In how many ways can a party of 4 or more be selected from 10 persons?

20. If ${}^{18}C_r = {}^{18}C_{r+2}$, find ${}^rC_5$.

21. Out of 25 consonants and 5 vowels how many words can be formed each consisting of 2 consonants and 3 vowels?

22. In a library there are 20 Latin and 6 Greek books; in how many ways can a group of 5 consisting of 3 Latin and 2 Greek books be placed on a shelf?

23. In how many ways can 12 things be divided equally among 4 persons?

24. From 3 capitals, 5 consonants, and 4 vowels, how many words can be made, each containing 3 consonants and 2 vowels, and beginning with a capital?

25. At an election three districts are to be canvassed by 10, 15, and 20 men respectively. If 45 men volunteer, in how many ways can they be allotted to the different districts?

26. In how many ways can 4 Latin and 1 English book be placed on a shelf so that the English book is always in the middle, the selection being made from 7 Latin and 3 English books?

27. A boat is to be manned by eight men, of whom 2 can only row on bow side and 1 can only row on stroke side; in how many ways can the crew be arranged?

28. There are two works each of 3 volumes, and two works each of 2 volumes; in how many ways can the 10 books be placed on a shelf so that volumes of the same work are not separated?

29. In how many ways can 10 examination papers be arranged so that the best and worst papers never come together?

30. An eight-oared boat is to be manned by a crew chosen from 11 men, of whom 3 can steer but cannot row, and the rest can row but cannot steer. In how many ways can the crew be arranged, if two of the men can only row on bow side?

31. Prove that the number of ways in which p positive and n negative signs may be placed in a row so that no two negative signs shall be together is ${}^{p+1}C_n$.

32. If ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$, find r.

33. How many different signals can be made by hoisting 6 differently coloured flags one above the other, when any number of them may be hoisted at once?

34. If ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225 : 11$, find r.

149. Hitherto, in the formulæ we have proved, the things have been regarded as *unlike*. Before considering cases in which some one or more sets of things may be *like*, it is necessary to point out exactly in what sense the words *like* and *unlike* are used. When we speak of things being *dissimilar*, *different*, *unlike*, we imply that the things are *visibly unlike*, so as to be easily distinguishable from each other. On the other hand we shall always use the term *like* things to denote such as are alike to the eye and cannot be distinguished from each other. For instance, in Ex. 2, Art. 148, the consonants and the vowels may be said each to consist of a group of things united by a common characteristic, and thus in a certain sense to be of the same kind; but they cannot be regarded as like things, because there is an individuality existing among the things of each group which makes them easily distinguishable from each other. Hence, in the final stage of the example we considered each group to consist of five *dissimilar* things and therefore capable of $\lfloor 5$ arrangements among themselves. [Art. 141 Cor.]

150. Suppose we have to find all the possible ways of arranging 12 books on a shelf, 5 of them being Latin, 4 English, and the remainder in different languages.

The books in each language may be regarded as belonging to one class, united by a common characteristic; but if they were distinguishable from each other, the number of permutations would be $\lfloor 12$, since for the purpose of arrangement among themselves they are essentially different.

If, however, the books in the same language are not distinguishable from each other, we should have to find the number of ways in which 12 things can be arranged among themselves, when 5 of them are exactly alike of one kind, and 4 exactly alike of a second kind: a problem which is not directly included in any of the cases we have previously considered.

151. *To find the number of ways in which* n *things may be arranged among themselves, taking them all at a time, when* p *of the things are exactly alike of one kind,* q *of them exactly alike of another kind,* r *of them exactly alike of a third kind, and the rest all different.*

Let there be n letters; suppose p of them to be a, q of them to be b, r of them to be c, and the rest to be unlike.

Let x be the required number of permutations; then if in *any one* of these permutations the p letters a were replaced by p unlike letters different from any of the rest, from this single permutation, without altering the position of any of the remaining letters, we could form $\underline{|p}$ new permutations. Hence if this change were made in each of the x permutations we should obtain $x \times \underline{|p}$ permutations.

Similarly, if the q letters b were replaced by q unlike letters, the number of permutations would be

$$x \times \underline{|p} \times \underline{|q}.$$

In like manner, by replacing the r letters c by r unlike letters, we should finally obtain $x \times \underline{|p} \times \underline{|q} \times \underline{|r}$ permutations.

But the things are now all different, and therefore admit of $\underline{|n}$ permutations among themselves. Hence

$$x \times \underline{|p} \times \underline{|q} \times \underline{|r} = \underline{|n};$$

that is,

$$x = \frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}};$$

which is the required number of permutations.

Any case in which the things are not all different may be treated similarly.

Example 1. How many different permutations can be made out of the letters of the word *assassination* taken all together?

We have here 13 letters of which 4 are s, 3 are a, 2 are i, and 2 are n. Hence the number of permutations

$$= \frac{13!}{4!\,3!\,2!\,2!}$$

$$= 13 . 11 . 10 . 9 . 8 . 7 . 3 . 5$$

$$= 1001 \times 10800 = 10810800.$$

Example 2. How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1, so that the odd digits always occupy the odd places?

The odd digits 1, 3, 3, 1 can be arranged in their four places in

$$\frac{4!}{2!\,2!} \text{ ways} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The even digits 2, 4, 2 can be arranged in their three places in

$$\frac{3!}{2!} \text{ ways} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Each of the ways in (1) can be associated with each of the ways in (2).

Hence the required number $= \frac{4!}{2!\,2!} \times \frac{3!}{2!} = 6 \times 3 = 18.$

152. *To find the number of permutations of* n *things* r *at a time, when each thing may be repeated once, twice,......up to* r *times in any arrangement.*

Here we have to consider the number of ways in which r places can be filled up when we have n different things at our disposal, each of the n things being used as often as we please in any arrangement.

The first place may be filled up in n ways, and, when it has been filled up in any one way, the second place may also be filled up in n ways, since we are not precluded from using the same thing again. Therefore the number of ways in which the first two places can be filled up is $n \times n$ or n^2. The third place can also be filled up in n ways, and therefore the first three places in n^3 ways.

Proceeding in this manner, and noticing that at any stage the index of n is always the same as the number of places filled up, we shall have the number of ways in which the r places can be filled up equal to n^r.

Example. In how many ways can 5 prizes be given away to 4 boys, when each boy is eligible for all the prizes?

Any one of the prizes can be given in 4 ways; and then any one of the remaining prizes can also be given in 4 ways, since it may be obtained by the boy who has already received a prize. Thus two prizes can be given away in 4^2 ways, three prizes in 4^3 ways, and so on. Hence the 5 prizes can be given away in 4^5, or 1024 ways.

153. *To find the total number of ways in which it is possible to make a selection by taking some or all of* n *things.*

Each thing may be dealt with in two ways, for it may either be taken or left; and since either way of dealing with any one thing may be associated with either way of dealing with each one of the others, the number of selections is

$$2 \times 2 \times 2 \times 2 \ldots\ldots \text{to } n \text{ factors.}$$

But this includes the case in which all the things are left, therefore, rejecting this case, the total number of ways is $2^n - 1$.

This is often spoken of as "the total number of combinations" of n things.

Example. A man has 6 friends; in how many ways may he invite one or more of them to dinner?

He has to select some or all of his 6 friends; and therefore the number of ways is $2^6 - 1$, or 63.

This result can be verified in the following manner.

The guests may be invited singly, in twos, threes,......; therefore the number of selections $= {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$

$$= 6 + 15 + 20 + 15 + 6 + 1 = 63.$$

154. *To find for what value of* r *the number of combinations of* n *things* r *at a time is greatest.*

Since $${}^nC_r = \frac{n(n-1)(n-2)\ldots\ldots(n-r+2)(n-r+1)}{1 \cdot 2 \cdot 3 \ldots\ldots (r-1)\, r},$$

and $${}^nC_{r-1} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{1 \cdot 2 \cdot 3 \ldots\ldots (r-1)};$$

$$\therefore\quad {}^nC_r = {}^nC_{r-1} \times \frac{n-r+1}{r}.$$

The multiplying factor $\dfrac{n-r+1}{r}$ may be written $\dfrac{n+1}{r} - 1$, which shews that it decreases as r increases. Hence as r receives the values 1, 2, 3...... in succession, ${}^nC_r$ is continually increased until $\frac{n+1}{r} - 1$ becomes equal to 1 or less than 1.

Now $$\frac{n+1}{r} - 1 > 1,$$

so long as $$\frac{n+1}{r} > 2;$$

that is, $$\frac{n+1}{2} > r.$$

We have to choose the greatest value of r consistent with this inequality.

(1) Let n be even, and equal to $2m$; then

$$\frac{n+1}{2} = \frac{2m+1}{2} = m + \frac{1}{2};$$

and for all values of r up to m inclusive this is greater than r. Hence by putting $r = m = \frac{n}{2}$, we find that the greatest number of combinations is ${}^nC_{\frac{n}{2}}$.

(2) Let n be odd, and equal to $2m+1$; then

$$\frac{n+1}{2} = \frac{2m+2}{2} = m + 1;$$

and for all values of r up to m inclusive this is greater than r; but when $r = m+1$ the multiplying factor becomes equal to 1, and

$${}^nC_{m+1} = {}^nC_m; \text{ that is, } {}^nC_{\frac{n+1}{2}} = {}^nC_{\frac{n-1}{2}};$$

and therefore the number of combinations is greatest when the things are taken $\frac{n+1}{2}$, or $\frac{n-1}{2}$ at a time; the result being the same in the two cases.

155. The formula for the number of combinations of n things r at a time may be found without assuming the formula for the number of permutations.

Let ${}^nC_r$ denote the number of combinations of n things taken r at a time; and let the n things be denoted by the letters $a, b, c, d, \ldots\ldots$.

Take away a; then with the remaining letters we can form ${}^{n-1}C_{r-1}$ combinations of $n-1$ letters taken $r-1$ at a time. With each of these write a; thus we see that of the combinations of n things r at a time, the number of those which contain a is ${}^{n-1}C_{r-1}$; similarly the number of those which contain b is ${}^{n-1}C_{r-1}$; and so for each of the n letters.

Therefore $n \times {}^{n-1}C_{r-1}$ is equal to the number of combinations r at a time which contain a, together with those that contain b, those that contain c, and so on.

But by forming the combinations in this manner, each particular one will be repeated r times. For instance, if $r=3$, the combination abc will be found among those containing a, among those containing b, and among those containing c. Hence

$$ {}^{n}C_{r} = {}^{n-1}C_{r-1} \times \frac{n}{r}. $$

By writing $n-1$ and $r-1$ instead of n and r respectively,

$$ {}^{n-1}C_{r-1} = {}^{n-2}C_{r-2} \times \frac{n-1}{r-1}. $$

Similarly, $$ {}^{n-2}C_{r-2} = {}^{n-3}C_{r-3} \times \frac{n-2}{r-2}, $$

....

$$ {}^{n-r+2}C_{2} = {}^{n-r+1}C_{1} \times \frac{n-r+2}{2}; $$

and finally, $$ {}^{n-r+1}C_{1} = n-r+1. $$

Multiply together the vertical columns and cancel like factors from each side; thus

$$ {}^{n}C_{r} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r(r-1)(r-2)\ldots\ldots 1}. $$

156. *To find the total number of ways in which it is possible to make a selection by taking some or all out of* p + q + r + ... *things, whereof* p *are alike of one kind,* q *alike of a second kind,* r *alike of a third kind; and so on.*

The p things may be disposed of in $p+1$ ways; for we may take 0, 1, 2, 3, ... p of them. Similarly the q things may be disposed of in $q+1$ ways; the r things in $r+1$ ways; and so on.

Hence the number of ways in which all the things may be disposed of is $(p+1)(q+1)(r+1)\ldots\ldots$.

But this includes the case in which none of the things are taken; therefore, rejecting this case, the total number of ways is

$$(p+1)(q+1)(r+1)\ldots\ldots-1.$$

157. A general formula expressing the number of permutations, or combinations, of n things taken r at a time, when the things are not all different, may be somewhat complicated; but a particular case may be solved in the following manner.

Example. Find the number of ways in which (1) a selection, (2) an arrangement, of four letters can be made from the letters of the word *proportion.*

There are 10 letters of six different sorts, namely $o, o, o;\ p, p;\ r, r;\ t;\ i;\ n$.

In finding groups of four these may be classified as follows:

(1) Three alike, one different.

(2) Two alike, two others alike.

(3) Two alike, the other two different.

(4) All four different.

(1) The selection can be made in 5 ways; for each of the five letters, p, r, t, i, n, can be taken with the single group of the three like letters o.

(2) The selection can be made in ${}^3C_2$ ways; for we have to choose two out of the three pairs $o, o;\ p, p;\ r, r$. This gives 3 selections.

(3) This selection can be made in 3×10 ways; for we select one of the 3 pairs, and then two from the remaining 5 letters. This gives 30 selections.

(4) This selection can be made in ${}^6C_4$ ways, as we have to take 4 different letters to choose from the six o, p, r, t, i, n. This gives 15 selections.

Thus the total number of selections is $5+3+30+15$; that is, 53.

In finding the different arrangements of 4 letters we have to permute in all possible ways each of the foregoing groups.

(1) gives rise to $5\times\dfrac{|\underline{4}}{|\underline{3}}$, or 20 arrangements.

(2) gives rise to $3\times\dfrac{|\underline{4}}{|\underline{2}\,|\underline{2}}$, or 18 arrangements.

(3) gives rise to $30\times\dfrac{|\underline{4}}{|\underline{2}}$, or 360 arrangements.

(4) gives rise to $15\times|\underline{4}$, or 360 arrangements.

Thus the total number of arrangements is $20+18+360+360$; that is, 758.

EXAMPLES. XI. b.

1. Find the number of arrangements that can be made out of the letters of the words

(1) *independence*, (2) *superstitious*,
(3) *institutions.*

2. In how many ways can 17 billiard balls be arranged, if 7 of them are black, 6 red, and 4 white?

3. A room is to be decorated with fourteen flags; if 2 of them are blue, 3 red, 2 white, 3 green, 2 yellow, and 2 purple, in how many ways can they be hung?

4. How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3?

5. Find the number of arrangements which can be made out of the letters of the word *algebra*, without altering the relative positions of vowels and consonants.

6. On three different days a man has to drive to a railway station, and he can choose from 5 conveyances; in how many ways can he make the three journeys?

7. I have counters of n different colours, red, white, blue,......; in how many ways can I make an arrangement consisting of r counters, supposing that there are at least r of each different colour?

8. In a steamer there are stalls for 12 animals, and there are cows, horses, and calves (not less than 12 of each) ready to be shipped; in how many ways can the shipload be made?

9. In how many ways can n things be given to p persons, when there is no restriction as to the number of things each may receive?

10. In how many ways can five things be divided between two persons?

11. How many different arrangements can be made out of the letters in the expression $a^3b^2c^4$ when written at full length?

12. A letter lock consists of three rings each marked with fifteen different letters; find in how many ways it is possible to make an unsuccessful attempt to open the lock.

13. Find the number of triangles which can be formed by joining three angular points of a quindecagon.

14. A library has a copies of one book, b copies of each of two books, c copies of each of three books, and single copies of d books. In how many ways can these books be distributed, if all are out at once?

15. How many numbers less than 10000 can be made with the eight digits 1, 2, 3, 0, 4, 5, 6, 7?

16. In how many ways can the following prizes be given away to a class of 20 boys: first and second Classical, first and second Mathematical, first Science, and first French?

17. A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest; what is the total number of signals that can be made?

18. In how many ways can 7 persons form a ring? In how many ways can 7 Englishmen and 7 Americans sit down at a round table, no two Americans being together?

19. In how many ways is it possible to draw a sum of money from a bag containing a sovereign, a half-sovereign, a crown, a florin, a shilling, a penny, and a farthing?

20. From 3 cocoa nuts, 4 apples, and 2 oranges, how many selections of fruit can be made, taking at least one of each kind?

21. Find the numbér of different ways of dividing mn things into n equal groups.

22. How many signals can be made by hoisting 4 flags of different colours one above the other, when any number of them may be hoisted at once? How many with 5 flags?

23. Find the number of permutations which can be formed out of the letters of the word *series* taken three together?

24. There are p points in a plane, no three of which are in the same straight line with the exception of q, which are all in the same straight line; find the number (1) of straight lines, (2) of triangles which result from joining them.

25. There are p points in space, no four of which are in the same plane with the exception of q, which are all in the same plane; find how many planes there are each containing three of the points.

26. There are n different books, and p copies of each; find the number of ways in which a selection can be made from them.

27. Find the number of selections and of arrangements that can be made by taking 4 letters from the word *expression.*

28. How many permutations of 4 letters can be made out of the letters of the word *examination?*

29. Find the sum of all numbers greater than 10000 formed by using the digits 1, 3, 5, 7, 9, no digit being repeated in any number.

30. Find the sum of all numbers greater than 10000 formed by using the digits 0, 2, 4, 6, 8, no digit being repeated in any number.

31. If of $p+q+r$ things p be alike, and q be alike, and the rest different, shew that the total number of combinations is

$$(p+1)(q+1)2^r-1.$$

32. Shew that the number of permutations which can be formed from $2n$ letters which are either a's or b's is greatest when the number of a's is equal to the number of b's.

33. If the $n+1$ numbers $a, b, c, d, \ldots\ldots$ be all different, and each of them a prime number, prove that the number of different factors of the expression $a^m bcd\ldots\ldots$ is $(m+1)2^n-1$.